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Parametric representation of integral operators for $x > 0$

Kapil Kumar[a,b], **Naokant Deo**[a,*], **Durvesh Kumar Verma**[c]

[a]*Department of Applied Mathematics, Delhi Technological University, India*
[b]*Department of Mathematics, Hansraj College, University of Delhi, India*
[c]*Department of Mathematics, Miranda House, University of Delhi, India*

Abstract. In the present paper, we introduce a new sequence of integral operators with a certain parameter α that can be used to approximate the functions over the interval $(0, \infty) = \mathbb{R}^*$. Firstly, we obtain the moments for the proposed operators. Next, we estimate some direct results, which include the rate of convergence, the asymptotic formula, and point-wise convergence in terms of modulus of continuity; weighted approximation for these operators is given, and some results related to the $A-$statistical convergence of the operators are obtained. Ultimately, in order to validate the findings, we employ numerical illustrations and visual depictions.

1. Introduction

Approximation theory is a fundamental area of mathematics that has wide-ranging applications in various fields, including numerical analysis, signal processing, computer graphics, and more. Linear positive operators are an essential component of this theory, as they provide tools to approximate complex functions using simpler ones. There are many developed positive linear operators that are described over bounded intervals, such as the Bernstein operators described on $[0, 1]$. There are also many linear positive operators that are described over unbounded intervals, such as Baskakov operators described on $[0, \infty)$. In 1957, the classical Baskakov operators (see [6]) were introduced, which are defined as follows:

$$V_n(\Psi; x) = \sum_{\kappa=0}^{\infty} \binom{n+\kappa-1}{\kappa} x^{\kappa} (1+x)^{-n-\kappa} \Psi\left(\frac{\kappa}{n}\right), \ x \in [0, \infty) \text{ and } \Psi \in C[0, \infty).$$

Basic facts on the Baskakov operators and their generalizations can be found in ([2, 4, 5, 17, 23, 28]). Recently, many researchers in the field of approximation theory have constructed modified Baskakov operators in different ways over unbounded intervals (cf. [10, 11, 13, 21, 23, 25, 30, 31]).

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* Corresponding author: Naokant Deo
Email addresses: kapil_2k19phdam502@dtu.ac.in, kapil@hrc.du.ac.in (Kapil Kumar), naokantdeo@dce.ac.in (Naokant Deo), durvesh.kv.du@gmail.com (Durvesh Kumar Verma)
ORCID iDs: https://orcid.org/0009-0004-8024-6787 (Kapil Kumar), https://orcid.org/0000-0001-7079-4211 (Naokant Deo), https://orcid.org/0000-0003-4750-8234 (Durvesh Kumar Verma)

In 1941, G.M. Mirakyan, extending the Bernstein operators from a finite interval to an infinite interval, defined a sequence of linear positive operators in the form:

$$K_n(\Psi;x) = e^{-nx}\sum_{\kappa=0}^{\infty}\frac{(nx)^{\kappa}}{\kappa!}\Psi\left(\frac{\kappa}{n}\right),\ x\in[0,\infty),$$

for the function $\Psi \in C[0,\infty)$. The $(K_n)_{n\geq 1}$ linear positive operators were also studied separately by O. Szász [29]. This operator is known in the literature as the Szász–Mirakyan operators.
Recently, W.T. Cheng et al. [9] introduced a new modification Baskakov operators on $(0,\infty)$ as follows:
For $\Psi \in C(\mathbb{R}^*)$ and $x \in \mathbb{R}^*$, then

$$L_n^*(\Psi;x) = \frac{1}{n}\sum_{\kappa=0}^{\infty}\binom{n+\kappa-1}{\kappa}(\kappa-nx)^2x^{\kappa-1}(1+x)^{-n-\kappa-1}\Psi\left(\frac{\kappa}{n}\right).$$

Gupta and Srivastava [26] developed simultaneous approximation by Baskakov-Szász type operators and investigated the degree of approximation and rate of convergence for these operators. Deo and Kumar [14] presented the Durrmeyer variant of Apostol-Genocchi-Baskakov operators for which they obtained some direct results. The interested reader is directed to consult some of the relevant literature (refer to ([1, 3, 12, 24, 27])) for further information.
Motivated by the above work, we propose a new sequence of operators as follows:
For $\Psi \in C(\mathbb{R}^*)$ and $0 < \alpha < 1$,

$$Z_n^{(\alpha)}(\Psi;x) = \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)\Psi(\alpha t)dt, \tag{1}$$

where $b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right) = \binom{n+\kappa-1}{\kappa}\left(\kappa-\frac{nx}{\alpha}\right)^2\left(\frac{x}{\alpha}\right)^{\kappa-1}\left(1+\frac{x}{\alpha}\right)^{-n-\kappa-1}$ and $s_{n,k}(x) = e^{-nx}\frac{(nx)^{\kappa}}{\kappa!},\ x\in\mathbb{R}^*$.
The objective of this study is to formulate a new sequence of integral operators with a certain parameter α. The profile of the paper being presented is outlined as follows: In the first section, we formulate a new sequence of integral operators with a certain parameter α on $\mathbb{R}^*$. In Section 2, we discuss the moments that are associated with the new operators. In the subsequent section 3, we conducted an estimation of the direct theorem and Voronovskaja-type asymptotic formula and engaged in a discussion regarding the rate of convergence and weighted approximation. In Section 5, we obtain some results related to the $A-$statistical convergence of the operators. This section might discuss the convergence properties of the operators under certain statistical conditions. In the final section, we present graphical representations and numerical illustrations to demonstrate the convergence of the proposed operators to the function Ψ. This section likely provides visual and numerical evidence to support the theoretical results presented in earlier sections.

