Filomat 39:14 (2025), 4757–4764
https://doi.org/10.2298/FIL2514757D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sufficient conditions for a Lorentzian manifold to be a generalized Robertson-Walker spacetime

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Abstract. The aim of this article is to find the conditions under which a Lorentzian manifold is a generalized Robertson-Walker spacetime. We illustrate that a Lorentzian manifold admitting a semi-symmetric metric connection represents a generalized Robertson-Walker spacetime under certain restrictions on the Ricci tensor and torsion tensor. As a consequence, we establish that such a manifold becomes a static spacetime.

1. Introduction

According to general relativity, a space-time is a Lorentzian manifold (M^4, g) with the signature $(+,+,+,-)$ that allows a vector that is globally time-oriented. Perfect fluids(PFs), the source of Einstein's field equations, play an intriguing role in general relativity. Scientists have been studying plasma physics, nuclear physics, and astrophysics with PF models nowadays. A wide range of methods have been used by numerous geometers to study space-times in ([2], [8], [9]) and numerous other places.

The generalized Robertson-Walker (GRW) spacetime is nothing but a n- dimensional $(n \geq 4)$ Lorentzian manifold which can also be expressed as a warped product $-I \times_{\varphi^2} \overset{*}{M}$, where open interval I contained in $\mathbb{R}$, $\overset{*}{M}$ is an $(n-1)$-dimensional Riemannian manifold and $\varphi > 0$ indicates the scale factor. In 1995, the foregoing concept was introduced by Alías et al. [1]. To learn more about GRW spacetimes, see ([5], [17], [18]).

If M^4 is a PF spacetime, then the non-zero Ricci tensor $\mathcal{R}_{jk}$ fulfills

$$\mathcal{R}_{jk} = a g_{jk} + b u_j u_k, \tag{1}$$

in which a, b stand for scalars and the velocity vector u_k is unit and time-like, that is, $u_k u^k = -1$, $u^k = g^{lk} u_l$. The EMT in a PF spacetime [20], is demonstrated by

$$T_{jk} = \rho g_{jk} + (\rho + \mu) u_j u_k, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 53B05; Secondary 53B30; 53C50; 53Z05.
Keywords. Semi-symmetric metric connection; concircular vector; generalized Robertson-Walker spacetime; static spacetime.
Received: 05 January 2025; Accepted: 17 February 2025
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ρ and μ indicate the isotropic pressure and the energy density, respectively.

In M^4, conformal curvature tensor C_{hijk} is written by

$$C_{hijk} = \mathcal{R}_{hijk} - \frac{1}{2}\left\{g_{ij}\mathcal{R}_{hk} - g_{ik}\mathcal{R}_{hj} + g_{lk}\mathcal{R}_{ij} - g_{hj}\mathcal{R}_{ik}\right\} + \frac{\mathcal{R}}{6}\left\{g_{hk}g_{ij} - g_{hj}g_{ik}\right\}. \tag{3}$$

It is well-known that [12]

$$\nabla_l C^l_{ijk} = \frac{1}{2}\left\{\left(\nabla_k\mathcal{R}_{ij} - \nabla_j\mathcal{R}_{ik}\right) - \frac{1}{6}\left(g_{ij}\nabla_k\mathcal{R} - g_{ik}\nabla_j\mathcal{R}\right)\right\}. \tag{4}$$

In 1924, the concept of semi-symmetric linear connection was presented by Friedmann and Schouten [13]. In a Riemannian manifold, the concept of metric connection with torsion tensor was initiated by Hayden [15]. After that, Yano [25] continued to study semi-symmetric metric connection (SSMC). SSMC performs a significant role in general relativity theory as well as in Riemannian geometry.

