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Lifting triple linear vector fields to Weil like functors on triple vector bundles

Wlodzimierz M. Mikulski[a]

[a] *Institute of Mathematics, Jagiellonian University, ul. Lojasiewicza 6, Cracow, Poland*

Abstract. Given a Weil algebra A, the concept of A-admissible systems $\diamond$ is introduced. The complete description is given of the Weil like functors (i.e. product preserving gauge bundle functors) F on the category of triple vector bundles in terms of the A^F-admissible systems $\diamond^F$. Given a Weil like functor F on the category of triple vector bundles, the complete description of natural operators C lifting triple linear vector fields Z on a triple vector bundle K to vector fields CZ on FK is presented.

1. Introduction

We assume that any manifold and any map between manifolds considered in the paper is smooth (i.e. of class C^∞).

Double vector bundles were introduced in [21] and studied or applied e.g. in [2, 9, 12–14]. Triple vector bundles were introduced in [13]. The definition of triple vector bundles, we use in the paper, is presented in Section 2. Let [3]-$\mathcal{VB}$ be the category of triple vector bundles.

The general concept of (gauge) bundle functors can be found in [8]. In the present paper we need the concept of Weil like functors (i.e. product preserving gauge bundle functors (ppgb-functors)) F on the category [3]-$\mathcal{VB}$, only. Respective definitions concerning ppgb-functors on [3]-$\mathcal{VB}$ can be found in Section 3.

Let A be a Weil algebra. Roughly speaking, an A-admissible system is a collection $\diamond$ of A-modules $U_1, ..., U_8$ being finite dimensional as real vector spaces together with a system of A-bilinear maps $\diamond^{(\nu,\mu,\kappa)}$: $U_\nu \times U_\mu \to U_\kappa$ satisfying respective conditions, see Definition 3.5.

The main result of the present paper is the complete description of the ppgb-functors F on the category [3]-$\mathcal{VB}$ in terms of the admissible systems. Namely, given an A-admissible system $\diamond$, we construct canonically the ppgb-functor $F^\diamond$ on [3]-$\mathcal{VB}$, see Example 3.10. Conversely, given a ppgb-functor F on [3]-$\mathcal{VB}$, we construct canonically the A^F-admissible system $\diamond^F$, see Example 5.1. Next, in Section 6, we prove that $F = F^{\diamond^F}$ modulo the isomorphism.

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Email address: wlodzimierz.mikulski@im.uj.edu.pl (Wlodzimierz M. Mikulski)
ORCID iD: ttps://orcid.org/0000-0002-2905-0461 (Wlodzimierz M. Mikulski)

In Sections 4 and 6, we observe that any ppgb-functor F on [3]-$\mathcal{VB}$ has values in [3]-$\mathcal{VB}$. So, we can compose ppgb-functors F^1 and F^2 on [3]-$\mathcal{VB}$ and obtain ppgb-functor $F^2 \circ F^1$ on [3]-$\mathcal{VB}$. In Section 7, we prove that $\diamond^{F^2\circ F^1} = \diamond^{F^1} \otimes \diamond^{F^2}$. In particular, F^1 and F^2 commute.

A triple linear vector field on a [3]-$\mathcal{VB}$-object K is a vector field Z on K such that the flow of Z is formed by (locally defined) [3]-$\mathcal{VB}$-isomorphisms. Thus, if F is a ppgb-functor on [3]-$\mathcal{VB}$, we have the (usual) flow operator $\mathcal{F}$ lifting triple linear vector fields Z on a [3]-$\mathcal{VB}$-object K into vector fields $\mathcal{F}(Z)$ on FK. This $\mathcal{F}$ is a gauge natural operator in the sense of [8].

In Section 11, after preparations in Sections 8—10, given a ppgb-functor F on [3]-$\mathcal{VB}$, we present the complete description of all gauge-natural operators C (like the flow operator) lifting triple linear vector fields Z on a [3]-$\mathcal{VB}$-object K into vector fields $C(Z)$ on FK.

The Weil like functors on double vector bundles are described in [17]. The Weil like functors on some important categories over manifolds are described e.g. in [1, 3, 7, 8, 10, 15–19, 22, 23]. Natural operators lifting vector fields are studied e.g. in [4–7, 11, 17, 20].

From now on, let

$$Q^o := \{(8,6),(8,7),(6,5),(7,5),(8,4),(4,3),(6,2),(7,3),(5,1),(4,2),(3,1),(2,1)\} \text{ and}$$

$$Q^{oo} := \{(2,3,4),(2,5,6),(3,5,7),(2,7,8),(3,6,8),(4,5,8)\}\,.$$

The category of fibred manifolds and their fibred maps will be denoted by $\mathcal{FM}$. All algebra homomorphism considered in this paper are assumed to be unital.

2. The category of triple vector bundles

Definition 2.1. *An almost triple vector bundle is a system $K = (K_8, K_7, ..., K_1)$ of vector bundles $K_i = (K_i, \tau_{(i,j)}, K_j)$ for any $(i, j) \in Q^o$ such that the diagram with the vertices $K_8, K_7, ..., K_1$ and the arrows $\tau_{(i,j)} : K_i \to K_j$ for $(i, j) \in Q^o$ is commutative, where Q^o is the set as in Introduction. (For the convenience, we propose to draw this (cubic) diagram with vertices $K_1(0,0,0)$, $K_2(1,0,0)$, $K_3(0,1,0)$, $K_4(1,1,0)$, $K_5(0,0,1)$, $K_6(1,0,1)$, $K_7(0,1,1)$, $K_8(1,1,1)$ in $\mathbf{R}^3$.) We call $K = K_8$ the total space of K (for the simplicity of notation we will use the same letter for an almost triple vector bundle and for its total space) and $M = K_1$ the base of K and $p_K = \tau_{(5,1)} \circ \tau_{(6,5)} \circ \tau_{(8,6)} : K \to M$ the projection of K.*

If $K^1 = (K^1_8, K^1_7, ..., K^1_1)$ is an another almost triple vector bundle then a morphism $f : K \to K^1$ is a system $f = (f_8, f_7, ..., f_2, f_1)$ of maps $f_i : K_i \to K^1_i$ for $i = 1, ..., 8$ such that (f_i, f_j) is a vector bundle map $\tau_{(i,j)} \to \tau^1_{(i,j)}$ for any $(i, j) \in Q^o$. We call $\underline{f} = f_1 : M \to M^1$ the base map of f. For the simplicity of notation we will use the same notation for a morphism and for its corresponding map between total spaces.

Example 2.2. *For any $m = (m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) \in \mathbf{N}^8$ (where $\mathbf{N} = \{0, 1, 2, ...\}$, we have the trivial almost triple vector bundle $K = \mathbf{R}^{[m]}$ such that $K_8 := \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{m_3} \times \mathbf{R}^{m_4} \times \mathbf{R}^{m_5} \times \mathbf{R}^{m_6} \times \mathbf{R}^{m_7} \times \mathbf{R}^{m_8}$, $K_7 := \mathbf{R}^{m_1}\times\mathbf{R}^{m_3}\times\mathbf{R}^{m_5}\times\mathbf{R}^{m_7}$, $K_6 := \mathbf{R}^{m_1}\times\mathbf{R}^{m_2}\times\mathbf{R}^{m_5}\times\mathbf{R}^{m_6}$, $K_5 := \mathbf{R}^{m_1}\times\mathbf{R}^{m_5}$, $K_4 := \mathbf{R}^{m_1}\times\mathbf{R}^{m_2}\times\mathbf{R}^{m_3}\times\mathbf{R}^{m_4}$, $K_3 := \mathbf{R}^{m_1}\times\mathbf{R}^{m_3}$, $K_2 := \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$, $K_1 := \mathbf{R}^{m_1}$ and $\tau_{(i,j)} : K_i \to K_j$ for $(i, j) \in Q^o$ are the obvious canonical projections.*

Definition 2.3. *An almost triple vector bundle K is called a triple vector bundle if there is $m \in \mathbf{N}^8$ such that K is locally isomorphic to $\mathbf{R}^{[m]}$ (from Example 2.2), i.e. for any $x \in M$ there exists an open neighborhood $U \subset M$ of x such that $K_{|U} = \mathbf{R}^{[m]}$ modulo an isomorphism of almost triple vector bundles.*

From now on, [3]-$\mathcal{VB}$ denotes the category of all triple vector bundles and their almost triple vector bundle morphisms.

Remark 2.4. *Some triple vector bundles appear naturally in differential geometry. For example let $D = (D, E_r, E_l, M, \tau_r, \tau_l, p_r, p_l)$ be a double vector bundle where $\tau_r : D \to E_r$ and $\tau_l : D \to E_l$ and $p_r : E_r \to M$ and $p_l : E_l \to M$ is the system of vector bundle projections of D. Applying the tangent functor T to D we obtain the triple vector bundle $K = TD$, where $\tau_{(i,j)} : K_i \to K_j$ for $(i, j) \in Q^o$ is defined as follows. We put $K_8 := TD$, $K_7 := TE_l$, $K_6 := TE_r$, $K_5 := TM$, $K_4 := D$, $K_3 := E_l$, $K_2 := E_r$ and $K_1 := M$. Next we put $\tau_{(8,7)} := T\tau_l$, $\tau_{(8,6)} := T\tau_r$, $\tau_{(6,5)} := Tp_r$,*

$\tau_{(7,5)} := Tp_l$, $\tau_{(8,4)} := p_{TD}$, $\tau_{(7,3)} := p_{TE_l}$, $\tau_{(6,2)} := p_{TE_r}$, $\tau_{(5,1)} := p_{TM}$, $\tau_{(4,2)} := \tau_r$, $\tau_{(4,3)} := \tau_l$, $\tau_{(2,1)} := p_r$, $\tau_{(3,1)} := p_l$. *In particular, if $p : E \to M$ is a vector bundle, then $D = TE$ is a double vector bundle, where $D = TE$, $E_r = E$, $E_l = TM$, $\tau_r = p_{TE}$, $\tau_l = Tp$, $p_r = p$, $p_l = p_{TM}$. Then we have the triple vector bundle $TTE := TD$, where $D = TE$. Putting $E = TM$, we obtain the triple vector bundle $TTTM$. Putting $E = T^*M$, we obtain the triple vector bundle TTT^*M.*

Proposition 2.5. *Let $m = (m_1, ..., m_8)$ and $\tilde m = (\tilde m_1, ..., \tilde m_8)$ be arbitrary 8-tuples of non-negative integers. Any [3]-$\mathcal{VB}$-map $f : \mathbf{R}^{[m]} \to \mathbf{R}^{[\tilde m]}$ is of the form*

$$\tilde x_1^{\tilde i_1} \circ f = a_1^{\tilde i_1}(x_1)\ ,\ \ \tilde x_2^{\tilde i_2} \circ f = b_{2i_2}^{\tilde i_2}(x_1)x_2^{i_2}\ ,\ \ \tilde x_3^{\tilde i_3} \circ f = c_{3i_3}^{\tilde i_3}(x_1)x_3^{i_3}\ ,$$

$$\tilde x_4^{\tilde i_4} \circ f = d_{4i_2i_3}^{\tilde i_4}(x_1)x_2^{i_2}x_3^{i_3} + e_{4i_4}^{\tilde i_4}(x_1)x_4^{i_4}\ ,\ \ \tilde x_5^{\tilde i_5} \circ f = A_{5i_5}^{\tilde i_5}(x_1)x_5^{i_5}\ ,$$

$$\tilde x_6^{\tilde i_6} \circ f = B_{6i_2i_5}^{\tilde i_6}(x_1)x_2^{i_2}x_5^{i_5} + C_{6i_6}^{\tilde i_6}(x_1)x_6^{i_6}\ ,\ \ \tilde x_7^{\tilde i_7} \circ f = D_{7i_3i_5}^{\tilde i_7}(x_1)x_3^{i_3}x_5^{i_5} + E_{7j_7}^{\tilde i_7}(x_1)x_7^{i_7}\ ,$$