2. Preliminaries

Lemma 2.1. *For $Z_n^{(\alpha)}(e_i;x)$, where $e_i(t) = t^i$ and $i = 0,1,2,3$. We have*

$$Z_n^{(\alpha)}(e_0;x) = 1,$$
$$Z_n^{(\alpha)}(e_1;x) = x + \frac{2(x+\alpha)}{n},$$
$$Z_n^{(\alpha)}(e_2;x) = x^2 + \frac{x(7x+8\alpha)}{n} + \frac{6(x+\alpha)^2}{n^2},$$
$$Z_n^{(\alpha)}(e_3;x) = x^3 + \frac{3x^2(5x+6\alpha)}{n} + \frac{2x(19x^2+45\alpha x+27\alpha^2)}{n^2} + \frac{24(x+\alpha)^3}{n^3}.$$

Proof. From (1), we have

$$\begin{aligned}
Z_n^{(\alpha)}(e_0;x) &= \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)dt \\
&= \frac{1}{n}\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right) = 1,
\end{aligned}$$

$$\begin{aligned}
Z_n^{(\alpha)}(e_1;x) &= \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} \alpha t s_{n,\kappa}(t)dt \\
&= \alpha\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\frac{n^{\kappa}}{\kappa!}\int_0^{\infty} e^{-nt}t^{(\kappa+2)-1}dt \\
&= \frac{\alpha}{n^2}\sum_{\kappa=0}^{\infty}(\kappa+1) b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right) \\
&= \frac{\alpha + x(n+2)}{n} + \frac{\alpha}{n} \\
&= x + \frac{2(x+\alpha)}{n},
\end{aligned}$$

$$\begin{aligned}
Z_n^{(\alpha)}(e_2;x) &= \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} (\alpha t)^2 s_{n,\kappa}(t)dt \\
&= \alpha^2\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\frac{n^{\kappa}}{\kappa!}\int_0^{\infty} e^{-nt}t^{(\kappa+3)-1}dt \\
&= \alpha^2\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\frac{n^{\kappa}}{\kappa!}\frac{\Gamma(\kappa+3)}{n^{\kappa+3}} \\
&= \frac{\alpha^2}{n^3}\sum_{\kappa=0}^{\infty}(\kappa^2+3\kappa+2) b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right) \\
&= x^2 + \frac{x(7x+8\alpha)}{n} + \frac{6(x+\alpha)^2}{n^2},
\end{aligned}$$

$$\begin{aligned}
Z_n^{(\alpha)}(e_3;x) &= \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} (\alpha t)^3 s_{n,\kappa}(t)dt \\
&= \alpha^3\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\frac{n^{\kappa}}{\kappa!}\int_0^{\infty} e^{-nt}t^{(\kappa+4)-1}dt \\
&= \alpha^3\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\frac{n^{\kappa}}{\kappa!}\frac{\Gamma(\kappa+4)}{n^{\kappa+4}} \\
&= \frac{\alpha^3}{n^4}\sum_{\kappa=0}^{\infty}(\kappa^3+6\kappa^2+11\kappa+6) b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right) \\
&= x^3 + \frac{3x^2(5x+6\alpha)}{n} + \frac{2x(19x^2+45\alpha x+27\alpha^2)}{n^2} + \frac{24(x+\alpha)^3}{n^3}.
\end{aligned}$$

□

Lemma 2.2. *From Lemma 2.1 and simple computation, we get*

$$Z_n^{(\alpha)}((t-x);x) = \frac{2(x+\alpha)}{n},$$
$$Z_n^{(\alpha)}((t-x)^2;x) = \frac{x(3x+4\alpha)}{n} + \frac{6(x+\alpha)^2}{n^2}.$$

Let $\mathcal{S}_B^{\mathbb{R}^*}$ be a normed space, where

$$\mathcal{S}_B^{\mathbb{R}^*} = \{\Psi \in C(\mathbb{R}^*) : \Psi \text{ is bounded over } \mathbb{R}^*\}$$

endowed with the norm

$$\|\Psi\| = \sup_{x\in\mathbb{R}^*} |\Psi(x)|. \tag{2}$$

Lemma 2.3. *For each $\Psi \in \mathcal{S}_B^{\mathbb{R}^*}$ and $x \in \mathbb{R}^*$, then*

$$|Z_n^{(\alpha)}(\Psi;x)| \leq \|\Psi\|. \tag{3}$$

3. Direct Result and Asymptotic Formula

Now, we discuss some important results in this section.
For $\Psi \in \mathcal{S}_B^{\mathbb{R}^*}$, the Peetre's K-functional is given by

$$K_2(\Psi,\delta) = \inf_{g\in W_{C^2(\mathbb{R}^*)}} \{\|\Psi - g\| + \delta\|g''\|\}, \text{ where } \delta > 0, \text{ and}$$

$$W_{C^2(\mathbb{R}^*)} = \{g \in \mathcal{S}_B^{\mathbb{R}^*} : g', g'' \in \mathcal{S}_B^{\mathbb{R}^*}\}.$$

By DeVore and Lorentz ([15] p.177, Theorem 2.4), there exists a positive constant B such that

$$K_2(\Psi,\delta) \leq B\omega_2(\Psi, \sqrt{\delta}), \tag{4}$$

$$\text{where } \omega_2(\Psi, \sqrt{\delta}) = \sup_{0<\epsilon\leq\sqrt{\delta}} \left(\sup_{x,x+\epsilon,x+2\epsilon\in\mathbb{R}^*} |\Psi(x+2\epsilon) - 2\Psi(x+\epsilon) + \Psi(x)| \right)$$

is the second order of modulus of continuity of Ψ.
The expression

$$\omega(\Psi,\delta) = \sup_{|t-x|\leq\delta} |\Psi(t) - \Psi(x)| \tag{5}$$

is modulus of continuity of Ψ, where $x, t \in \mathbb{R}^*$.

