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Schur complement-based error bounds for linear complementarity problems of B_π^R-matrices

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Abstract. An error bound for B_π^R-matrices linear complementarity problems (LCPs) is given by García-Esnaola and Peña in the paper (*Calcolo*, 54(3), 813-822, 2017). However, this bound is not effective for B_π^R-matrices with a non-positive vector π. In this paper, based on the Schur complement, some error bounds involving a parameter for LCPs of B_π^R-matrices with a non-positive vector π are presented, and the optimal values of these error bounds are also determined. Numerical examples are performed to illustrate the effectiveness of the obtained bounds.

1. Introduction

A real square matrix A is called a P-matrix if all its principal minors are positive [1]. It is well-known that the class of P-matrices has important applications in many practical problems, especially in linear complementarity problems [2]. Here, the linear complementarity problem is to find a vector $x \in \mathbb{R}^n$ such that

$$Mx + q \geq 0, \ \ x \geq 0, \ \ (Mx+q)^T x = 0 \tag{1}$$

or to show that no such vector x exists, where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The problem (1) is usually denoted by LCP(M, q), and often arises from the various scientific computing, economics, and engineering areas such as quadratic programming, the Nash equilibrium point of a bimatrix game, traffic equilibriums, for details, see [1, 2].

For the LCP(M, q), it has a unique solution for any $q \in \mathbb{R}^n$ if and only if M is a P-matrix, and the estimation problem of the corresponding error bound has received great attention in recent years and has been studied

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extensively [3, 4, 8, 10, 11, 13–16, 18, 21, 22, 24]. Among them, an important error bound for the LCP(M, q) is discovered by Chen and Xiang in [3]:

$$\|x - x^*\|_\infty \le \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where x^* is the solution of the LCP(M, q), $D = diag(d_i)$ with $0 \le d_i \le 1$ for all $i \in N := \{1, \dots, n\}$, and the min operator $r(x) = \min(x, Mx + q)$ denotes the componentwise minimum of two vectors. Later, to avoid the high-cost computations of the inverse matrix, many researchers focused on the estimation of

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty, \tag{2}$$

and derived various refined error bounds for the LCP(M, q) when M is a subclass of P-matrices, see [5–7, 9, 12, 19, 23, 25] and references therein.

In [17], Neumann et al., motivated by conditions that arise from results on mean first passage times matrices in Markov chains, defined the so-called class of B_π^R-matrices, which is a subclass of P-matrices and contains B-matrices.

Definition 1.1. *Let $\pi = [\pi_1, \dots, \pi_n]^T$ be a vector satisfying $0 < \sum_{j=1}^n \pi_j \le 1$, $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a matrix with positive row sums, and let $R = [R_1, \dots, R_n]^T$ be the vector formed by the row sums of M. Then M is a B_π^R-matrix if, for each $i \in N$,*

$$\pi_j R_i > m_{ij}, \text{ for all } j \ne i.$$

Subsequently, García-Esnaola and Peña in [10] provided an error bound for the LCP(M, q) when M is a B_π^R-matrix.

Theorem 1.2. [10] *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a positive vector $\pi = [\pi_1, \dots, \pi_n]^T$ and let $M = B^+(\varepsilon) + C(\varepsilon)$, where*

$$B^+(\varepsilon) = \begin{bmatrix} m_{11}-\pi_1 R_1 & \cdots & m_{1j}-(\pi_j-\varepsilon)R_1 & \cdots & m_{1n}-\pi_n R_1 \\ \vdots & & \vdots & & \vdots \\ m_{n1}-\pi_1 R_n & \cdots & m_{nj}-(\pi_j-\varepsilon)R_n & \cdots & m_{nn}-\pi_n R_n \end{bmatrix} =: [b_{ij}]$$

with ε is a positive integer satisfying $\pi_j - \varepsilon > 0$ and

$$m_{ij} - (\pi_j - \varepsilon)R_i < 0, \ \forall\, i \ne j, \ \text{for some } j \in N.$$

Then

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \le \frac{\max\limits_{i\in N}\{\frac{1}{\pi_i} - 1\}}{\min\{\beta(\varepsilon), 1\}}, \tag{3}$$

where $\beta(\varepsilon) := \min\limits_{i\in N}\{\beta_i\}$ and $\beta_i := b_{ii} - \sum\limits_{j\ne i} |b_{ij}|$.

Observe that the bound (3) in Theorem 1.2 works only for B_π^R-matrices with a positive vector π, that is, the bound (3) is not effective for B_π^R-matrices with a non-positive vector π, and to the best of the authors' knowledge, for the later case, the corresponding error bound for the LCP(M, q) remains unclear. In this paper, by using the Schur complement, we derive error bounds involving a parameter ω for linear complementarity problems when the matrix involved is a B_π^R-matrix for a non-positive vector π, and then determine completely the optimal values of these error bounds by using the monotonicity of functions of this parameter. Numerical examples demonstrate the effectiveness of our theoretical results.

2. Error bounds for LCPs of B_π^R-matrices with a non-positive vector π

We start with some definitions. Let I be the identity matrix of order n. A Z-matrix is a real matrix whose off-diagonal elements are non-positive. A matrix $M = [m_{ij}] \in \mathbb{C}^{n\times n}$ is a strictly diagonally dominant (SDD) matrix if $|m_{ii}| > \sum_{j\neq i} |m_{ij}|$ for all $i, j \in N$ [1].

Given a B_π^R-matrix with a non-positive vector π, since $0 < \sum_{i=1}^n \pi_i \leq 1$, it follows that there exists an index $j \in N$ such that $\pi_j > 0$. Therefore, we divide the following three cases to bound (2) when M is a B_π^R-matrix, and for other cases of a non-positive π, the corresponding error bounds can similarly be analyzed.

Case (a) $\pi = [\pi_1, \ldots, \pi_n]^T$ is a vector with only $\pi_i \leq 0$ for all $i \in \{1, 2, \ldots, s\} \subset N$;
Case (b) $\pi = [\pi_1, \ldots, \pi_n]^T$ is a vector with only $\pi_i \leq 0$ for some $i \in N$;
Case (c) $\pi = [\pi_1, \ldots, \pi_n]^T$ is a vector with only $\pi_s \leq 0,\ \pi_t \leq 0$ for $s, t \in N$ and $s < t$.

2.1. *Error bounds for case (a)*

Consider a B_π^R-matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ for a vector $\pi = [\pi_1, \ldots, \pi_n]^T$ only with $\pi_i \leq 0$ for all $i \in \{1, 2, \ldots, s\} \subset N$, we can decompose M as

$$M = B^+(\omega) + C(\omega), \tag{4}$$

where

$$B^+(\omega) = \begin{bmatrix} m_{11} & \cdots & m_{1s} & m_{1,s+1} - \pi_{s+1}R_1 & \cdots & m_{1,j_0} - \omega R_1 & \cdots & m_{1n} - \pi_n R_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ m_{n1} & \cdots & m_{ns} & m_{n,s+1} - \pi_{s+1}R_n & \cdots & m_{n,j_0} - \omega R_n & \cdots & m_{nn} - \pi_n R_n \end{bmatrix}$$

and

$$C(\omega) = \begin{bmatrix} 0 & \cdots & 0 & \pi_{s+1}R_1 & \cdots & \omega R_1 & \cdots & \pi_n R_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \pi_{s+1}R_n & \cdots & \omega R_n & \cdots & \pi_n R_n \end{bmatrix}$$

with ω is an adjustable parameter.

The following lemma provides that $B^+(\omega)$ is an SDD Z-matrix with positive diagonal entries by selecting ω.

Lemma 2.1. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a vector $\pi = [\pi_1, \ldots, \pi_n]^T$ only with $\pi_i \leq 0$ for all $i \in \{1, 2, \ldots, s\} \subset N$, and $B^+(\omega)$ be the matrix of (4). If there exists an index $j_0 \in \{s+1, \ldots, n\}$ such that*

$$\max_{i\neq j_0} \left\{ \frac{m_{i,j_0}}{R_i} \right\} < 1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i,$$

then, for each $\omega \in [\max_{i\neq j_0}\{\frac{m_{i,j_0}}{R_i}\}, 1 - \sum_{i=s+1,i\neq j_0}^{n} \pi_i)$, $B^+(\omega)$ is an SDD Z-matrix with positive diagonal entries.

Proof. By the assumptions and Definition 1.1, it follows that for each $i \in N$,

$$\pi_j R_i > m_{ij},\ \forall\ j \neq i,\ j \in N \text{ and } m_{ik} < 0,\ \forall\ k \neq i,\ k \in \{1, 2, \ldots, s\},$$

which together with $\omega \geq \max_{i\neq j_0}\{\frac{m_{i,j_0}}{R_i}\}$ implies that $B^+(\omega)$ is a Z-matrix. Since $\omega < 1 - \sum_{i=s+1,i\neq j_0}^{n} \pi_i$, it follows that for each $i \in N$,

$$m_{i1} + \cdots + m_{is} + m_{i,s+1} - \pi_{s+1}R_i + \cdots + m_{i,j_0} - \omega R_i + \cdots + m_{in} - \pi_n R_i = R_i - \Bigg(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega\Bigg) R_i = R_i \Bigg(1 - \sum_{j=s+1, j\neq j_0}^{n} \pi_j - \omega\Bigg) > 0,$$

and thus the row sums of $B^+(\omega)$ are positive. Hence, $B^+(\omega)$ is an SDD matrix with positive diagonal entries. The proof is complete. □

In what follows, we list some notations and lemmas which will be used later. Given a B_π^R-matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ and a parameter ω, denote

$$\theta_{max} := \max\left\{0, \max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\}\right\},\ \eta_1 := \max_{i\in\{s+1,\ldots,n\}\setminus\{j_0\}}\left\{\frac{1}{\pi_i} - 1\right\}; \tag{5}$$

$$\eta_2 := 1 + \frac{\left(\sum_{j=s+1, j\neq j_0}^{n} \pi_j\right)\cdot \max_{i\in N}\{R_i\}}{\min\{\bar{\beta}(0), 1\}},\ \eta_3 := \max_{i\in\{s+1,\ldots,n\}\setminus\{j_0\}}\left\{\frac{\sum_{j=s+1, j\neq j_0}^{n} \pi_j}{\pi_i} - 1\right\}; \tag{6}$$

and

$$\varphi(\omega) := 1 + \frac{\left(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega\right)\cdot \max_{i\in N}\{R_i\}}{\min\{\bar{\beta}(\omega), 1\}}, \tag{7}$$

where $\bar{\beta}(\omega) = \min_{i\in N}\{R_i\}(1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i - \omega)$.

