Filomat 39:14 (2025), 4807–4819
https://doi.org/10.2298/FIL2514807B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the Hankel transform of the number of closed walks on regular trees

Radica R. Bojičić[a,*], Marko D. Petković[b], Predrag M. Rajković[c], Paul Barry[d]

[a]*Faculty of Economy, University of Priština, Kosovska Mitrovica, Serbia*
[b]*Faculty of Sciences and Mathematics, University of Niš, Serbia*
[c]*Faculty of Mechanical Engineering, University of Niš, Serbia*
[d]*School of Science South East Technological University, Ireland*

Abstract. We give a closed form expression for the Hankel transform of the sequence which counts closed walks of length $2n$ on $d-$regular trees, and several additional properties of that sequence. The method based on continued fractions and orthogonal polynomials is used. In addition, we give closed form expressions for the Hankel transform of two more related sequences.

1. Introduction

Catalan numbers $\{C_n\}_{n\geq 0} = \{1, 1, 2, 5, 14, 42, \ldots\}$ represent one of the most frequently studied sequences in mathematics. They have many algebraic and combinatorial interpretations, from counting product brackets in non-associative algebra to counting various trees and set partitions [7, 14, 15]. They also appear in the GUE (Gaussian Unitary Ensemble) matrix model as the leading coefficient of certain polynomials, a link closely related to plane trees and partition interpretations of non-intersecting sets [19]. Here are some of the combinatorial interpretations of Catalan numbers, related to the number of paths in the certain structures [8]:

1. A *plane tree* is a rooted tree with a specified order for the descendants of each vertex. The sequence $\{C_n\}$ counts the number of plane trees with $n+1$ vertices.
2. A *Dyck path* of length n is a directed path from $(0, 0)$ to $(n, 0)$ in two dimensional grid $\mathbb{N}_0^2$ that uses only steps of type $(1, 1)$ and $(1, -1)$ and never goes below the $x-$axis. The sequence $\{C_n\}$ counts the number of Dick paths of length $2n$.
3. A binary tree is an empty graph or plane tree in which each node has at most two children. Then $\{C_n\}$ counts the number of planar binary trees with n vertices.

2020 *Mathematics Subject Classification*. Primary 11B83; Secondary 05A19, 33C45, 11B37, 11B65.
Keywords. Transform of related sequences, closed walks on regular trees, series reversion, weight function transformations.
Received: 30 December 2024; Revised: 03 March 2025; Accepted: 17 March 2025
Communicated by Dijana Mosić
* Corresponding author: Radica R. Bojičić
Email addresses: tallesboj@gmail.com (Radica R. Bojičić), dexterofnis@gmail.com (Marko D. Petković), pedja.rajk@gmail.com (Predrag M. Rajković), pbarry@wit.ie (Paul Barry)
ORCID iDs: https://orcid.org/0000-0002-4192-5646 (Radica R. Bojičić), https://orcid.org/0000-0002-6862-1968 (Marko D. Petković), https://orcid.org/ 0000-0002-2914-0985 (Predrag M. Rajković), https://orcid.org/0000-0003-1909-8725 (Paul Barry)

These interpretations, as well as more than 200 others, are found in the recently published monograph by Richard Stanley [19].

One can also count the number of closed vertex walks in a regular tree using the Catalan numbers, showing another combinatorial structure that is counted using this sequence of numbers. These counts will be the main focus of this paper.

Let G be an infinite $d-$regular tree. Denote by the $a(n;d)$ the number of closed walks of length $n \in \mathbb{N}$ that start and end at the vertex $v \in V(G)$. The well-known result [18] provides the generating function approach to compute $a(n;d)$. A new result [9] gives a combinatorial alternative approach, which relates the number of closed walks to the Catalan triangle and also to Borel triangles. It is the sequence of numbers closely related to Catalan numbers, which have recently appeared in several studies regarding commutative algebra, combinatorics and discrete geometry, Cambrian Hopf algebras [4], quantum physics [12] and permutation patterns [17].

It is known (see for example [1, 10, 18]) that the generating function for the sequence $\{a(n;d)\}_{n\in\mathbb{N}}$ that counts closed walks of length n (at a vertex) on a d-regular tree is given by

$$G(x;d) = \sum_{n=0}^{\infty} a(n,d)\, x^n = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)x^2}} \qquad \left(d \in \mathbb{N}\backslash\{1\};\ |x| < \frac{1}{2\sqrt{d-1}}\right). \tag{1}$$

Equivalently, $G(x;d)$ can be written in the form

$$G(x;d) = \frac{d-2-d\sqrt{1-4(d-1)x^2}}{2(d^2x^2-1)}.$$

Since $G(x,d)$ is an even function, i.e., $G(-x,d) = G(x,d)$, we conclude that

$$a(2n+1;d) = 0 \qquad (n \in \mathbb{N}_0).$$

Remark 1.1. In the special case, for $d = 2$, we get

$$G(x;2) = \frac{1}{\sqrt{1-4x^2}} \qquad \left(|x| < \frac{1}{2}\right),$$

which is the generating function of the sequence

$$a(0,2) = 1, \qquad a(n,2) = \begin{cases} 0, & n\text{- odd}, \\ \binom{n}{n/2}, & n\text{- even} \end{cases} \qquad (n \in \mathbb{N}).$$

The Hankel transform of this sequence is given by $\{2^n\}$ (see for example [14, 16] and Section 3 in this paper for a definition of the Hankel transform).