Theorem 3.1. *For $\Psi \in \mathcal{S}_B^{\mathbb{R}^*}$ and $x \in \mathbb{R}^*$, then*

$$|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| \leq B\omega_2(\Psi, \sqrt{\delta/2}) + \omega\left(\Psi, \frac{2(x+\alpha)}{n}\right),$$

where B is a positive constant and $\delta = Z_n^{(\alpha)}((t-x)^2;x) + \left(\frac{2(x+\alpha)}{n}\right)^2$.

Proof. Firstly, we introduce a new sequence of positive linear operators

$$\mathcal{H}_n(\Psi;x) = Z_n^{(\alpha)}(\Psi;x) - \Psi\left(x + \frac{2(x+\alpha)}{n}\right) + \Psi(x). \tag{6}$$

Let $g \in W_{C^2(\mathbb{R}^*)}$, then by Taylor's theorem,

$$g(t) = g(x) + (t-x)g^{'}(x) + \frac{1}{2}\int_x^t (t-u)g^{''}(u)du. \tag{7}$$

Apply $\mathcal{H}_n$ on (7),

$$\mathcal{H}_n(g;x) = g(x) + g^{'}(x)\mathcal{H}_n(t-x;x) + \frac{1}{2}\mathcal{H}_n\left(\int_x^t (t-u)g^{''}(u)du;x\right).$$

By Lemma 2.2 and (6),

$$\mathcal{H}_n(t-x;x) = 0.$$

Therefore,

$$\begin{aligned}|\mathcal{H}_n(g;x) - g(x)| &\leq \mathcal{H}_n\left(\int_x^t (t-u)|g^{''}(u)|du;x\right)\\ &\leq ||g^{''}||\left|\mathcal{H}_n\left(\int_x^t (t-u)du;x\right)\right|.\end{aligned}$$

By (6),

$$|\mathcal{H}_n(g;x) - g(x)| \leq ||g^{''}||\left(\left|(Z_n^{(\alpha)}\left(\int_x^t (t-u)du;x\right)\right| + \left|\int_x^{x+\frac{2(x+\alpha)}{n}}\left(x + \frac{2(x+\alpha)}{n} - u\right)du\right|\right)$$

$$\leq ||g^{''}||\left(Z_n^{(\alpha)}(t-x)^2;x) + \left(\frac{2(x+\alpha)}{n}\right)^2\right) = \delta||g^{''}||. \tag{8}$$

We have $$\begin{aligned}Z_n^{(\alpha)}(\Psi;x) - \Psi(x) &= \mathcal{H}_n(\Psi - g;x) - (\Psi - g)(x) + \mathcal{H}_n(g;x) - g(x)\\ &\quad + \Psi\left(x + \frac{2(x+\alpha)}{n}\right) - \Psi(x).\end{aligned}$$

Form (2), (5), (8), and Lemma 2.3, we get

$$\begin{aligned}|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| &\leq |\mathcal{H}_n(\Psi - g;x) - (\Psi - g)(x)| + |\mathcal{H}_n(g;x) - g(x)|\\ &\quad + \left|\Psi\left(x + \frac{2(x+\alpha)}{n}\right) - \Psi(x)\right|\\ &\leq 2||\Psi - g|| + \delta||g^{''}|| + \omega\left(\Psi, \frac{2(x+\alpha)}{n}\right).\end{aligned}$$

Taking infimum of g on $W_{C^2(\mathbb{R}^*)}$ of the right hand side of the inequality,

$$|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| \leq K_2\left(\Psi, \frac{\delta}{2}\right) + \omega\left(\Psi, \frac{2(x+\alpha)}{n}\right).$$

By (4),

$$|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| \leq B\omega_2(\Psi, \sqrt{\delta/2}) + \omega\left(\Psi, \frac{2(x+\alpha)}{n}\right).$$

□

Now we study to obtain the degree of approximation with the help of Ditzian-Toitik moduli of smoothness: By [16], let

$$\omega^2_{\phi^\lambda}(\Psi,\delta) = \sup_{0<\epsilon\leq\delta}\left(\sup_{x,x+\epsilon\phi^\lambda,x-2\epsilon\phi^\lambda\in\mathbb{R}^*}\left|\Psi(x+\epsilon\phi^\lambda) - 2\Psi(x) + \Psi(x-\epsilon\phi^\lambda)\right|\right),$$

and corresponding K-functional is given by

$$K_{2,\phi^\lambda}(\Psi,\delta^2) = \inf_{g'\in A.C._{loc}(\mathbb{R}^*)}\{\|\Psi - g\| + \delta^2\|\phi^{2\lambda}g''\|\},$$

and $\mathfrak{D}^2_\lambda = \{g \in \mathcal{S}_B(\mathbb{R}^*) : g' \in A.C._{loc}(\mathbb{R}^*), \|\phi^{2\lambda}g''\| < \infty\}$, where $\phi^2(x) = x, 0 \leq \lambda \leq 1$.