Lemma 2.2. *[14, Lemma 3] Let $\gamma > 0$. Then for any $x \in [0, 1]$,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}}.$$

Lemma 2.3. *[9] If $P := [p_1, \ldots, p_n]^T e$, where $e = [1, \ldots, 1]$ and $p_1, \ldots, p_n \geq 0$, then*

$$(I + P)^{-1} = \begin{bmatrix} 1 - \frac{p_1}{1+\sum_{i=1}^{n} p_i} & -\frac{p_1}{1+\sum_{i=1}^{n} p_i} & \cdots & -\frac{p_1}{1+\sum_{i=1}^{n} p_i} \\ \vdots & \vdots & & \vdots \\ -\frac{p_2}{1+\sum_{i=1}^{n} p_i} & 1 - \frac{p_2}{1+\sum_{i=1}^{n} p_i} & \cdots & -\frac{p_2}{1+\sum_{i=1}^{n} p_i} \\ \vdots & \vdots & & \vdots \\ -\frac{p_n}{1+\sum_{i=1}^{n} p_i} & -\frac{p_n}{1+\sum_{i=1}^{n} p_i} & \cdots & 1 - \frac{p_n}{1+\sum_{i=1}^{n} p_i} \end{bmatrix}.$$

Lemma 2.4. *[26] Let A be the partitioned matrix:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices. If A_{11} is nonsingular, then the Schur complement of the matrix A_{11} in the partitioned matrix A is defined by $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$, and

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A/A_{11})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A/A_{11})^{-1} \\ -(A/A_{11})^{-1}A_{21}A_{11}^{-1} & (A/A_{11})^{-1} \end{bmatrix}. \tag{8}$$

The following theorem is our main result, which gives some error bounds for the LCP(M, q) when M is a B_π^R-matrix for **Case (a)**.

Theorem 2.5. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a vector $\pi = [\pi_1, \ldots, \pi_n]^T$ only with $\pi_i \leq 0$ for all $i \in \{1, \ldots, s\} \subset N$, and $\theta_{max} < 1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i$ for some $j_0 \in \{s + 1, \ldots, n\}$, where θ_{max} is given by (5).*

- *If* $\theta_{max} \neq 0$, *then for each* $\omega \in [\theta_{max}, 1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i)$,

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{\max\{\varphi(\omega), \eta_1\}}{\min\{\bar{\beta}(\omega), 1\}}; \tag{9}$$

- *If* $\theta_{max} = 0$, *then for* $\omega = \theta_{max}$,

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{\max\{\eta_2, \eta_3\}}{\min\{\bar{\beta}(0), 1\}}, \tag{10}$$

and for each $\omega \in (\theta_{max}, 1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i)$,

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{\max\{\varphi(\omega), \eta_1\}}{\min\{\bar{\beta}(\omega), 1\}}, \tag{11}$$

where η_1, η_2 *and* η_3 *are given by (5) and (6), respectively,* $\varphi(\omega)$ *is given by (7), and* $\bar{\beta}(\omega) = \min_i\{\bar{\beta}_i\}$ *with* $\bar{\beta}_i = R_i(1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i - \omega)$.

Proof. Denote $M_D := I - D + DM$, where $D = diag(d_i)$ with $0 \leq d_i \leq 1$ for all $i \in N$, and $M = B^+(\omega) + C(\omega)$ given as in (4). Then

$$M_D = I - D + D\,(B^+(\omega) + C(\omega)) = B_D^+ + C_D,$$

where $B_D^+ = I - D + DB^+(\omega)$ and $C_D = DC(\omega)$. It follows from Lemma 2.1 that $B^+(\omega)$ is an *SDD* Z-matrix with positive diagonal entries, and so is B_D^+. Thus, B_D^+ is a nonsingular M-matrix, and

$$M_D^{-1} = (B_D^+(I + (B_D^+)^{-1}C_D))^{-1} = \left(I + (B_D^+)^{-1}C_D\right)^{-1}(B_D^+)^{-1},$$

implying that

$$\|M_D^{-1}\|_\infty \leq \|\left(I + (B_D^+)^{-1}C_D\right)^{-1}\|_\infty \cdot \|(B_D^+)^{-1}\|_\infty. \tag{12}$$

We first bound $\|(B_D^+)^{-1}\|_\infty$. Let $B^+(\omega) =: [b_{ij}]$ and $B_D^+ =: [\bar{b}_{ij}]$. Notice that

$$\bar{b}_{ij} = \begin{cases} 1 - d_i + d_i b_{ij}, & i = j, \\ d_i b_{ij}, & i \neq j, \end{cases}$$

and B_D^+ is an SDD Z-matrix with positive diagonal entries. Then, for each $i \in N$,

$$\begin{aligned} \bar{b}_{ii} - \sum_{j\neq i} |\bar{b}_{ij}| = 1 - d_i + d_i \left(b_{ii} - \sum_{j\neq i} |b_{ij}|\right) &= 1 - d_i + d_i \left[R_i \left(1 - \sum_{j=s+1, j\neq j_0}^{n} \pi_j - \omega\right)\right] \text{(by (4))} \\ &= 1 - d_i + d_i\bar{\beta}_i \\ &> 0. \end{aligned}$$

By Lemma 2.2 and Theorem 1 of [20], we can see that

$$\|(B_D^+)^{-1}\|_\infty \leq \frac{1}{\min\limits_{i\in N}\{\bar{b}_{ii} - \sum\limits_{j\neq i} |\bar{b}_{ij}|\}} = \frac{1}{\min\limits_{i\in N}\{1 - d_i + d_i\bar{\beta}_i\}} \leq \frac{1}{\min\{\min\{\bar{\beta}_i, 1\}\}} = \frac{1}{\min\{\bar{\beta}(\omega), 1\}}, \tag{13}$$

where $\bar{\beta}(\omega) = \min_{i\in N}\{\bar{\beta}_i\}$.

We next bound $\|\left(I + (B_D^+)^{-1}C_D\right)^{-1}\|_\infty$. Since B_D^+ is a nonsingular M-matrix, it holds that $(B_D^+)^{-1} =: [\tilde{b}_{ij}] \geq 0$. Observe that $C_D = DC(\omega)$. Then,

$$I + (B_D^+)^{-1}C_D = \left[\begin{array}{ccc:ccccc} 1 & \cdots & 0 & \pi_{s+1}a_1 & \cdots & \omega a_1 & \cdots & \pi_n a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \pi_{s+1}a_s & \cdots & \omega a_s & \cdots & \pi_n a_s \\ \hdashline 0 & \cdots & 0 & 1+\pi_{s+1}a_{s+1} & \cdots & \omega a_{s+1} & \cdots & \pi_n a_{s+1} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \pi_{s+1}a_n & \cdots & \omega a_n & \cdots & 1+\pi_n a_n \end{array}\right] =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{14}$$

where $a_i := \sum_{j=1}^n \tilde{b}_{ij} d_j R_j \geq 0$ for all $i \in N$.

According to (14) and $\omega \geq \theta_{max} := \max\{0, \max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\}\}$, we next divide two cases: $\omega > 0$ and $\omega = 0$ to bound $\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty$.

<u>The first case</u>. Suppose that $\omega > 0$. Note that

$$B_{22} = I + [a_{s+1}, \ldots, a_{j_0}, \ldots, a_n]^T e \cdot \mathrm{diag}(\pi_{s+1}, \ldots, \omega, \ldots, \pi_n) =: I + CD,$$

where $[a_{s+1}, \ldots, a_{j_0}, \ldots, a_n]^T e$ and $D = \mathrm{diag}(\pi_{s+1}, \ldots, \omega, \ldots, \pi_n)$. Clearly, D is nonsingular, and so $I + CD = D^{-1}(I + DC)D$ and

$$B_{22}^{-1} = (I + CD)^{-1} = D^{-1}(I + DC)^{-1}D = D^{-1}(I + \bar{C})^{-1}D, \tag{15}$$

where $\bar{C} = DC = [\bar{a}_{s+1}, \ldots, \bar{a}_{j_0}, \ldots, \bar{a}_n]^T e$ with $\bar{a}_i = \pi_i a_i \geq 0$ for all $i \in \{s+1, \ldots, n\} \setminus \{j_0\}$ and $\bar{a}_{j_0} = \omega a_{j_0} \geq 0$. Hence, by Lemma 2.3 and (15), it holds that

$$B_{22}^{-1} = \begin{bmatrix} 1 - \frac{\bar{a}_{s+1}}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & \frac{\omega}{\pi_{s+1}}\left(-\frac{\bar{a}_{s+1}}{1+\sum_{i=s+1}^n \bar{a}_i}\right) & \cdots & \frac{\pi_n}{\pi_{s+1}}\left(-\frac{\bar{a}_{s+1}}{1+\sum_{i=s+1}^n \bar{a}_i}\right) \\ \vdots & & \vdots & & \vdots \\ \frac{\pi_{s+1}}{\omega}\left(-\frac{\bar{a}_{j_0}}{1+\sum_{i=s+1}^n \bar{a}_i}\right) & \cdots & 1 - \frac{\bar{a}_{j_0}}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & \frac{\pi_n}{\omega}\left(-\frac{\bar{a}_{j_0}}{1+\sum_{i=s+1}^n \bar{a}_i}\right) \\ \vdots & & \vdots & & \vdots \\ \frac{\pi_{s+1}}{\pi_n}\left(-\frac{\bar{a}_n}{1+\sum_{i=s+1}^n \bar{a}_i}\right) & \cdots & \frac{\omega}{\pi_n}\left(-\frac{\bar{a}_n}{1+\sum_{i=s+1}^n \bar{a}_i}\right) & \cdots & 1 - \frac{\bar{a}_n}{1+\sum_{i=s+1}^n \bar{a}_i} \end{bmatrix}. \tag{16}$$