Let ∇ be a Levi-Civita connection on a 4-dimensional Lorentzian manifold M^4. A linear connection $\widetilde{\nabla}$ on M^4 is named a SSMC ([13], [15]) if the metric tensor g of M^4 and the torsion tensor T of the connection $\widetilde{\nabla}$ obeys:

$$T^h_{ij} = \delta^h_i p_j - \delta^h_j p_i \tag{5}$$

and

$$\widetilde{\nabla}_k g_{ij} = 0, \tag{6}$$

where p_i is a unit time-like vector (that is, $p_ip^i = -1$, $p^i = g^{ij}p_j$) and p_i is named the associated vector of the SSMC. The components of SSMC are presented by [25]

$$\Gamma^h_{ij} = \left\{{h \atop ij}\right\} + \delta^h_i p_j - g_{ij}p^h \tag{7}$$

and

$$\widetilde{\nabla}_k p_j = \nabla_k p_j - p_k p_j - g_{jk}, \tag{8}$$

where $\left\{{h \atop ij}\right\}$ and Γ^h_{ij} are the Christoffel symbols of ∇ and $\widetilde{\nabla}$, respectively.

By the aid of (7) and (8), the curvature tensors $\widetilde{\mathcal{R}}_{hijk}$ and $\mathcal{R}_{hijk}$ of $\widetilde{\nabla}$ and ∇ respectively, are related by [25]

$$\widetilde{\mathcal{R}}_{hijk} = \mathcal{R}_{hijk} - g_{hk}A_{ij} + g_{ik}A_{hj} - g_{ij}A_{hk} + g_{hj}A_{ik}, \tag{9}$$

where

$$A_{ij} = \nabla_i p_j - p_i p_j - \frac{1}{2}g_{ij}. \tag{10}$$

Several authors ([4], [7], [10], [14], [16], [27], [28], [29]) have investigated SSMC in various ways.

Motivated by the above studies, in this article we consider a Lorentzian manifold admitting a SSMC whose torsion tensor T is pseudo-symmetric in the sense of Chaki and obtained several important results about a GRW spacetime.

2. Preliminaries

Contracting h and i in (5), we notice that

$$T^h_{hj} = 3p_j. \tag{11}$$

This implies

$$\widetilde{\nabla}_k T^h_{hj} = 3\widetilde{\nabla}_k p_j. \tag{12}$$

Since $p_i p^i = -1$, therefore $p^i(\nabla_k p_i) + p_i\left(\nabla_k p^i\right) = 0$, which implies

$$p^i(\nabla_k p_i) = 0. \tag{13}$$

The results shown below are necessary for the forthcoming results.

Theorem A. [11] A necessary and sufficient condition that the Ricci tensor $\widetilde{\mathcal{R}}_{hk}$ of a SSMC $\widetilde{\nabla}$ with the associated vector p_i to be symmetric is that p_i is closed.

Theorem B. [25] A necessary and sufficient condition for a Lorentzian manifold admitting a SSMC $\widetilde{\nabla}$ to be conformally flat is that $\widetilde{\mathcal{R}}_{hijk}$ vanishes.

Theorem C. [5] A necessary and sufficient condition for a Lorentzian manifold to be a GRW spacetime is that the spacetime has a time-like concircular vector v_i: $\nabla_k v_i = \Psi g_{ik}$, Ψ being a scalar.

3. SSMC with pseudo-symmetric torsion tensor

Definition 3.1. *[3] A Lorentzian manifold* M^4 *is named pseudo-symmetric if* $\mathcal{R}_{hijk}$ *obey:*

$$\nabla_l \mathcal{R}_{hijk} = 2\lambda_l \mathcal{R}_{hijk} + \lambda_h \mathcal{R}_{lijk} + \lambda_i \mathcal{R}_{hljk} + \lambda_j \mathcal{R}_{hilk} + \lambda_k \mathcal{R}_{hijl},$$

where λ_l *is a non-zero covariant vector, named the associated vector.*

Here, we consider a Lorentzian manifold allowing a SSMC whose torsion tensor T is pseudo-symmetric in the sense of Chaki [3], that is,

$$\widetilde{\nabla}_k T^h_{ij} = 2a_k T^h_{ij} + a_i T^h_{kj} + a_j T^h_{ik} + a^h T_{kij}, \tag{14}$$

where a_k is a non-zero vector, named the associated vector and $T_{kij} = g_{hk} T^h_{ij}$.

Contracting h and i in the foregoing equation (14), we obtain

$$\widetilde{\nabla}_k T^h_{hj} = 2a_k T^h_{hj} + a_h T^h_{kj} + a_j T^h_{hk} + a^h T_{khj}. \tag{15}$$

Using (5) and (11) in (15), we infer

$$\widetilde{\nabla}_k p_j = \frac{8}{3} a_k p_j + \frac{2}{3} a_j p_k - \frac{f}{3} g_{hk}, \tag{16}$$

where $f = a^h p_h$.