We have

$$K_{2,\phi^\lambda}(\Psi,\delta^2) \sim \omega^2_{\phi^\lambda}(\Psi,\delta).$$

Theorem 3.2. *For $\Psi \in \mathcal{S}_B^{\mathbb{R}^*}$ and $x \in \mathbb{R}^*$, then*

$$|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| \leq 4\omega^2_{\phi^\lambda}\left(\Psi, \frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{2n}}\right) + \omega\left(\Psi, \frac{2x+1}{n}\right).$$

Proof. Consider

$$\mathcal{H}_n(\Psi;x) = Z_n^{(\alpha)}(\Psi;x) - \Psi\left(x + \frac{2(x+\alpha)}{n}\right) + \Psi(x). \tag{9}$$

Let $g \in \mathfrak{D}^2_\lambda$, then by Taylor's theorem,

$$g(t) = g(x) + (t-x)g'(x) + \frac{1}{2}\int_x^t (t-u)g''(u)du. \tag{10}$$

Apply $\mathcal{H}_n$ on (10),

$$\mathcal{H}_n(g;x) = g(x) + g'(x)\mathcal{H}_n(t-x;x) + \frac{1}{2}\mathcal{H}_n\left(\int_x^t (t-u)g''(u)du;x\right). \tag{11}$$

By Lemma 2.2 and (9),

$$\mathcal{H}_n(t-x;x) = 0.$$

We have

$$|\mathcal{H}_n(\Psi; x)| \leq 3\|\Psi\|, \tag{12}$$

and

$$Z_n^{(\alpha)}((t-x)^2; x) \leq \frac{1}{n}\delta_n^2(x),$$

where $\delta_n^2(x) = \phi^2(x)(9\phi^2(x) + 16\alpha) + 6\alpha^2$.
From ([16], p. 141), for $t < u < x$, we have

$$\frac{|t-u|}{\phi^{2\lambda}(u)} \leq \frac{|t-x|}{\phi^{2\lambda}(x)} \text{ and } \frac{|t-u|}{\delta_n^{2\lambda}(u)} \leq \frac{|t-x|}{\delta_n^{2\lambda}(x)}. \tag{13}$$

By (9) and (11),

$$|\mathcal{H}_n(g; x) - g(x)| \leq \left| Z_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du; x \right) \right| + \left| \int_x^{x+\frac{2(x+\alpha)}{n}} \left(x + \frac{2(x+\alpha)}{n} - u \right) g''(u)du \right|,$$

using (13),

$$\leq \|\delta_n^{2\lambda} g''\| Z_n^{(\alpha)}\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)}; x \right) + \frac{\|\delta_n^{2\lambda} g''\|}{\delta_n^{2\lambda}(x)} \left(\frac{2(x+\alpha)}{n} \right)^2.$$

$$\begin{aligned}
|\mathcal{H}_n(g; x) - g(x)| &\leq \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda} g''\| \left(Z_n^{(\alpha)}(t-x)^2; x \right) + \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda} g''\| \left(Z_n^{(\alpha)}(t-x); x \right)^2 \\
&\leq \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda} g''\| \frac{\delta_n^2(x)}{n} + \delta_n^{-2\lambda}(x)\|\delta_n^{2\lambda} g''\| \frac{\delta_n^2(x)}{n}.
\end{aligned}$$

Hence,

$$|\mathcal{H}_n(g; x) - g(x)| \leq \frac{2\delta_n^{2(1-\lambda)}(x)}{n} \|\delta_n^{2\lambda} g''\|. \tag{14}$$

Using (2), (12), and (14),

$$\begin{aligned}
|\mathcal{H}_n(\Psi; x) - \Psi(x)| &\leq |\mathcal{H}_n(\Psi - g; x)| + |\mathcal{H}_n(g; x) - g(x)| + |\Psi(x) - g(x)| \\
&\leq 4\|\Psi - g\| + |\mathcal{H}_n(g; x) - g(x)| \\
&\leq 4\|\Psi - g\| + \frac{2\delta_n^{2(1-\lambda)}(x)}{n} \|\delta_n^{2\lambda} g''\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|Z_n^{(\alpha)}(\Psi; x) - \Psi(x)| &\leq |\mathcal{H}_n(\Psi; x) - \Psi(x)| + \left| \Psi\left(x + \frac{2(x+\alpha)}{n} \right) - \Psi(x) \right| \\
&\leq 4\omega_{\phi^\lambda}^2\left(\Psi, \frac{\delta_n^{(1-\lambda)}(x)}{\sqrt{2n}} \right) + \omega\left(\Psi, \frac{2(x+\alpha)}{n} \right)
\end{aligned}$$

□

Now, we give the Voronovskaja asymptotic formula for the operators $Z_n^{(\alpha)}$.