Since $B_{11}^{-1} = I$ and $B_{21} = 0$, it follows from (8) and (16) that

$$(I + (B_D^+)^{-1}C_D)^{-1} = \begin{bmatrix} I & -B_{12}B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{bmatrix}, \tag{17}$$

where

$$-B_{12}B_{22}^{-1} = \begin{bmatrix} -\pi_{s+1}\frac{a_1}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\omega\frac{a_1}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\pi_n\frac{a_1}{1+\sum_{i=s+1}^n \bar{a}_i} \\ -\pi_{s+1}\frac{a_2}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\omega\frac{a_2}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\pi_n\frac{a_2}{1+\sum_{i=s+1}^n \bar{a}_i} \\ \vdots & & \vdots & & \vdots \\ -\pi_{s+1}\frac{a_s}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\omega\frac{a_s}{1+\sum_{i=s+1}^n \bar{a}_i} & \cdots & -\pi_n\frac{a_s}{1+\sum_{i=s+1}^n \bar{a}_i} \end{bmatrix}.$$

By (17) and $\bar{a}_i \geq 0$ for all $i = s+1, \ldots, n$, it follows that

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = 1 + \frac{a_i}{1 + \sum_{i=s+1}^n \bar{a}_i} \cdot \left(\sum_{j=s+1, j\neq j_0}^n \pi_j + \omega\right) \tag{18}$$

for some $i \in \{1, \ldots, s\}$, or

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = \frac{\bar{a}_i}{1 + \sum_{i=s+1}^n \bar{a}_i} \cdot \Big(\sum_{j=s+1, j\neq i, j_0}^{n} \frac{\pi_j}{\pi_i} + \frac{\omega}{\pi_i} - 1 \Big) + 1 \tag{19}$$

for some $i \in \{s+1, \ldots, n\} \setminus \{j_0\}$, or

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = 1 + \frac{a_{j_0}}{1 + \sum_{i=s+1}^n \bar{a}_i} \cdot \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j - \omega \Big). \tag{20}$$

We are now in a position to bound $\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty$. Notice that $a_i := \sum_{j=1}^n \tilde{b}_{ij} d_j R_j \geq 0$ for all $i \in N$. It is straightforward to see that formula (18) can be bounded above by

$$\begin{aligned} 1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega \Big) a_i = 1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega \Big) \sum_{j=1}^{n} \tilde{b}_{ij} d_j R_j &\leq 1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega \Big) \cdot \sum_{j=1}^{n} \tilde{b}_{ij} \cdot \max_{i\in N}\{R_i\} \\ &\leq 1 + \frac{\Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j + \omega \Big) \cdot \max_{i\in N}\{R_i\}}{\min\{\bar{\beta}(\omega), 1\}} \text{ (by (13))} \\ &=: \varphi(\omega), \end{aligned}$$

formula (19) can be bounded above by

$$\Big(\sum_{j=s+1, j\neq i, j_0}^{n} \frac{\pi_j}{\pi_i} + \frac{\omega}{\pi_i} \Big) + 1 = \frac{\sum_{j=s+1, j\neq j_0}^{n} \pi_j - \pi_i + \omega}{\pi_i} + 1 < \frac{\sum_{j=s+1, j\neq j_0}^{n} \pi_j + 1 - \sum_{j=s+1, j\neq j_0}^{n} \pi_j - \pi_i}{\pi_i} = \frac{1}{\pi_i} - 1,$$

and that formula (20) can be bounded above by

$$1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j \Big) a_i = 1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j \Big) \sum_{j=1}^{n} \tilde{b}_{ij} d_j R_j \leq 1 + \frac{\Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j \Big) \cdot \max_{i\in N}\{R_i\}}{\min\{\bar{\beta}(\omega), 1\}} =: \phi(\omega).$$

Obverse that $\phi(\omega) \leq \varphi(\omega)$ for any $\omega \in [\theta_{max}, 1 - \sum_{i=s+1, i\neq j_0}^{n} \pi_i)$ and $\eta_1 = \max_{i\in\{s+1,\cdots,n\}\setminus\{j_0\}} \{\frac{1}{\pi_i} - 1\}$. Hence,

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty \leq \max\left\{\varphi(\omega), \phi(\omega), \eta_1\right\} = \max\left\{\varphi(\omega), \eta_1\right\}. \tag{21}$$

Due to (12), (13) and (21), we get (9) and (11).

The second case. Suppose that $\omega = 0$. Then,

$$I + (B_D^+)^{-1}C_D =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $B_{11} = I$, $B_{21} = 0$,

$$B_{12} = \begin{bmatrix} \pi_{s+1}a_1 & \cdots & \pi_{j_0-1}a_1 & 0 & \pi_{j_0+1}a_1 & \cdots & \pi_n a_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \pi_{s+1}a_s & \cdots & \pi_{j_0-1}a_s & 0 & \pi_{j_0+1}a_1 & \cdots & \pi_n a_s \end{bmatrix},$$

and

$$B_{22}=\left[\begin{array}{ccc|cccc} 1+\pi_{s+1}a_{s+1} & \cdots & \pi_{j_0-1}a_{s+1} & 0 & \pi_{j_0+1}a_{s+1} & \cdots & \pi_n a_{s+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \pi_{s+1}a_{j_0-1} & \cdots & 1+\pi_{j_0-1}a_{j_0-1} & 0 & \pi_{j_0+1}a_{j_0-1} & \cdots & \pi_n a_{j_0-1} \\ \hline \pi_{s+1}a_{j_0} & \cdots & \pi_{j_0-1}a_{j_0} & 1 & \pi_{j_0+1}a_{j_0} & \cdots & \pi_n a_{j_0} \\ \pi_{s+1}a_{j_0+1} & \cdots & \pi_{j_0-1}a_{j_0+1} & 0 & 1+\pi_{j_0+1}a_{j_0+1} & \cdots & \pi_n a_{j_0+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \pi_{s+1}a_n & \cdots & \pi_{j_0-1}a_n & 0 & \pi_{j_0+1}a_n & \cdots & 1+\pi_n a_n \end{array}\right]$$

$$=:\left[\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array}\right].$$

To get $(I+(B_D^+)^{-1}C_D)^{-1}$, we first compute B_{22}^{-1}. Similar to the skills for computing (16), we know that

$$C_{11}^{-1}=\left[\begin{array}{cccc} 1-\frac{\bar a_{s+1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i} & \frac{\pi_{s+2}}{\pi_{s+1}}\left(-\frac{\bar a_{s+1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) & \cdots & \frac{\pi_{j_0-1}}{\pi_{s+1}}\left(-\frac{\bar a_{s+1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) \\ \frac{\pi_{s+1}}{\pi_{s+2}}\left(-\frac{\bar a_{s+2}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) & 1-\frac{\bar a_{s+2}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i} & \cdots & \frac{\pi_{j_0-1}}{\pi_{s+2}}\left(-\frac{\bar a_{s+2}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) \\ \vdots & \vdots & & \vdots \\ \frac{\pi_{s+1}}{\pi_{j_0-1}}\left(-\frac{\bar a_{j_0-1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) & \frac{\pi_{s+2}}{\pi_{j_0-1}}\left(-\frac{\bar a_{j_0-1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i}\right) & \cdots & 1-\frac{\bar a_{j_0-1}}{1+\sum\limits_{i=s+1}^{j_0-1}\bar a_i} \end{array}\right], \tag{22}$$

and thus

$$B_{22}/C_{11}=C_{22}-C_{21}C_{11}^{-1}C_{12}=\left[\begin{array}{c|ccc} 1 & \frac{\pi_{j_0+1}a_{j_0}}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} & \cdots & \frac{\pi_n a_{j_0}}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} \\ \hline 0 & 1+\frac{\pi_{j_0+1}a_{j_0+1}}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} & \cdots & \frac{\pi_n a_{j_0+1}}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\pi_{j_0+1}a_n}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} & \cdots & 1+\frac{\pi_n a_n}{1+\sum_{i=s+1}^{j_0-1}\bar a_i} \end{array}\right].$$

By (14) and (17), it follows that

$$(B_{22}/C_{11})^{-1}=\left[\begin{array}{ccccc} 1 & -\frac{\pi_{j_0+1}a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & -\frac{\pi_{j_0+2}a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & \cdots & -\frac{\pi_n a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} \\ 0 & 1-\frac{\pi_{j_0+1}a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & -\frac{\pi_{j_0+2}a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & \cdots & -\frac{\pi_n a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{j_0+1}a_n}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & -\frac{\pi_{j_0+1}a_n}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} & \cdots & 1-\frac{\pi_n a_n}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar a_i} \end{array}\right].$$

Then, from (8), we can see that

$$B_{22}^{-1}=\left[\begin{array}{cc} C_{11}^{-1}+C_{11}^{-1}C_{12}(B_{22}/C_{11})^{-1}C_{21}C_{11}^{-1} & -C_{11}^{-1}C_{12}(B_{22}/C_{11})^{-1} \\ -(B_{22}/C_{11})^{-1}C_{21}C_{11}^{-1} & (B_{22}/C_{11})^{-1} \end{array}\right], \tag{23}$$

where

$$C_{11}^{-1}+C_{11}^{-1}C_{12}(B_{22}/C_{11})^{-1}C_{21}C_{11}^{-1}=\begin{bmatrix} 1-\frac{\pi_{s+1}a_{s+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & -\frac{\pi_{s+2}a_{s+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{s+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ -\frac{\pi_{s+1}a_{s+2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & 1-\frac{\pi_{s+2}a_{s+2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{s+2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ -\frac{\pi_{s+1}a_{j_0-1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & -\frac{\pi_{s+2}a_{j_0-1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & 1-\frac{\pi_{j_0-1}a_{j_0-1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \end{bmatrix},$$

$$-C_{11}^{-1}C_{12}(B_{22}/C_{11})^{-1}=\begin{bmatrix} 0 & -\frac{\pi_{j_0+1}a_{s+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{s+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ 0 & -\frac{\pi_{j_0+1}a_{s+2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{s+2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{j_0+1}a_{j_0-1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{j_0-1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \end{bmatrix},$$

and

$$-(B_{22}/C_{11})^{-1}C_{21}C_{11}^{-1}=\begin{bmatrix} -\frac{\pi_{s+1}a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & -\frac{\pi_{s+2}a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{j_0}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ -\frac{\pi_{s+1}a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & -\frac{\pi_{s+2}a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{j_0+1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ -\frac{\pi_{s+1}a_{n}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & -\frac{\pi_{s+2}a_{n}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{n}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \end{bmatrix}.$$

