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# Critical exponent for a free boundary problem with a nonlocal reaction term

**Linfei Shi[a,*], Tianzhou Xu[b], Jinjin Mao[c]**

[a] *School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, Henan, People's Republic of China*
[b] *School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, People's Republic of China*
[c] *School of Science, Nanjing University of Posts and Telecommunications, Nanjing, Jiangsu, People's Republic of China*

**Abstract.** This paper concerns the dynamics of a reaction–diffusion equation with an exponential nonlocal reaction term and free boundaries in one-dimensional space. We first give some sufficient conditions of the finite time blow-up and vanishing of the solution. Then, a sharp threshold trichotomy result for distinguishing blow-up, vanishing, and transition solutions is established.

## 1. Introduction

Numerous physical and biological processes are well known to be explained by the following reaction–diffusion equation:

$$u_t = u_{xx} + F(u(x,t)),$$

where $u(x,t)$ indicates the population density of a biological species or the chemical reaction temperature [4, 5], $F$ represents the net birth rate or net heat source, and $u_{xx}$ denotes diffusion. The results in previous literature show that if the initial temperature is high enough, it will increase to a very high temperature in a finite time, which is referred to as the blow-up phenomenon. Analogously, a high initial population density can lead to the blow-up phenomenon.

Fujita [8] proposed "critical exponent" to solve the problem of blow-up to solutions. Following this work, the problem has received a lot of attention. In [14], Meier considered the problem with temporal weight

$$\begin{cases} u_t - u_{xx} = e^{\beta t}u^p, & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u = u_0(x) > 0, & x \in \Omega,\ t = 0, \end{cases}$$

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Communicated by Marko Nedeljkov
* Corresponding author: Linfei Shi
*Email addresses:* shilinfeixtu@163.com (Linfei Shi), xutianzhou@bit.edu.cn (Tianzhou Xu), 3120205703@bit.edu.cn (Jinjin Mao)
ORCID iDs: https://orcid.org/0000-0003-1163-2444 (Linfei Shi), https://orcid.org/0000-0002-1650-9798 (Tianzhou Xu)

admits the critical exponent $P_c = 1 + \beta/\lambda_1$, in which $\lambda_1$ indicates the first Dirichlet eigenvalue to the Laplace operator in a bounded domain $\Omega$. Then, some results on the critical exponent of Fujita type were investigated, such as nonlocal diffusion equations [9, 21] and parabolic systems [1, 15, 23].

In recent years, scholars have been interested in the free boundary problem of the reaction–diffusion equation and have achieved fruitful results[3, 14, 17, 18, 24]. In particular, Zhou and Lin [25] considered a double fronts free boundary problem with a nonlocal reaction term in one-dimensional space, they showed the blow-up happens if the initial value is large enough and gave the existence of a global fast solution with small enough initial data and a slow solution with suitably large initial data. Yang [20] obtained similar results based on [25]. The blow-up and global existence of the solutions to several nonlocal diffusion problems with critical exponent are analyzed in [13]. Wang et al. [19] studied the free boundary problem with a spatially exponential time-weighted source, mainly considering the asymptotic behavior of the solution, and showed a sharp threshold trichotomy result based on the initial data.

By the inspiration of the above papers, the purpose of this paper is mainly to present an analysis of the free boundary with an exponential nonlocal reaction term for the reaction-diffusion equation shown below:

$$\begin{cases} u_t - du_{xx} = au + be^{\beta t}\int_{r(t)}^{s(t)} u^p dx, & r(t) < x < s(t),\ t > 0, \\ u(s(t),t) = 0,\ s'(t) = -\mu u_x(s(t),t), & t > 0, \\ u(r(t),t) = 0,\ r'(t) = -\mu u_x(r(t),t), & t > 0, \\ -r(0) = s(0) = s_0,\ u(x,0) = u_0(x), & -s_0 \le x \le s_0, \end{cases} \tag{1}$$

where $s_0$, $\mu$, $d$, and $b$ are some given positive constants, $a \in \mathbb{R}$, $p > 1$, and $x = r(t), s(t)$ are spreading fronts. For $s_0 \in (0,\infty)$, $u_0 \in \mathcal{X}(s_0)$, where

$$\mathcal{X}(s_0) := \left\{u_0 \in C^2([-s_0,s_0]) : u_0 > 0 \text{ in } (-s_0,s_0) \text{ with } u(-s_0) = u(s_0) = 0\right\}. \tag{2}$$

The intra-species growth rate and dispersal coefficient of the species are represented $a$ and $d > 0$, respectively. $u(x,t)$ denotes the population density for the species in one-dimensional space and the population size of a invasive or new species is expressed by $u_0(x)$, which has an initial region $[-s_0, s_0]$. We further enrich the results of [25] by considering the effect of the critical exponent on the blow-up for the solution based on [25].