Theorem 3.3. *For functions* $\Psi, \Psi', \Psi'' \in \mathcal{S}_B^{\mathbb{R}^*}$ *and* $x \in \mathbb{R}^*$, *then*

$$\lim_{n\to\infty} n[Z_n^{(\alpha)}(\Psi;x) - \Psi(x)] = 2(x+\alpha)\Psi'(x) + \frac{x(3x+4\alpha)}{2}\Psi''(x).$$

Proof. By Taylor's theorem,

$$\Psi(t) = \Psi(x) + (t-x)\Psi'(x) + \frac{(t-x)^2}{2}\Psi''(x) + \epsilon_B(x;t)(t-x)^2, \tag{15}$$

where $\epsilon_B(x;t)$ is the Peano form of the remainder, $\epsilon_B(x;t) \in \mathcal{S}_B^{\mathbb{R}^*}$.
We have $\epsilon_B(x;t) = \frac{\Psi(t)-\Psi(x)-(t-x)\Psi'(x)-\frac{(t-x)^2}{2!}\Psi''(x)}{(t-x)^2}$,
and

$$\lim_{t\to x}\epsilon_B(x;t) = 0. \tag{16}$$

Apply $Z_n^{(\alpha)}$ on (15),

$$\begin{aligned} Z_n^{(\alpha)}(\Psi(t);x) = \Psi(x) &+ \Psi'(x)Z_n^{(\alpha)}(t-x;x) + \frac{\Psi''(x)}{2}Z_n^{(\alpha)}((t-x)^2;x)\\ &+ Z_n^{(\alpha)}(\epsilon_B(x;t)(t-x)^2;x). \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} n[Z_n^{(\alpha)}(\Psi(t);x) - \Psi(x)] &= \Psi'(x)\lim_{n\to\infty}(nZ_n^{(\alpha)}(t-x;x))\\ &\quad + \frac{\Psi''(x)}{2}\lim_{n\to\infty}(nZ_n^{(\alpha)}((t-x)^2;x))\\ &\quad + \lim_{n\to\infty}(nZ_n^{(\alpha)}(\epsilon_B(x;t)(t-x)^2;x))\\ &= 2(x+\alpha)\Psi'(x) + \frac{x(3x+4\alpha)}{2}\Psi''(x) + \mathcal{R}_n, \end{aligned}$$

where $\mathcal{R}_n = \lim_{n\to\infty}(nZ_n^{(\alpha)}(\epsilon_B(x;t)(t-x)^2;x))$.
Using Cauchy-Bunyakovsky-Schwarz Inequality,

$$nZ_n^{(\alpha)}(\epsilon_B(x;t)(t-x)^2;x) \le \sqrt{n^2 Z_n^{(\alpha)}(\epsilon_B^2(x;t);x)}\sqrt{Z_n^{(\alpha)}((t-x)^4;x)}.$$

We observe that by (16) if $n\to\infty$, then $t\to x$ and $\lim_{t\to x}\epsilon_B(x;t) = 0$. It follows that

$$\lim_{n\to\infty}(n^2 Z_n^{(\alpha)}(\epsilon_B^2(x;t);x)) = 0 \text{ uniformly with respect to } x \in \mathbb{R}^*.$$

So, $\mathcal{R}_n = \lim_{n\to\infty}(nZ_n^{(\alpha)}(\epsilon_B(x;t)(t-x)^2;x)) = 0$. Therefore, the desired result is proved. □

Theorem 3.4. *For* $\Psi \in \mathcal{S}_B^{\mathbb{R}^*}$ *and* $x \in \mathbb{R}^*$, *then*

$$|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)| \le 2\omega(\Psi,\delta), \text{where } \delta = \sqrt{Z_n^{(\alpha)}((t-x)^2;x)}.$$

Proof. By the property of the modulus of continuity,

$$\begin{aligned} |\Psi(t)-\Psi(x)| &\le \omega(\Psi,|t-x|)\\ &\le \left(1+\frac{1}{\delta}|t-x|\right)\omega(\Psi,\delta). \end{aligned}$$

Now, $$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq \sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)\,|\Psi(\alpha t)-\Psi(x)|\,dt$$
$$\leq \left(1+\frac{1}{\delta}\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)\,|\alpha t-x|\,dt\right)\omega(\Psi,\delta).$$

By Cauchy-Bunyakovsky-Schwarz Inequality,

$$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq \left(1+\frac{1}{\delta}\sqrt{\left(\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)dt\right)}\sqrt{\sum_{\kappa=0}^{\infty} b_{n,\kappa}^{(\alpha)}\left(\frac{x}{\alpha}\right)\int_0^{\infty} s_{n,\kappa}(t)\,(|\alpha t-x|)^2dt}\right)$$
$$\times\,\omega(\Psi,\delta)$$
$$=\left(1+\frac{1}{\delta}\sqrt{(Z_n^{(\alpha)}((t-x)^2;x)}\right)\omega(\Psi,\delta)$$
$$=2\omega(\Psi,\delta).$$

□

Now, the rate of convergence of $Z_n^{(\alpha)}$ with help of the Lipschitz class $Lip_K(\eta),\ \eta>0$ is obtained. If $\Psi\in Lip_K(\eta)$, then function Ψ satisfies the inequality

$$|\Psi(t)-\Psi(x)| \leq K|t-x|^{\eta},\ x,t\in\mathbb{R}^*, \tag{17}$$

where K is a positive constant.