Since $B_{11}=I$, $B_{21}=0$, we know from (8) and (23) that

$$(I+(B_D^{+})^{-1}C_D)^{-1}=\begin{bmatrix} I & -B_{12}B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{bmatrix}, \tag{24}$$

where

$$-B_{12}B_{22}^{-1}=\begin{bmatrix} -\frac{\pi_{s+1}a_{1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & 0 & -\frac{\pi_{j_0+1}a_{1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{1}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ -\frac{\pi_{s+1}a_{2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & 0 & -\frac{\pi_{j_0+1}a_{2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{2}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -\frac{\pi_{s+1}a_{s}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{j_0-1}a_{s}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & 0 & -\frac{\pi_{j_0+1}a_{s}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} & \cdots & -\frac{\pi_{n}a_{s}}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i} \end{bmatrix}.$$

By (24), $a_i \geq 0$ for all $i \in N$ and $\bar{a}_i \geq 0$ for all $i=s+1,\ldots,n$, it follows that

$$\|(I+(B_D^{+})^{-1}C_D)^{-1}\|_\infty = 1+\frac{a_i}{1+\sum\limits_{i=s+1,i\neq j_0}^{n}\bar{a}_i}\cdot\left(\sum_{j=s+1,j\neq j_0}^{n}\pi_j\right) \tag{25}$$

for some $i \in \{1, \dots, s\}$, or

$$\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty = \frac{\bar{a}_i}{1 + \sum\limits_{i=s+1, i\neq j_0}^{n} \bar{a}_i} \cdot \Big(\sum_{j=s+1, j\neq i, j_0}^{n} \frac{\pi_j}{\pi_i} - 1 \Big) + 1 \tag{26}$$

for some $i \in \{s+1, \dots, n\} \setminus \{j_0\}$, or

$$\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty = 1 + \frac{a_{j_0}}{1 + \sum\limits_{i=s+1, i\neq j_0}^{n} \bar{a}_i} \cdot \Big(\sum_{j=s+1, j\neq j_0} \pi_j \Big). \tag{27}$$

Now, we turn to bound $\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty$. Because $a_i = \sum_{j=1}^{n} \tilde{b}_{ij} d_j R_j \geq 0$ and $\bar{a}_i \geq 0$ for all $i = s+1, \dots, n$, it holds that formula (25) and (27) can be bounded above by

$$1 + a_i \cdot \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j \Big) = 1 + \Big(\sum_{j=s+1, j\neq j_0}^{n} \pi_j \Big) \sum_{j=1}^{n} \hat{b}_{ij} d_j R_j \leq 1 + \frac{\Big(\sum\limits_{j=s+1, j\neq j_0}^{n} \pi_j \Big) \cdot \max\{R_j\}}{\min\{\bar{\beta}(0), 1\}} =: \eta_2,$$

and that formula (26) can be bounded above by

$$\sum_{j=s+1, j\neq i, j_0}^{n} \frac{\pi_j}{\pi_i} = \frac{\sum\limits_{j=s+1, j\neq j_0}^{n} \pi_j - \pi_i}{\pi_i} \leq \max_{i\in\{s+1, \cdots, n\}\setminus\{j_0\}} \left\{ \frac{\sum\limits_{j=s+1, j\neq j_0}^{n} \pi_j}{\pi_i} - 1 \right\} =: \eta_3.$$

Hence,

$$\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty \leq \max\{\eta_2, \eta_3\}, \tag{28}$$

and (10) follows from (12), (13) and (28). This completes the proof. □

The following theorem gives the optimal values of bounds (9), (10) and (11), which depend only on the entries of B_π^R-matrices.

Theorem 2.6. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a vector $\pi = [\pi_1, \dots, \pi_n]^T$ only with $\pi_i \leq 0$ for all $i \in \{1, \dots, s\} \subset N$, and $\theta_{max} < 1 - \sum\limits_{i=s+1, i\neq j_0}^{n} \pi_i$ for some $j_0 \in \{s+1, \dots, n\}$, where θ_{max} is given by (5). Let $I_1 := [\theta_{max}, 1 - \sum\limits_{i=s+1, i\neq j_0}^{n} \pi_i)$ and $I_1^o := I_1 \setminus \{\theta_{max}\}$.*

- *If $\theta_{max} \neq 0$, then*

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \min_{\omega\in I_1} \left\{ \frac{\max\{\varphi(\omega), \eta_1\}}{\min\{\bar{\beta}(\omega), 1\}} \right\} = \frac{\max\{\varphi(\theta_{max}), \eta_1\}}{\min\{\bar{\beta}(\theta_{max}), 1\}}; \tag{29}$$

- *If $\theta_{max} = 0$, then*

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \min \left\{ \frac{\max\{\eta_2, \eta_3\}}{\min\{\bar{\beta}(0), 1\}}, \min_{\omega\in I_1^o} \left\{ \frac{\max\{\varphi(\omega), \eta_1\}}{\min\{\bar{\beta}(\omega), 1\}} \right\} \right\} = \frac{\max\{\eta_2, \eta_3\}}{\min\{\bar{\beta}(0), 1\}}, \tag{30}$$

where η_1, η_2 and η_3 are given by (5) and (6), respectively, $\varphi(\omega)$ is given by (7), and $\bar{\beta}(\omega) = \min_i\{\bar{\beta}_i\}$ with $\bar{\beta}_i = R_i(1 - \sum\limits_{i=s+1, i\neq j_0}^{n} \pi_i - \omega)$.

Proof. Note that

$$\varphi(\omega) := 1 + \frac{\left(\sum\limits_{j=s+1,j\neq j_0}^{n} \pi_j + \omega\right) \cdot \max\limits_{i\in N}\{R_i\}}{\min\{\bar{\beta}(\omega), 1\}},$$

and $\bar{\beta}(\omega) = \min\limits_{i\in N}\{R_i\}(1 - \sum\limits_{i=s+1,i\neq j_0}^{n} \pi_i - \omega)$ is decreasing for $\omega \in I_1$. Then,

$$\min_{i\in I_1} \varphi(\omega) = \varphi(\theta_{max}),$$

which implies that (29) holds. In addition, observe that for any $\omega \in I_1^o$,

$$\eta_2 := 1 + \frac{\left(\sum\limits_{j=s+1,j\neq j_0}^{n} \pi_j\right) \cdot \max\limits_{i\in N}\{R_i\}}{\min\{\bar{\beta}(0), 1\}} \leq \varphi(\omega),$$

and

$$\eta_3 := \max_{i\in\{s+1,\cdots,n\}\setminus\{j_0\}} \left\{ \frac{\sum\limits_{j=s+1,j\neq j_0}^{n} \pi_j}{\pi_i} - 1 \right\} \leq \max_{i\in\{s+1,\cdots,n\}\setminus\{j_0\}} \left\{ \frac{1}{\pi_i} - 1 \right\} =: \eta_1.$$

Hence,

$$\frac{\max\{\eta_2, \eta_3\}}{\min\{\bar{\beta}(0), 1\}} \leq \min_{\omega\in I_1^o} \left\{ \frac{\max\{\varphi(\omega), \eta_1\}}{\min\{\bar{\beta}(\omega), 1\}} \right\}.$$

This means that (30) follows. The proof is complete. □

Example 2.7. *Consider matrices*

$$M_1 = \begin{bmatrix} 11.2 & -7 & 15 & 0.8 \\ -8 & 20 & 8 & 4 \\ -8 & -8 & 36 & 4 \\ -8 & -8 & 4 & 32 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 1.4 & -0.4 & -0.6 & 0.6 \\ -0.4 & 1 & -0.2 & 0.6 \\ -0.4 & -0.4 & 1.2 & 0.6 \\ -0.4 & -0.4 & -0.2 & 2 \end{bmatrix}.$$

It is easy to verify that M_1 is a B_π^R-matrix with $\pi = [-0.25, -0.25, 1, 0.2]^T$ and $R = [20, 24, 24, 20]^T$. By some calculations, for $j_0 = 3$, we have

$$\theta_{max} = \max\left\{ \max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\}, 0\right\} = 0.75 < 0.8 = 1 - \sum_{i=3,i\neq j_0}^{n} \pi_i,$$

$\bar{\beta}(0.75) = 1$, $\varphi(0.75) = 23.8$, and $\eta_1 = 4$, which satisfy the hypotheses of Theorem 2.6. Therefore, by the bound (29) it holds that

$$\max_{d\in[0,1]^4} \|(I - D + DM_1)^{-1}\|_\infty \leq \frac{\max\{\varphi(0.75), \eta_1\}}{\min\{\bar{\beta}(0.75), 1\}} = 23.8.$$

Obviously, M_2 is a B_π^R-matrix for $\pi = [-0.2, -0.2, 0.5, 0.8]^T$, and $R = [1, 1, 1, 1]^T$. By computations,

$$\theta_{max} = \max\left\{ \max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\}, 0\right\} = 0 < \frac{1}{5} = 1 - \sum_{i=3,i\neq j_0}^{n} \pi_i \text{ for } j_0 = 3,$$

$\bar{\beta}(0) = \frac{1}{5}$, $\eta_2 = 5$, and $\eta_3 = 0$, which satisfy the hypotheses of Theorem 2.6. Hence, from bound (30) it holds that

$$\max_{d\in[0,1]^4} \|(I - D + DM_2)^{-1}\|_\infty \le \frac{\max\{\eta_2, \eta_3\}}{\min\{\bar{\beta}(0), 1\}} = 25.$$

In contrast, since $\pi_1 < 0$ and $\pi_2 < 0$ for M_1 and M_2, which does not satisfy hypothesises of Theorem 1.2, so we cannot use the bound (3) to estimate $\max\limits_{d\in[0,1]^4} \|(I - D + DM)^{-1}\|_\infty$ when $M = M_1$ or $M = M_2$.

2.2. *Error bounds for case (b)*

Consider a B_π^R-matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ for a vector $\pi = [\pi_1, \dots, \pi_n]^T$ only with $\pi_s \le 0$ for some $s \in N$, we can decompose M as

$$M = B^+(\omega) + C(\omega), \tag{31}$$

where

$$B^+(\omega) = \begin{bmatrix} m_{11} - \pi_1 R_1 & \cdots & m_{12} - \omega R_1 & \cdots & m_{1s} & \cdots & m_{1n} - \pi_n R_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{n1} - \pi_1 R_n & \cdots & m_{n2} - \omega R_n & \cdots & m_{ns} & \cdots & m_{nn} - \pi_n R_n \end{bmatrix}$$

and

$$C(\omega) = \begin{bmatrix} \pi_1 R_1 & \cdots & \omega R_1 & \cdots & 0 & \cdots & \pi_n R_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \pi_1 R_n & \cdots & \omega R_n & \cdots & 0 & \cdots & \pi_n R_n \end{bmatrix}$$

with ω is an adjustable parameter.