In this paper, we will be concerned with the blow-up of the solution for (1). Since the solution may blow up in finite time, the maximum existence time is denoted by the following:

$$T_{\max} := \sup\{T > 0 : \text{ the classical solution to (1) exists on } [0,T) \text{ for the initial data } u_0\}.$$

If $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty([g(t),h(t)])} = \infty$, which implies that the solution $u$ of (1) blows up in finite time and $T_{\max}$ is the blow-up time. Theorem 3.4 shows that when $T_{\max} < \infty$ the solution blows up in finite time.

The structure of this paper is shown below. We first present some basic results that will be applied later in this paper in Section 2. Section 3 gives some sufficient conditions for blow-up and vanishing by utilizing the comparison principle and constructing an appropriate upper solution or lower solution. Section 4 is devoted to showing a sharp threshold trichotomy of the initial data size by using the indirect demonstration and comparison principle to distinguish the blow-up, vanishing, and transition solutions.

## 2. Preliminary

Several lemmas and theorems that will be used later in this paper are given in this section.

**Theorem 2.1.** *For any given $u_0 \in \mathcal{X}$ and $\alpha \in (0,1)$, there exists a constant $T > 0$ such that the problem* (1) *admits a unique positive solution*

$$(u,r,s) \in C^{1+\alpha,(1+\alpha)/2}(\overline{G}_T) \times C^{1+\alpha/2}([0,T]) \times C^{1+\alpha/2}([0,T]).$$

*Furthermore,*

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(\bar{G}_T)} + \|g\|_{C^{1+\alpha/2}([0,T])} + \|h\|_{C^{1+\alpha/2}([0,T])} \le C, \tag{3}$$

*where* $G_T := \{(x,t) \in \mathbb{R}^2 : x \in [r(t), s(t)],\ t \in (0,T]\}$, *C and T depend only on* $s_0$, $\alpha$, $\|u_0\|_{C^2([-s_0,s_0])}$.

*Proof.* As in [24], we first straighten the free boundary with the introduction of the transformation

$$y = \frac{2s_0 x}{s(t) - r(t)} + \frac{s_0(s(t) + r(t))}{s(t) - r(t)},\ v(y,t) = u(x,t).$$

Then we get

$$\begin{cases} v_t - Av_y - Bv_{yy} = av + De^{\beta t}\int_{-s_0}^{s_0} v^p(y,t)dy, & -s_0 < y < s_0,\ t > 0, \\ v = 0,\ s'(t) = -\frac{2s_0\mu}{s(t)-r(t)}\frac{\partial v}{\partial y}, & t > 0, y = s_0, \\ v = 0,\ r'(t) = -\frac{2s_0\mu}{s(t)-r(t)}\frac{\partial v}{\partial y}, & t > 0, y = -s_0, \\ -r(0) = s(0) = s_0,\ v(y,0) = v_0(y) := u_0(y), & -s_0 \le y \le s_0, \end{cases} \tag{4}$$

where $A = y\frac{s'(t)-r'(t)}{s(t)-r(t)} + s_0\frac{s'(t)+r'(t)}{s(t)-r(t)}$, $B = \frac{4s_0^2 d}{(s(t)-r(t))^2}$, $D = \frac{b(s(t)-r(t))}{2s_0}$. It is easy to show that this transformation makes $x = s(t)$ and $x = r(t)$ into $y = s_0$ and $y = -s_0$, respectively, and that this equation becomes more complicated because the coefficients of the first equation in (4) include $s(t)$ and $r(t)$.

We omit the rest of the proof because it is based on [5, 25] by utilizing the contraction mapping theorem with minor modifications. □

**Lemma 2.2.** *(The Comparison Principle) Let* $T \in (0, T_{\max}), \bar{r}, \bar{s} \in C^1([0,T]), \bar{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T := \{(x,t) \in \mathbb{R}^2 : x \in (\bar{r}(t), \bar{s}(t)),\ t \in (0,T]\}$ *and*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} \ge a\bar{u} + be^{\beta t}\int_{\bar{r}(t)}^{\bar{s}(t)} \bar{u}^p dx, & \bar{r}(t) < x < \bar{s}(t),\ t > 0, \\ \bar{u}(\bar{s}(t),t) = 0,\ \bar{s}'(t) \ge -\mu\bar{u}_x(\bar{s}(t),t), & t > 0, \\ \bar{u}(\bar{r}(t),t) = 0,\ \bar{r}'(t) \le -\mu\bar{u}_x(\bar{r}(t),t), & t > 0. \end{cases} \tag{5}$$

*If* $(u, r, s)$ *is the solution to the problem* (1), *and satisfies*

$$\bar{r}(0) \le -s_0,\ \bar{s}(0) \ge s_0,\ \text{and } \bar{u}(x,0) \ge u_0(x) \text{ in } [-s_0, s_0],$$

*then*

$$\bar{r}(t) \le r(t),\ \bar{s}(t) \ge s(t) \text{ in } (0,T],$$

*and for* $(x,t) \in (r(t), s(t)) \times (0,T]$ *there is* $\bar{u}(x,t) \ge u(x,t)$.