Theorem 3.5. *If $x\in\mathbb{R}^*$ and $\Psi\in S_B^{\mathbb{R}^*}$ belongs to the class $Lip_K(\eta)$, then*

$$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq K(\Phi_n(x))^{\frac{\eta}{2}},\text{where } \Phi_n(x)=\frac{x(3x+4\alpha)}{n}+\frac{6(x+\alpha)^2}{n^2}.$$

Proof. By (17),

$$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq Z_n^{(\alpha)}(|\Psi(t)-\Psi(x)|;x)$$
$$\leq Z_n^{(\alpha)}(K|t-x|^{\eta};x).$$

By Cauchy-Bunyakovsky-Schwarz Inequality,

$$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq K[Z_n^{(\alpha)}((t-x)^2;x)]^{\frac{\eta}{2}}.$$

Using Lemma 2.2,

$$|Z_n^{(\alpha)}(\Psi;x)-\Psi(x)| \leq K\left[\frac{x(3x+4\alpha)}{n}+\frac{6(x+\alpha)^2}{n^2}\right]^{\frac{\eta}{2}}=K(\Phi_n(x))^{\frac{\eta}{2}}.$$

□

4. Weighted approximation

Let $B_\zeta(\mathbb{R}^*)$ be a normed space by

$$B_\zeta(\mathbb{R}^*)=\{\Psi:\mathbb{R}^*\to\mathbb{R}:|\Psi(x)|\leq M_\Psi\zeta(x),\ x\in\mathbb{R}^*\},$$

endowed with the norm

$$\|\Psi\|_\zeta=\sup_{x\in\mathbb{R}^*}\frac{|\Psi(x)|}{\zeta(x)},$$

where positive constant M_Ψ depends on Ψ and $\zeta(x)=1+x^2$.
Additionally, we define the following spaces,

i. $C_\zeta(\mathbb{R}^*) = \{\Psi \in B_\zeta(\mathbb{R}^*) : \Psi$ is continuous function over $\mathbb{R}^*\}$,

ii. $C^*_\zeta(\mathbb{R}^*) = \{\Psi \in C_\zeta(\mathbb{R}^*) : \lim\limits_{x\to\infty} \dfrac{\Psi(x)}{\zeta(x)}$ exists in $\mathbb{R}\}$.

Theorem 4.1. *For each $\Psi \in C^*_\zeta(\mathbb{R}^*)$, then*

$$\lim_{n\to\infty} \|Z_n^{(\alpha)}(\Psi;.) - \Psi\|_\zeta = 0.$$

Proof. Using [22], to prove this theorem, it is sufficient to verify the following conditions

$$\lim_{n\to\infty} \|Z_n^{(\alpha)}(e_i;x) - x^i\|_\zeta = 0, \ i = 0,1,2. \tag{18}$$

Since $Z_n^{(\alpha)}(e_0;x) = 1$, so for $i = 0$ (18) holds.
By Lemma 2.1,

$$\begin{aligned}\|Z_n^{(\alpha)}(e_1;x) - x\|_\zeta &= \sup_{x\in\mathbb{R}^*} \frac{|Z_n^{(\alpha)}(e_1;x) - x|}{\zeta(x)} \\ &\le \frac{1}{n} \sup_{x\in\mathbb{R}^*}\left(\frac{2(x+\alpha)}{\zeta(x)}\right) \to 0 \text{ as } n\to\infty.\end{aligned}$$

The condition (18) holds for $i = 1$.
Again by Lemma 2.1,

$$\begin{aligned}\|Z_n^{(\alpha)}(e_2;x) - x^2\|_\zeta &= \sup_{x\in\mathbb{R}^*} \frac{|Z_n^{(\alpha)}(e_2;x) - x^2|}{\zeta(x)} \\ &\le \frac{1}{n} \sup_{x\in\mathbb{R}^*}\left(\frac{x(7x+8\alpha)}{\zeta(x)}\right) + \frac{1}{n^2} \sup_{x\in\mathbb{R}^*}\left(\frac{6(x+\alpha)^2}{\zeta(x)}\right).\end{aligned}$$

Clearly, $\|Z_n^{(\alpha)}(e_2;x) - x^2\|_\zeta \to 0$ as $n \to \infty$, the condition (18) holds for $i = 2$.
Hence, the theorem is proved. □

Theorem 4.2. *For each $\Psi \in C_\zeta(\mathbb{R}^*)$ and $\nu > 1$, then*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^*} \frac{|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)|}{(\zeta(x))^\nu} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}\sup_{x\in\mathbb{R}^*} \frac{|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)|}{(\zeta(x))^\nu} &\le \sup_{x\le x_0} \frac{|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)|}{(\zeta(x))^\nu} + \sup_{x\ge x_0} \frac{|Z_n^{(\alpha)}(\Psi;x) - \Psi(x)|}{(\zeta(x))^\nu} \\ &\le \|Z_n^{(\alpha)}(\Psi;.) - \Psi\|_{\zeta^\nu} + \|\Psi\|_\zeta \sup_{x\ge x_0} \frac{|Z_n^{(\alpha)}(1+t^2;x)|}{(\zeta(x))^\nu} \\ &\quad + \sup_{x\ge x_0} \frac{|\Psi(x)|}{(\zeta(x))^\nu}.\end{aligned}$$

By Theorem 4.1, the second term from the left in the above inequality tends to 0 as $n \to \infty$ and for the fixed x_0, if we choose x_0 large enough, then the terms $\|\Psi\|_\zeta \sup\limits_{x\ge x_0} \dfrac{|Z_n^{(\alpha)}(1+t^2;x)|}{(\zeta(x))^\nu}$ and $\sup\limits_{x\ge x_0} \dfrac{|\Psi(x)|}{(\zeta(x))^\nu}$ can be made small enough. Thus, the desired proof is proved. □

Now, we recall the concept of $A-$statistical convergence. Let $x = (x_m)$ be a sequence, and let $A = (a_{nm})$ be an infinite summability matrix. If the series

$$(Ax)_n = \sum_m a_{ni} x_m$$

is convergent for all $n \in \mathbb{N}$, then $Ax = ((Ax)_n)$ is the $A-$transformation of x. If the sequence Ax converges to a number l, then x is $A-$ summable to l. The matrix A is said to be regular if the condition $\lim_n (Ax)_n = l$ holds, whenever $\lim_n x_n = l$ [7].