A sufficient condition, similar to that in Lemma 2.1, can easily be obtained such that $B^+(\omega)$ is an SDD matrix with positive diagonal entries.

Lemma 2.8. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ is a B_π^R-matrix for a vector $\pi = [\pi_1, \dots, \pi_n]^T$ only with $\pi_s \le 0$ for some $s \in N$, and $B^+(\omega)$ be the matrix of (31). If there exists an index $j_0 \in N \setminus \{s\}$ such that*

$$\max_{i\ne j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\} < 1 - \sum_{i\ne j_0,s} \pi_i,$$

then, for each $\omega \in [\max\limits_{i\ne j_0}\{\frac{m_{i,j_0}}{R_i}\}, 1 - \sum\limits_{i\ne j_0,s} \pi_i)$, $B^+(\omega)$ is an SDD Z-matrix with positive diagonal entries.

By Lemma 2.8, we next address error bounds for the linear complementarity problems of B_π^R-matrices under **Case (b)**. First, some notations are needed. Denote

$$\xi_1 := \frac{\sum\limits_{i\ne j_0,s} \pi_i}{\max\limits_{i\ne j_0}\{\frac{m_{i,j_0}}{R_i}\}},\ \xi_2 := \max_{i\in N\setminus\{j_0,s\}}\left\{\frac{1}{\pi_i} - 1\right\}, \tag{32}$$

and

$$\xi(\omega) = 1 + \frac{\max\limits_{i\in N}\{R_i\}}{\min\{\tilde{\beta}(\omega), 1\}}, \tag{33}$$

where $\tilde{\beta}(\omega) := \min\limits_{i\in N}\{\tilde{\beta}_i\}$ and $\tilde{\beta}_i = R_i(1 - \sum\limits_{i\ne j_0,s} \pi_i - \omega)$.

Theorem 2.9. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a vector $\pi = [\pi_1, \dots, \pi_n]^T$ only with $\pi_s \le 0$ for some $s \in N$, and $0 \ne \theta_{max} < 1 - \sum\limits_{i\ne j_0,s} \pi_i$ for some $j_0 \in N \setminus \{s\}$, where θ_{max} is given by (5). Then for each $\omega \in I_2 := [\theta_{max}, 1 - \sum\limits_{i\ne j_0,s} \pi_i)$,*

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \le \frac{\max\{\xi_1, \xi_2, \xi(\omega)\}}{\min\{\tilde{\beta}(\omega), 1\}},$$

where ξ_1, ξ_2 and $\xi(\omega)$ are given by (32) and (33), respectively, and $\tilde{\beta}(\omega) := \min_i\{R_i\}(1 - \sum\limits_{i\ne j_0,s} \pi_i - \omega)$. Furthermore,

$$\min_{\omega\in I_2} \left\{ \frac{\max\{\xi_1, \xi_2, \xi(\omega)\}}{\min\{\tilde{\beta}(\omega), 1\}} \right\} = \frac{\max\{\xi_1, \xi_2, \xi(\theta_{max})\}}{\min\{\tilde{\beta}(\theta_{max}), 1\}}.$$

Proof. Let $M = B^+(\omega) + C(\omega)$ given by (31). Analogous to the proof of Theorem 2.5, we have

$$\|(I - D + DM)^{-1}\|_\infty \le \left\|\left(I + (B_D^+)^{-1}C_D\right)^{-1}\right\|_\infty \cdot \|(B_D^+)^{-1}\|_\infty, \tag{34}$$

and

$$\|(B_D^+)^{-1}\|_\infty \le \frac{1}{\min\{\tilde{\beta}(\omega), 1\}}, \tag{35}$$

where $B_D^+ = I - D + DB^+(\omega)$ and $C_D = DC(\omega)$.

We next bound $\left\|\left(I + (B_D^+)^{-1}C_D\right)^{-1}\right\|_\infty$. Let $Q := I + (B_D^+)^{-1}C_D$. Without loss of generality, we assume that there exists an index $j_0 = 1 \in N \setminus \{s\}$ such that

$$0 \ne \theta_{max} < 1 - \sum_{i\ne 1,s} \pi_i.$$

Let $(B_D^+)^{-1} =: [\tilde{b}_{ij}]$ with $\tilde{b}_{ij} \ge 0$ for all $i, j \in N$. Note that $C_D = DC(\omega)$. Then,

$$Q := \left[\begin{array}{cccc|cccc} 1+\omega a_1 & \pi_2 a_1 & \cdots & \pi_{s-1}a_1 & 0 & \pi_{s+1}a_1 & \cdots & \pi_n a_1 \\ \omega a_2 & 1+\pi_2 a_2 & \cdots & \pi_{s-1}a_2 & 0 & \pi_{s+1}a_2 & \cdots & \pi_n a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \omega a_{s-1} & \pi_2 a_{s-1} & \cdots & 1+\pi_{s-1}a_{s-1} & 0 & \pi_{s+1}a_{s-1} & \cdots & \pi_n a_{s-1} \\ \hline \omega a_s & \pi_2 a_s & \cdots & \pi_{s-1}a_s & 1 & \pi_{s+1}a_s & \cdots & \pi_n a_s \\ \omega a_{s+1} & \pi_2 a_{s+1} & \cdots & \pi_{s-1}a_{s+1} & 0 & 1+\pi_{s+1}a_{s+1} & \cdots & \pi_n a_{s+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \omega a_n & \pi_2 a_n & \cdots & \pi_{s-1}a_n & 0 & \pi_{s+1}a_n & \cdots & 1+\pi_n a_n \end{array}\right] =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $a_i := \sum_{j=1}^n \tilde{b}_{ij} d_j R_j \ge 0$ for all $i = 1, \dots, n$.

We now compute Q^{-1}. Similarly to the computation of (16), we have

$$B_{11}^{-1} = \begin{bmatrix} 1 - \frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \frac{\pi_2}{\omega}\left(-\frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) & \cdots & \frac{\pi_{s-1}}{\omega}\left(-\frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) \\ \frac{\omega}{\pi_2}\left(-\frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) & 1 - \frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \cdots & \frac{\pi_{s-1}}{\pi_2}\left(-\frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) \\ \vdots & \vdots & & \vdots \\ \frac{\omega}{\pi_{s-1}}\left(-\frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) & \frac{\pi_2}{\pi_{s-1}}\left(-\frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i}\right) & \cdots & 1 - \frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i} \end{bmatrix}$$

and

$$Q/B_{11}=B_{22}-B_{21}B_{11}^{-1}B_{12}=\left[\begin{array}{c|cccc} 1 & \frac{\pi_{s+1}a_s}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \cdots & \frac{\pi_n a_s}{1+\sum_{i=1}^{s-1}\bar{a}_i} \\ \hline 0 & 1+\frac{\pi_{s+1}a_{s+1}}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \cdots & \frac{\pi_n a_{s+1}}{1+\sum_{i=1}^{s-1}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\pi_{s+1}a_n}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \cdots & 1+\frac{\pi_n a_n}{1+\sum_{i=1}^{s-1}\bar{a}_i} \end{array}\right],$$

where $\bar{a}_1=\omega a_1$ and $\bar{a}_i=\pi_i a_i$ for $i\in N\setminus\{1,s\}$. By (14) and (17), we conclude that

$$(Q/B_{11})^{-1}=\begin{bmatrix} 1 & -\frac{\pi_{s+1}a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_{s+2}a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_n a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ 0 & 1-\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & \frac{\pi_{s+2}}{\pi_{s+1}}\Big(-\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i}\Big) & \cdots & \frac{\pi_n}{\pi_{s+1}}\Big(-\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i}\Big) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{\pi_{s+1}}{\pi_n}\Big(-\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i}\Big) & \frac{\pi_{s+2}}{\pi_n}\Big(-\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i}\Big) & \cdots & 1-\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i} \end{bmatrix}.$$

Hence, from (8) it holds that

$$Q^{-1}=\begin{bmatrix} B_{11}^{-1}+B_{11}^{-1}B_{12}(Q/B_{11})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(Q/B_{11})^{-1} \\ -(Q/B_{11})^{-1}B_{21}B_{11}^{-1} & (Q/B_{11})^{-1} \end{bmatrix}, \tag{36}$$

where

$$B_{11}^{-1}+B_{11}^{-1}B_{12}(Q/B_{11})^{-1}B_{21}B_{11}^{-1}=\begin{bmatrix} 1-\frac{\omega a_1}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_2 a_1}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_{s-1}a_1}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ -\frac{\omega a_2}{1+\sum\limits_{i\neq s}\bar{a}_i} & 1-\frac{\pi_2 a_2}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_{s-1}a_2}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ -\frac{\omega a_{s-1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_2 a_{s-1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & 1-\frac{\pi_{s-1}a_{s-1}}{1+\sum\limits_{i\neq s}\bar{a}_i} \end{bmatrix},$$

$$-B_{11}^{-1}B_{12}(Q/B_{11})^{-1}=\begin{bmatrix} 0 & -\frac{\pi_{s+1}a_1}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_n a_1}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ 0 & -\frac{\pi_{s+1}a_2}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_n a_2}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{s+1}a_{s-1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_n a_{s-1}}{1+\sum\limits_{i\neq s}\bar{a}_i} \end{bmatrix},$$

and

$$-(Q/B_{11})^{-1}B_{21}B_{11}^{-1}=\begin{bmatrix} -\frac{\omega a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_2 a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_{s-1}a_s}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ -\frac{\omega}{\pi_{s+1}}\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_2}{\pi_{s+1}}\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_{s-1}}{\pi_{s+1}}\frac{\bar{a}_{s+1}}{1+\sum\limits_{i\neq s}\bar{a}_i} \\ \vdots & \vdots & & \vdots \\ -\frac{\omega}{\pi_n}\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i} & -\frac{\pi_2}{\pi_n}\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i} & \cdots & -\frac{\pi_{s-1}}{\pi_n}\frac{\bar{a}_n}{1+\sum\limits_{i\neq s}\bar{a}_i} \end{bmatrix}.$$

According to (36) and $\bar{a}_i\geq 0$ for all $i\in N\setminus\{s\}$, it follows that

$$\|(I+(B_D^+)^{-1}C_D)^{-1}\|_\infty=1+\frac{\bar{a}_i}{1+\sum\limits_{i\neq s}\bar{a}_i}\left(\frac{\sum\limits_{i\neq 1,s}\pi_i}{\omega}-1\right) \tag{37}$$

for $i = 1$, or

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = 1 + \frac{\bar{a}_i}{1 + \sum\limits_{i \neq s} \bar{a}_i} \left(\frac{\omega + \sum\limits_{j \neq 1,i,s} \pi_j}{\pi_i} - 1 \right) \tag{38}$$

for some $i \in \{2, \cdots, s-1\}$, or

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = 1 + \frac{a_i}{1 + \sum\limits_{i \neq s} \bar{a}_i} \left(\omega + \sum_{i \neq 1,s} \pi_i \right) \tag{39}$$

for $i = s$, or

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty = \frac{\bar{a}_i}{1 + \sum\limits_{i \neq s} \bar{a}_i} \left(\frac{\sum\limits_{j \neq 1,i,s} \pi_j + \omega}{\pi_i} - 1 \right) + 1 \tag{40}$$

for some $i \in \{s+1, \cdots, n\}$.