$$\tilde x_8^{\tilde i_8} \circ f = H_{8i_2i_3i_5}^{\tilde i_8}(x_1)x_2^{i_2}x_3^{i_3}x_5^{i_5} + I_{8i_4i_5}^{\tilde i_8}(x_1)x_4^{i_4}x_5^{i_5} + J_{8i_3i_6}^{\tilde i_8}(x_1)x_3^{i_3}x_6^{i_6} + K_{8i_2i_7}^{\tilde i_8}(x_1)x_2^{i_2}x_7^{i_7} + L_{8i_8}^{\tilde i_8}(x_1)x_8^{i_8}$$

for arbitrary mappings $a_1^{\tilde i_1}, b_{2i_2}^{\tilde i_2}, c_{3i_3}^{\tilde i_3}, d_{4i_2i_3}^{\tilde i_4}, e_{4i_4}^{\tilde i_4}, A_{5i_5}^{\tilde i_5}, B_{6i_2i_5}^{\tilde i_6}, C_{6i_6}^{\tilde i_6}, D_{7i_3i_5}^{\tilde i_7}, E_{7i_7}^{\tilde i_7}, H_{8i_2i_3i_5}^{\tilde i_8}, I_{8i_4i_5}^{\tilde i_8}, J_{8i_3i_6}^{\tilde i_8}, K_{8i_2i_7}^{\tilde i_8}, L_{8i_8}^{\tilde i_8} : \mathbf{R}^{m_1} \to \mathbf{R}$, where $x_1 = (x_1^1, ..., x_1^{m_1})$ and where $x_\nu^{i_\nu}$ for $i_\nu = 1, ..., m_\nu$ and $\nu = 1, ..., 8$ are the usual coordinates on $\mathbf{R}^{[m]}$ and $\tilde x_\nu^{\tilde i_\nu}$ for $\tilde i_\nu = 1, ..., \tilde m_\nu$ and $\nu = 1, ..., 8$ are the usual coordinates on $\mathbf{R}^{[\tilde m]}$. In the above formulas the Einstein summation convention is used with respect to the indices $i_\nu = 1, ..., m_\nu$ for $\nu = 1, ..., 8$.

Proof. The proof is standard. □

Lemma 2.6. *Let $e_\nu = (0, ..., 1, ...0) \in \mathbf{N}^8$ (1 on ν-th position only) for $\nu = 1, ..., 8$, and let $(0) = (0, ..., 0) \in \mathbf{N}^8$. The sum map $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and the multiplication map $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ can be treated as the [3]-$\mathcal{VB}$-maps*

$$+ : \mathbf{R}^{[e_\nu]} \times \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}\ ,\ \text{and}\ \ \cdot : \mathbf{R}^{[e_1]} \times \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}\ \text{for}\ \nu = 1, ..., 8\ ,\ \text{and}$$

$$\cdot : \mathbf{R}^{[e_\nu]} \times \mathbf{R}^{[e_\mu]} \to \mathbf{R}^{[e_\kappa]}\ \text{for}\ (\nu, \mu, \kappa) \in Q^{oo}\ ,$$

where Q^{oo} is as in Introduction. The maps $1 : \mathbf{R}^0 \to \mathbf{R}$ and $0 : \mathbf{R}^0 \to \mathbf{R}$ can be treated as the 3-$\mathcal{VB}$-maps

$$1 : \mathbf{R}^{[(0)]} \to \mathbf{R}^{[e_1]}\ \text{and}\ 0 : \mathbf{R}^{[(0)]} \to \mathbf{R}^{[e_\nu]}\ \text{for}\ \nu = 1, ..., 8\ .$$

Proof. It follows immediately from Proposition 2.5. □

Lemma 2.7. *Let K be a triple vector bundle with the basis M and $x \in M$ be a point. Let*

$$\mathcal{G}_x(K, \mathbf{R}^{[m]}) := \text{the space of germs at } x \text{ of [3]-}\mathcal{VB}\text{-maps } K \to \mathbf{R}^{[m]}\ .$$

Then $\mathcal{G}_x(K, \mathbf{R}^{[e_1]})$ is the algebra (in obvious way) and $\mathcal{G}_x(K, \mathbf{R}^{[e_\nu]})$ for $\nu = 1, ..., 8$ is the free $\mathcal{G}_x(K, \mathbf{R}^{[e_1]})$-module (possible $\{0\}$), and we have the obvious $\mathcal{G}_x(K, \mathbf{R}^{[e_1]})$-bilinear maps

$$\bullet^{(\nu,\mu,\kappa)} : \mathcal{G}_x(K, \mathbf{R}^{[e_\nu]}) \times \mathcal{G}_x(K, \mathbf{R}^{[e_\mu]}) \to \mathcal{G}_x(K, \mathbf{R}^{[e_\kappa]})\ \text{for}\ (\nu, \mu, \kappa) \in Q^{oo}\ ,$$

where Q^{oo} is as in Introduction.

Proof. It is an immediate consequence of Lemma 2.6. □

3. Any admissible system induces canonically a ppgb-functor on [3]-$\mathcal{VB}$

The general concept of (gauge) bundle functors can be found in [8]. We need the following particular case of it.

Definition 3.1. *A gauge bundle functor on* [3]-$\mathcal{VB}$ *is a covariant functor* $F : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ *sending any triple vector bundle* K *with the base* M *into fibred manifold* $FK = \{p_{FK} : FK \to M\}$ *(with the base being the base of* K*) and any triple vector bundle map* $f : K \to K'$ *with the base map* $\underline{f} : M \to M'$ *into fibred map* $Ff : FK \to FK'$ *with the base map* $\underline{f} : M \to M'$ *(being the base map of* f*) and satisfying the following conditions:*

(i) (Localization condition) For a triple vector bundle K *with the basis* M *and any open subset* $U \subset M$ *the inclusion map* $i_{K|U} : K|U \to K$ *induces diffeomorphism* $Fi_{K|U} : F(K|U) \to p^{-1}_{FK}(U)$*, and*

(ii) (Regularity condition) F *transforms smoothly parametrized families of triple vector bundle maps into smoothly parametrized families of* $\mathcal{FM}$*-maps.*

Definition 3.2. *Given gauge bundle functors* F_1, F_2 *on* [3]-$\mathcal{VB}$*, a natural transformation* $\eta : F_1 \to F_2$ *is a system of base preserving fibred maps* $\eta_K : F_1K \to F_2K$ *for every triple vector bundle* K *satisfying* $F_2 f \circ \eta_K = \eta_{K'} \circ F_1 f$ *for every triple vector bundle morphism* $f : K \to K'$.

Definition 3.3. *A gauge bundle functor* F *on* [3]-$\mathcal{VB}$ *is a Weil like functor (product preserving gauge bundle functor (ppgb-functor)) if* $F(K_1 \times K_2) = F(K_1) \times F(K_2)$ *for any* [3]-$\mathcal{VB}$*-objects* K_1 *and* K_2.

Example 3.4. *A simple example of a ppgb-functor on* [3]-$\mathcal{VB}$ *is the tangent functor* T *sending any* [3]-$\mathcal{VB}$*-object* K *into the tangent bundle* TK *(over* M*) and any* 3-$\mathcal{VB}$*-map* $f : K \to K'$ *into the tangent map* $Tf : TK \to TK'$.

Definition 3.5. *Let* A *be a Weil algebra and* U_ν *for* $\nu = 1, ...8$ *be* A*-modules being finite dimensional over* $\mathbf{R}$*. Suppose that we have* A*-bilinear maps*

$$\diamond^{(\nu,\mu,\kappa)} : U_\nu \times U_\mu \to U_\kappa$$

for $(\nu, \mu, \kappa) \in Q^{oo}$ *, where* Q^{oo} *is as in Introduction. A system* $\diamond = \{(\diamond^{(\nu,\mu,\kappa)})_{(\nu,\mu,\kappa)\in Q^{oo}}, U_1, ..., U_8\}$ *(or shortly* $\diamond = (\diamond^{(\nu,\mu,\kappa)})_{(\nu,\mu,\kappa)\in Q^{oo}}$*) as above is called an* A*-admissible system if* $U_1 = A$ *(with the module multiplication equal to the multiplication of algebra* A*) and*

$$u_2 \diamond^{(2,7,8)} (u_3 \diamond^{(3,5,7)} u_5) = u_3 \diamond^{(3,6,8)} (u_2 \diamond^{(2,5,6)} u_5) = (u_2 \diamond^{(2,3,4)} u_3) \diamond^{(4,5,8)} u_5$$

for any $u_2 \in U_2$ *and* $u_3 \in U_3$ *and* $u_5 \in U_5$*, where* $x \diamond^{(\nu,\mu,\kappa)} y := \diamond^{(\nu,\mu,\kappa)}(x, y)$.

If $\tilde{A}$ *is an another Weil algebra and* $\tilde{\diamond}$ *is a* $\tilde{A}$*-admissible system, then a morphism* $\alpha : \diamond \to \tilde{\diamond}$ *is a system* $\alpha = (\alpha_o; \alpha_{(1)}, \alpha_{(2)}, ..., \alpha_{(8)})$ *consisting of an algebra morphism* $\alpha_o : A \to \tilde{A}$ *and module morphisms* $\alpha_{(\nu)} : U_\nu \to \tilde{U}_\nu$ *over* α_o *for* $\nu = 1, ..., 8$ *such that* $\alpha_{(1)} = \alpha_o$ *and* $\alpha_{(\kappa)} \circ \diamond^{(\nu,\mu,\kappa)} = \tilde{\diamond}^{(\nu,\mu,\kappa)} \circ (\alpha_{(\nu)} \times \alpha_{(\mu)})$ *for* $(\nu, \mu, \kappa) \in Q^{oo}$.

Example 3.6. *Let* A *be a Weil algebra and* $U_1 = U_2 = ...U_8 = A$ *and* $\diamond^{(\nu,\mu,\kappa)} := \cdot : A \times A \to A$ *be the multiplication of* A *for any* $(\nu, \mu, \kappa) \in Q^{oo}$*. Then* $\diamond = \{\diamond^{(\nu,,\mu,\kappa)}\}_{(\nu,\mu,\kappa)\in Q^{o,o}}$ *is an* A*-admissible system.*

Example 3.7. *Let* A *be a Weil algebra and* m_A *be the maximal ideal of* A *and* $U_1 = A$ *and* $U_2 = ...U_8 = m_A$ *and* $\diamond^{(\nu,\mu,\kappa)} : U_\nu \times U_\mu \to U_\kappa$ *be the restriction of the multiplication of* A *for any* $(\nu, \mu, \kappa) \in Q^{oo}$*. Then* $\diamond = \{\diamond^{(\nu,\mu,\kappa)}\}_{(\nu,\mu,\kappa)\in Q^{oo}}$ *is an* A*-admissible system.*

Example 3.8. *We can generalize Example 3.7 as follows. Let* A *be a Weil algebra and* I_i *for* $i = 1, ..., 8$ *be arbitrary ideals of* A *such that* $I_1 = A$ *and* $I_\nu \cdot I_\mu \subset I_\kappa$ *for* $(\nu, \mu, \kappa) \in Q^{oo}$*, and let* $U_i := I_i$ *for* $i = 1, ..., 8$ *and* $\diamond^{(\nu,\mu,\kappa)} : U_\nu \times U_\mu \to U_\kappa$ *be the restriction of the multiplication of* A *for any* $(\nu, \mu, \kappa) \in Q^{oo}$*. Then* $\diamond = \{\diamond^{(\nu,\mu,\kappa)}\}_{(\nu,\mu,\kappa)\in Q^{oo}}$ *is an* A*-admissible system.*

Example 3.9. *Let* $\diamond^1$ *be an* A^1*-admissible system and* $\diamond^2$ *be an* A^2*-admissible system. Then the tensor product* $\diamond^1 \otimes \diamond^2$ *(described in Section 7) is an* $A^1 \otimes A^2$*-admissible system.*

Suppose, we have an A-admissible system $\diamond$ as in Definition 3.5.

Using this admissible system, one can build a ppgb-functor $F^\diamond : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ as follows.