Denote by $a_2(n;d) = a(2n;d)$ the number of closed walks of length $2n$ on the $d-$regular tree. Since every closed walk on $d-$regular must be of the even length, we have $a(2n+1;d) = 0$, i.e.

$$a(2n;d) = a_2(n;d), \qquad a(2n+1;d) = 0 \qquad (n \in \mathbb{N}_0). \tag{2}$$

Let $G_2(x;d)$ be the generating function of the sequence $\{a_2(n;d)\}$. Directly from (2), we conclude that it satisfies the relation $G(x;d) = G_2(x^2;d)$.

Lemma 1.2. *The function $y = G_2(x;d)$ is a solution of the quadratic equation*

$$(d^2x-1)y^2 - (d-2)y + (d-1) = 0. \tag{3}$$

Proof. The discriminant of (3) is

$$D = (d-2)^2 - 4(d-1)(d^2x-1) = d^2\Big(1-4(d-1)x\Big).$$

Hence the first solution is

$$y_1 = \frac{d-2-d\sqrt{1-4(d-1)x}}{2(d^2x-1)} = G_2(x;d).\square$$

2. Preliminaries

In this section, we state some basic properties of the introduced sequences which will be useful for the further considerations. We also introduce the series reversion procedure and one more sequence generated by it.

Proposition 2.1. *Let*

$$c(x) = \frac{1-\sqrt{1-4x}}{2x} \qquad \left(|x| < \frac{1}{2}\right) \tag{4}$$

be the generating function of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. *Then*

$$G(x;d) = \frac{1}{1-dx^2\, c\Big((d-1)x^2\Big)}.$$

Proof. From (4), we can write

$$\sqrt{1-4x} = 1-2x\, c(x).$$

Putting $x \mapsto (d-1)x^2$ in the previous relation and including it into (1), we get

$$G(x;d) = \frac{2(d-1)}{d-2+d\Big(1-2(d-1)x^2\, c\Big((d-1)x^2\Big)\Big)}.$$

Simplifying the denominator, we finish the proof. $\square$

Let us consider the function

$$Q(x) = \frac{x(1+x)}{1+2x+dx^2}.$$

Its inverse function is

$$Q^{-1}(x) = \frac{\sqrt{1-4(d-1)x^2}+2x-1}{2(1-dx)} = \frac{xc((d-1)x^2)}{1-xc((d-1)x^2)}.$$

Definition 2.2. *For a given invertible function* $v = f(u)$ *with the property* $f(0) = 0$, *the series reversion is the sequence* $\{s_n\}_{n\in\mathbb{N}_0}$ *such that*

$$u = f^{-1}(v) = s_1v + s_2v^2 + \cdots + s_nv^n + \cdots,$$

where $u = f^{-1}(v)$ *is the inverse function of* $v = f(u)$. *Note that, since* $f(0) = 0$, *there must hold* $s_0 = f^{-1}(0) = 0$.

Lemma 2.3. *Let the sequence* $\{b(n;d)\}_{n\in\mathbb{N}_0}$ *be the series reversion of* $Q(x)$, *i.e*

$$Q^{-1}(x) = \sum_{n=1}^{\infty} b(n;d)x^n.$$

Then

$$a_2(n;d) = b(2n+1;d) \qquad (n \in \mathbb{N}). \tag{5}$$

Proof. Let us denote by

$$g(x) = \frac{1}{x} Q^{-1}(x) = \sum_{n=0}^{\infty} b(n+1;d)x^n. \tag{6}$$

Hence

$$g(x) = \frac{\sqrt{1-4(d-1)x^2}+2x-1}{2x(1-dx)} = \frac{c((d-1)x^2)}{1-x\,c\Big((d-1)x^2\Big)}, \tag{7}$$

and

$$\frac{g(x)+g(-x)}{2} = \sum_{n=0}^{\infty} b(2n+1;d)x^n = \frac{c\Big((d-1)x^2\Big)}{1-x^2\,c^2\Big((d-1)x^2\Big)},$$

wherefrom the relation (5) follows. □

By direct expansion into Taylor series of the corresponding generating functions, one may conclude that the following representations are valid:

$$b(n;d) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-2k}{n-k}\binom{n-1}{k}(d-1)^k, \tag{8}$$

$$a(n;d) = \frac{1+(-1)^n}{2}\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n-2k+1}{n-k+1}\binom{n}{k}(d-1)^k, \tag{9}$$

$$a_2(n;d) = \sum_{k=0}^{n} \frac{2n-2k+1}{2n-k+1}\binom{2n}{k}(d-1)^k = \delta_{n,0} + \sum_{j=1}^{n}\binom{2n-j}{n}\frac{j}{2n-j}d^j(d-1)^{n-j}. \tag{10}$$

Also, it can be shown that the sequence $a_2(n;d)$ satisfies the certain three-term recurrence relation, given by the following lemma.