Since $a_i := \sum_{j=1}^n \tilde{b}_{ij} d_j R_j \geq 0$ for all $i \in N$ and $\bar{a}_i \geq 0$ for all $i \in N \setminus \{s\}$, it holds that formula (37) can be bounded above by

$$\frac{\sum\limits_{i \neq 1,s} \pi_i}{\omega} \leq \frac{\sum\limits_{i \neq 1,s} \pi_i}{\max\limits_{i \neq 1}\{\frac{m_{i1}}{R_i}\}},$$

formula (38) and (40) can be bounded above by

$$\frac{\omega + \sum\limits_{j \neq 1,i,s} \pi_j}{\pi_i} \leq \frac{1 - \sum\limits_{j \neq 1,s} \pi_j + \sum\limits_{j \neq 1,i,s} \pi_j}{\pi_i} = \frac{1}{\pi_i} - 1,$$

and that formula (39) can be bounded above by

$$1 + \left(\omega + \sum_{i \neq 1,s} \pi_i \right) a_s \leq 1 + a_s = 1 + \sum_{j=1}^n \tilde{b}_{sj} d_j R_j \leq 1 + \sum_{j=1}^n \tilde{b}_{sj} \cdot \max_{i \in N}\{R_i\} \leq 1 + \frac{\max\limits_{i \in N}\{R_i\}}{\min\{\tilde{\beta}(\omega), 1\}}.$$

Notice that $j_0 = 1$, $\xi_1 := \frac{\sum\limits_{i \neq j_0,s} \pi_i}{\max\limits_{i \neq j_0}\{\frac{m_{i,j_0}}{R_i}\}}$, $\xi_2 := \max\limits_{i \in N \setminus \{j_0,s\}} \left\{ \frac{1}{\pi_i} - 1 \right\}$, and $\xi(\omega) := 1 + \frac{\max\limits_{i \in N}\{R_i\}}{\min\{\tilde{\beta}(\omega),1\}}$. Then,

$$\|(I + (B_D^+)^{-1}C_D)^{-1}\|_\infty \leq \max\{\xi_1, \xi_2, \xi(\omega)\},$$

which together with (34) and (35) imply that

$$\|M_D^{-1}\|_\infty \leq \frac{\max\{\xi_1, \xi_2, \xi(\omega)\}}{\min\{\tilde{\beta}(\omega), 1\}}.$$

Besides, since $\tilde{\beta}(\omega) = \min\limits_{i \in N}\{R_i\}(1 - \sum\limits_{i \neq j_0,s} \pi_i - \omega)$ is decreasing for $\omega \in I_2 := [\theta_{max}, 1 - \sum\limits_{i \neq j_0,s} \pi_i)$, it holds that

$$\max_{\omega \in I_2} \tilde{\beta}(\omega) = \tilde{\beta}(\theta_{max}) \text{ and } \min_{\omega \in I_2} \xi(\omega) = \xi(\theta_{max}),$$

which lead to

$$\min_{\omega \in I_2} \left\{ \frac{\max\{\xi_1, \xi_2, \xi(\omega)\}}{\min\{\tilde{\beta}(\omega), 1\}} \right\} = \frac{\max\{\xi_1, \xi_2, \xi(\theta_{max})\}}{\min\{\tilde{\beta}(\theta_{max}), 1\}}.$$

This completes the proof. □

Remark 2.10. *Using the same technique as in Theorem 2.5, for the B_π^R-matrix with $\theta_{max} = 0$ in Theorem 2.9, the corresponding error bound can also be analyzed. For reasons of space, we here limit our discussion to the case $\theta_{max} \neq 0$.*

Example 2.11. *Consider the following matrix*

$$M_3 = \begin{bmatrix} 9.1 & 0 & -0.1 & 0 \\ 0 & 9.1 & -0.1 & 0 \\ 2.9 & 2.9 & 0.3 & 2.9 \\ 0 & 0 & -0.1 & 9.1 \end{bmatrix}.$$

Observe that M_3 is a B_π^R-matrix with $\pi = [\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}]^T$ and $R = [9, 9, 9, 9]^T$. Since $\pi_3 = 0$, which does not satisfy the assumption of Theorem 1.2, so we cannot use the bound (3) to estimate $\max_{d\in[0,1]^4} \|(I - D + DM_3)^{-1}\|_\infty$. However, take $j_0 = 1$ such that $\pi_{j_0} = \frac{1}{3}$, we have

$$\theta_{max} = \max_{i\neq j_0}\{\frac{m_{i,j_0}}{R_i}\} = \frac{29}{90} < \frac{1}{3} = 1 - \sum_{i\neq j_0,s} \pi_i,$$

$\xi_1 = \frac{60}{29}$, $\xi_2 = 2$, $\tilde{\beta}(\theta_{max}) = \frac{1}{10}$, and $\xi(\theta_{max}) = 91$, which satisfy the hypotheses of Theorem 2.9, so by Theorem 2.9 it holds that

$$\max_{d\in[0,1]^4} \|(I - D + DM_3)^{-1}\|_\infty \leq \frac{\max\{\xi_1, \xi_2, \xi(\theta_{max})\}}{\min\{\tilde{\beta}(\theta_{max}), 1\}} = 910.$$

2.3. *Error bounds for case (c)*

Consider a B_π^R-matrix $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ for a vector $\pi = [\pi_1, \ldots, \pi_n]^T$ only with $\pi_s \leq 0, \pi_t \leq 0$ and $s < t$. Then we can write

$$M = B^+(\omega) + C(\omega), \tag{41}$$

where

$$B^+(\omega) = \begin{bmatrix} m_{11}-\pi_1R_1 & \cdots & m_{1,j_0}-\omega R_1 & \cdots & m_{1,s-1}-\pi_{s-1}R_1 & m_{1s} & m_{1,s+1}-\pi_{s+1}R_1 & \cdots & m_{1,t-1}-\pi_{t-1}R_1 & m_{1t} & m_{1,t+1}-\pi_{t+1}R_1 & \cdots & m_{1n}-\pi_nR_1 \\ \vdots & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ m_{n1}-\pi_1R_n & \cdots & m_{n,j_0}-\omega R_n & \cdots & m_{n,s-1}-\pi_{s-1}R_n & m_{ns} & m_{n,s+1}-\pi_{s+1}R_n & \cdots & m_{n,t-1}-\pi_{t-1}R_n & m_{nt} & m_{n,t+1}-\pi_{t+1}R_n & \cdots & m_{nn}-\pi_nR_n \end{bmatrix}$$

and

$$C(\omega) = \begin{bmatrix} \pi_1R_1 & \cdots & \omega R_1 & \cdots & \pi_{s-1}R_1 & 0 & \pi_{s+1}R_1 & \cdots & \pi_{t-1}R_1 & 0 & \pi_{t+1}R_1 & \cdots & \pi_nR_1 \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \pi_1R_n & \cdots & \omega R_n & \cdots & \pi_{s-1}R_n & 0 & \pi_{s+1}R_n & \cdots & \pi_{t-1}R_n & 0 & \pi_{t+1}R_n & \cdots & \pi_nR_n \end{bmatrix}$$

with ω is an adjustable parameter and $j_0 \in N \setminus \{s, t\}$.

Similarly to the proof the Lemma 2.1, a sufficient condition can be obtained such that $B^+(\omega)$ is an SDD matrix with positive diagonal entries.

Lemma 2.12. *Let $M = [m_{ij}] \in \mathbb{R}^{n\times n}$ be a B_π^R-matrix for a vector $\pi = [\pi_1, \ldots, \pi_n]^T$ only with $\pi_s \leq 0, \pi_t \leq 0, s < t$, and $B^+(\omega)$ be the matrix of (41). If there exists an index $j_0 \in N \setminus \{s, t\}$ such that*

$$\max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\} < 1 - \sum_{i\neq j_0,s,t} \pi_i,$$

then, for each $\omega \in [\max_{i\neq j_0}\{\frac{m_{i,j_0}}{R_i}\}, 1 - \sum_{i\neq j_0,s,t} \pi_i)$, $B^+(\omega)$ is an SDD Z-matrix with positive diagonal entries.

Based on Lemma 2.12, we next address error bounds for the linear complementarity problems of B_π^R-matrices under **Case (c)**. Before that, some notations are listed. For a B_π^R-matrix, denote

$$\zeta_1 := \frac{\sum\limits_{i\neq j_0,s,t} \pi_i}{\max\limits_{i\neq j_0}\{\frac{m_{i,j_0}}{R_i}\}},\ \zeta_2 := \max_{i\in N\setminus\{j_0,s,t\}}\left\{\frac{1}{\pi_i}-1\right\}, \tag{42}$$

and

$$\zeta(\omega) = 1+\frac{\max\limits_{i\in N}\{R_i\}}{\min\{\hat{\beta}(\omega),1\}}, \tag{43}$$

where $\hat{\beta}(\omega) := \min_i\{\hat{\beta}_i\}$ and $\hat{\beta}_i = R_i(1-\sum\limits_{i\neq j_0,s,t}\pi_i-\omega)$.