Example 3.10. *Let K be a triple vector bundle with the base M. Given a point $x \in M$, let $F^\diamond_x K$ be the space of all sequences $(\varphi; \psi_\nu)_{\nu=1,...,8}$ (i.e. sequences $(\varphi; \psi_1, \psi_2, ..., \psi_8)$) of algebra maps $\varphi : \mathcal{G}_x(K, \mathbf{R}^{[e_1]}) \to A$ and module maps $\psi_\nu : \mathcal{G}_x(K, \mathbf{R}^{[e_\nu]}) \to U_\nu$ over φ for $\nu = 1, ..., 8$ such that $\psi_1 = \varphi$ and*

$$\psi_\kappa(g \bullet^{(\nu,\mu,\kappa)} h) = \psi_\nu(g) \diamond^{(\nu,\mu,\kappa)} \psi_\mu(h) \quad \text{for all } g \in \mathcal{G}_x(K, \mathbf{R}^{[\nu]}) \text{ and } h \in \mathcal{G}_x(K, \mathbf{R}^{[e_\mu]})$$

for all $(\nu, \mu, \kappa) \in Q^{oo}$, where $g \bullet^{(\nu,\mu,\kappa)} h := \bullet^{(\nu,\mu,\kappa)}(g, h)$. Let $F^\diamond K := \bigcup_{x\in M} F^\diamond_x K$. Then $F^\diamond K$ is a fibred manifold (with the obvious projection $F^\diamond K \to M$). Given a local $[3]$-$\mathcal{VB}$- trivialization $(x^{i_\nu}_\nu) : K_{|\Omega} \tilde{=} \mathbf{R}^{[m]}$, where $m = (m_1, ..., m_8) \in \mathbf{N}^8$, we have the induced $\mathcal{FM}$-trivialization $(\widetilde{x^{i_\nu}_\nu}) : F^\diamond K_{|\Omega} \tilde{=} \prod_{\nu=1}^8 U_\nu^{m_\nu}$ such that

$$\widetilde{x^{i_\nu}_\nu}(w) := \psi_\nu(germ_x(x^{i_\nu}_\nu)) \in U_\nu \text{ for } w = (\varphi; \psi_1, ..., \psi_8) \in F^\diamond_x K \,, x \in \Omega \,,$$

where $i_\nu = 1, ..., m_\nu$ and $\nu = 1, ..., 8$ and $U_\nu^{m_\nu} = U_\nu \times ... \times U_\nu$ (m_ν-times). (That $(\widetilde{x^{i_\nu}_\nu})$ is bijective it is observed in Lemma 3.11.)

Every $[3]$-$\mathcal{VB}$-map $f : K \to K^1$ induces $\mathcal{FM}$-map $F^\diamond f : F^\diamond K \to F^\diamond K^1$ such that

$$F^\diamond(f)(w) := (\varphi \circ f^*; \psi_1 \circ f^*, ..., \psi_8 \circ f^*) \in F^\diamond_{\underline{f}(x)} K^1$$

for any $w = (\varphi; \psi_1, ..., \psi_8) \in F^\diamond_x K$, $x \in M = K_8$, where f^* is the pull-back with respect to f. That $F^\diamond(f)(w) \in F^\diamond_{\underline{f}(x)} K^1$ one can verify directly. (The local expression of $F^\diamond f$ is given in Lemma 3.12.)

The correspondence $F^\diamond : [3]$-$\mathcal{VB} \to \mathcal{FM}$ is a ppgb-functor. Using Lemma 3.12, we can see that $F^\diamond K$ is a triple vector bundle and $F^\diamond f$ is a $[3]$-$\mathcal{VB}$-morphism if K and f are. In other words $F^\diamond : [3]$-$\mathcal{VB} \to [3]$-$\mathcal{VB}$. (The last fact will be observed in intrinsic way in Example 4.1, too.)

If $\tilde{A}$ is an another Weil algebra and $\tilde{\diamond}$ an $\tilde{A}$-admissible system in question and $\alpha : \diamond \to \tilde{\diamond}$ is a morphism of admissible systems, $\alpha = (\alpha_o; \alpha_{(1)}, ..., \alpha_{(8)})$, then we have the natural transformation $\eta^\alpha : F^\diamond \to F^{\tilde{\diamond}}$ given by $(\varphi; , \psi_1, ..., \psi_8) \mapsto (\alpha_o \circ \varphi; \alpha_{(1)} \circ \psi_1, ..., \alpha_{(8)} \circ \psi_8)$. (The local expression of α is given in Lemma 3.12.)

Lemma 3.11. *Let $\diamond$ and $F^\diamond$ be as in Example 3.10. Let $(x^{i_\nu}_\nu) : K_{|\Omega} \tilde{=} \mathbf{R}^{[m]}$ be a local $[3]$-$\mathcal{VB}$- trivialization of an $[3]$-$\mathcal{VB}$-object K. Then $F^\diamond K_{|\Omega} = \prod_{\nu=1}^8 U_\nu^{m_\nu}$ modulo $(\widetilde{x^{i_\nu}_\nu})$.*

Proof. Given $x \in \Omega$, we can reconstruct $(\varphi; \psi_1, ...\psi_8) \in F^\diamond_x K$ from arbitrary given values $\psi_\nu(germ_x(x^{i_\nu}_\nu)) \in U_\nu$ for $i_\nu : 1, ..., m_\nu$ and $\nu = 1, ..., 8$. For example, we can reconstruct ψ_4 as follows. By Proposition 2.5, the basis in the free $\mathcal{G}_x(K, \mathbf{R}^{[e_1]})$-module $\mathcal{G}_x(K, \mathbf{R}^{[e_4]})$ is formed by $germ_x(x_2^{i_2} x_3^{i_3})$ and $germ_x(x_4^{i_4})$ for $i_2 = 1, ..., m_2$, $i_3 = 1, ..., m_3$ and $i_4 = 1, ..., m_4$. Using the formula $\psi_4(g \bullet^{(2,3,4)} h) = \psi_2(g) \diamond^{(2,3,4)} \psi_3(h)$ for $g = germ_x(x_2^{i_2})$ and $h = germ_x(x_3^{i_3})$ we derive that the values $\psi_4(germ_x(x_2^{i_2} x_3^{i_3}))$ are given. Then ψ_4 is given. Quite similarly one can reconstruct ψ_ν for $\nu = 5, 6, 7$. To reconstruct ψ_8, we must put

$$\psi_8(germ_x(x_2^{i_2} x_3^{i_3} x_5^{i_5})) = (\psi_2(germ_x(x_2^{i_2})) \diamond^{(2,3,4)} \psi_3(germ_x(x_3^{i_3}))) \diamond^{(4,5,8)} \psi_5(germ_x(x_5^{i_5})) \,,$$

$$\psi_8(germ_x(x_2^{i_2} x_3^{i_3} x_5^{i_5})) = \psi_2(germ_x(x_2^{i_2})) \diamond^{(2,7,8)} (\psi_3(germ_x(x_3^{i_3})) \diamond^{(3,5,7)} \psi_5(germ_x(x_5^{i_5}))) \,,$$

$$\psi_8(germ_x(x_2^{i_2} x_3^{i_3} x_5^{i_5})) = \psi_3(germ_x(x_3^{i_3})) \diamond^{(3,6,8)} (\psi_2(germ_x(x_2^{i_2})) \diamond^{(2,5,6)} \psi_5(germ_x(x_5^{i_5}))) \,.$$

Fortunately, the values of the right sides of the last three equalities are equal because of Definition 3.5. One can see that $(\varphi; \psi_1, ...\psi_8) \in F^\diamond_x K$. □

Lemma 3.12. *Let $\diamond$ and $F^\diamond$ and α and η^α be as in Example 3.10.*

(i) By the previous lemma, $F^\diamond \mathbf{R}^{[m]} = \prod_{\nu=1}^8 U_\nu^{m_\nu}$ modulo the trivialization induced by the usual trivialization on $\mathbf{R}^{[m]}$. In particular, $F^\diamond \mathbf{R}^{[m]}$ is the triple vector bundle.

(ii) Fix the bases in the real vector spaces U_ν. Let $p = (p_1, ..., p_8)$, where $p_\nu := dim_{\mathbf{R}}(U_\nu)$. Then for any $m = (m_1, ..., m_8) \in \mathbf{N}^8$ it holds $F^\diamond \mathbf{R}^{[m]} = \mathbf{R}^{[pm]}$ (modulo the identification), where $pm := (p_1 m_1, ..., p_8 m_8)$.

(iii) If $f : \mathbf{R}^{[m]} \to \mathbf{R}^{[\tilde{m}]}$ is a [3]-$\mathcal{VB}$-map, then so is $F^\diamond f : F\mathbf{R}^{[m]} \to F\mathbf{R}^{[\tilde{m}]}$. More detailed, if f is of the form as in Proposition 2.5, then $F^\diamond f$ is of the same form with $(\widehat{x_\nu^{i_\nu}})$ instead of $(x_\nu^{i_\nu})$ and with $(\widehat{\tilde{x}_\nu^{\tilde{i}_\nu}})$ instead of $(\tilde{x}_\nu^{\tilde{i}_\nu})$ and with $\lambda^A = T^A\lambda : U_1^{m_1} = T^A\mathbf{R}^{m_1} \to U_1 = T^A\mathbf{R}$ instead of $\lambda : \mathbf{R}^{m_1} \to \mathbf{R}$, where λ denotes arbitrary coefficient $a_1^{\tilde{i}_1}, b_{2i_2}^{\tilde{i}_2}, c_{3i_3}^{\tilde{i}_3}, d_{4i_2i_3}^{\tilde{i}_4}, e_{4i_4}^{\tilde{i}_4}, A_{5i_5}^{\tilde{i}_5}, B_{6i_2i_5}^{\tilde{i}_6}, C_{6i_6}^{\tilde{i}_6}, D_{7i_3i_5}^{\tilde{i}_7}, E_{7i_7}^{\tilde{i}_7}, H_{8i_2i_3i_5}^{\tilde{i}_8}, I_{8i_4i_5}^{\tilde{i}_8}, J_{8i_3i_6}^{\tilde{i}_8}, K_{8i_2i_7}^{\tilde{i}_8}, L_{8i_8}^{\tilde{i}_8} : \mathbf{R}^{m_1} \to \mathbf{R}$ of f, and where $\diamond^{(\nu,\mu,\kappa)}$ is not indicated for any $(\nu, \mu, \kappa) \in Q^{oo}$.

(iv) Given $m = (m_1, ..., m_8) \in \mathbf{N}^8$, $\eta^\alpha_{\mathbf{R}^{[m]}} = \prod_{\nu=1}^8 \alpha_{(\nu)}^{m_\nu} : \prod_{\nu=1}^8 U_\nu^{m_\nu} \to \prod_{\nu=1}^8 \tilde{U}_\nu^{m_\nu}$.

Proof. **ad**(i) Clearly, $\tilde{K} := \prod_{\nu=1}^8 U_\nu^{m_\nu}$ is the triple vector bundle, where $\tilde{K}_8 := U_1^{m_1} \times U_2^{m_2} \times U_3^{m_3} \times U_4^{m_4} \times U_5^{m_5} \times U_6^{m_6} \times U_7^{m_7} \times U_8^{m_8}$, $\tilde{K}_7 := U_1^{m_1} \times U_3^{m_3} \times U_5^{m_5} \times U_7^{m_7}$, $\tilde{K}_6 := U_1^{m_1} \times U_2^{m_2} \times U_5^{m_5} \times U_6^{m_6}$, $\tilde{K}_5 = U_1^{m_1} \times U_5^{m_5}$, $\tilde{K}_4 := U_1^{m_1} \times U_2^{m_2} \times U_3^{m_3} \times U_4^{m_4}$, $\tilde{K}_3 := U_1^{m_1} \times U_3^{m_3}$, $\tilde{K}_2 := U_1^{m_1} \times U_2^{m_2}$, $\tilde{K}_1 := U_1^{m_1}$ and $\tilde{\tau}_{(i,j)} : \tilde{K}_i \to \tilde{K}_j$ are the canonical projections for $(i, j) \in Q^o$.

ad(ii) It is clear because $U_\nu = \mathbf{R}^{p_\nu}$ (modulo the base identification) for $\nu = 1, ..., 8$.

ad(iii)-(iv) For example, we verify that

$$\widehat{\tilde{x}_4^{\tilde{i}_4}} \circ F^\diamond f = d_{4i_2i_3}^{\tilde{i}_4 A}(\widehat{x_1})\widehat{x_2^{i_2}}\widehat{x_3^{i_3}} + e_{4i_4}^{\tilde{i}_4 A}(\widehat{x_1})\widehat{x_4^{i_4}} ,$$

where we do not indicate $\diamond^{(2,3,4)}$ in $\widehat{x_2^{i_2}}\widehat{x_3^{i_3}}$ and the module multiplications, and where (of course) $\widehat{x_1} = (\widehat{x_1^1}, \dots, \widehat{x_1^{m_1}})$.