*Proof. Since the proof of Lemma 2.2 is the same as that of Lemma 3.5 in [5], we omit it here.* □

**Remark 2.3.** *The triple* $(\bar{u}, \bar{r}, \bar{s})$ *of Lemma 2.2 is frequently referred to as the upper solution of (1). Inverting all the above inequalities defines the lower solution can be defined similarly.*

**Lemma 2.4.** *Let* $(u, r, s)$ *is the solution to* (1) *defined in* $t \in [0, T_0)$ *for* $T_0 \in (0, T_{\max})$, *and there exists* $C_1(T_0) > 0$ *such that*

$$u(x,t) \le C_1,\ \text{for } x \in [r(t), s(t)] \text{ and } t \in [0, T_0).$$

*Then there exists a constant* $C(S) > 0$, *and S is a fixed number with* $T_0 < S$ *such that*

$$0 < s'(t),\ -r'(t) \le C(S), \tag{6}$$

*and*

$$s_0 < s(t), \quad -r(t) \le \frac{[2C(T_0)]^2}{m_2},\ \forall t \in [0, T_0), \tag{7}$$

*where* $m_2 = 8be^{\beta T_0}\mu^2 C_1^{p+1}$.

*Proof.* This proof of this Lemma is similar to [25], so we skip some steps and mainly write the modifications. Let

$$\tilde{w}(x,t) = C_1\left[2H(s(t)-x) - H^2(s(t)-x)^2\right]$$

for some suitable $H$ over the region

$$\Omega_H := \left\{(x,t) : x \in (s(t)-H^{-1}, s(t)),\ t \in (0,T_0)\right\}.$$

For any given $T < T_0$, $(x,t) \in \Omega_{H_T} := \left\{(x,t) : x \in (s(t)-H^{-1}, s(t)),\ t \in (0,T)\right\}$, we have

$$a\tilde{w} + be^{\beta t}\int_{r(t)}^{s(t)} \tilde{w}^p dx \le |a|C_1 + be^{\beta T_0}C_1^p G(T),$$

where $G(T) := s(T) - r(T) < +\infty$. If $H^2 \ge \frac{1}{2d}(|a| + be^{\beta T_0}C_1^{p-1})$, one has

$$\tilde{w}_t - d\tilde{w}_{xx} \ge 2dC_1H^2 \ge a\tilde{w} + be^{\beta t}\int_{r(t)}^{s(t)} \tilde{w}^p dx \ \text{ in } \ \Omega_{H_T},$$

Without loss of generality, we suppose that $C_1$ is sufficiently large such that

$$\frac{1}{2d}(|a| + be^{\beta T_0}C_1^{p-1}G(T)) \ge \frac{4\|u_0\|_{C^1([0,s_0])}}{3C_1}.$$

Hence, setting

$$H := \sqrt{\frac{1}{2d}}\left[(|a| + be^{\beta T_0}C_1^{p-1}G(T))\right]^{\frac{1}{2}} \ge \frac{4\|u_0\|_{C^1([0,s_0])}}{3C_1},$$

it holds that

$$\tilde{w}(x,0) \ge \frac{3}{4}C_1,\ u_0(x) \le \|u_0\|_{C^1([0,s_0])}H^{-1} \le \frac{3}{4}C_1$$

for $x \in [s_0 - H^{-1}, s_0 - (2H)^{-1}]$. Hence, $u_0(x) \le \tilde{w}(x,0)$ for $x \in [s_0 - H^{-1}, s_0]$.

By (7), we know

$$s'(t) = -\mu u_x(s(t),t) \le 2HC_1\mu \ \text{ for } t \in [0,T]. \tag{8}$$

Analogously, one can yield

$$-r'(t) = \mu u_x(r(t),t) \le 2HC_1\mu \ \text{ for } t \in [0,T].$$

We next combine $s'(t)$ and $-r'(t)$ to get

$$G'(T) \le (m_1 + m_2G(T))^{\frac{1}{2}},$$

where $m_1 = 8d^{-1}|a|\mu^2C_1^2,\ m_2 = 8be^{\beta T_0}\mu^2C_1^{p+1}$. Which implies

$$\left[(m_1 + m_2G(T))^{\frac{1}{2}}\right]' \le \frac{m_2}{2}, \tag{9}$$

and

$$(m_1 + m_2G(T))^{\frac{1}{2}} \le \frac{m_2}{2}T_0 + (m_1 + m_2(2s_0))^{\frac{1}{2}} \ \text{ for } T \in [0,T_0). \tag{10}$$

Since $T_0 < S$, by (8), (9) and (10), we easily derive

$$0 < s'(t),\ -r'(t) \le C(S),$$

where $C(S) = \frac{m_2}{4}T_0 + \frac{1}{2}(m_1 + m_2(2s_0))^{\frac{1}{2}}$ and

$$s_0 < s(t), -r(t) \le \frac{[2C(T_0)]^2}{m_2}.$$

We finish the proof for this theorem based on the arbitrariness of $T$. □

**Theorem 2.5.** *The double free frontiers $s(t)$ and $r(t)$ of problem* (1) *are strictly monotone increasing and decreasing, respectively. It means that*

$$r' < 0,\ s' > 0 \ \text{ for } 0 < t \le T.$$

*for any solution in* $(0, T]$.