Lemma 2.4. *The following recurrence relation is true*

$$n\cdot a_2(n;d) = \Big((d^2+4d-4)n-6(d-1)\Big)\cdot a_2(n-1,d) - 2(d-1)d^2(2n-3)\cdot a_2(n-2,d).$$

Proof. Because of the formula (10), it is easier to consider the case $d \to d+1$. Now, the assumed right-hand side is given by

$$RHS = RHS_1 - 2(2n-3)RHS_2,$$

where

$$RHS_1 = \Big(((d+1)^2+4(d+1)-4)n-6d\Big)\cdot a_2(n-1;d+1),$$
$$RHS_2 = d(d+1)^2\cdot a_2(n-2;d+1).$$

Then the first term has the form

$$\begin{aligned} RHS_1 &= \Big(nd^2+6(n-1)d+n\Big)\cdot a_2(n-1;d+1) \\ &= \Big(nd^2+6(n-1)d+n\Big)\cdot \sum_{k=0}^{n-1}\frac{2n-2k-1}{2n-k-1}\binom{2n-2}{k}d^k \\ &= n\sum_{i=2}^{n+1}\frac{2n-2i+3}{2n-i+1}\binom{2n-2}{i-2}d^i + 6(n-1)\sum_{j=1}^{n}\frac{2n-2j+1}{2n-j}\binom{2n-2}{j-1}d^j \\ &\qquad + n\sum_{k=0}^{n-1}\frac{2n-2k-1}{2n-k-1}\binom{2n-2}{k}d^k. \end{aligned}$$

The second term, without the coefficient $-2(2n-3)$, is given by

$$\begin{aligned}
RHS_2 &= (d^3+2d^2+d)\cdot a_2(n-2,d+1)\\
&= (d^3+2d^2+d)\cdot\sum_{k=0}^{n-2}\frac{2n-2k-3}{2n-k-3}\binom{2n-4}{k}d^k\\
&= \sum_{i=3}^{n+1}\frac{2n-2i+3}{2n-i}\binom{2n-4}{i-3}d^i+2\sum_{j=2}^{n}\frac{2n-2j+1}{2n-j-1}\binom{2n-4}{j-2}d^j\\
&\qquad+\sum_{k=1}^{n-1}\frac{2n-2k-1}{2n-k-2}\binom{2n-4}{k-1}d^k.
\end{aligned}$$

Finally, for any $k\in\mathbb{N}_0$, the coefficient with d^k in the expression RHS, is given by

$$\begin{aligned}
&n\frac{2n-2k+3}{2n-k+1}\binom{2n-2}{k-2}+6(n-1)\frac{2n-2k+1}{2n-k}\binom{2n-2}{k-1}+n\frac{2n-2k-1}{2n-k-1}\binom{2n-2}{k}\\
&\quad-2(2n-3)\left(\frac{2n-2k+3}{2n-k}\binom{2n-4}{k-3}+2\frac{2n-2k+1}{2n-k-1}\binom{2n-4}{k-2}+\frac{2n-2k-1}{2n-k-2}\binom{2n-4}{k-1}\right).
\end{aligned}$$

After some elementary transformations and using the properties of the binomial coefficients, this term is simplified to

$$\frac{2n-2k+1}{2n-k+1}\binom{2n}{k},$$

which completes the proof. □.

3. The Hankel transform

One of the important transformations of the number sequences is the Hankel transform.

Definition 3.1. *The Hankel transform of a given sequence $a=\{a_0,a_1,a_2,\ldots\}$ is the sequence of Hankel determinants $\mathcal{H}(a)=h=\{h_0,h_1,h_2,\ldots\}$ where $h_n=|a_{i+j}|_{i,j=0}^{n}$, i.e*

$$a=\{a_n\}_{n\in\mathbb{N}_0}\quad\to\quad\mathcal{H}(a)=h=\{h_n\}_{n\in\mathbb{N}_0}:\quad h_n=\begin{vmatrix}a_0 & a_1 & \cdots & a_n\\ a_1 & a_2 & & a_{n+1}\\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n}\end{vmatrix}\tag{11}$$

Remark 3.2. This transform is sometimes named as *Hankel sequence of determinants* in order to make difference with the other Hankel transformations [2] and operators [23]. Here, we suppose that the definition is clear enough to make difference with the similarly named transforms.

One method for computing Hankel transform of some sequences is *the method based on continued fractions and the orthogonal polynomials* (see for example [5, 16]). It is applicable to the *moment sequences*, i.e. ones which can be written by

$$a_n=\int_{-\infty}^{+\infty}x^n w(x)\,dx,$$

where $w:\mathbb{R}\to\mathbb{R}^+$ is the *weight function*. It induces the functional $\mathcal{U}[f]$ defined by

$$\mathcal{U}[f]=\int_{-\infty}^{+\infty}f(x)w(x)\,dx\qquad\big(f(x)\in\mathbb{C}(\mathbb{R})\big).$$

Then one may write $a_n = \mathcal{U}[x^n]$ for $n \in \mathbb{N}_0$.