Let K be a subset of positive integers and let $A = (a_{nm})$ be a non-negative regular summability matrix. Then K is said to have $A-$density $\delta_A(K)$, if the limit $\delta_A(K) = \lim_n \sum_{m \in K} a_{nm} \mathcal{X}K(m)$ exists [8, 19], where $\mathcal{X}K$ is the characteristic function of K. The sequence $x = (x_m)$ is said to be $A-$statistically convergent to l, if for any $\epsilon > 0$,

$$\lim_n \sum_{m:|x_m - l| \geq \epsilon} a_{nm} = 0.$$

In this case, we write $st_A - lim\ x = l$ [18, 20]. If $A = I$, then $I-$statistical convergence reduces to ordinary convergence.

5. $A-$statistical approximation

In this section, we estimate the A-statistical convergence of the given operators $Z_n^{(\alpha)}$ to the Identity operator on the weighted spaces.

Corollary 5.1. *Let $A = (a_{nm})$ be a non−negative regular summability matrix and let ζ_1, ζ_2 two weight functions such that*

$$\lim_{|x| \to \infty} \frac{\zeta_1(x)}{\zeta_2(x)} = 0. \tag{19}$$

Let us suppose that $(L_n)_{n \geq 1}$ is a sequence of positive linear operators from $C_{\zeta_1}(\mathbb{R}^*)$ into $B_{\zeta_2}(\mathbb{R}^*)$. One has

$$st_A - \lim_n \|L_n f - f\|_{\zeta_2} = 0 \text{ for all } f \in C_{\zeta_1}(\mathbb{R}^*) \text{ iff}$$
$$st_A - \lim_n \|L_n Z_\mu - Z_\mu\|_{\zeta_1} = 0,$$

where $Z_\mu(x) = \frac{x^\mu \zeta_1(x)}{1+x^2}$, $\mu = 0, 1, 2$.

Corollary 5.2. *Let $A = (a_{nm})$ be a non−negative regular summability matrix and let (L_n) be a sequence of positive linear operators acting from $C_{\zeta_0}(\mathbb{R}^*)$ into $B_{\zeta_\lambda}(\mathbb{R}^*)$, $\lambda > 0$ one has*

$$st_A - \lim_n \|L_n f - f\|_{\zeta_\lambda} = 0, \quad f \in C_{\zeta_0}(\mathbb{R}^*)$$

iff

$$st_A - \lim_n \|L_n e_m - e_m\|_{\zeta_0} = 0, \quad m = 0, 1, 2, \tag{20}$$

where $\zeta_0(x) = x^2 + 1$ and $\zeta_\lambda(x) = x^{2+\lambda} + 1$, $\lambda > 0$.

Now, we prove the Korovkin-type statistical theorem for $Z_n^{(\alpha)}$.

Theorem 5.3. *Let $A = (a_{nm})$ be a non-negative regular summability matrix and $st_A - \lim_{n\to\infty} \frac{c}{n} = 0$, where $c > 0$. Then for each $\Psi \in C^*_\zeta(\mathbb{R}^*)$, we have*

$$st_A - \lim_{n\to\infty} \|Z_n^{(\alpha)}\Psi - \Psi\|_{\zeta_\lambda} = 0$$

where $\zeta_0(x) = x^2 + 1$ and $\zeta_\lambda(x) = x^{2+\lambda} + 1, \lambda > 0$.

Proof. Applying Corollary 5.2, it is sufficient to prove that the operators $Z_n^{(\alpha)}$ satisfy the condition given in (20). From the Lemma 2.1,

$$st_A - \lim_n \|Z_n^{(\alpha)}(e_0;.) - e_0\|_{\zeta_0} = 0,$$
$$st_A - \lim_n \|Z_n^{(\alpha)}(e_1;.) - e_1\|_{\zeta_0} = 0.$$

Using Lemma 2.1,

$$\begin{aligned} st_A - \lim_n \|Z_n^{(\alpha)}(e_2;.) - x^2\|_{\zeta_0} &= \frac{1}{n^2} sup \left\{ \frac{(7n+6)x^2}{1+x^2} + \frac{4\alpha(2n+3)x}{1+x^2} + \frac{6\alpha^2}{1+x^2} \right\} \\ &\leq \frac{1}{n^2}\left((7n+6) + 4\alpha(2n+3) + 6\alpha^2\right) = K_n. \end{aligned}$$