To do it we take a point $w = (\varphi; \psi_1, \psi_2, \psi_3, \psi_4, ..., \psi_8) \in F_x^\diamond \mathbf{R}^{[m]}$. Then

$$\widehat{\tilde{x}_4^{\tilde{i}_4}} \circ F^\diamond f(w) = \widehat{\tilde{x}_4^{\tilde{i}_4}}(\varphi \circ f^*, ..., \psi_4 \circ f^*, ...) = \psi_4(f^*(germ_x(\tilde{x}_4^{\tilde{i}_4})))$$

$$= \psi_4(germ_x(\tilde{x}_4^{\tilde{i}_4} \circ f)) = \psi_4(germ_x(d_{4i_2i_3}^{\tilde{i}_4}(x_1)x_2^{i_2}x_3^{i_3} + e_{4i_4}^{\tilde{i}_4}(x_1)x_4^{i_4}))$$

$$= \varphi(germ_x(d_{4i_2i_3}^{\tilde{i}_4}(x_1)))\psi_2(germ_x(x_2^{i_2}))\psi_3(germ_x(x_3^{i_3})) + \varphi(germ_x(e_{4i_4}^{\tilde{i}_4}(x_1)))\psi_4(germ_x(x_4^{i_4}))$$

$$= (d_{4i_2i_3}^{\tilde{i}_4 A}(\widehat{x_1})\widehat{x_2^{i_2}}\widehat{x_3^{i_3}} + e_{4i_4}^{\tilde{i}_4 A}(\widehat{x_1})\widehat{x_4^{i_4}})(w) ,$$

where $x_1 = (x_1^1, ..., x_1^{m_1})$.

The proofs of the other formulas in question are quite similar. □

4. The functors $F^\diamond$ have values in [3]-$\mathcal{VB}$

Let $\diamond$ be an A-admissible system and let $F^\diamond : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ be the ppgb-functor corresponding to $\diamond$ as in Example 3.10. Given a [3]-$\mathcal{VB}$-object K, we can make $F^\diamond K$ to be 3-$\mathcal{VB}$-object (in geometrical way) as follows.

Example 4.1. *Let K be a [3]-$\mathcal{VB}$-object with the base M. Given a point $x \in M$ and $i = 1, ..., 8$, we have $\tau_x^{(i)} : F_x^\diamond K \to F_x^\diamond K$ such that*

$$\tau_x^{(8)}(v) := v\ ,\ \tau_x^{(7)}(v) := (\varphi; \psi_1, 0, \psi_3, 0, \psi_5, 0, \psi_7, 0)\ ,$$

$$\tau_x^{(6)}(v) := (\varphi; \psi_1, \psi_2, 0, 0, \psi_5, \psi_6, 0, 0)\ ,\ \tau_x^{(5)}(v) := (\varphi; \psi_1, 0, 0, 0, \psi_5, 0, 0, 0)\ ,$$

$$\tau_x^{(4)}(v) := (\varphi; \psi_1, \psi_2, \psi_3, \psi_4, 0, 0, 0, 0)\ ,\ \tau_x^{(3)}(v) := (\varphi; \psi_1, 0, \psi_3, 0, 0, 0, 0, 0)\ ,$$

$$\tau_x^{(2)}(v) := (\varphi; \psi_1, \psi_2, 0, 0, 0, 0, 0, 0)\ ,\ \tau_x^{(1)}(v) := (\varphi; \psi_1, 0, 0, 0, 0, 0, 0, 0)$$

for all $v = (\varphi; \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8) \in F_x^\diamond K$. *(Since* $\tau_x^{(i)}(v) \in F_x^\diamond K$ *for any* $v \in F_x^\diamond K$, *then* $\tau_x^{(i)}$ *is defined correctly.) Let* $\tau_K^{(i)} : F^\diamond K \to F^\diamond K$ *be the resulting maps and let* $\hat{K}_i := \mathrm{im}(\tau_K^{(i)}) \subset F^\diamond K$ *for* $i = 1, ..., 8$. *For any* $(i, j) \in Q^o$, *it holds* $\tau_K^{(j)}(\hat{K}_i) \subset \hat{K}_j$. *So, we can define* $\hat{\tau}_{(i,j)} : \hat{K}_i :\to \hat{K}_j$ *to be the restriction* $\hat{\tau}_{(i,j)} := \tau^{(j)}_{|\hat{K}_i}$. *It turns out that for any* $(i, j) \in Q^o$, $\hat{K}_i$ *is the vector bundle with the basis* $\hat{K}_j$ *and projection* $\hat{\tau}_{(i,j)} : \hat{K}_i \to \hat{K}_j$. *(For example, if* $(i, j) = (6, 5)$ *and* $v = (\varphi; \psi_1, 0, 0, 0, \psi_5, 0, 0, 0) \in (\hat{K}_5)_x$, $x \in M$, *then the sum map in the fibre* $(\hat{K}_6)_v$ *is defined by*

$$(\varphi; \psi_1, \psi_2^1, 0, 0, \psi_5, \psi_6^1, 0, 0) + (\varphi, \psi_1, \psi_2^2, 0, 0, \psi_5, \psi_6^2, 0, 0) = (\varphi; \psi_1, \psi_2^1 + \psi_2^2, 0, 0, \psi_5, \psi_6^1 + \psi_6^2, 0, 0)$$

and the scalar multiplication by $\lambda \in \mathbf{R}$ *in this fibre is defined by*

$$\lambda \cdot (\varphi; \psi_1, \psi_2^1, 0, 0, \psi_5, \psi_6^1, 0, 0) = (\varphi; \psi_1, \lambda\psi_2^1, 0, 0, \psi_5, \lambda\psi_6^1, 0, 0)\ ,$$

where (for example) the sum $\psi_2^1 + \psi_2^2$ *is the one of the vector space of A-module homomorphisms* $\mathcal{G}_x(K, \mathbf{R}^{[e_2]}) \to U_2$ *and similarly for* $\lambda\psi_2^1$.*) Finally, it turns out that the system*

$$\hat{K} = (\hat{K}_8, \hat{K}_7, ..., \hat{K}_1)$$

of vector bundles $\hat{K}_i = (\hat{K}_i, \hat{\tau}_{(i,j)}, \hat{K}_j)$ *for* $(i, j) \in Q^o$ *is a triple vector bundle. (These facts can be easily verified by using a local* $[3]$*-*$\mathcal{VB}$*- trivialization* $(x_\nu^{i_\nu}) : K_{|\Omega} \tilde{=} \mathbf{R}^{[m]}$. *Indeed, denoting (for simplicity)* $K_{|\Omega}$ *by* K, *one can see that modulo the induced* $\mathcal{FM}$*-trivialization* $(\widehat{x_\nu^{i_\nu}}) : F^\diamond K \tilde{=} \prod_{\nu=1}^8 U_\nu^{m_\nu}$ *and modulo the obvious "identity" isomorphism, there are*

$$\hat{K}_8 = U_1^{m_1} \times U_2^{m_2} \times U_3^{m_3} \times U_4^{m_4} \times U_5^{m_5} \times U_6^{m_6} \times U_7^{m_7} \times U_8^{m_8}\ ,$$

$$\hat{K}_7 = U_1^{m_1} \times \{0\} \times U_3^{m_3} \times \{0\} \times U_5^{m_5} \times \{0\} \times U_7^{m_7} \times \{0\} = U_1^{m_1} \times U_3^{m_3} \times U_5^{m_5} \times U_7^{m_7}\ ,$$

$$\hat{K}_6 = U_1^{m_1} \times U_2^{m_2} \times \{0\} \times \{0\} \times U_5^{m_5} \times U_6^{m_6} \times \{0\} \times \{0\} = U_1^{m_1} \times U_2^{m_2} \times U_5^{m_5} \times U_6^{m_6},$$

$$\hat{K}_5 = U_1^{m_1} \times \{0\} \times \{0\} \times \{0\} \times U_5^{m_5} \times \{0\} \times \{0\} \times \{0\} = U_1^{m_1} \times U_5^{m_5}\ ,$$

$$\hat{K}_4 = U_1^{m_1} \times U_2^{m_2} \times U_3^{m_3} \times U_4^{m_4} \times \{0\} \times \{0\} \times \{0\} \times \{0\} = U_1^{m_1} \times U_2^{m_2} \times U_3^{m_3} \times U_4^{m_4}\ ,$$

$$\hat{K}_3 = U_1^{m_1} \times \{0\} \times U_3^{m_3} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} = U_1^{m_1} \times U_3^{m_3}\ ,$$

$$\hat{K}_2 = U_1^{m_1} \times U_2^{m_2} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} = U_1^{m_1} \times U_2^{m_2}\ ,$$

$$\hat{K}_1 = U_1^{m_1} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} = U_1^{m_1}\ ,$$

and $\hat{\tau}_{(i,j)} : \hat{K}_i \to \hat{K}_j$ *is the canonical projection for any* $(i, j) \in Q^o$, *and* $\hat{K} = \tilde{K}$, *where* $\tilde{K}$ *is the* $[3]$*-*$\mathcal{VB}$*-objects from the proof of Lemma 3.12(i).)*

Let K^1 *be an another* $[3]$*-*$\mathcal{VB}$*-object and* $f : K \to K^1$ *be a* $[3]$*-*$\mathcal{VB}$*-morphism. Denote* $\hat{f} := F^\diamond f : \hat{K} \to \hat{K}^1$. *For any* $i = 1, ..., 8$ *we have* $\hat{f}(\hat{K}_i) \subset \hat{K}_i^1$ *and we define* $\hat{f}_i : \hat{K}_i \to \hat{K}_i^1$ *by* $\hat{f}_i := \hat{f}_{|\hat{K}_i}$. *Let* $\hat{f} = (\hat{f}_8, \hat{f}_7, ..., \hat{f}_1)$. *It turn out that* $\hat{f} : \hat{K} \to \hat{K}^1$ *is a* $[3]$*-*$\mathcal{VB}$*-morphism.*

Thus, given a $[3]$*-*$\mathcal{VB}$*-object* K, *we have introduced intrinsically the* $[3]$*-*$\mathcal{VB}$*-structure in* $F^\diamond K$ *and observed that for any* $[3]$*-*$\mathcal{VB}$*-morphism* $f : K \to K^1$, $F^\diamond f : F^\diamond K \to F^\diamond K^1$ *is a* $[3]$*-*$\mathcal{VB}$*-morphism. In other word, we have proved intrinsically that* $F^\diamond : [3]\text{-}\mathcal{VB} \to [3]\text{-}\mathcal{VB}$.

Let $\tilde{\diamond}$ be an another $\tilde{A}$-admissible system and $\alpha : \diamond \to \tilde{\diamond}$ be a morphism of admissible systems as in Definition 3.5. Let $\eta^\alpha : F^\diamond \to F^{\tilde{\diamond}}$ be the natural transformation corresponding to α (as in Example 3.10). Let K be a [3]-$\mathcal{VB}$-object with basis M. Put $\hat{K}^\diamond := \hat{K} = (\hat{K}_8, \hat{K}_7, ...\hat{K}_1)$, where $\hat{K}$ is as above. Let $\hat{K}^{\tilde{\diamond}} = (\hat{K}_8^{\tilde{\diamond}}, \hat{K}_7^{\tilde{\diamond}}, ..., \hat{K}_1^{\tilde{\diamond}})$ be the [3]-$\mathcal{VB}$-object being defined as $\hat{K}^\diamond$ by using $\tilde{\diamond}$ instead of $\diamond$. Denote $\eta := \eta^\alpha_K : \hat{K}^\diamond \to \hat{K}^{\tilde{\diamond}}$. We can see that for any $i = 1, ..., 8$, there is $\eta(\hat{K}_i^\diamond) \subset \hat{K}_i^{\tilde{\diamond}}$, and we can define $\hat{\eta}_i : \hat{K}_i^\diamond \to \hat{K}_i^{\tilde{\diamond}}$ to be the restriction $\hat{\eta}_i := \eta_{|\hat{K}_i^\diamond}$. Let $\hat{\eta} = (\hat{\eta}_8, \hat{\eta}_7, ..., \hat{\eta}_1)$. It turns out that $\hat{\eta} : \hat{K}^\diamond \to \hat{K}^{\tilde{\diamond}}$ is a [3]-$\mathcal{VB}$-morphism.