It is known (for example, see [22], [21] or [11]) that the Hankel determinant h_n of order n of such sequence $a = \{a_n\}_{n\in\mathbb{N}_0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1} \tag{12}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cdots}}} \,. \tag{13}$$

The sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ are also the coefficients in the three-term recurrence relation (see for example [3])

$$Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x)$$

satisfied by the monic orthogonal polynomials $\{Q_n(x)\}_{n\in\mathbb{N}_0}$, with respect to the functional $\mathcal{U}[f]$ (or scalar product defined by $\langle f, g\rangle = \mathcal{U}[f\ g]$). The coefficients α_n and β_n can be determined by (see for example [3] or [6]):

$$\alpha_n = \frac{\mathcal{U}[x\, Q_n^2(x)]}{\mathcal{U}[Q_n^2(x)]} \qquad \beta_n = \frac{\mathcal{U}[Q_n^2(x)]}{\mathcal{U}[Q_{n-1}^2(x)]} \qquad (n \in \mathbb{N}_0)\,. \tag{14}$$

In order to effectively use the procedure described above, we need to do perform the following two steps:

1. Determine the weigh function $w(x)$ corresponding to the target sequence;
2. Compute the analytical expressions for the elements of the sequence $\{\beta_n\}_{n\in\mathbb{N}}$, corresponding to (14);
3. Use the expression (12) to compute the element h_n.

The key advantage of the proposed method is that equation (12) allows direct computation of the Hankel determinant in closed form, provided closed-form expressions exist for the coefficients α_n and β_n defined by (14). This holds for a broad class of moment sequences a_n, i.e. orthogonal polynomials.

The second step (i.e. computation of β_n) of above procedure is usually done using the transformation formulas for the coefficients α_n and β_n, for the corresponding transformation of the weight function. The statements of the next lemma are proved in [3] and [6] and shows some examples of the transformation formulas, which we use in the following sections.

Lemma 3.3 ([3, 6]). *Let*

$$w(x) \ \mapsto \ \{\alpha_n,\ \beta_n\}_{n\in\mathbb{N}_0}, \qquad \tilde{w}(x) \ \mapsto \ \{\tilde{\alpha}_n,\ \tilde{\beta}_n\}_{n\in\mathbb{N}_0}\,.$$

Then

(i) $\tilde{w}(x) = Cw(x) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \alpha_n, \quad \tilde{\beta}_0 = C\beta_0, \quad \tilde{\beta}_n = \beta_n\ (n \in \mathbb{N})\}\,;$

(ii) $\tilde{w}(x) = w(ax + b) \quad \Rightarrow \quad \{\tilde{\alpha}_n = \frac{\alpha_n - b}{a}, \quad \tilde{\beta}_0 = \frac{\beta_0}{|a|}, \quad \tilde{\beta}_n = \frac{\beta_n}{a^2}\ (n \in \mathbb{N})\}\,;$

(iii) *If*

$$\dot{w}(x) = \frac{\tilde{w}(x)}{x - c} \qquad (c \notin supp(\tilde{w})),$$

then

$$\dot{\alpha}_0 = \tilde{\alpha}_0 + r_0, \qquad \dot{\alpha}_k = \tilde{\alpha}_k + r_k - r_{k-1},$$
$$\dot{\beta}_0 = -r_{-1}, \qquad \dot{\beta}_k = \tilde{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}),$$

where

$$r_{-1} = -\int_{\mathbb{R}} \dot{w}(x)\, dx, \qquad r_n = c - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots)\,.$$

(iv) *If*

$$\hat{w}(x) = \frac{\tilde{w}(x)}{\delta - x} \qquad (\delta > x,\ \forall x \in supp(\tilde{w})),$$

then

$$\hat{\alpha}_0 = \tilde{\alpha}_0 - r_0, \qquad \hat{\alpha}_k = \tilde{\alpha}_k + r_k - r_{k-1},$$
$$\hat{\beta}_0 = -r_{-1}, \qquad \hat{\beta}_k = \tilde{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}),$$

where

$$r_{-1} = -\int_{\mathbb{R}} \hat{w}(x)\,dx, \qquad r_n = -\delta - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots).$$

Here by $supp(w)$ we denote the set $supp(w) = \{x \in \mathbb{R} \mid w(x) \neq 0\}$ and refer to as the *support* of the weight function $w(x)$.

4. The Hankel transform of closed walks on the *d*-regular tree

In this section, we derive the closed-form expression for the Hankel transform of the sequences $a(n, d)$ and $a_2(n, d)$, which is the main result of the paper.

The first step is to represent $a(n, d)$ and $a_2(n; d)$ as the moment sequences of the certain weight function. We will do it by representing $b(n + 1; d)$ as the moment sequence, and then using (2) and (5).