For a given $\epsilon > 0$, we define the following sets:

$$\mathcal{S} = \left\{ m : \left\| Z_n^{(\alpha)}(e_2;.) - e_2 \right\|_{\zeta_0} \right\} \geq \epsilon$$
$$\mathcal{S}_1 = \left\{ m : \left\| \frac{7n+6}{n^2} \right\|_{\zeta_0} \right\} \geq \frac{\epsilon}{3}$$
$$\mathcal{S}_2 = \left\{ m : \left\| \frac{4\alpha(2n+3)}{n^2} \right\|_{\zeta_0} \right\} \geq \frac{\epsilon}{3}$$
$$\mathcal{S}_3 = \left\{ m : \left\| \frac{6\alpha^2}{n^2} \right\|_{\zeta_0} \right\} \geq \frac{\epsilon}{3}$$

Then, we see that $\mathcal{S} \subseteq \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. Therefore, we get

$$\sum_{n:\|Z_n^{(\alpha)}(e_2;.)-e_2\|_{\zeta_0} \geq \epsilon} a_{nm} \leq \sum_{m\in S_1} a_{nm} + \sum_{m\in S_2} a_{nm} + \sum_{m\in S_2} a_{nm},$$

taking the limit $m \to \infty$ in above, we get the result $st_A - \lim_n \|Z_n^{(\alpha)}(e_2;.) - e_2\|_{\zeta_0} = 0.$ □

6. Graphical and numerical analysis

We present graphical representations in Figures 1 and 3 for the convergence of the proposed operators to $\Psi(x)$ over the intervals with different values of n. In Tables 1 and 2, we compute the absolute error $\epsilon_n^{(\alpha)}(x) = |Z_n^{(\alpha)}(\Psi; x) - \Psi(x)|$ to $\Psi(x)$ for various values of x over $[1, 5]$, respectively. The errors are represented graphically in Figures 2 and 4. All of the computational processes are performed on an Intel Core i5 by running a code implemented in Wolfram Mathematica software with version 12.0.

Example 6.1. *Let us consider test function* $\Psi(x) = e^{-3x} sin(10x) + 1,\ n = 15, 30, 45,$ *and* $\alpha = 0.1.$

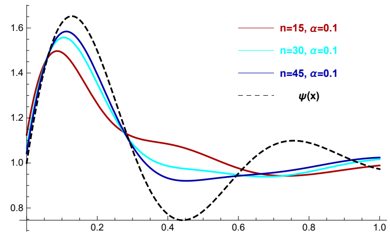

Figure 1: Convergence of $Z_{15}^{(0.1)}(red), Z_{30}^{(0.1)}(cyan), Z_{45}^{(0.1)}(blue)$ and test function $\Psi(x)$(black).

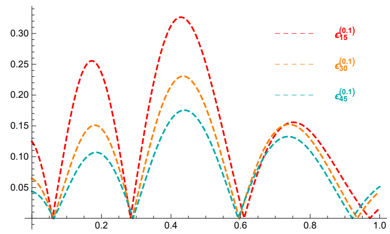

Figure 2: Graph of errors $\epsilon_{15}^{(0.1)}(red), \epsilon_{30}^{(0.1)}(orange), \epsilon_{45}^{(0.1)}(cyan)$.

Table 1: Table for absolute error of the proposed operators.

x	$n = 15$	$n = 30$	$n = 45$
1	0.0161353000	0.0436026000	0.0512909000
2	0.0023755400	0.0019702600	0.0017053600
3	0.0001179590	0.0001190890	0.0001270290
4	0.0000044089	0.0000045528	0.0000045589
5	0.0000000979	0.0000000797	0.0000000800

Example 6.2. *Let us consider test function* $\Psi(x) = e^{-3x}x^2,\ n = 25, 50, 75$ *and* $\alpha = 0.001.$

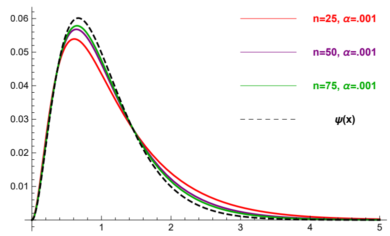

Figure 3: Convergence of $Z_{25}^{(0.001)}(red), Z_{50}^{(0.001)}(purple), Z_{75}^{(0.001)}(green)$ and test function $\Psi(x)$(black).

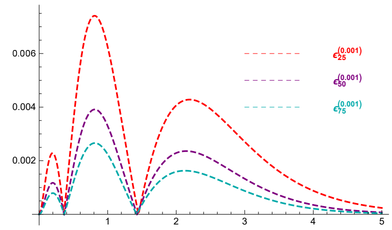

Figure 4: Graph of errors $\epsilon_{25}^{(0.001)}(red), \epsilon_{50}^{(0.001)}(purple), \epsilon_{75}^{(0.001)}(cyan)$.

Table 2: Table for absolute error of the proposed operators.

x	$n = 25$	$n = 50$	$n = 75$
1	0.0063075000	0.0033297200	0.0022616700
2	0.0040841000	0.0023021600	0.0015977400
3	0.0027189300	0.0013316400	0.0008756770
4	0.0008829090	0.0003510320	0.0002121500
5	0.0002393830	0.0000720062	0.0000386964

7. Conclusion

We delved into the study of approximation by a new sequence of integral operators with a certain parameter α on $\mathbb{R}^*$. Our investigation encompasses a comprehensive analysis of various approximation properties, including the rate of convergence, Voronovskaja-type asymptotic formula, and the statistical convergence for the operators. The adaptability and convergence rate of these proposed operators are directly impacted by the selection of n and α. Additionally, we enhance the understanding of our findings by providing graphical representations of the proposed operators under diverse selections of n and α.

Declarations

The authors declare that there are no conflicts of interest.

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