*Proof.* We ignore the details because the proof of this theorem is analogous to the proof for Theorem 2.2 in [25]. □

We conclude that $r(t)$ and $s(t)$ are not only monotonic but also bounded.

## 3. Conditions of blow-up and vanishing

We primarily provide several sufficient conditions of blow-up or vanishing in this section. We first employ Kaplan's first eigenvalue approach [11] to give some sufficient conditions for the finite-time blow-up of the solution to (1). Now let's consider the associated eigenvalue problem:

$$\begin{cases} -d\psi_{xx} = \lambda\psi, & -s_0 < x < s_0, \\ \psi(-s_0) = \psi(s_0) = 0. \end{cases} \tag{11}$$

Let $\psi_1(x) := ksin\frac{\pi(x+s_0)}{2s_0}$ be the first eigenfunction with respect to the first eigenvalue $\lambda_1 = \frac{d\pi^2}{4s_0^2}$ for (11). It is possible to select $k$ such that $\int_{-s_0}^{s_0} \psi_1(x)dx = 1$. If $p > 1$, then (1) allows a critical exponent $P_\beta^* := 1 + \frac{\beta}{\lambda_1 - a}$.

**Theorem 3.1.** *Assume $p > 1$, $s_0 > 0$, $\beta > 0$, $\phi \in \mathcal{X}(s_0)$. Then the solution to problem (1) with $u_0(x) = \sigma\phi(x)$ will blow up in finite time while satisfying one of the following conditions*

(i) $1 < p \le P_\beta^*$ *and* $\sigma > 0$,

(ii) $p > P_\beta^*$ *and* $\sigma > \sigma^* := \left[\frac{(p-1)(\lambda_1 - a) - \beta}{(p-1)bA_0}\right]^{\frac{1}{p-1}} K_0^{-1}$, *where* $A_0 = \left(\int_{-s_0}^{s_0} \psi_1^{\frac{p}{p-q}} dx\right)^{\frac{q-p}{q}}$, $p > q \ge 1$, $K_0 = \int_{-s_0}^{s_0} \phi\psi_1 dx$.

*Furthermore,* $T_{\max} \le \widetilde{C}\sigma^{-(p-1)}$ *with* $\widetilde{C} > 0$ *and depends on the value of* $\alpha$, $a$, $p$, $\beta$, $s_0$ *and* $\phi$.

*Proof.* We present an auxiliary problem as follows

$$\begin{cases} \tilde{v}_t - d\tilde{v}_{xx} = a\tilde{v} + be^{\beta t}\int_{-s_0}^{s_0} \tilde{v}^p dx, & -s_0 < x < s_0,\ 0 < t < \widetilde{T}_{\max}, \\ \tilde{v}(-s_0, t) = \tilde{v}(s_0, t) = 0, & 0 < t < \widetilde{T}_{\max}, \\ \tilde{v}(x, 0) = \sigma\phi(x), & -s_0 \le x \le s_0, \end{cases} \tag{12}$$

where $\widetilde{T}_{\max}$ is the maximum existence time that makes $\tilde{v}(x,t)$ exists in $(0, \widetilde{T}_{\max})$. We use the comparison principle to derive $T_{\max} \le \widetilde{T}_{\max}$ and $u(x, t) \ge \tilde{v}(x, t)$ on $[-s_0, s_0] \times [0, T_{\max})$. It is sufficient to demonstrate that when $\tilde{v}$ blows up in finite time, $u$ will also blow up. Inspired by [12], we build the auxiliary functional below

$$K(t) = \int_{-s_0}^{s_0} \tilde{v}(x, t)\psi_1(x)dx.$$

Next, we use Jensen's integral inequality, Green's identity and Hölder inequality to deduce

$$\begin{aligned} K_t = \int_{-s_0}^{s_0} \tilde{v}_t\psi_1 dx &= \int_{-s_0}^{s_0} (d\tilde{v}_{xx} + a\tilde{v})\psi_1 dx + be^{\beta t}\left(\int_{-s_0}^{s_0} \psi_1 dx\right)\left(\int_{-s_0}^{s_0} \tilde{v}^p dx\right) \\ &= (a - \lambda_1)K + be^{\beta t}\int_{-s_0}^{s_0} \tilde{v}^p dx \\ &\ge (a - \lambda_1)K + be^{\beta t}\left(\int_{-s_0}^{s_0} \psi_1^{\frac{p}{p-q}} dx\right)^{\frac{q-p}{q}}\left(\int_{-s_0}^{s_0} \tilde{v}^q\psi_1 dx\right)^{\frac{p}{q}} \\ &\ge (a - \lambda_1)K + be^{\beta t}A_0K^p, \end{aligned} \tag{13}$$

where $A_0 = \left(\int_{-s_0}^{s_0} \psi_1^{\frac{p}{p-q}} dx\right)^{\frac{q-p}{q}}$ is a constant, and $p > q \geq 1$.