Thus we have observed intrinsically that given a morphism $\alpha : \diamond \to \tilde{\diamond}$ of admissible systems, the corresponding natural transformation $\eta^\alpha_K : F^\diamond K \to F^{\tilde{\diamond}} K$ is a [3]-$\mathcal{VB}$-morphism for any [3]-$\mathcal{VB}$-object K.

5. Any ppgb-functor on [3]-$\mathcal{VB}$ induces canonically an admissible system

Example 5.1. *Let $F : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ be a ppgb-functor. We put*

$$A^F := F\mathbf{R}^{[e_1]} \, , \; U^F_\nu := F\mathbf{R}^{[e_\nu]} \, , \; \nu = 1, ..., 8 \, .$$

Then A^F is the Weil algebra. (Indeed, it is the Weil algebra of the product preserving bundle functor $\tilde{F} : \mathcal{M}f \to \mathcal{FM}$ (on the category $\mathcal{M}f$ of manifolds and their maps) given by $\tilde{F}M = FM$ and $\tilde{F}M = Ff$, where manifolds M are treated as the [3]-$\mathcal{VB}$-objects with all arrows being the identity maps of M. We recall that the sum map of A^F is $F(+) : F(\mathbf{R}^{[e_1]} \times \mathbf{R}^{[e_1]}) = A^F \times A^F \to F\mathbf{R}^{[e_1]} = A^F$ and the multiplication map of A^F is $F(\cdot) : A^F \times A^F \to A^F$, where the sum map $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and the multiplication map $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are treated as [3]-$\mathcal{VB}$-maps $+, \cdot : \mathbf{R}^{[e_1+e_1]} = \mathbf{R}^{[e_1]} \times \mathbf{R}^{[e_1]} \to \mathbf{R}^{[e_1]}$, the unity map of A^F is $F(1)$ and the null map is $F(0)$, where the unity map $1 : \mathbf{R}^0 \to \mathbf{R}$ and the zero map $0 : \mathbf{R}^0 \to \mathbf{R}$ are treated as [3]-$\mathcal{VB}$-maps $1, 0 : \mathbf{R}^{[(0)]} \to \mathbf{R}^{[e_1]}$, see Lemma 2.6.)

Similarly, U^F_ν is the A^F-module. (The A^F-module operations of U^F_ν are $F(+) : U^F_\nu \times U^F_\nu \to U^F_\nu$ and $F(\cdot) : A^F \times U^F_\nu \to U^F_\nu$, where the sum and multiplication maps $+$ and $\cdot$ are treated as [3]-$\mathcal{VB}$-maps $+ : \mathbf{R}^{[e_\nu]} \times \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}$ and $\cdot : \mathbf{R}^{[e_1]} \times \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}$, the zero map of U^F_ν is $F(0)$, where $0 : \mathbf{R}^0 \to \mathbf{R}$ is treated as the [3]-$\mathcal{VB}$-map $0 : \mathbf{R}^{[(0)]} \to \mathbf{R}^{[e_\nu]}$. That the operations satisfy respective module properties, one can verify by applying functor F to the algebraic properties of $+, \cdot, 0, 1$.)

For any $(\nu, \mu, \kappa) \in Q^{oo}$, we put

$$\diamond^{F,(\nu,\mu,\kappa)} := F(\cdot) : U^F_\nu \times U^F_\mu \to U^F_\kappa \, ,$$

where the multiplication map $\cdot : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is treated as the [3]-$\mathcal{VB}$-maps $\cdot : \mathbf{R}^{[e_\nu]} \times \mathbf{R}^{[e_\mu]} \to \mathbf{R}^{[e_\kappa]}$, where Q^{oo} is as in Definition 3.5. Then $\diamond^{F,(\nu,\mu,\kappa)}$ is A^F-bilinear. (This fact can be verified by using the same method as for the operations of U^F_ν.) Applying F to the associativity and commutativity of $\cdot$ we easily obtain $u_2 \diamond^{F,(2,7,8)} (u_3 \diamond^{F,(3,5,7)} u_5) = u_3 \diamond^{F,(3,6,8)} (u_2 \diamond^{F,(2,5,6)} u_5) = (u_2 \diamond^{F,(2,3,4)} u_3) \diamond^{F,(4,5,8)} u_5$ for any $u_2 \in U^F_2$ and $u_3 \in U^F_3$ and $u_5 \in U^F_5$. Thus we have the A^F-admissible system

$$\diamond^F := (\diamond^{F,(\nu,\mu,\kappa)})_{(\nu,\mu,\kappa) \in Q^{oo}} \, .$$

For example, if $F = T : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ is the tangent functor (from Example 3.4) then $A^T = \mathbf{D}$, the algebra of dual numbers, $U^T_\nu = \mathbf{D}$, the $\mathbf{D}$-module (in obvious way), and $\diamond^{T,(\nu,\mu,\kappa)} : U^T_\nu \times U^T_\mu \to U^T_\kappa$ is equal to the multiplication $\cdot : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ for $(\nu, \mu, \kappa) \in Q^{oo}$.

If $F^1 : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ is an another ppgb-functor and $\eta : F \to F^1$ is a natural transformation we define a system $\alpha^\eta = (\alpha^\eta_o; \alpha^\eta_{(1)}, ..., \alpha^\eta_{(8)})$ by

$$\alpha^\eta_o := \eta_{\mathbf{R}^{[e_1]}} \text{ and } \alpha^\eta_{(\nu)} := \eta_{\mathbf{R}^{[e_\nu]}} \text{ for } \nu = 1, ..., 8 \, .$$

Then $\alpha^\eta : \diamond^F \to \diamond^{F^1}$ is a morphism of admissible system (because natural transformations commute with Ff and $F^1 f$ for[3]-$\mathcal{VB}$-maps f (for $f = +$ or $f = \cdot$ in the cases where $+$ and $\cdot$ are [3]-$\mathcal{VB}$-maps, in particular)).

6. The complete description of ppgb-functors on [3]-$\mathcal{VB}$ be means of admissible systems

We prove the following classification theorem.

Theorem 6.1. *(i) Given a ppgb-functor $F : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ we have $F = F^{\diamond^F}$ modulo canonically depending on F natural isomorphism Θ^F of ppgb-functors. In particular, $F : [3]\text{-}\mathcal{VB} \to [3]\text{-}\mathcal{VB}$.*

(ii) For any admissible system $\diamond$ (in the sense of Definition 3.5) we have $\diamond = \diamond^{F^\diamond}$ modulo canonically depending on $\diamond$ isomorphism $\mathcal{T}^\diamond$ of admissible systems.

Proof. **ad**(i) Let K be a triple vector bundle with basis M. Let $y \in F_xK$, $x \in M$.

At first, we define $\varphi^y : \mathcal{G}_x(K, \mathbf{R}^{[e_1]}) \to A^F = F\mathbf{R}^{[e_1]}$ by

$$\varphi^y(u) := F(g)(y), \;\; u = germ_x(g) \in \mathcal{G}_x(K, \mathbf{R}^{[e_1]}) \, .$$

It is an algebra homomorphism because

$$\varphi^y(uu^1) = F(gg^1)(y) = F(\cdot(g, g^1))(y) = F(\cdot)(Fg(y), F(g^1)(y)) = \varphi^y(u)\varphi^y(u^1)$$

for all $u = germ_x(g), u^1 = germ_x(g^1) \in \mathcal{G}_x(K, \mathbf{R}^{[e_1]})$, and similarly $\varphi^y(u + u^1) = \varphi^y(u) + \varphi^y(u^1)$ and $\varphi^y(1) = 1$.

Next, given $\nu = 1, ..., 8$ we define $\psi^y_\nu : \mathcal{G}_x(K, \mathbf{R}^{[e_\nu]}) \to U^F_\nu = F\mathbf{R}^{[e_\nu]}$ by

$$\psi^y_\nu(u) := F(g)(y) \, , \;\; u = germ_x(g) \in \mathcal{G}_x(K, \mathbf{R}^{[e_\nu]})$$

and by (almost) the same procedure as above we can see that ψ^y_ν is a module homomorphism over φ^y.

Then , we can see that

$$(\varphi^y; \psi^y_1, ..., \psi^y_8) \in F^{\diamond^F}_x K \, .$$

Indeed, if $(\nu, \mu, \kappa) \in Q^{oo}$, then $\psi^y_\kappa(g \bullet^{(\nu,\mu,\kappa)} h) = \psi^y_\nu(g) \diamond^{F,(\nu,\mu,\kappa)} \psi^y_\mu(h)$ for all $g \in \mathcal{G}_x(K, \mathbf{R}^{[e_\nu]})$ and $h \in \mathcal{G}_x(K, \mathbf{R}^{[e_\mu]})$ because

$$\psi^y_\kappa(g \bullet^{(\nu,\mu,\kappa)} h) = F(\cdot(g, h))(y) = F(\cdot)(Fg(y), Fh(y)) = \psi^y_\nu(g) \diamond^{F,(\nu,\mu,\kappa)} \psi^y_\mu(h) \, .$$

Thus we have the natural transformation $\Theta^F : F \to F^{\diamond^F}$ defined by

$$\Theta^F_K(y) := (\varphi^y; \psi^y_1, ..., \psi^y_8) \in F^{\diamond^F}_x K \, , \; y \in F_xK \, , \; x \in M \, .$$

We can show that Θ^F_K is a diffeomorphism for any $[3]\text{-}\mathcal{VB}$-object K as follows.

Applying $[3]\text{-}\mathcal{VB}$-trivialization, we can assume that $K = \mathbf{R}^{[m]}$. Since F and $F^{\diamond^F}$ are product preserving and $K = \mathbf{R}^{[m]}$ is the (multi) product of $\mathbf{R}^{[e_\nu]}$ for $\nu = 1, ..., 8$, using Lemma 6.3, we can assume that $K = \mathbf{R}^{[e_\nu]}$, where $\nu = 1, ..., 8$. Then we can consider the composition $\widehat{x^1_\nu} \circ \Theta^F_K : F\mathbf{R}^{[e_\nu]} \to U^F_\nu = F\mathbf{R}^{[e_\nu]}$, where $\widehat{x^1_\nu}$ is the trivialization induced by the $[3]\text{-}\mathcal{VB}$-trivialization $x^1_\nu = id : \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}$, see Example 3.10. This composition is the identity map of $F\mathbf{R}^{[e_\nu]} = U^F_\nu$. Indeed,

$$\widehat{x^1_\nu} \circ \Theta^F_K(y) = \widehat{x^1_\nu}(\varphi^y) = \varphi^y(germ_x(x^1_\nu)) = F(x^1_\nu)(y) = F(id)(y) = y$$

for any $y \in F_xK$, $x \in$ the base of $\mathbf{R}^{[\nu]}$. That is why, Θ^F_K is a diffeomorphism.

So, we have proved that $F = F^{\diamond^F}$ modulo the natural isomorphism. Now, since $F^{\diamond^F} : [3]\text{-}\mathcal{VB} \to [3]\text{-}\mathcal{VB}$ (see, Section 4), then $F : [3]\text{-}\mathcal{VB} \to [3]\text{-}\mathcal{VB}$, as well.

ad(ii) Let $\diamond$ be an A-admissible system as in Definition 3.5. Let $\tilde{F} = F^\diamond : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ be the ppgb-functor corresponding to $\diamond$ as in Example 3.10. Let $\tilde{\diamond} := \diamond^{\tilde{F}}$ be the admissible system corresponding to $\tilde{F}$ as in Example 5.1 (with $\tilde{F}$ instead of F). We define an isomorphism of admissible systems $\mathcal{T}^\diamond : \tilde{\diamond} \to \diamond$ as follows.