Recall that the generating function of $\{b(n + 1; d)\}_{n\in\mathbb{N}_0}$ is given by ((6) and (7)):

$$g(x) = x^{-1}Q^{-1}(x) = \frac{\sqrt{1 - 4(d-1)x^2} + 2x - 1}{2x(1 - dx)}.$$

We also use the following well-known theorem:

Theorem 4.1. (Stieltjes-Perron inversion formula) [3, 13] *If $g(z) = \sum_{n=0}^{+\infty} \mu_n z^n$ is the generating function of the real sequence $\{\mu_n\}$ and $F(z) = z^{-1}G(z^{-1})$, then $\mu_n = \int_{-\infty}^{+\infty} x^n w(x)\,dx$ where $w(t) = \lambda'(t)$ and the function $\lambda(t)$ is defined by*

$$\lambda(t) = -\frac{1}{2\pi i}\lim_{y\to 0^+}\int_0^t \Big[F(x + iy) - F(x - iy)\Big]\,dx.$$

If additionally $F(\bar{z}) = \overline{F(z)}$, then one can simpler write:

$$\lambda(t) = -\frac{1}{\pi}\lim_{y\to 0^+}\int_0^t \Im F(x + iy)\,dx.$$

Now one can state and prove the required weight function representations.

Theorem 4.2. *The sequence $\{b(n + 1; d)\}$ can be represented by*

$$b(n+1; d) = \frac{1}{2\pi}\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^n \frac{\sqrt{4(d-1) - x^2}}{d - x}\,dx.$$

Proof. We use Stieltjes-Perron inversion formula i.e. Theorem 4.1. Let

$$F(z) = z^{-1}g(z^{-1}) = \frac{\sqrt{4(1-d) + z^2} - z + 2}{2(z - d)}$$

and denote by $z_0 = 2\sqrt{d-1}$. Then $-z_0$ and z_0 are the branch points of $\rho(z) = \sqrt{z^2 - 4(d-1)}$. We take a regular branch satisfying $\arg(z - z_0) = \arg(z + z_0) = 0$ for $z \in (z_0, +\infty)$ (branch is defined on $\mathbb{C} \setminus (-z_0, z_0)$). By direct evaluation, we compute the following primitive function

$$F_1(z) = \int F(z)\, dz = \frac{1}{2}\rho(z) - \log(\rho(z) - z) - (d-2)\log(\rho(z) + 2(d-1) - z) - \frac{z}{2}.$$

According to Theorem 4.1, we have

$$\lambda(t) - \lambda(0) = -\frac{1}{\pi}\lim_{y\to 0+} \Im F_1(x + iy)$$

After the analysis of the regular branches of $\Im F_1(x + iy)$ and the limit value when $y \to 0+$, one conclude that $\lambda(t)$ is constant in $\mathbb{R} \setminus [-z_0, z_0]$, while

$$\begin{aligned} w(t) &= \lambda'(t) \\ &= -\frac{1}{\pi}\frac{d}{dt}\left[\frac{1}{2}\hat{\rho}(t) + \arctan\left[\frac{\hat{\rho}(t)}{t}\right] + (d-2)\arctan\left[\frac{\hat{\rho}(t)}{t - 2(d-1)}\right]\right] \\ &= \frac{\hat{\rho}(t)}{2\pi(d-t)}. \end{aligned}$$

for $t \in [-z_0, z_0]$ where $\hat{\rho}(t) = \sqrt{4(d-1) - t^2}$. This completes the proof of the theorem. □

Using the previous theorem, one directly obtains

$$a_2(n;d) = b(2n+1;d) = \frac{1}{2\pi}\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^{2n}\frac{\sqrt{4(d-1) - x^2}}{d - x}dx = \frac{(d-1)^n}{2\pi}\int_{-2}^{2} x^{2n}\frac{d(d-1)\sqrt{4 - x^2}}{d^2 - (d-1)x^2}dx,$$

wherefrom

$$a_2(n;d) = \frac{1}{2\pi}\int_0^{4(d-1)} x^n \frac{d\sqrt{x(4(d-1) - x)}}{x(d^2 - x)}dx.$$

By similar approach, we conclude that

$$a(n;d) = \frac{d}{2\pi}\int_{-2\sqrt{d-1}}^{2\sqrt{d-1}} x^n \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2}dx \qquad (n \in \mathbb{N}_0). \tag{15}$$

Now we are ready to state and prove the first main result.

Theorem 4.3. *The Hankel transform of the sequence $\{a(n;d)\}_{n\in\mathbb{N}_0}$, defined by (15), is given by*

$$h(0;d) = 1, \quad h(n;d) = d^n (d-1)^{\binom{n}{2}} \qquad (n \in \mathbb{N}).$$

Proof. We will start from the monic orthogonal polynomials $\{S_n(x)\}$ with respect to the

$$w^*(x) = \sqrt{1 - x^2}, \qquad x \in (-1, 1)\,.$$

These polynomials are monic Chebyshev polynomials of the second kind:

$$S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1 - x^2}}\,.$$

They satisfy the three-term recurrence relation (see for example [3]):

$$S_{n+1}(x) = (x - \alpha_n^*)\, S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \ldots),$$

with initial values

$$S_{-1}(x) = 0, \qquad S_0(x) = 1,$$

where

$$\alpha_n^* = 0 \quad (n \geq 0) \qquad \text{and} \qquad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \qquad (n \geq 1).$$