From (8) and (16), it follows that

$$\nabla_k p_j = \frac{8}{3} a_k p_j + \frac{2}{3} a_j p_k + p_k p_j + (1 - \frac{f}{3}) g_{jk}. \tag{17}$$

Suppose the Ricci tensor of the SSMC is symmetric, that is, $\widetilde{\mathcal{R}}_{ij} = \widetilde{\mathcal{R}}_{ji}$. Then by Theorem A, p_i is closed and hence $\nabla_k p_j = \nabla_j p_k$.

Therefore, from equation (17), we get $a_j p_k = a_k p_j$, which implies

$$a_j = -(p^k a_k) p_j = -f_1 p_j, \tag{18}$$

where $f_1 = p^k a_k$ and $p_i p^i = -1$.

With the help of equation (18), the equation (17) becomes

$$\nabla_k p_j = (1 - \frac{10}{3} f_1) p_k p_j + (1 - \frac{f}{3}) g_{jk}.$$

which entails

$$\nabla_k p_j = f_2 p_k p_j + f_3 g_{jk}, \tag{19}$$

where $f_2 = (1 - \frac{10}{3} f_1)$ and $f_3 = (1 - \frac{f}{3})$.

Thus, we obtain

$$\nabla_k p_j = p_k w_j + f_3 g_{jk}, \tag{20}$$

where $w_j = f_2 p_j$.

Here, p_j is closed and hence $w_j = f_2 p_j$ is closed, provided $f_1 = p^k a_k$ is constant.

A vector field p_j on a Lorentzian manifold or Riemannian manifold is called torse-forming [26] if

$$\nabla_k p_j = w_k p_j + f_4 g_{jk}, \tag{21}$$

where f_4 and w_k denote a scalar function and a 1-form, respectively.

If the 1-form w_k is closed or locally a gradient, that is, if a scalar function μ exists on an appropriate coordinate domain of the manifold such that $w_k = \nabla_k \mu$ holds on this set, then such a vector is called concircular [6]. Mantica, Suh and De in [19], proved the following:

Consider (M, g) as a n-dimensional $(n > 3)$ Lorentzian manifold that admits a unit concircular vector field of the type (20), then on this set, there exists an appropriate coordinate domain of M such that the spacetime is a GRW spacetime.

From the above result and (20), we can acquire:

Theorem 3.2. *If a Lorentzian manifold allows a SSMC whose Ricci tensor is symmetric and the torsion tensor is pseudo-symmetric, then the manifold becomes a GRW spacetime, provided $p^h a_h$= constant.*

Suppose the curvature tensor $\widetilde{\mathcal{R}}_{hijk}$ of the SSMC $\widetilde{\nabla}$ vanishes and $a_i = p_i$. Then obviously the Ricci tensor $\widetilde{\mathcal{R}}_{ij}$ vanishes.

From equation (9) we get

$$\mathcal{R}_{ij} = 2A_{ij} + A g_{ij}, \tag{22}$$

where $A = g^{ij} A_{ij}$.

Also, in this case equation (17) becomes

$$\nabla_i p_j = \frac{13}{3} p_i p_j + \frac{4}{3} g_{ij}, \tag{23}$$

since $a_i p^i = p^i p_i = -1$.

Using equations (10) and (23) in equation (22), we infer

$$\mathcal{R}_{ij} = \frac{20}{3} p_i p_j + (\frac{5}{3} + A) g_{ij}. \tag{24}$$

Again using equation (23) in equation (10), we obtain $A = g^{ij} A_{ij} = 0$. Thus, equation (24) becomes

$$\mathcal{R}_{ij} = \frac{20}{3} p_i p_j + \frac{5}{3} g_{ij}, \tag{25}$$

which implies $\mathcal{R} = 0$.

Since $\mathcal{R} = 0$, we have

$$\nabla_k \mathcal{R} = 0, \tag{26}$$

where $\mathcal{R}$ is the Ricci scalar.