Write $\tilde{\diamond} = (\tilde{\diamond}^{(\nu,\mu,\kappa)})_{(\nu,\mu,\kappa)\in Q^{oo}}$, where $\tilde{\diamond}^{(\nu,\mu,\kappa)} : \tilde{U}_\nu \times \tilde{U}_\mu \to \tilde{U}_\kappa$ for $(\nu, \mu, \kappa) \in Q^{oo}$ are $\tilde{A}$-bilinear maps satisfying the respective conditions. Next, given $\nu = 1, ..., 8$, let $\tilde{\alpha}_{(\nu)} : \tilde{U}_\nu \to U_\nu$ be such that

$$\tilde{\alpha}_{(\nu)}(\psi) := \psi_\nu(germ_x(x^1_\nu)) \in U_\nu$$

for all $\psi = (\varphi; \psi_1, ..., \psi_8) \in F^\diamond_x \mathbf{R}^{[e_\nu]}$, $x \in$ the base of $\mathbf{R}^{[e_\nu]}$, where $x^1_\nu : \mathbf{R}^{[e_\nu]} \to \mathbf{R}^{[e_\nu]}$ is the usual trivialization (the identity map). We put $\tilde{\alpha}_o := \tilde{\alpha}_{(1)} : \tilde{A} \to A$ and $\tilde{\alpha} := (\tilde{\alpha}_o; \tilde{\alpha}_{(1)}, ..., \tilde{\alpha}_{(8)})$. Then $\tilde{\alpha} : \tilde{\diamond} \to \diamond$ is a morphism (and even isomorphism) of admissible systems because modulo the induced trivialization $\hat{x}^1_\nu : \tilde{U}_\nu = F^\diamond \mathbf{R}^{[e_\nu]} \tilde{=} U_\nu$ it looks as the identity morphism $\tilde{\diamond} \to \tilde{\diamond}$ (we propose to use Lemma 3.12(iii) to express $\tilde{\diamond}$ in this induced trivialization). Let $\mathcal{T}^\diamond := \tilde{\alpha}$.

Clearly, $\diamond = \diamond^{F^\diamond}$ modulo the isomorphism $\mathcal{T}^\diamond$ of admissible systems. □

Proposition 6.2. *The described in Example 3.10 correspondence $\diamond \mapsto F^\diamond$ induces the bijection $[\diamond] \mapsto [F^\diamond]$ between the isomorphic classes of admissible systems and the isomorphism classes of ppgb-functors on* [3]-$\mathcal{VB}$. *The inverse bijection is induced by the described in Example 5.1 correspondence $F \mapsto \diamond^F$.*

Proof. The correspondence $[\diamond] \mapsto [F^\diamond]$ is well defined. For, if $\alpha : \diamond \to \tilde{\diamond}$ is an isomorphism, then so is $\eta^\alpha : F^\diamond \to F^{\tilde{\diamond}}$ (from Example 3.10). The correspondence $[F] \to [\diamond^F]$ is well defined, too. For, if $\eta : F \to \tilde{F}$ is a natural isomorphism, then so is $\alpha^\eta : \diamond^F \to \diamond^{\tilde{F}}$ (from Example 5.1). The correspondences $[F] \to [\diamond^F]$ and $[\diamond] \to [\diamond^F]$ are mutually inverse. For, by Theorem 6.1, $F = F^{\diamond^F}$ modulo the isomorphism Θ^F and $\diamond = \diamond^{F^\diamond}$ modulo the isomorphism $\mathcal{T}^\diamond$. □

Lemma 6.3. *Let $\eta : F \to F^1$ be a natural transformation between ppgb-functors on* [3]-$\mathcal{VB}$. *If $K = K^1 \times K^2$ is the product of* [3]-$\mathcal{VB}$*-objects K^1 and K^2, then $\eta_K = \eta_{K^1} \times \eta_{K^2}$ (modulo the product preserving identifications $FK = FK^1 \times FK^2$ and $F^1K = F^1K^1 \times F^1K^2$).*

Proof. Let $p_1 : K \to K^1$ and $p_2 : K \to K^2$ be the product projections. They are [3]-$\mathcal{VB}$-morphisms. Then $F^1(p_1)(\eta_K(v_1, v_2)) = \eta_{K^1}(F(p_1)(v_1, v_2)) = \eta_{K^1}(v_1)$ and $F^1(p_2)(\eta_K(v_1, v_2)) = \eta_{K^2}(v_2)$ for any $v = (v_1, v_2) \in FK = FK^1 \times FK^2$ (because η is a natural transformation). That is why $\eta_K(v_1, v_2) = (\eta_{K^1}(v_1), \eta_{K^2}(v_2))$. □

Lemma 6.4. *Let $\eta^1, \eta^2 : F \to F^1$ be two natural transformations of ppgb-functors on* [3]-$\mathcal{VB}$. *If $\eta^1_{\mathbf{R}^{[e_\nu]}} = \eta^2_{\mathbf{R}^{[e_\nu]}}$ for $\nu = 1, ..., 8$, then $\eta^1_K = \eta^2_K$ for any* [3]-$\mathcal{VB}$*-object K.*

Proof. Let K be a [3]-$\mathcal{VB}$-object. Natural transformations commute with [3]-$\mathcal{VB}$-trivialization. Then one can assume $K = \mathbf{R}^{[m]}$. Now, since $\mathbf{R}^{[m]}$ is the multi product of $\mathbf{R}^{[e_\nu]}$ for $\nu = 1, ..., 8$, our lemma is a simple consequence of Lemma 6.3. □

Proposition 6.5. *(i) Let F and F^1 be ppgb-functors on* [3]-$\mathcal{VB}$. *The described in Example 5.1 correspondence $\eta \mapsto \alpha^\eta$ is the bijection between the natural transformations $F \to F^1$ and the morphisms $\diamond^F \to \diamond^{F^1}$ of the corresponding admissible systems.*

(ii) Let $\diamond$ and $\tilde{\diamond}$ be admissible systems. There is the bijection between the morphisms $\diamond \to \tilde{\diamond}$ and the natural transformations $F^\diamond \to F^{\tilde{\diamond}}$ of the corresponding ppgb-functors.

Proof. **ad**(i) The correspondence $\eta \mapsto \alpha^\eta$ is injective. For, if $\eta^1 : F \to F^1$ is a natural transformation such that $\eta \neq \eta^1$, then $\alpha^\eta \neq \alpha^{\eta_1}$ because of Lemma 6.4.

We can prove that the correspondence $\eta \mapsto \alpha^\eta$ is surjective as follows.

Consider a morphism $\alpha : \diamond^F \to \diamond^{F^1}$ of admissible systems. Let $\eta^\alpha : F^{\diamond^F} \to F^{\diamond^{F^1}}$ be the descibed in Example 3.10 (for $\diamond^F$ and $\diamond^{F^1}$ instead of $\diamond$ and $\tilde{\diamond}$) natural transformation corresponding to α. Since $F = F^{\diamond^F}$ (modulo the isomorphism Θ^F from the proof of Theorem 6.1(i)) and $F^1 = F^{\diamond^{F^1}}$ (modulo the isomorphism Θ^{F^1}), then $\eta^\alpha : F \to F^1$ (modulo these isomorphisms). Put $\eta := \eta^\alpha$. Then $\alpha^\eta = \alpha$.

ad(ii) By part (i) of this proposition, the correspondence $\eta \mapsto \alpha^\eta$ is the bijection between the natural transformations $F^\diamond \to F^{\tilde{\diamond}}$ and the morphisms $\diamond^{F^\diamond} \to \diamond^{F^{\tilde{\diamond}}}$. On the other hand $\diamond = \diamond^{F^\diamond}$ modulo isomorphism $\mathcal{T}^\diamond$ (from Theorem 6.1(ii)) and $\tilde{\diamond} = \diamond^{F^{\tilde{\diamond}}}$ modulo $\mathcal{T}^{\tilde{\diamond}}$. □

7. Composition

Let F^1 and F^2 be ppgb-functors on [3]-$\mathcal{VB}$. Let $\diamond^{F^1}$ be the A^{F_1}-admissible system of F^1 and $\diamond^{F^2}$ be A^{F^2}-admissible system of F^2. By Theorem 6.1, F^1, F^2 : [3]-$\mathcal{VB} \to$ [3]-$\mathcal{VB}$. So, we can compose F^1 and F^2. The composition $F := F^2 \circ F^1$ is again a ppgb-functor on [3]-$\mathcal{VB}$. Let $\diamond^F$ the A^F-admissible system of $F = F^2 \circ F^1$.

Lemma 7.1. *We have $A^F = A^{F^1} \otimes A^{F^2}$ (the tensor product over $\mathbf{R}$) and the multiplication is given by $(a^1 \otimes a^2)(b^1 \otimes b^2) = (a^1 b^1) \otimes (a^2 b^2)$ for any $a^1, b^1 \in A^{F^1}$ and $a^2, b^2 \in A^{F^2}$.*

Proof. We know that A^F, A^{F^1} and A^{F^2} are the Weil algebras of the product preserving bundle functors $\tilde{F}, \tilde{F}^1, \tilde{F}^2 : \mathcal{M}f \to \mathcal{FM}$ (on the category $\mathcal{M}f$ of manifolds and their maps) given by $\tilde{F}M = FM$, $\tilde{F}^1 M = F^1 M$, $\tilde{F}^2 M = F^2 M$, where manifolds M are treated as the [3]-$\mathcal{VB}$-objects with all arrows being the identity maps of M. We can see that $\tilde{F} = \tilde{F}^2 \circ \tilde{F}^1$. Then the lemma is the well known result on Weil functors, see [7, 8]. □

Lemma 7.2. *Let $\nu = 1, ..., 8$. We have $U^F_\nu = U^{F^1}_\nu \otimes U^{F^2}_\nu$ (the tensor product over $\mathbf{R}$) and the module action of $A^F = A^{F^1} \otimes A^{F^2}$ on U^F_ν is given by $(a^1 \otimes a^2)(u^1 \otimes u^2) = (a^1 u^1) \otimes (a^2 u^2)$ for any $a^1 \in A^{F^1}$, $a^2 \in A^{F^2}$, $u^1 \in U^{F^1}_\nu$ and $u^2 \in U^{F^2}_\nu$.*

Proof. Let $p^{(1)} := \dim_{\mathbf{R}}(A^{F^1})$, $p^{(2)} := \dim_{\mathbf{R}}(A^{F^2})$, $q^{(1)} := \dim_{\mathbf{R}}(U^{F^1}_\nu)$ and $q^{(2)} := \dim_{\mathbf{R}}(U^{F^2}_\nu)$. Let $\{v^{(1)}_i\}_{i=1,...,p^{(1)}}$ be the basis (over $\mathbf{R}$) of A^{F^1} and $\{v^{(2)}_j\}_{j=1,...,p^{(2)}}$ be the basis of A^{F^2} and $\{w^{(1)}_k\}_{k=1,...,q^{(1)}}$ be the basis of $U^{F^1}_\nu$ and $\{w^{(2)}_l\}_{l=1,...,q^{(2)}}$ be the basis of $U^{F^2}_\nu$. Identifying any $x = \sum_i x^i v^{(1)}_i \in A^{F^1}$ with $x = (x^i) \in \mathbf{R}^{p^{(1)}}$, we have $A^{F^1} = \mathbf{R}^{p^{(1)}}$. Similarly, $A^{F^2} = \mathbf{R}^{p^{(2)}}$, $U^{F^1}_\nu = \mathbf{R}^{q^{(1)}}$ and $U^{F^2}_\nu = \mathbf{R}^{q^{(2)}}$. Then, using Lemma 3.12, $\mathbf{R}^{p^{(1)}} = A^{F^1} = F^1 \mathbf{R}^{[e_1]} = \mathbf{R}^{[p^{(1)} e_1]} = (\mathbf{R}^{[e_1]})^{p^{(1)}}$ and $\mathbf{R}^{q^{(1)}} = (\mathbf{R}^{[e_\nu]})^{q^{(1)}}$, and then $F^2 \mathbf{R}^{p^{(1)}} = (A^{F^2})^{p^{(1)}}$ and $F^2 \mathbf{R}^{q^{(1)}} = (U^{F^2}_\nu)^{q^{(1)}}$.