Utilizing the basic theory of ODE, we derive

$$K^{1-p} \leq \left\{\sigma^{1-p} K_0^{1-p} - \frac{bA_0(1-p)}{\beta - (p-1)(\lambda_1 - a)} \left[1 - e^{[\beta-(p-1)(\lambda_1-a)]t}\right]\right\} e^{(p-1)(\lambda_1-a)t}, \tag{14}$$

which indicates that

$$K^{p-1} \geq \frac{1}{\left\{\sigma^{1-p} K_0^{1-p} - \frac{bA_0(1-p)}{\beta-(p-1)(\lambda_1-a)} \left[1 - e^{[\beta-(p-1)(\lambda_1-a)]t}\right]\right\} e^{(p-1)(\lambda_1-a)t}}. \tag{15}$$

Thus, we conclude the following two cases:

(i) If $1 < p < P_\beta^*$, we get $\beta - (p-1)(\lambda_1 - a) > 0$ and $\frac{bA_0(1-p)}{\beta-(p-1)(\lambda_1-a)} < 0$. For all $\sigma > 0$, $\sigma^{1-p} K_0^{1-p} > 0$, and (15) yields that $\tilde{v}(t, x)$ blows up in finite time. Analogously, when $p = P_\beta^* < \infty$, (15) with L'Hopital Rule together gives

$$K^{p-1} \geq \frac{1}{[\sigma^{1-p} K_0^{1-p} - bA_0(p-1)t] e^{(p-1)(\lambda_1-a)t}}, \tag{16}$$

this yields that blowup happens again.

(ii) If $p > P_\beta^*$, it holds that

$$\beta - (p-1)(\lambda_1 - a) < 0, \ \frac{bA_0(1-p)}{\beta - (p-1)(\lambda_1 - a)} > 0.$$

Therefore, when

$$\sigma^{1-p} K_0^{1-p} - \frac{bA_0(1-p)}{\beta - (p-1)(\lambda_1 - a)} > 0,$$

i.e.,

$$\sigma > \sigma^* = \left[\frac{(p-1)(\lambda_1 - a) - \beta}{bA_0(1-p)}\right]^{\frac{1}{p-1}} K_0^{-1},$$

We observe that $\tilde{v}(x, t)$ blows up in finite time by using (15).

(14) and (15) imply that $\widetilde{T}_{\max} \leq \widetilde{C} \sigma^{-(p-1)}$, where constant $\widetilde{C} > 0$ is dependent on $a$, $p$, $\beta$, $s_0$ and $\phi$, and conclude the same estimate by the comparison principle. It concludes the argument. □

**Remark 3.2.** *Define*

$$\Lambda := \Lambda(s_0, \phi) = \begin{cases} \sigma_1 = 0, & \text{if } 1 < p \leq P_\beta^*, \\ \sigma_2 = \sigma^*, & \text{if } p > P_\beta^*, \end{cases} \tag{17}$$

*with $\lambda_1 = \lambda_1(s_0)$. $\Lambda$ is bounded. Then the solution of problem (11) with $u_0 = \sigma\phi$ will blow up when $\sigma > \Lambda$.*

Furthermore, we can deduce that $1 < p \leq P_\beta^*$ is equivalent to $s_0 \geq \frac{\pi}{2} \sqrt{\frac{d(p-1)}{\beta + a(p-1)}} = L$ (where $p > 1$). Hence, we get the result below.

**Corollary 3.3.** *Suppose $p > 1$, $a > 0$ and $\beta > 0$. If $s_0 \geq L$, then the solution to the problem (1) will blow up in finite time.*

The proof of the below theorem is almost identical to [25, Theorem 3.2], i.e., when the maximal existence interval is bounded, the solution blows up in $L^\infty$ norm.

**Theorem 3.4.** *Suppose $[0, T_{\max})$ is the maximum existence time interval of the solution $(u, r, s)$ for the problem (1). If $T_{\max} < \infty$, then*

$$\lim_{t\to T_{\max}} \|u(\cdot, t)\|_{L^\infty([r(t),\, s(t)])} = \infty. \tag{18}$$

**Theorem 3.5.** *Assume $p > 1$, $a \in \mathbb{R}$, $\beta > 0$, $s_0 > 0$, and $u_0 \in \mathcal{X}(s_0)$. Then $T_{\max} = \infty$, $I_\infty := (r_\infty, s_\infty)$ is a finite interval and $\lim_{t\to\infty} \|u(\cdot, t)\|_{L^\infty([r(t),\, s(t)])} = 0$ when satisfying one of the following conditions*