As $\widetilde{\mathcal{R}}_{hijk} = 0$, from Theorem B, we obtain $C_{hijk} = 0$. This implies $\nabla_m C^m_{jkl} = 0$ [12], that is,

$$\nabla_k \mathcal{R}_{jl} - \nabla_l \mathcal{R}_{jk} = \frac{1}{6}\left\{g_{lj}\left(\nabla_k \mathcal{R}\right) - g_{jk}\left(\nabla_l \mathcal{R}\right)\right\}. \tag{27}$$

From (26) and (27), it follows that

$$\nabla_k \mathcal{R}_{jl} - \nabla_l \mathcal{R}_{jk} = 0. \tag{28}$$

Utilizing (25) in (28), we arrive at

$$p_l\left(\nabla_k p_j\right) + p_j\left(\nabla_k p_l\right) - p_k\left(\nabla_l p_j\right) - p_j\left(\nabla_l p_k\right) = 0. \tag{29}$$

Multiplying (29) with p^j and using (13), we have

$$\nabla_k p_l - \nabla_l p_k = 0. \tag{30}$$

Again, multiplying (30) with p^l and using (13), we infer

$$p^l\left(\nabla_l p_k\right) = 0. \tag{31}$$

From (29) and (30), it follows that

$$p_l\left(\nabla_k p_j\right) - p_k\left(\nabla_l p_j\right) = 0. \tag{32}$$

Multiplying (32) with p^l and using (31), we find

$$\nabla_k p_j = 0. \tag{33}$$

A spacetime is called static ([21], [22], p. 283) if p_i is Killing and irrotational. We know that in a smooth vector u following relation holds,

$$£_u g_{lk} = \nabla_l u_k + \nabla_k u_l,$$

£ is the Lie differentiation. As $\nabla_k p_i = 0$, $£_p g_{ki} = 0$, which means that p_i is Killing. Also, $\nabla_k p_i = 0$ gives p_i is irrotational. Hence, it is static. Thus, we provide:

Corollary 3.3. *If a Lorentzian manifold admits a SSMC with $p_i = a_i$, vanishing curvature tensor and pseudo-symmetric torsion tensor, then the manifold represents a static spacetime.*

4. SSMC with recurrent torsion tensor

Here, we choose a Lorentzian manifold admitting a SSMC whose torsion tensor T is recurrent [23], that is,

$$\widetilde{\nabla}_k T^h_{ij} = b_k T^h_{ij}, \tag{34}$$

b_k is a non-zero vector.

Now contracting h and i in (34), we acquire

$$\widetilde{\nabla}_k T^h_{hj} = b_k T^h_{hj}, \tag{35}$$

Equations (11), (12) and (35) together imply

$$\widetilde{\nabla}_k p_j = b_k p_j. \tag{36}$$

Using (36) in (8), we obtain

$$\nabla_k p_j = b_k p_j + p_k p_j + g_{jk}. \tag{37}$$

From (10) and (37), it follows that

$$A_{ij} = b_i p_j + \frac{1}{2} g_{ij}. \tag{38}$$

Equations (9) and (38) give us

$$\begin{aligned}\widetilde{\mathcal{R}}_{hijk} = \mathcal{R}_{hijk} &- \left(b_i p_j + \frac{1}{2} g_{ij}\right) g_{hk} + \left(b_h p_j + \frac{1}{2} g_{hj}\right) g_{ik} \\ &- \left(b_h p_k + \frac{1}{2} g_{hk}\right) g_{ij} + \left(b_i p_k + \frac{1}{2} g_{ik}\right) g_{hj}.\end{aligned} \tag{39}$$

Multiplying (39) with g^{ij}, we obtain

$$\widetilde{\mathcal{R}}_{hk} = \mathcal{R}_{hk} - \left(b_i p^i + 2\right) g_{hk} - 2\left(b_h p_k + \frac{1}{2} g_{hk}\right). \tag{40}$$

Interchanging h and k in (40), we reach

$$\widetilde{\mathcal{R}}_{kh} = \mathcal{R}_{kh} - \left(b_i p^i + 2\right) g_{kh} - 2\left(b_k p_h + \frac{1}{2} g_{kh}\right). \tag{41}$$