We can write $v^{(1)}_i w^{(1)}_k = \sum_{k_1} c^{k_1}_{ik} w^{(1)}_{k_1}$ and $v^{(2)}_j w^{(2)}_l = \sum_{l_1} d^{l_1}_{jl} w^{(2)}_{l_1}$, where $c^{k_1}_{ik}$ and $d^{l_1}_{jl}$ are the real numbers. Then the multiplication map $F^1(\cdot) : A^{F^1} \times U^{F^1}_\nu = \mathbf{R}^{p^{(1)}} \times \mathbf{R}^{q^{(1)}} \to \mathbf{R}^{q^{(1)}} = U^{F^1}_\nu$ satisfies $F^1(\cdot)(x, y) = (\sum_{i,k} c^{k_1}_{ik} x^i y^k)$ for $x = (x^i) \in \mathbf{R}^{p^{(1)}}$ and $(y^k) \in \mathbf{R}^{q^{(1)}}$. Then $F(\cdot) = F^2(F^1(\cdot)) : (A^{F^2})^{p^{(1)}} \times (U^{F^2}_\nu)^{q^{(1)}} \to (U^{F^2}_\nu)^{q^{(1)}}$, and (by Lemma 3.12) we have the quite similar formula $F(\cdot)(x, y) = (\sum_{i,k} c^{k_1}_{ik} x^i y^k)$ for $x = (x^i) \in (A^{F^2})^{p^{(1)}}$ and $y = (y^k) \in (U^{F^2}_\nu)^{q^{(1)}}$. Then $F(\cdot) : \mathbf{R}^{p^{(2)} p^{(1)}} \times \mathbf{R}^{q^{(2)} q^{(1)}} \to \mathbf{R}^{q^{(2)} q^{(1)}}$ and $F(\cdot)((x^{i\alpha}), (y^{k\beta})) = (\sum_{i,k,\alpha,\beta} c^{k_1}_{ik} d^{l_1}_{\alpha\beta} x^{i\alpha} y^{k\beta})$. It means that $F(\cdot) : (A^{F^1} \otimes A^{F^2}) \times (U^{F^1}_\nu \otimes U^{F^2}_\nu) \to U^{F^1}_\nu \otimes U^{F^2}_\nu$ and $F(\cdot)(a^1 \otimes a^2, u^1 \otimes u^2) = (a^1 u^1) \otimes (a^2 u^2)$ for $a^1 \in A^{F^1}$, $a^2 \in A^{F^2}$, $u^1 \in U^{F^1}_\nu$ and $u^2 \in U^{F^2}_\nu$, where $A^{F^1} \otimes A^{F^2} = \mathbf{R}^{p^{(1)} p^{(2)}}$ modulo the basis $(v^{(1)}_i \otimes v^{(2)}_j)$ and $U^{F^1}_\nu \otimes U^{F^2}_\nu = \mathbf{R}^{q^{(1)} q^{(2)}}$ modulo the basis $(w^{(1)}_k \otimes w^{(2)}_l)$. □

Quite similarly we can deduce

Lemma 7.3. *Given $(\nu, \mu, \kappa) \in Q^{oo}$, we have*

$$(u^1_\nu \otimes u^2_\nu) \diamond^{F,(\nu,\mu,\kappa)} (u^1_\mu \otimes u^2_\mu) = (u^1_\nu \diamond^{F^1,(\nu,\mu,\kappa)} u^1_\mu) \otimes (u^2_\nu \diamond^{F^2,(\nu,\mu,\kappa)} u^2_\mu)$$

for any $u^1_\nu \in U^{F^1}_\nu$, $u^2_\nu \in U^{F^2}_\nu$, $u^1_\mu \in U^{F^1}_\mu$, $u^2_\mu \in U^{F^2}_\mu$.

Consequently, we obtain

Theorem 7.4. *For any ppgb-functors F^1 and F^2 on [3]-$\mathcal{VB}$ we have the composition $F^2 \circ F^1 : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ of F_1 and F^2. This composition is a ppgb-functor on [3]-$\mathcal{VB}$ and we have $\diamond^{F_2 \circ F^1} = \diamond^{F^1} \otimes \diamond^{F^2}$, where the "tensor product" is explained in Lemmas 7.1—7.3. In particular, because of the exchanging isomorphism of the tensor product, any two ppgb-functors on [3]-$\mathcal{VB}$ commute.*

8. The canonical affinors af(c)

Let F be a ppgb-functor on [3]-$\mathcal{VB}$ and $T : [3]\text{-}\mathcal{VB} \to \mathcal{FM}$ be the tangent functor. The composition TF of T and F is again a ppgb-functor on [3]-$\mathcal{VB}$. Let $\diamond^F$ be the A^F-admissible system corresponding to F and $\diamond^T$ be the A^T-admissible system corresponding to T.

Lemma 8.1. *Let $\diamond^{TF}$ be the A^{TF}-admissible system of TF. Then $A^{TF} = A^F \otimes \mathbf{D} = A^F \times A^F$ is the Weil algebra with the algebra multiplication*

$$(a, a')(b, b') = (ab, a'b + ab') ,$$

and $U^{TF}_\nu = U^F_\nu \otimes \mathbf{D} = U^F_\nu \times U^F_\nu$ (for $\nu = 1, ..., 8$) is the $(A^F \times A^F)$-module with the module multiplication

$$(a, a')(u, u') = (au, a'u + au') ,$$

and $\diamond^{TF,(\nu,\mu,\kappa)} : U^{TF}_\nu \times U^{TF}_\mu \to U^{TF}_\kappa$ (for $(\nu, \mu, \kappa) \in Q^{oo}$) is the $(A^F \times A^F)$-bilinear map satisfying

$$(u, u') \diamond^{TF,(\nu,\mu,\kappa)} (v, v') = (u \diamond^{F,(\nu,\mu,\kappa)} v, u' \diamond^{F,(\nu,\mu,\kappa)} v + u \diamond^{F,(\nu,\mu,\kappa)} v') ,$$

where u, u', v, v', a, a' are elements of respective sets.

Proof. In Example 5.1, we observed that $A^T = U^T_1 = ... = V^T_8 = \mathbf{D}$ and $\diamond^{T,(\nu,\mu,\kappa)}$ (for $(\nu, \mu, \kappa) \in Q^{oo}$) is the multiplication of $\mathbf{D}$. Then, applying Theorem 7.4, we complete the proof. □

Proposition 8.2. *Let F be as above and K be a* [3]-$\mathcal{VB}$*-object. For any $c \in A^F$, there exists some* [3]-$\mathcal{VB}$*-natural affinor* af$(c) : TFK \to TFK$ *on FK such that the tangent bundle TFK of FK is the A^F-module bundle over FK with the fiber multiplication* $cy :=$ af$(c)(y)$.

Proof. Given $c \in A^F$, we define $\alpha^c_o : A^F \times A^F \to A^F \times A^F$ and $\alpha^c_{(\nu)} : U^F_\nu \times U^F_\nu \to U^F_\nu \times U^F_\nu$ for $\nu = 1, ..., 8$ by

$$\alpha^c_o(a, a') = (a, ca') , \; \alpha^c_{(\nu)}(u, u') = (u, cu') .$$

Then $(\alpha^c_o; \alpha^c_{(1)}, ...\alpha^c_{(8)}) : \diamond^{TF} \to \diamond^{TF}$ is a morphism of admissible systems. Let

$$\mathrm{af}(c) : TF \to TF$$

be the corresponding natural transformation. In the induced trivialization, we have

$$\mathrm{af}(c)(x, y) = (x, cy) \in \prod_{\nu=1}^{8} (U^F_\nu)^{m_\nu} \times \prod_{\nu=1}^{8} (U^F_\nu)^{m_\nu}$$

for any $(x, y) \in \prod_{\nu=1}^{8} (U^F_\nu)^{m_\nu} \times \prod_{\nu=1}^{8} (U^F_\nu)^{m_\nu}$. Then af(c) is an affinor on FK. One can easily see that TFK is the A-module bundle over FK with the fibre multiplication of $TFK \to FK$ given by $cy =$ af$(c)(y)$, $c \in A$, $y \in TFK$. □

9. The canonical vector fields Op(D)

Let $\diamond$ be an A-admissible system in the sense of Definition 3.5.

Definition 9.1. *A derivation of $\diamond$ is a system $D = (\tilde{\delta}_1, ..., \tilde{\delta}_8)$ of* **R***-linear maps $\tilde{\delta}_\nu : U_\nu \to U_\nu$ such that*

$$\tilde{\delta}_\nu(au_\nu) = a\tilde{\delta}_\nu(u_\nu) + \tilde{\delta}_1(a)u_\nu$$

for any $a \in A = U_1$ and any $u_\nu \in U_\nu$ and $\nu = 1, ..., 8$ and such that

$$\tilde{\delta}_\kappa(u_\nu \diamond^{(\nu,\mu,\kappa)} u_\mu) = \tilde{\delta}_\nu(u_\nu) \diamond^{(\nu,\mu,\kappa)} u_\mu + u_\nu \diamond^{(\nu,\mu,\kappa)} \tilde{\delta}_\mu(u_\mu)$$

for any $u_\nu \in U_\nu$ and any $u_\mu \in U_\mu$ and any $(\nu, \mu, \kappa) \in Q^{oo}$.

Proposition 9.2. *Let $\diamond$ be an A-admissible system (as above) and $F = F^\diamond$ be the ppgb-functor on* [3]-$\mathcal{VB}$ *corresponding to $\diamond$. Let K be a* [3]-$\mathcal{VB}$*-object. Any derivation D of $\diamond$ induces canonically the vector field (denoted by* Op(D)*) on FK.*

Proof. Let $\alpha_o : A \to A \times A$ and $\alpha_{(\nu)} : U_\nu \to U_\nu \times U_\nu$ for $\nu = 1, ..., 8$ be defined by

$$\alpha_o(a) = (a, \tilde{\delta}_1(a)) , \; \alpha_{(\nu)}(u_\nu) = (u_\nu, \tilde{\delta}_\nu(u_\nu))$$

for any $a \in A$ and any $u_\nu \in U_\nu$. Put $\alpha = (\alpha_o; \alpha_{(1)}, ..., \alpha_{(8)})$. Then $\alpha : \diamond \to \diamond \otimes \diamond^T$ is a morphism of admissible systems. Let $\eta^\alpha : FK \to TFK$ be the natural transformation corresponding to this morphism. By the local expression of η^α, presented in Example 3.10, one can easily see that $\eta^\alpha : FK \to TFK$ is a vector field. We denote it by Op(D). □

10. The natural vector fields

From now on, given a 8-tuple $m \in \mathbf{N}^8$, [3]-$\mathcal{VB}_{[m]}$ denotes the category of all triple vector bundles locally isomorphic with $\mathbf{R}^{[m]}$ and their [3]-$\mathcal{VB}$-isomorphisms onto open sub-objects.

Definition 10.1. *Let F be a ppgb-functor on [3]-$\mathcal{VB}$ and let m be an 8-tuple of non-negative integers. A [3]-$\mathcal{VB}_{[m]}$-natural vector field on F is a [3]-$\mathcal{VB}_{[m]}$-invariant family L of vector fields $L \in \mathcal{X}(FK)$ for any [3]-$\mathcal{VB}_{[m]}$-object K, where the invariance of L means that $TFf \circ L = L \circ Ff$ for any [3]-$\mathcal{VB}_{[m]}$-map $f : K \to K'$.*

Proposition 10.2. *Let F be a ppgb-functor on [3]-$\mathcal{VB}$ and let m be an 8-tuple of positive integers. Let L be [3]-$\mathcal{VB}_{[m]}$-natural vector field on F. Then $L = \mathrm{Op}(D)$ for some derivation D of the A^F-admissible system $\diamond^F$ corresponding to F.*

Proof. By the invariance of L with respect to [3]-$\mathcal{VB}_{[m]}$-trivialization, the family L is determined by the vector field L on $F\mathbf{R}^{[m]} = \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu}$, where (of course) $(U_\nu^F)^{m_\nu} = U_\nu^F \times ... \times U_\nu^F$ (m_ν times).