Theorem 2.13. *Suppose that $M=[m_{ij}]\in\mathbb{R}^{n\times n}$ is a B_π^R-matrix for a vector $\pi=[\pi_1,\ldots,\pi_n]^T$ only with $\pi_s\leq 0, \pi_t\leq 0, s<t$, and $0\neq\theta_{max}<1-\sum\limits_{i\neq j_0,s,t}\pi_i$ for some $j_0\in N\setminus\{s,t\}$, where θ_{max} is given by (5). Then for each $\omega\in I_3:=[\theta_{max},1-\sum\limits_{i\neq j_0,s,t}\pi_i)$,*

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty\leq\frac{\max\{\zeta_1,\zeta_2,\zeta(\omega)\}}{\min\{\hat{\beta}(\omega),1\}},$$

where ζ_1, ζ_2 and $\zeta(\omega)$ are given by (42) and (43), respectively, and $\hat{\beta}(\omega) := \min\limits_{i\in N}\{R_i\}(1-\sum\limits_{i\neq j_0,s,t}\pi_i-\omega)$. Furthermore,

$$\min_{\omega\in I_3}\left\{\frac{\max\{\zeta_1,\zeta_2,\zeta(\omega)\}}{\min\{\hat{\beta}(\omega),1\}}\right\}=\frac{\max\{\zeta_1,\zeta_2,\zeta(\theta_{max})\}}{\min\{\hat{\beta}(\theta_{max}),1\}}.$$

Proof. Let $M=B^+(\omega)+C(\omega)$ given by (41), $B_D^+=I-D+DB^+(\omega)$ and $C_D=DC(\omega)$. Then, similar to the proof of Theorem 2.5, we can see that

$$\|(I-D+DM)^{-1}\|_\infty\leq\|\left(I+(B_D^+)^{-1}C_D\right)^{-1}\|_\infty\cdot\|(B_D^+)^{-1}\|_\infty, \tag{44}$$

and

$$\|(B_D^+)^{-1}\|_\infty\leq\frac{1}{\min\{\hat{\beta}(\omega),1\}}. \tag{45}$$

We next bound $\|\left(I+(B_D^+)^{-1}C_D\right)^{-1}\|_\infty$. Let $Q:=I+(B_D^+)^{-1}C_D$ and $(B_D^+)^{-1}=:[\tilde{b}_{ij}]$ with $\tilde{b}_{ij}\geq 0$ for all $i,j\in N$. Without loss of generality, we assume that there exists an index $j_0=1\in N\setminus\{s,t\}$ such that $0\neq\theta_{max}<1-\sum\limits_{i\neq j_0,s,t}\pi_i$. Then,

$$Q=\begin{bmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix},$$

where

$$B_{11}=\begin{bmatrix} 1+\omega a_1 & \pi_2 a_1 & \cdots & \pi_{s-1}a_1\\ \omega a_2 & 1+\pi_2 a_2 & \cdots & \pi_{s-1}a_2\\ \vdots & \vdots & & \vdots\\ \omega a_{s-1} & \pi_2 a_{s-1} & \cdots & 1+\pi_{s-1}a_{s-1}\end{bmatrix},$$

$$B_{12}=\begin{bmatrix} 0 & \pi_{s+1}a_1 & \cdots & \pi_{t-1}a_1 & 0 & \pi_{t+1}a_1 & \cdots & \pi_n a_1 \\ 0 & \pi_{s+1}a_2 & \cdots & \pi_{t-1}a_2 & 0 & \pi_{t+1}a_2 & \cdots & \pi_n a_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \pi_{s+1}a_{s-1} & \cdots & \pi_{t-1}a_{s-1} & 0 & \pi_{t+1}a_{s-1} & \cdots & \pi_n a_{s-1} \end{bmatrix},$$

$$B_{21}=\begin{bmatrix} \omega a_s & \pi_2 a_s & \cdots & \pi_{s-1}a_s \\ \omega a_{s+1} & \pi_2 a_{s+1} & \cdots & \pi_{s-1}a_{s+1} \\ \vdots & \vdots & & \vdots \\ \omega a_n & \pi_2 a_n & \cdots & \pi_{s-1}a_n \end{bmatrix},$$

and

$$B_{22}=\begin{bmatrix} 1 & \pi_{s+1}a_s & \cdots & \pi_{t-1}a_s & 0 & \pi_{t+1}a_s & \cdots & \pi_n a_s \\ 0 & 1+\pi_{s+1}a_{s+1} & \cdots & \pi_{t-1}a_{s+1} & 0 & \pi_{t+1}a_{s+1} & \cdots & \pi_n a_{s+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \pi_{s+1}a_{t-1} & \cdots & 1+\pi_{t-1}a_{t-1} & 0 & \pi_{t+1}a_{t-1} & \cdots & \pi_n a_{t-1} \\ 0 & \pi_{s+1}a_t & \cdots & \pi_{t-1}a_t & 1 & \pi_{t+1}a_t & \cdots & \pi_n a_t \\ 0 & \pi_{s+1}a_{t+1} & \cdots & \pi_{t-1}a_{t+1} & 0 & 1+\pi_{t+1}a_{t+1} & \cdots & \pi_n a_{t+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \pi_{s+1}a_n & \cdots & \pi_{t-1}a_n & 0 & \pi_{t+1}a_n & \cdots & 1+\pi_n a_n \end{bmatrix}$$

with $a_i := \sum_{j=1}^{n} \tilde{b}_{ij}d_jR_j \geq 0$ for all $i \in N$. By Lemma 2.3 and 2.4, we have

$$B_{11}^{-1}=\begin{bmatrix} 1-\frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \frac{\pi_2}{\omega}(-\frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i}) & \cdots & \frac{\pi_{s-1}}{\omega}(-\frac{\bar{a}_1}{1+\sum_{i=1}^{s-1}\bar{a}_i}) \\ \frac{\omega}{\pi_2}(-\frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i}) & 1-\frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i} & \cdots & \frac{\pi_{s-1}}{\pi_2}(-\frac{\bar{a}_2}{1+\sum_{i=1}^{s-1}\bar{a}_i}) \\ \vdots & \vdots & & \vdots \\ \frac{\omega}{\pi_{s-1}}(-\frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i}) & \frac{\pi_2}{\pi_{s-1}}(-\frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i}) & \cdots & 1-\frac{\bar{a}_{s-1}}{1+\sum_{i=1}^{s-1}\bar{a}_i} \end{bmatrix}, \tag{46}$$

and

$$Q/B_{11}=\left[\begin{array}{cccc|cccc} 1 & \pi_{s+1}c_s & \cdots & \pi_{t-1}c_s & 0 & \pi_{t+1}c_s & \cdots & \pi_n c_s \\ 0 & 1+\pi_{s+1}c_{s+1} & \cdots & \pi_{t-1}c_{s+1} & 0 & \pi_{t+1}c_{s+1} & \cdots & \pi_n c_{s+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \pi_{s+1}c_{t-1} & \cdots & 1+\pi_{t-1}c_{t-1} & 0 & \pi_{t+1}c_{t-1} & \cdots & \pi_n c_{t-1} \\ \hline 0 & \pi_{s+1}c_t & \cdots & \pi_{t-1}c_t & 1 & \pi_{t+1}c_t & \cdots & \pi_n c_t \\ 0 & \pi_{s+1}c_{t+1} & \cdots & \pi_{t-1}c_{t+1} & 0 & 1+\pi_{t+1}c_{t+1} & \cdots & \pi_n c_{t+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \pi_{s+1}c_n & \cdots & \pi_{t-1}c_n & 0 & \pi_{t+1}c_n & \cdots & 1+\pi_n c_n \end{array}\right]$$
$$=: \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where $c_k = \frac{a_k}{1+\sum_{i=1}^{s-1}\bar{a}_i} \geq 0$ for all $k = s, \ldots, n$ with $\bar{a}_1 = a_1\omega$ and $\bar{a}_i = \pi_i a_i$ for $i = 2, \ldots, s-1$.

We now compute $(Q/B_{11})^{-1}$. According to (14) and (17), it holds that

$$C_{11}^{-1}=\begin{bmatrix} 1 & -\frac{\pi_{s+1}c_s}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & -\frac{\pi_{s+2}c_s}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_s}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \\ 0 & 1-\frac{\pi_{s+1}c_{s+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & -\frac{\pi_{s+2}c_{s+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_{s+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{s+1}c_{t-1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & -\frac{\pi_{s+2}c_{t-1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & 1-\frac{\pi_{t-1}c_{t-1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \end{bmatrix},$$

and

$$(Q/B_{11})/C_{11}=\begin{bmatrix} 1 & \frac{\pi_{t+1}c_t}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \frac{\pi_{t+2}c_t}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & \frac{\pi_{n}c_t}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \\ 0 & 1+\frac{\pi_{t+1}c_{t+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \frac{\pi_{t+2}c_{t+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & \frac{\pi_{n}c_{t+1}}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{\pi_{t+1}c_n}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \frac{\pi_{t+2}c_n}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} & \cdots & 1+\frac{\pi_{n}c_n}{1+\sum\limits_{i=s+1}^{t-1}\bar{c}_i} \end{bmatrix},$$

where $\bar{c}_i=\pi_i c_i\geq 0$ for $i=s+1,\ldots,t-1$. So,

$$((Q/B_{11})/C_{11})^{-1}=\begin{bmatrix} 1 & -\frac{\pi_{t+1}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{n}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ 0 & 1-\frac{\pi_{t+1}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{n}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{t+1}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & 1-\frac{\pi_{n}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \end{bmatrix}.$$

where $\bar{c}_i=\pi_i c_i$ for $i=s+1,\ldots,t-1,t+1,\ldots,n$. Then, from (8), we can deduce that

$$(Q/B_{11})^{-1}=\begin{bmatrix} C_{11}^{-1}+C_{11}^{-1}C_{12}((Q/B_{11})/C_{11})^{-1}C_{21}C_{11}^{-1} & -C_{11}^{-1}C_{12}((Q/B_{11})/C_{11})^{-1} \\ -((Q/B_{11})/C_{11})^{-1}C_{21}C_{11}^{-1} & ((Q/B_{11})/C_{11})^{-1} \end{bmatrix}, \tag{47}$$

where

$$C_{11}^{-1}+C_{11}^{-1}C_{12}((Q/B_{11})/C_{11})^{-1}C_{21}C_{11}^{-1}=\begin{bmatrix} 1 & -\frac{\pi_{s+1}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ 0 & 1-\frac{\pi_{s+1}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{s+1}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & 1-\frac{\pi_{t-1}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \end{bmatrix},$$

$$-C_{11}^{-1}C_{12}((Q/B_{11})/C_{11})^{-1}=\begin{bmatrix} 0 & -\frac{\pi_{t+1}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{n}c_s}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ 0 & \frac{\pi_{t+1}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{n}c_{s+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{t+1}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{t+2}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{n}c_{t-1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \end{bmatrix},$$

and

$$-((Q/B_{11})/C_{11})^{-1}C_{21}C_{11}^{-1} = \begin{bmatrix} 0 & -\frac{\pi_{s+1}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_t}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ 0 & -\frac{\pi_{s+1}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_{t+1}}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{s+1}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & -\frac{\pi_{s+2}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} & \cdots & -\frac{\pi_{t-1}c_n}{1+\sum\limits_{i=s+1,i\neq t}^{n}\bar{c}_i} \end{bmatrix}.$$