For the weight function

$$\tilde{w}(x) = w^*\Big(\frac{x}{2\sqrt{d-1}}\Big) = \sqrt{1 - \Big(\frac{x}{2\sqrt{d-1}}\Big)^2}, \quad x \in (-2\sqrt{d-1}, 2\sqrt{d-1}),$$

by Lemma 3.3, in the case of (ii), it is valid

$$\tilde{\alpha}_n = 0 \ (n \in \mathbb{N}_0), \qquad \tilde{\beta}_0 = \pi\sqrt{d-1}, \qquad \tilde{\beta}_n = d-1 \quad (n \in \mathbb{N})\,.$$

Let

$$\hat{w}(x) = \frac{\tilde{w}(x)}{d-x}\,.$$

Taking $\delta = d$ in the case of (iv) of Lemma 3.3, we get

$$r_{-1} = -\frac{\pi}{\sqrt{d-1}}, \qquad r_n = -\delta - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots)\,.$$

By mathematical induction, we can prove

$$r_{-1} = -\frac{\pi}{\sqrt{d-1}}, \qquad r_n = -1 \qquad (n \in \mathbb{N}_0).$$

Hence

$$\begin{aligned} &\hat{\alpha}_0 = 1, && \hat{\alpha}_k = 0, \\ &\hat{\beta}_0 = \frac{\pi}{\sqrt{d-1}}, && \hat{\beta}_k = d-1 \qquad (k \in \mathbb{N}). \end{aligned}$$

Similarly, let us consider

$$\dot{w}(x) = \frac{\hat{w}(x)}{x+d} = \frac{1}{d^2 - x^2} \cdot \sqrt{1 - \Big(\frac{x}{2\sqrt{d-1}}\Big)^2}, \quad x \in (-2\sqrt{d-1}, 2\sqrt{d-1})\,.$$

Taking $c = -d$ in case of (**iii**) of Lemma 3.3, we get

$$r_{-1} = \frac{-\pi}{d\sqrt{d-1}}, \qquad r_n = -d - \hat{\alpha}_n - \frac{\hat{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots)\,.$$

By mathematical induction, we can prove

$$r_{-1} = \frac{-\pi}{d\sqrt{d-1}}, \qquad r_n = -1 \qquad (n \in \mathbb{N}_0).$$

Hence

$$\begin{aligned} &\dot{\alpha}_0 = \hat{\alpha}_0 + r_0 = 0, && && \dot{\alpha}_k = 0, \\ &\dot{\beta}_0 = \frac{\pi}{d\sqrt{d-1}}, && \dot{\beta}_1 = d, && \dot{\beta}_k = d-1 \qquad (k \in \mathbb{N};\ k \geq 2). \end{aligned}$$

Introduction of the weight

$$w(x) = \frac{d\sqrt{d-1}}{\pi}\dot{w}(x),$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $d\sqrt{d-1}/\pi$. Now, it is

$$\beta_0 = \frac{d\sqrt{d-1}}{\pi}\dot{\beta}_0 = 1, \qquad \beta_1 = \dot{\beta}_1 = d, \qquad \beta_n = \dot{\beta}_n = d-1 \quad (n \in \mathbb{N};\ n \geq 2)\,.$$

Applying the formula (12), we have

$$h(1;d) = 1, \qquad h(n;d) = d^{n-1}(d-1)^{(n-2)+\cdots+2+1} = d^{n-1}\cdot(d-1)^{\binom{n-1}{2}},$$

what finishes the proof. □

Example 4.4. For $d = 5$, we have that the sequence

$$\{a(n,5)\}_{n\in\mathbb{N}_0} = \{1, 0, 5, 0, 45, 0, \ldots\},$$

has the Hankel transform

$$\{h(n;5)\}_{n\in\mathbb{N}_0} = \{1, 5, 100, 8000, 2560000, \ldots\} = \{1, 5, 5^2\cdot 4, 5^3\cdot 4^3, 5^4\cdot 4^6, \ldots\}.$$

In the similar manner, we can prove the next theorem which is the second main result of the paper.

Theorem 4.5. *The Hankel transform of the sequence*

$$a_2(n;d) = \frac{d}{2\pi}\int_0^{4(d-1)} x^n \frac{\sqrt{x(4(d-1)-x)}}{x(d^2-x)}dx \qquad (d > 1;\ n \in \mathbb{N}_0)$$

is given by

$$h_2(n,d) = d^{n-1}\cdot(d-1)^{(n-1)^2} \qquad (n \in \mathbb{N})\,.$$

Proof. Let $P_n(x) = P_n^{(1/2,-1/2)}(x)$ ($n \in \mathbb{N}_0$) be a special Jacobi polynomial, which is also known as *the Chebyshev polynomial of the fourth kind*. The sequence of these polynomials is orthogonal with respect to

$$w^*(x) = w^{(1/2,-1/2)}(x) = \sqrt{\frac{1-x}{1+x}}, \qquad x \in (-1,1)\,.$$

These polynomials can be expressed (Szegö [20]) by

$$P_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{2^n\sin\frac{1}{2}\theta}\,.$$

They satisfy the three-term recurrence relation (Chihara [3]):

$$P_{n+1}(x) = (x-\alpha_n^*)\,P_n(x) - \beta_n^* P_{n-1}(x) \quad (n = 0, 1, \ldots),$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1,$$

where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0, \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \qquad (n \in \mathbb{N})\,.$$