Then $L : \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu} \to (\prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu}) \times (\prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu})$, and we can write

$$L(u) = (u, (\delta_\mu^{i_\mu}(u))_{i_\mu=1,...,m_\mu\ ,\ \mu=1,...,8})\ ,$$

where $\delta_\mu^{i_\mu} : \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu} \to U_\mu^F$ are the maps and $u = (u_\nu^{i_\nu})_{i_\nu=1,...,m_\nu,\nu=1,...,8} \in \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu}$.

Let $(x_\nu^{i_\nu})$ be the usual trivialization on $\mathbf{R}^{[m]}$. Because of the invariance of L with respect to [3]-$\mathcal{VB}_{[m]}$-maps

$$(\tau_\nu^{i_\nu} x_\nu^{i_\nu}) : \mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$$

for positive real numbers $\tau_\nu^{i_\nu}$ and the homogeneous function theorem we can derive that given $\mu = 1, ..., 8$ and $i_\mu = 1, ..., m_\mu$ the map $\delta_\mu^{i_\mu} : \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu} \to U_\mu^F$ is of the form

$$\delta_\mu^{i_\mu}(u) = \delta_\mu^{i_\mu}(u_\mu^{i_\mu})\ ,\ u = (u_\nu^{i_\nu})_{i_\nu=1,...,m_\nu,\nu=1,...,8} \in \prod_{\nu=1}^{8}(U_\nu^F)^{m_\nu}$$

for some (denoted by the same symbol) $\mathbf{R}$-linear map $\delta_\mu^{i_\mu} : U_\mu^F \to U_\mu^F$.

Given $\mu = 1, ..., 8$, by the invariance of L with respect to the maps $\mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$ permuting the coordinates $x_\mu^{i_\mu}$ for $i_\mu = 1, ..., m_\mu$ and not changing the others (they are [3]-$\mathcal{VB}$-maps) we get that

$$\delta_\mu^1 = ... = \delta_\mu^{m_\mu} = \tilde{\delta}_\mu$$

for some $\mathbf{R}$-linear map $\tilde{\delta}_\mu : U_\mu^F \to U_\mu^F$.

So, we have

$$L = \prod_{\mu=1}^{8}(\delta_\mu)^{m_\mu} : \prod_{\mu=1}^{8}(U_\mu^F)^{m_\mu} \to \prod_{\mu=1}^{8}(U_\mu^F \times U_\mu^F)^{m_\mu}\ ,$$

where $\delta_\mu : U_\mu^F \to U_\mu^F \times U_\mu^F$ is defined by $\delta_\mu(u_\mu) = (u_\mu, \tilde{\delta}_\mu(u_\mu))$, $u_\mu \in U_\mu^F$.

By Proposition 2.5, given $\nu = 1, ..., 8$, there is a [3]-$\mathcal{VB}_{[m]}$-map $f : \mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$ (defined on some open dense subset of $\mathbf{R}^{[m]}$) such that

$$x_\nu^1 \circ f = x_\nu^1 + x_1^1 x_\nu^1\ .$$

If $\nu \neq 1$, then by the invariance of L with respect to f (and Lemma 3.12(iii) for $\diamond^{TF}$ and $\diamond^F$ instead $\diamond$) we get

$$\delta_\nu(u_\nu + au_\nu) = \delta_\nu(u_\nu) + \delta_1(a)\delta_\nu(u_\nu)$$

for any $a \in A^F = U_1^F$ and any $u_\nu \in U_\nu^F$. Then

$$(au_\nu, \tilde{\delta}_\nu(au_\nu)) = \delta_\nu(au_\nu) = \delta_1(a)\delta_\nu(u_\nu) = (a, \tilde{\delta}_1(a)) \cdot (u_\nu, \tilde{\delta}_\nu(u_\nu)) = (au_\nu, \tilde{\delta}_1(a)u_\nu + a\tilde{\delta}_\nu(u_\nu))\ ,$$

and then

$$\tilde{\delta}_\nu(au_\nu) = a\tilde{\delta}_\nu(u_\nu) + \tilde{\delta}_1(a)u_\nu$$

for any $a \in A^F = U_1^F$ and $u_\nu \in U_\nu^F$.

If $\nu = 1$, then (similarly) $\delta_1(a^2) = (\delta_1(a))^2$. Then by the polarization, $\delta_1(ab) = \delta_1(a)\delta_1(b)$, and then

$$\tilde{\delta}_1(ab) = a\tilde{\delta}_1(b) + \tilde{\delta}_1(a)b$$

for any $a, b \in A^F = U_1^F$.

Quite similarly, given $(\nu, \mu, \kappa) \in Q^{oo}$, there exists a [3]-$\mathcal{VB}_{[m]}$-map $f : \mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$ (defined on some open dense subset of $\mathbf{R}^{[m]}$) such that $x_\kappa^1 \circ f = x_\kappa^1 + x_\nu^1 x_\mu^1$. Then using the invariance of L with respect to this f we can easily (similarly as above) derive that

$$\tilde{\delta}_\kappa(u_\nu \diamond^{F,(\nu,\mu,\kappa)} u_\mu) = \tilde{\delta}_\nu(u_\nu) \diamond^{F,(\nu,\mu,\kappa)} u_\mu + u_\nu \diamond^{F,(\nu,\mu,\kappa)} \tilde{\delta}_\mu(u_\mu)$$

for any $u_\nu \in U_\nu^F$ and any $u_\mu \in U_\mu^F$ and any $(\nu, \mu, \kappa) \in Q^{oo}$.

Then $D := (\tilde{\delta}_1, ..., \tilde{\delta}_8)$ is a derivation of $\diamond^F$, and $L = \mathrm{Op}(D)$. □

11. Lifting triple linear vector fields

Definition 11.1. *A triple linear vector field on an* [3]-$\mathcal{VB}$*-object K is a vector field Z on K such that the flow of Z is formed by (locally defined)* [3]-$\mathcal{VB}$*-morphisms.*

Let F be a ppgb-functor on [3]-$\mathcal{VB}$. Let m be an 8-tuple of positive integers.

Definition 11.2. *An* [3]-$\mathcal{VB}_{[m]}$*-natural gauge operator lifting triple linear vector fields Z on K into vector fields C(Z) on FK is a* [3]-$\mathcal{VB}_{[m]}$*-invariant family C of regular operators*

$$C : \mathcal{X}_{[3]-LIN}(K) \to \mathcal{X}(FK)$$

for any [3]-$\mathcal{VB}_{[m]}$*-object K, where* $\mathcal{X}_{[3]-LIN}(K)$ *is the space of all triple linear vector fields on K and* $\mathcal{X}(FK)$ *is the space of all vector fields on FK. The* [3]-$\mathcal{VB}_{[m]}$*-invariance of C means that if triple linear vector fields* $Z_1 \in \mathcal{X}_{[3]-LIN}(K_1)$ *and* $Z_2 \in \mathcal{X}_{[3]-LIN}(K_2)$ *are f-related (i.e.* $Tf \circ Z_1 = Z_2 \circ f$*) for some* [3]-$\mathcal{VB}_{[m]}$*-map* $f : K_1 \to K_2$*, then* $C(Z_1)$ *and* $C(Z_2)$ *are Ff-related. The regularity of C means that C transforms smoothly parametrized families of triple linear vector fields into smoothly parametrized families of vector fields.*

Example 11.3. *The flow operator* $\mathcal{F}$ *transforming any* $Z \in \mathcal{X}_{[3]-LIN}(K)$ *into* $\mathcal{F}Z \in \mathcal{X}(FK)$ *is a natural operator in the sense of Definition 11.2. (We recall that* $\mathcal{F}Z$ *is given by the flow* $\{F\varphi_t\}$*, where* $\{\varphi_t\}$ *is the flow of Z.)*

We have the following generalization of the result of I. Kolář [6].

Theorem 11.4. *Let* $m = (m_1, ..., m_8)$ *be a 8-tuple of positive integers. Let F be a ppgb-functor on* [3]-$\mathcal{VB}$ *and* $\diamond^F$ *be its* A^F*-admissible system. Any* [3]-$\mathcal{VB}_{[m]}$*-natural gauge operator C lifting triple linear vector fields Z on K into vector fields C(Z) on FK is of the form*

$$C(Z) = \mathrm{af}(c) \circ \mathcal{F}Z + \mathrm{Op}(D)$$

for a (unique) element $c \in A^F$ *and a (unique) derivation D of* $\diamond^F$*.*

Proof. The proof of this theorem is the respective modification of the proof of Theorem 8.2 in [17]. Below, we present this modification for the reader convenience.

Let C be an operator in question. Then $C = (C - C(0)) + C(0)$. By Proposition 10.2, $C(0) = \mathrm{Op}(D)$. So, we may assume $C(0) = 0$. Let $(x_\nu^{i_\nu})_{i_\nu=1,...,m_\nu,\ \nu=1,...,8}$ be the usual coordinates on $\mathbf{R}^{[m]}$. Denote $x^1 = x_1^1$. By Lemma 11.5, C is determined by $C(\frac{\partial}{\partial x^1})$. Define $\overline{C} : \mathbf{R} \times (U_1^F)^{m_1} \times ... \times (U_8^F)^{m_8} \to (U_1^F)^{m_1} \times ... \times (U_8^F)^{m_8}$ by

$$((u_1, ..., u_8), \overline{C}(t, u_1, ..., u_8)) = C(t\frac{\partial}{\partial x^1})(u_1, ..., u_8)\ ,$$

$t \in \mathbf{R}$, $u_1 \in (U_1^F)^{m_1}, ..., u_8 \in (U_8^F)^{m_8}$. Using the invariance of C with respect to the homotheties $\tau\mathrm{id} : \mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$ for $\tau \neq 0$ and using the homogeneous function theorem, we derive that $\overline{C}$ is $\mathbf{R}$-linear. Then, because of the assumption $C(0) = 0$, we have $\overline{C}(1, u_1, ..., u_8) = \overline{C}(1) \in (U_1^F)^{m_1} \times ... \times (U_8^F)^{m_8}$. Then using the invariance of C with respect to the [3]-$\mathcal{VB}_{[m]}$-maps $(x^1, \tau x_1^2, ..., \tau x_1^{m_1}, ...\tau x_8^1, ..., \tau x_8^{m_8}) : \mathbf{R}^{[m]} \to \mathbf{R}^{[m]}$ for $\tau \neq 0$, we derive $\overline{C}(1) \in U_1^F \times \{0\} = A^F \times \{0\}$. Then the vector space of all C in question is of dimension $\leq dim_{\mathbf{R}}(A^F)$. Then $C(Z) = \mathrm{af}(c) \circ \mathcal{F}Z$ for a unique $c \in A^F$ because of the dimension argument. □

We else prove the following lemma we used in the proof of Theorem 11.4.

Lemma 11.5. *Let Z be a triple linear vector fields on K such that the underlying vector field $\underline{Z}$ on the basis M is non-zero at a point $x_o \in M$. Then there exists a local* [3]-$\mathcal{VB}$*-coordinate system* $(x_\nu^{i_\nu})_{i_\nu=1,...,m_\nu,\ \nu=1,...,8}$ *on K with centrum x_o with $x^1 = x_1^1$ such that $Z = \frac{\partial}{\partial x^1}$.*

Proof. The proof is quite the same as the one in the manifold case. We may assume that $K = \mathbf{R}^{[m]}$ and $x_o = 0 \in \mathbf{R}^{m_1}$ and $\underline{Z}_{|0} = \frac{\partial}{\partial x^1}_{|0}$. Let $\{\varphi_t\}$ be the flow of Z. Then $\Phi : K \to K$ given by $\Phi(x^1, x_1^2, ..., x_8^{m_8}) = \varphi_{x^1}(0, x_1^2, ..., x_8^{m_8})$ for $(x^1, x_1^2, ..., x_8^{m_8}) \in \mathbf{R}^{[m]}$ is a local [3]-$\mathcal{VB}$-isomorphism sending $\frac{\partial}{\partial x^1}$ to Z. □

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