Denote $q =: (1 + \sum\limits_{i=1}^{s-1} \bar{a}_i)(1 + \sum\limits_{i=s+1,i\neq t}^{n} \bar{c}_i)$. By (8), (46) and (47), it follows that

$$Q^{-1} = \begin{bmatrix} B_{11}^{-1} + B_{11}^{-1}B_{12}(Q/B_{11})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(Q/B_{11})^{-1} \\ -(Q/B_{11})^{-1}B_{21}B_{11}^{-1} & (Q/B_{11})^{-1} \end{bmatrix}, \tag{48}$$

where

$$B_{11}^{-1} + B_{11}^{-1}B_{12}(Q/B_{11})^{-1}B_{21}B_{11}^{-1} = \begin{bmatrix} 1 - \frac{\bar{a}_1}{q} & \frac{\pi_2}{\omega}(-\frac{\bar{a}_1}{q}) & \cdots & \frac{\pi_{s-1}}{\omega}(-\frac{\bar{a}_1}{q}) \\ \frac{\omega}{\pi_2}(-\frac{\bar{a}_2}{q}) & 1 - \frac{\bar{a}_2}{q} & \cdots & \frac{\pi_{s-1}}{\pi_2}(-\frac{\bar{a}_2}{q}) \\ \vdots & \vdots & & \vdots \\ \frac{\omega}{\pi_{s-1}}(-\frac{\bar{a}_{s-1}}{q}) & \frac{\pi_2}{\pi_{s-1}}(-\frac{\bar{a}_{s-1}}{q}) & \cdots & 1 - \frac{\bar{a}_{s-1}}{q} \end{bmatrix},$$

$$-B_{11}^{-1}B_{12}(Q/B_{11})^{-1} = \begin{bmatrix} 0 & -\frac{\pi_{s+1}a_1}{q} & \cdots & -\frac{\pi_{t-1}a_1}{q} & 0 & -\frac{\pi_{s+1}a_1}{q} & \cdots & -\frac{\pi_n a_1}{q} \\ 0 & -\frac{\pi_{s+1}a_2}{q} & \cdots & -\frac{\pi_{t-1}a_2}{q} & 0 & -\frac{\pi_{s+1}a_2}{q} & \cdots & -\frac{\pi_n a_2}{q} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & -\frac{\pi_{s+1}a_{s-1}}{q} & \cdots & -\frac{\pi_{t-1}a_{s-1}}{q} & 0 & -\frac{\pi_{t+1}a_{s-1}}{q} & \cdots & -\frac{\pi_n a_{s-1}}{q} \end{bmatrix},$$

and

$$-(Q/B_{11})^{-1}B_{21}B_{11}^{-1} = \begin{bmatrix} -\frac{\omega a_s}{q} & -\frac{\pi_2 a_s}{q} & \cdots & -\frac{\pi_{s-1}a_s}{q} \\ -\frac{\omega a_{s+1}}{q} & -\frac{\pi_2 a_{s+1}}{q} & \cdots & -\frac{\pi_{s-1}a_{s+1}}{q} \\ \vdots & \vdots & & \vdots \\ -\frac{\omega a_n}{q} & -\frac{\pi_2 a_n}{q} & \cdots & -\frac{\pi_{s-1}a_n}{q} \end{bmatrix}.$$

Owing to (48) and $\bar{a}_i \geq 0$ for all $i \in N \setminus \{s, t\}$, we get that

$$\|(I + (B_D^{+})^{-1}C_D)^{-1}\|_\infty = 1 + \frac{\bar{a}_i}{q}\left(\frac{\sum\limits_{i\neq 1,s,t}\pi_i}{\omega} - 1\right) \tag{49}$$

for $i = 1$, or

$$\|(I + (B_D^{+})^{-1}C_D)^{-1}\|_\infty = 1 + \frac{\bar{a}_i}{q}\left(\frac{\omega + \sum\limits_{j\neq 1,i,s,t}\pi_j}{\pi_i} - 1\right) \tag{50}$$

for some $i \in N \setminus \{1, s, t\}$, or

$$\|(I + (B_D^{+})^{-1}C_D)^{-1}\|_\infty = \frac{a_i}{q}\Big(\omega + \sum_{j\neq 1,s,t}\pi_j\Big) + 1 \tag{51}$$

for some $i \in \{s, t\}$.

Since $q > 0$, $a_i \geq 0$ for all $i \in N$, and $\bar{a}_i \geq 0$ for $i \in N \setminus \{s, t\}$, it follows that formula (49) can be bounded above by

$$1 + \left(\frac{\sum\limits_{i \neq 1, s, t} \pi_i}{\omega} - 1 \right) = \frac{\sum\limits_{i \neq 1, s, t} \pi_i}{\omega} \leq \frac{\sum\limits_{i \neq 1, s, t} \pi_i}{\max\limits_{i \neq 1} \{\frac{m_{i1}}{R_i}\}},$$

formula (50) can be bounded above by

$$1 + \left(\frac{\omega + \sum\limits_{j \neq 1, i, s, t} \pi_j}{\pi_i} - 1 \right) = \frac{\omega + \sum\limits_{j \neq 1, i, s, t} \pi_j}{\pi_i} \leq \frac{1 - \sum\limits_{j \neq 1, s, t} \pi_j + \sum\limits_{j \neq 1, i, s, t} \pi_j}{\pi_i} = \frac{1}{\pi_i} - 1,$$

and that formula (51) can be bounded above by

$$1 + \Big(\omega + \sum_{j \neq 1, s, t} \pi_j \Big) a_i \leq 1 + a_i = 1 + \sum_{j=1}^{n} \tilde{b}_{ij} d_j R_j \leq 1 + \sum_{j=1}^{n} \tilde{b}_{ij} \cdot \max_{i \in N} \{R_i\} \leq 1 + \frac{\max\limits_{i \in N} \{R_i\}}{\min\{\hat{\beta}(\omega), 1\}}.$$

Notice that $j_0 = 1$, $\zeta_1 := \frac{\sum_{i \neq j_0, s, t} \pi_i}{\max_{i \neq j_0} \{\frac{m_{i,j_0}}{R_i}\}}$, $\zeta_2 := \max\limits_{i \in N \setminus \{j_0, s, t\}} \left\{ \frac{1}{\pi_i} - 1 \right\}$, and $\zeta(\omega) := 1 + \frac{\max_{i \in N} \{R_i\}}{\min\{\hat{\beta}(\omega), 1\}}$. Then,

$$\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty \leq \max \{\zeta_1, \zeta_2, \zeta(\omega)\},$$

which together with (44) and (45) imply that

$$\|M_D^{-1}\|_\infty \leq \frac{\max \{\zeta_1, \zeta_2, \zeta(\omega)\}}{\min\{\hat{\beta}(\omega), 1\}}.$$

Additionally, because $\hat{\beta}(\omega) = \min\limits_{i \in N} \{R_i\} (1 - \sum\limits_{i \neq j_0, s, t} \pi_i - \omega)$ is decreasing for $\omega \in I_3 := [\theta_{max}, 1 - \sum\limits_{i \neq j_0, s, t} \pi_i)$, it follows that

$$\max_{\omega \in I_3} \hat{\beta}(\omega) = \hat{\beta}(\theta_{max}) \text{ and } \min_{\omega \in I_3} \zeta(\omega) = \zeta(\theta_{max}).$$

Hence,

$$\min_{\omega \in I_3} \left\{ \frac{\max\{\zeta_1, \zeta_2, \zeta(\omega)\}}{\min\{\hat{\beta}(\omega), 1\}} \right\} = \frac{\max \{\zeta_1, \zeta_2, \zeta(\theta_{max})\}}{\min\{\hat{\beta}(\theta_{max}), 1\}}.$$

This completes the proof. □

Remark 2.14. *Using the same skills in the proof of Theorem 2.5, for B_π^R-matrices with $\theta_{max} = 0$ in Theorem 2.13, the corresponding error bound can also be analyzed, for reasons of space, we here only discuss the case of $\theta_{max} \neq 0$.*

Example 2.15. *Consider the following matrix*

$$M_4 = \begin{bmatrix} \frac{8}{3} & -1 & \frac{1}{3} & -1 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & -1 \\ \frac{1}{3} & -1 & \frac{8}{3} & -1 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

It is easy to validate that M_4 is a B_π^R-matrix with $\pi = [\frac{19}{20}, -\frac{1}{5}, \frac{1}{2}, -\frac{1}{4}]^T$ and $R = [1,1,1,1]^T$. Note that $\pi_2 < 0$ and $\pi_4 < 0$, which does not satisfy the assumption of Theorem 1.2, so we cannot use the bound (3) to estimate $\max\limits_{d\in[0,1]^4} \|(I - D + DM_4)^{-1}\|_\infty$. However, take $j_0 = 1$ such that $\pi_{j_0} = \frac{19}{20}$, we have

$$\theta_{max} = \max\left\{0, \max_{i\neq j_0}\left\{\frac{m_{i,j_0}}{R_i}\right\}\right\} = \frac{1}{3} < \frac{1}{2} = 1 - \sum_{i\neq 1,2,4} \pi_i,$$

$\zeta_1 = \frac{3}{2}$, $\zeta_2 = 1$, $\hat{\beta}(\frac{1}{3}) = \frac{1}{6}$, and $\zeta(\frac{1}{3}) = 7$, which satisfy the hypotheses of Theorem 2.13. Then, due to Theorem 2.13, we get

$$\max_{d\in[0,1]^4} \|(I - D + DM_4)^{-1}\|_\infty \leq \frac{\max\left\{\zeta_1, \zeta_2, \zeta(\frac{1}{3})\right\}}{\min\left\{\hat{\beta}\left(\frac{1}{3}\right), 1\right\}} = 42.$$

Conflicts of interest. The authors declare no potential conflict of interest.

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