For the weight function

$$\tilde{w}(x) = w\Big(\frac{x}{2(d-1)} - 1\Big) = \sqrt{\frac{4(d-1)-x}{x}}, \qquad x \in (0, 4(d-1)),$$

applying the case (**ii**) of Lemma 3.3, we find coefficients

$$\tilde{\alpha}_0 = d-1, \quad \tilde{\alpha}_n = 2(d-1) \quad (n \geq 1) \qquad \tilde{\beta}_0 = 2(d-1)\pi, \quad \tilde{\beta}_n = (d-1)^2 \qquad (n \geq 1)\ .$$

Further, we will define the weight function

$$\hat{w}(x) = \frac{\tilde{w}(x)}{d^2 - x} = \frac{1}{d^2 - x}\ \sqrt{\frac{4(d-1)-x}{x}}, \qquad x \in (0, 4(d-1))\ .$$

Applying the case (**iv**) from Lemma 3.3, we find

$$r_{-1} = -\frac{2\pi}{d}, \qquad r_0 = -(2d-1), \qquad r_n = -2(d-1) - \frac{(d-1)^2}{r_{n-1}} \qquad (n \in \mathbb{N}).$$

By mathematical induction, we can prove

$$r_{-1} = -\frac{2\pi}{d}, \qquad r_n = -(d-1) \qquad (n \in \mathbb{N}_0).$$

Hence

$$\begin{aligned} \hat{\alpha}_0 &= d, & \hat{\alpha}_k &= 2(d-1), \\ \hat{\beta}_0 &= \frac{2\pi}{d}, & \hat{\beta}_k &= (d-1)^2 \qquad (k \in \mathbb{N}). \end{aligned}$$

At last, we will define the weight function

$$\bar{w}(x) = \frac{d}{2\pi}\hat{w}(x) = \frac{d}{2\pi}\frac{1}{d^2 - x}\ \sqrt{\frac{4(d-1)-x}{x}}, \qquad x \in (0, 4(d-1))\ .$$

Applying the case (**i**) from Lemma 3.3, we find

$$\begin{aligned} \bar{\alpha}_0 &= d, & \bar{\alpha}_k &= 2(d-1), \\ \bar{\beta}_0 &= 1, & \bar{\beta}_k &= (d-1)^2 \qquad (k \in \mathbb{N}). \end{aligned}$$

Applying the formula (12), we have

$$h(1;d) = 1, \qquad h(n;d) = d^{n-1}(d-1)^{(n-2)+\cdots+2+1} = d^{n-1}\cdot(d-1)^{\binom{n-1}{2}},$$

and

$$h_2(n,d) = d^{n-1}\cdot(d-1)^{(n-1)^2} \qquad (n \in \mathbb{N})\ .$$

what finishes the proof. □

Example 4.6. For $d = 7$, we have the sequence

$$\{a_2(n,7)\}_{n\in\mathbb{N}_0} = \{1, 7, 91, 1435, 24955, 460747, \ldots\},$$

which has the Hankel transform equal to

$$\{h_2(n,7)\}_{n\in\mathbb{N}_0} = \{1, 42, 63504, 3456649728, \ldots\} = \{1, 6\cdot 7, 6^4\cdot 7^2, 6^9\cdot 7^3, \ldots\}\ .$$

Finally, we derive the Hankel transform of the sequence $\{b(n+1;d)\}$ using the similar procedure and transformations.

Theorem 4.7. *The Hankel transform of the sequence* $\{b(n+1;d)\}$ *is given by*

$$h_b(n;d) = (d-1)^{\binom{n}{2}} \qquad (n \in \mathbb{N})\ .$$

Proof. We will start again from the monic Chebyshev polynomials of the second kind $\{S_n(x)\}$ with respect to the

$$w^*(x) = \sqrt{1-x^2}, \qquad x \in (-1,1),$$

and the sequences of coefficients

$$\alpha_n^* = 0 \quad (n \geq 0) \qquad \text{and} \qquad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \qquad (n \geq 1).$$

For the weight function

$$\tilde{w}(x) = w^*\Big(\frac{x}{2\sqrt{d-1}}\Big) = \sqrt{1-\Big(\frac{x}{2\sqrt{d-1}}\Big)^2}, \quad x \in (-2\sqrt{d-1}, 2\sqrt{d-1}),$$

by case (**i**) from Lemma 3.3, it is valid

$$\tilde{\alpha}_n = 0 \ (n \in \mathbb{N}_0), \qquad \tilde{\beta}_0 = \pi\sqrt{d-1}, \qquad \tilde{\beta}_n = d-1 \quad (n \in \mathbb{N}).$$

Now, we should consider the weight function

$$\hat{w}(x) = \frac{\tilde{w}(x)}{d-x}.$$

Taking $\delta = d$ in case (**iv**) from Lemma 3.3, we get

$$r_{-1} = -\frac{\pi}{\sqrt{d-1}}, \qquad r_n = -d - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots).$$

By mathematical induction, we can prove

$$r_{-1} = -\frac{\pi}{\sqrt{d-1}}, \qquad r_n = -1 \qquad (n \in \mathbb{N}_0).$$