Subtracting (41) from (40) using $\widetilde{\mathcal{R}}_{hk} = \widetilde{\mathcal{R}}_{kh}$, we have

$$b_k p_h = b_h p_k. \tag{42}$$

Multiplying (42) with p^h, we have

$$b_k = -\left(b_h p^h\right) p_k, \tag{43}$$

that is,

$$b_k = -f_4 p_k, \qquad \text{where} \quad f_4 = b_i p^i. \tag{44}$$

Equations (37) and (44) reveal that

$$\nabla_k p_j = (1 - f_4)\, p_k p_j + g_{jk}, \tag{45}$$

which implies

$$\nabla_k p_j = \omega_k p_j + g_{jk}, \qquad \text{where} \quad \omega_k = (1 - f_4)\, p_k. \tag{46}$$

Now,

$$\nabla_i \omega_k - \nabla_k \omega_i = (1 - f_4)\{\nabla_i p_k - \nabla_k p_i\} + p_i\,(\nabla_k f_4) - p_k\,(\nabla_i f_4)\,. \tag{47}$$

As $\widetilde{\mathcal{R}}_{hk} = \widetilde{\mathcal{R}}_{kh}$, from Theorem A, $\nabla_i p_k = \nabla_k p_i$. Using this in (47), we get

$$\nabla_i \omega_k - \nabla_k \omega_i = p_i (\nabla_k f_4) - p_k (\nabla_i f_4). \tag{48}$$

If f_4 = constant, that is, $b_i p^i$ = constant, then ω_k is closed.
Since ω_k is closed, there is a scalar function λ which is constructed on an appropriate coordinate domain of M^4 such that [24]

$$\omega_k = \nabla_k \lambda \tag{49}$$

holds. Putting $V_i = p_i e^{-\lambda}$, we obtain

$$\nabla_k V_i = e^{-\lambda} \{(\nabla_k p_i) - p_i (\nabla_k \lambda)\}. \tag{50}$$

Equations (46), (49) and (50) reflect that

$$\nabla_k V_i = e^{-\lambda} g_{ik}. \tag{51}$$

Now, $V_i V^i = \left(p_i e^{-\lambda}\right)\left(p^i e^{-\lambda}\right) = -e^{-2\lambda} < 0$. So, V_i is a time-like concircular vector.
Thus from Theorem C, we conclude:

Theorem 4.1. *If a Lorentzian manifold admits a SSMC with $b_i p^i$ = constant, symmetric Ricci tensor and recurrent torsion tensor, then the manifold represents a* GRW *spacetime.*

Suppose $\widetilde{\mathcal{R}}_{ij} = 0$. Then from equation (9) it follows that

$$\mathcal{R}_{ij} = 2A_{ij} + A g_{ij}, \tag{52}$$

where

$$A = g^{ij} A_{ij}, \quad A_{ij} = \nabla_i p_j - p_i p_j - \frac{1}{2} g_{ij}. \tag{53}$$

From equation (38), we get

$$\mathcal{R}_{ij} = 2(b_i p_j + \frac{1}{2} g_{ij}) + A g_{ij}. \tag{54}$$

Since $\mathcal{R}_{ij}$ is symmetric, the above equation gives $b_i p_j = b_j p_i$ which readily entails

$$b_i = -f_5 p_i, \qquad \text{where} \quad f_5 = p^j b_j. \tag{55}$$

Then using the last equation in equation (54), we infer

$$\mathcal{R}_{ij} = -2 f_5 \, p_i p_j + (A+1) g_{ij}. \tag{56}$$

Therefore, we state:

Theorem 4.2. *If a Lorentzian manifold allows a SSMC whose Ricci tensor vanishes and torsion tensor is recurrent, then the manifold represents a PF-spacetime.*

5. Discussion

In Applied Mathematics, general relativity is the greatest achievement. General relativity is widely acknowledged as the most sophisticated and difficult theory of physics ever discovered. In literature, there are several necessary and sufficient conditions under which a Lorentzian manifold will be a GRW spacetime or PF-spacetime. In [8], the authors have established two necessary and sufficient conditions under which a Lorentzian manifold will be a PF-spacetime. In this article, we find a sufficient condition under which a Lorentzian manifold to be a GRW spacetime. In future, we or perhaps other researchers will search under which condition a Lorentzian manifold will be a GRW spacetime or PF-spacetime.

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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