(i) $a \le 0$, $\|u_0\|_{L^\infty} \le \frac{1}{2}\min\{[d/(128bs_0^3)]^{\frac{1}{p-1}}, d/8\mu\}$,

(ii) $a > 0$, $s_0 < L$ *and* $\|u_0\|_{L^\infty}$ *is sufficiently small.*

*Furthermore, $s_\infty < \infty$, $r_\infty > -\infty$ and there exist several real numbers $C$ and $\delta > 0$ depending on $u_0$ such that*

$$\|u(t)\|_\infty \le Ce^{-\delta t},\ t \ge 0. \tag{19}$$

*Proof.* This theorem is a direct result of the comparison principle. We first prove condition (i). It is sufficient to construct an appropriate global supersolution. Inspired by [6], we let

$$\upsilon(t) = 2s_0(2 - e^{-\gamma t}),\ \ \zeta(t) = -\upsilon(t),\ \ t \ge 0,\ \ M(y) = 1 - y^2,\ \ -1 \le y \le 1,$$

and

$$v(x, t) = \varepsilon e^{-\alpha t} M(x/\upsilon),\ \ \zeta(t) \le x \le \upsilon(t),\ \ t > 0,$$

here $\gamma, \varepsilon, \alpha > 0$ will be given later. A direct calculation shows that

$$\begin{aligned} & v_t - dv_{xx} - av - be^{\beta t}\int_{\zeta(t)}^{\upsilon(t)} v^p dx \\ &= \varepsilon e^{-\alpha t}\left[-\alpha M - x\upsilon'\upsilon^{-2}M' - d\upsilon^{-2}M'' - aM - be^{[\beta-\alpha(p-1)]t}\int_{\zeta(t)}^{\upsilon(t)} \varepsilon^{p-1}M^p dx\right] \\ &\ge \varepsilon e^{-\alpha t}\left[-\alpha + \frac{d}{8s_0^2} - 8s_0 b\varepsilon^{p-1}\right] \end{aligned}$$

for all $t > 0$ and $\zeta(t) \le x \le \upsilon(t)$.

Moreover, we get $\upsilon' = 2\gamma s_0 e^{-\gamma t} > 0$ and $-v_x(\upsilon(t), t) = 2\varepsilon e^{-\alpha t}\upsilon^{-1}(t)$. We select $\gamma = \alpha = \frac{d}{16s_0^2}, \varepsilon \le \varepsilon_0 = \min\left\{[d/(128bs_0^3)]^{\frac{1}{p-1}}, d/8\mu\right\}$ to obtain

$$\begin{cases} v_t - dv_{xx} \ge av + be^{\beta t}\int_{\zeta(t)}^{\upsilon(t)} v^p dx, & \zeta(t) < x < \upsilon(t), \\ v(\upsilon(t), t) = 0,\ \upsilon'(t) \ge -\mu u_x(\upsilon(t), t), & t > 0, \\ v(\zeta(t), t) = 0,\ \zeta'(t) \le -\mu u_x(\zeta(t), t), & t > 0, \\ \zeta(0) = -2s_0 < -s_0,\ \upsilon(0) = 2s_0 > s_0. \end{cases} \tag{20}$$

We select $\varepsilon = 2\|u_0\|_\infty \le \varepsilon_0$ and suppose $\|u_0\|_\infty \le \frac{1}{2}\min\{[d/(128bs_0^3)]^{\frac{1}{p-1}}, d/8\mu\}$, then we get $u_0(x) \le v(x, 0)$ for $-s_0 \le x \le s_0$. As long as u exists, it is obvious to derive $s(t) < \upsilon(t)$, $r(t) > \zeta(t)$ and $u(x, t) < v(x, t)$ for $r(t) \le x \le s(t)$ by applying the comparison principle. In particular, by the continuation property (18), we determine u exists globally.

We follow up by proving condition (ii). Because $s_0 < L$, there exists some small $\rho, \kappa > 0$ such that

$$\frac{d\pi^2}{4(1+\rho)^2 s_0^2} \ge a + 2s_0 b\kappa^{p-1}(1+\rho) + \rho \text{ and } \pi\mu\kappa \le \rho^2 s_0^2. \tag{21}$$

Set

$$g(t) := s_0\left(1 + \rho - \frac{\rho}{2}e^{-\rho t}\right),\ w(x, t) := \kappa e^{-\rho t}\cos\left(\frac{\pi x}{2g(t)}\right).$$

*Obviously $w(-g(t),t) = w(g(t),t) = 0$. For $t > 0$ and $x \in [-g(t), g(t)]$, a direct calculation yields*

$$w_t - dw_{xx} - aw - be^{\beta t}\int_{-g(t)}^{g(t)} w^p dx \geq \left(-\rho + \frac{d\pi^2}{4k^2(t)} - a - 2s_0 b\kappa^{(p-1)}(1+\rho)\right) w \geq 0.$$