Hence

$$\hat{\alpha}_0 = 1, \qquad \hat{\alpha}_k = 0,$$
$$\hat{\beta}_0 = \frac{\pi}{\sqrt{d-1}}, \qquad \hat{\beta}_k = d-1 \qquad (k \in \mathbb{N}).$$

Multiplying the weight function by constant

$$w(x) = \frac{\sqrt{d-1}}{\pi}\hat{w}(x),$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $\sqrt{d-1}/\pi$. Now, it is

$$\beta_0 = \frac{\sqrt{d-1}}{\pi}\hat{\beta}_0 = 1, \qquad \beta_n = \hat{\beta}_n = d-1 \quad (n \in \mathbb{N}).$$

Applying the formula (12), we have

$$h_b(1;d) = 1, \qquad h_b(n;d) = (d-1)^{(n-1)+(n-2)+\cdots+2+1} = (d-1)^{\binom{n}{2}},$$

what finishes the proof. □

Example 4.8. For $d = 9$, we have the sequence

$$\{b(n;10)\}_{n\in\mathbb{N}_0} = \{1, 1, 10, 19, 190, 442, \ldots\},$$

and the Hankel transform is given by

$$\{h_b(n;10)\}_{n\in\mathbb{N}_0} = \{1, 9, 729, 531441, 3486784401, \ldots\} = \{1, 9, 9^3, 9^6, 9^{10}, \ldots\}.$$

5. Conclusion

In this paper, we applied a method based on continued fractions and orthogonal polynomials to derive the Hankel transform of the sequence that enumerates tree walks in a d-regular tree, and some related sequences. Moreover, we derived several interesting properties of these sequences. This approach provides an elegant framework for computing the Hankel transform of combinatorial sequences by leveraging results from the theory of orthogonal polynomials. A natural direction for future research is to explore whether the same approach can be applied to other combinatorial sequences arising in path enumeration on graphs (for instance, sequences counting walks in non-regular trees, directed acyclic graphs, etc.).

Acknowledgements

The authors are thankful to the the referees for their comments and suggestions.

Authors contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Funding

The second author gratefully acknowledge the support from Serbian Ministry of Science, Technological Development and Innovation, grant number 451-03-137/2025-03/200124.

References

[1] L. Bartholdi, *Counting Paths in Graphs*, L'Enseignement Mathématique **45**(1) (2002), 83–131.
[2] J. J. Betancor, *On Hankle transformable distribution spaces*, Publications de l'institut mathematiques, Nouvelle serie, tome bf 65 (79), 1999, 123–141.
[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York,(1978).
[4] G. Chatel, V. Pilaud, *Cambrian hopf algebras*, Adv. Math. **311**, (2017), 598–633.
[5] A. Cvetković, P. Rajković and M. Ivković, *Catalan Numbers, the Hankel Transform and Fibonacci Numbers*, J. Integer Seq. **5**, (2002), Article 02.1.3.
[6] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press - Oxford, 2003.
[7] M. Goubi, *On a generalization of Catalan polynomials*, FU. Math. Inform. **33**(2), (2018), 163–176.
[8] P. E. Gunnells, *Generalized Catalan numbers from Hypergraphs*, Electron. J. Comb. **28**(1), (2021), #P1.52.
[9] L. C. Kavi, M. W. Newman, *Counting closed walks in infinite regular trees using Catalan and Borel's triangles*, arXiv:2212.08795v1, (2022).
[10] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92**, (1959), 336–354.
[11] C. Krattenthaler, *Advanced determinant calculus: A complement,* Linear Algebra Appl. **411**, (2005), 68–166.
[12] A. Lakshminarayan, Z. Puchala, K. Życzkowski, *Diagonal unitary entangling gates and contradiagonal quantum states*, Phys. Rev. A. **90**(3), (2014), 032303.
[13] W. Lang, *On sums of powers of zeros of polynomials*, J. Comput. Appl. Math. **89:2**, 237–256.
[14] L. Mu, Y. Wang, Y.N. Yeh, *Hankel determinants of linear combinations of consecutive Catalan-like numbers*, Discrete Math. **340**(6), (2017), 1389-1396.
[15] A.O. Öztürk, F. Kaplan, *Some properties of bivariate Fibonacci and Lucas quaternion polynomials*, FU. Math. Inform. **35**(1), (2020), 73–87.
[16] P. M. Rajković, M. D. Petković, P. Barry *The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers,* Integr. Transf. Spec. Funct. **18**(4), (2007), 285–296.
[17] J. B. Remmel, *Consecutive up-down patterns in up-down permutations*, Electron. J. Comb., **21**(3), (2014), 1–35.
[18] E. Rowland, D. Zeilberger, *On the number of walks on a regular cayley tree*, arXiv:0903.1877, (2009).
[19] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, New York, (2015).
[20] G. Szegö, *Orthogonal Polynomials*, AMS, 4th. ed., **23** (Colloquium publications), (1975).
[21] X. Viennot, *Une theorie combinatoire des polynomes orthogonaux generaux*, UQAM, Montreal, Quebec, 1983.
[22] H. S. Wall, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.
[23] D. R. Yafaev, *Hankle and Toeplitz operators: continuos and discrete representations*, Opuscula Math. **37**(1) (2017), 189–218.