*By choosing s we get*

$$\mu w_x(-g(t),t) = -\mu w_x(g(t),t) = \frac{\pi\mu\kappa}{2g(t)}e^{-\rho(t)} \leq \frac{\pi\mu\kappa}{2s_0}e^{-\rho(t)} \leq \frac{\rho^2 s_0}{2}e^{-\rho(t)} = g'(t).$$

*Thus, if $w(x,0) \geq u_0(x)$, $(w(x,t), -g(t), g(t))$ is a supersolution of (1) in $[-s_0, s_0]$.*

*We select $\sigma_1 := \kappa\cos\frac{\pi}{2+\rho}$ depending on $\mu, s_0$. It follows that when $\|u_0\|_{L^\infty} \leq \sigma_1$ since $s_0 < g(0) = s_0(1+\frac{\rho}{2})$, we know $u_0(x) \leq \sigma_1 \leq w(x,0)$ in $[-s_0, s_0]$. Using Lemma 2.2, we obtain*

$$s(t) \leq g(t) \leq s_0(1+\rho),\ s_\infty < \infty.$$

*As a result, $u \to 0$ as $t \to \infty$ locally uniformly in $I_\infty$, which is a finite interval. This proof is thus complete.* □

We need the following lemma to elaborate on the asymptotic behavior of the problem (1), which gives a comprehensive description of the solution to (1). It can be proved to combine with the theorem 3.5 and then using [Lemma 4.1, [25]] in the same way, so we ignore it here.

**Lemma 3.6.** *Suppose $p > 1$, $a > 0$, and $\beta > 0$. Let $T_{\max}$ be the maximum existence time of the global solution $(u, r, s)$ to (1). If $T_{\max} = \infty$, then u is bounded, and $I_\infty$ is a finite interval of length not bigger than $2L$. Furthermore, it holds that*

$$\lim_{t\to\infty} \max_{r(t)<x<s(t)} u(x,t) = 0.$$

## 4. Sharp threshold trichotomy

Based on the conclusions in the previous sections, a sharp threshold trichotomy theorem is proved by us in this section.

**Theorem 4.1.** *Assume that $s_0 > 0$, $\phi \in \mathcal{X}(s_0)$, and $(u, r, s)$ is the solution to (1) with $u_0 = \sigma\phi(x)$ for some $\sigma > 0$, and $T_{\max}$ is the maximal existence time. Then there exists $\sigma^* = \sigma^*(p, \phi) \in [0, \infty)$ in which the following holds:*

- (i) *When $\sigma > \sigma^*$, $(u, r, s)$ blows up in finite time.*
- (ii) *When $0 < \sigma < \sigma^*$, $(u, r, s)$ is a global vanishing solution, which implies that $T_{\max} = \infty$, $I_\infty$ is a finite interval not bigger than the length of $2L$ and*

$$\lim_{t\to\infty} \max_{r(t)<x<s(t)} u(x,t) = 0. \tag{22}$$

- (iii) *When $\sigma = \sigma^*$, $(u, r, s)$ is a global transition solution, which implies that $T_{\max} = \infty$, $I_\infty$ is a finite interval of length exact equal to $2L$, and (22) follows.*

*Proof.* We find that blow-up happens when $s_0 \geq L$ by Corollary 3.3. Hence in this case for any $\phi \in \mathcal{X}(s_0)$ we have $\sigma^* = 0$. In what follows we consider the case $s_0 < L$. Additionally, $(u^\sigma, r^\sigma, s^\sigma)$ is denoted as the solution, $r^\sigma_\infty$, $s^\sigma_\infty$ and $T^\sigma_{\max}$ that allows emphasis on the dependence of the solution to the initial data if necessary.

Define the following set:

$$\Theta := \{\sigma > 0 : T^\sigma_{\max} = \infty \text{ and } I^\sigma_\infty \text{ is a finite interval not bigger than the length of } 2L\}.$$

$\Theta$ is nonempty and belongs to $[0, \Lambda]$ from Theorem 3.1 and Remark 3.2, in which $\Lambda$ is defined in (17). Let $\sigma^* := \sigma^*(a, \phi, p) = \sup\Theta$. In the following proof we let $T^{\sigma^*}_{\max} = \infty$. By the continuous dependence[16], we yield that for any fixed $t \in [0, T^{\sigma^*}_{\max})$, $u^\sigma$ approaches to $u^{\sigma^*}$ in $L^\infty(-\infty, +\infty)$ as $\sigma \nearrow \sigma^*$, with here $u(x,t)$ extend to 0 on $(-\infty, r(t)) \cup (s(t), +\infty)$. Because $T^{\sigma^*}_{\max} = \infty$ for all $0 < \sigma < \sigma^*$ by the Lemma 3.6, we deduce $T^{\sigma^*}_{\max} = \infty$.

Next, we will prove that $s^{\sigma^*}_\infty - r^{\sigma^*}_\infty = 2L$ and $I^{\sigma^*}_\infty$ is a finite interval. The continuous dependence of the solution to (1) on the initial value yields that there exists $T > 0$ sufficiently large such that if $\varepsilon > 0$ is small enough, the solution $(u^{\sigma^*+\varepsilon},\ r^{\sigma^*+\varepsilon},\ s^{\sigma^*+\varepsilon})$ of (1) with $u_0 = (\sigma^* + \varepsilon)\phi$ satisfies

$$s^{\sigma^*+\varepsilon}(T) - r^{\sigma^*+\varepsilon}(T) < 2L.$$

Which implies that $s^{\sigma^*+\varepsilon} - r^{\sigma^*+\varepsilon} \le 2L$. It is contradiction to the definition of $\sigma^*$.

Furthermore, we demonstrate that there exists only a unique $\sigma^*$ such that $I^{\sigma^*}_\infty$ is a finite interval whose exact length is $2L$ and $T^{\sigma^*}_{\max} = \infty$. If the conclusion does not hold, then there exists $\sigma_1^* > \sigma_2^*$ such that for $\sigma^* = \sigma_i^*(i = 1, 2)$ transition occurs, and this infers that the solution to (1) with $u_0 = \sigma_i^*\phi$ denoted by $(u^{\sigma_i^*},\ r^{\sigma_i^*},\ s^{\sigma_i^*})$ satisfies

$$s^{\sigma_i^*}_\infty - r^{\sigma_i^*}_\infty = 2L, i = 1, 2.$$

We find that for fixed $T_0 > 0$ using the comparison principle, which yields

$$[r^{\sigma_2^*}(T_0),\ s^{\sigma_2^*}(T_0)] \subset (r^{\sigma_1^*}(T_0),\ s^{\sigma_1^*}(T_0)),$$

and

$$u^{\sigma_2^*}(x, T_0) < u^{\sigma_1^*}(x, T_0).$$

for all $r^{\sigma_2^*}(T_0) \le x \le s^{\sigma_2^*}(T_0)$. Set

$$\Gamma = \left\{\varepsilon > 0 : u^{\sigma_1^*}(x, T_0) > u^{\sigma_2^*}(x - \varepsilon, T_0), \forall x \in [r^{\sigma_2^*}(T_0 + \varepsilon), s^{\sigma_2^*}(T_0 + \varepsilon)] \subset (r^{\sigma_1^*}(T_0), s^{\sigma_1^*}(T_0))\right\}.$$

$\Gamma$ is bounded is obvious. Write $\varepsilon_0 = \sup \Gamma$, and let

$$\hat{u}(x,t) = u^{\sigma_2^*}(x - \varepsilon_0, t + T_0, ),$$

and

$$\hat{s}(t) = s^{\sigma_2^*}(t + T_0) + \varepsilon_0,\ \hat{r}(t) = r^{\sigma_2^*}(t + T_0) + \varepsilon_0.$$

Therefore, $(\hat{u}, \hat{r}, \hat{s})$ is the unique solution of (1) with

$$\hat{u}_0 = u^{\sigma_2^*}(x - \varepsilon_0, T_0),\ \hat{s}(0) = s^{\sigma_2^*}(T_0) + \varepsilon_0,\ \hat{r}(0) = r^{\sigma_2^*}(T_0) + \varepsilon_0.$$

According to the comparison principle and the definition of $\varepsilon_0$ we deduce that

$$s^{\sigma_1^*}_\infty - r^{\sigma_1^*}_\infty \ge \hat{s}_\infty - \hat{r}^{\sigma_2^*}_\infty > 2L,$$

which contradicts the definition of $\sigma^*$. Therefore, it can be obtained that the transition occurs only when $\sigma = \sigma^*$.

Finally, conditions (i)-(iii) are derived by combining Corollary 3.3, Lemma 3.6 and all the results above. The proof is completed. □

## 5. Discussion

In this paper, we analyze the blow-up problem of a free boundary problem with an exponential nonlocal reaction term. Firstly, we establish some sufficient conditions to determine a finite time solution blasting and the existence of a global extinction solution by using the comparison principle and constructing an appropriate upper and lower solution in Section 3, which is to study the asymptotic behavior of the solution. Furthermore, for $a \ge 0$ and $a < 0$, we proof that the two free boundaries converge in the finite limit when the initial data is small, the results are different. In Section 4, on the basis of the initial data size, a sharp threshold trichotomy result is given. At that time, the solution vanishes when $\sigma < \sigma^*$, blows up in finite time when $\sigma > \sigma^*$, and transition occurs when $\sigma = \sigma^*$.

The Laplace operator is known to be a local and symmetric operator. Nevertheless, in the real world, there are some species movement mechanism based on nonlocal neighboring regions, and some species move to favorable environments due to rivers, climate and other factors. For example, the nonlocal operator in [2] and the Laplace operator with advection terms in [7, 10, 22] will emerge with more complex dynamics. Therefore, we will also consider in future studies the nonlocal and nonsymmetric versions of the problem (1).

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