Filomat 39:14 (2025), 4821–4842
https://doi.org/10.2298/FIL2514821K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fibonacci matrices from tridiagonal symmetric Toeplitz forms and their application in cryptographic Hill cipher schemes

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Abstract. This study focuses on the derivation of closed-form expressions for the entries of the matrix powers $S_4^n(x,y)$, where $S_4(x,y)$ is a tridiagonal symmetric Toeplitz matrix associated with Fibonacci numbers F_n and F_{n+1}. Specific cases of the ordered pair (x,y), including (F_{s+1},F_s), $(F_{-s},F_{-(s+1)})$, and $(F_{-(s+1)},F_{-s})$, are investigated to characterize when $S_4(x,y)$ becomes a Fibonacci matrix. Using these closed-form expressions, we derive and analyze key matrix properties, including trace, determinant, and row sums. These results not only offer explicit evaluations of fundamental matrix characteristics, but also contribute to the theoretical understanding of Fibonacci matrices. As an application, the derived Fibonacci matrices are employed as key matrices in a Affine Hill cipher algorithm, highlighting their applicability in symmetric key cryptographic systems.

1. Introduction

The Fibonacci numbers F_n are defined by the second-order linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n follow the same recurrence relation, but with different initial values: $L_0 = 2$ and $L_1 = 1$. Both sequences admit closed-form expressions known as Binet formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation associated with the recurrence relation. Notably, α is known as the golden ratio, and it follows that $\beta = -\alpha^{-1}$.

In the literature, various fundamental identities involving Fibonacci and Lucas numbers have been established using techniques from number theory, geometry, matrix theory, and other mathematical fields.

2020 *Mathematics Subject Classification*. Primary 11B39 mandatory; Secondary 15A16, 40C05.
Keywords. Fibonacci Matrix, Matrix Methods, Fibonacci Number, Lucas Number, Golden Ratio.
Received: 21 November 2024; Revised: 12 May 2025; Accepted: 03 June 2025
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Among these methods, Binet formulas play a crucial role in deriving elegant expressions and relationships, including the following identities:

$$F_{-n} = (-1)^{n+1}F_n, \qquad L_{-n} = (-1)^n L_n, \tag{2}$$
$$L_n = F_{n-1} + F_{n+1}, \qquad 5F_n = L_{n-1} + L_{n+1}, \tag{3}$$

as well as several connections between Fibonacci numbers and powers of the golden ratio:

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2}, \qquad \beta^n = \beta^{n-1} + \beta^{n-2}, \tag{4}$$
$$\alpha^n = \alpha F_n + F_{n-1}, \qquad \beta^n = \beta F_n + F_{n-1}, \tag{5}$$
$$F_{n+1} - \alpha F_n = \beta^n, \qquad F_{n+1} - \beta F_n = \alpha^n. \tag{6}$$

These identities not only reveal the deep algebraic structure of the sequences, but also highlight the remarkable interplay between recursive sequences and irrational numbers such as the golden ratio.

There exists a wide range of mathematical and geometric studies centered around the golden ratio and the Fibonacci numbers. Prominent areas of interest include the intrinsic properties of the golden ratio, its role in solving geometric problems in both two and three dimensions, and its deep connections with the sequences $\{F_n\}$ and $\{L_n\}$, along with their various generalizations. Moreover, these numbers appear in numerous interdisciplinary applications in fields such as mathematics, physics, biology, and engineering, further highlighting their broad scientific relevance and utility [17, 23, 27].

In geometry, the golden ratio appears in various elegant constructions. One such example is the golden triangle, an isosceles triangle in which the ratio of a leg to the base equals the golden ratio. Furthermore, the regular pentagon and the fifth roots of unity are closely related to the golden ratio. This connection is illustrated in Table 1, where the values $2\cos\left(\frac{j\pi}{5}\right)$ for $j = 1, 2, 3, 4$ yield expressions involving the golden ratio and its conjugate:

Table 1: The golden ratio and the fifth roots of unity

j	1	2	3	4
$2\cos\left(\frac{j\pi}{5}\right)$	α	$-\beta$	β	$-\alpha$

where, $\alpha = \frac{1+\sqrt{5}}{2}$ denotes the golden ratio and $\beta = 1 - \alpha$ is its algebraic conjugate. For more on this topic, see [17, 23, 27]. A well-known matrix closely associated with Fibonacci numbers is the Fibonacci matrix:

$$Q = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad n \geq 1. \tag{7}$$

The identity in Eq. 7 can be readily verified by mathematical induction. Several classical results follow from the properties of Q, such as Cassini identity: $\det(Q^n) = F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, and the characteristic equation of Q^n, $|Q^n - \lambda I| = \lambda^2 - L_n\lambda + (-1)^n$, whose roots correspond to the Binet expressions: $\alpha^n, \beta^n = \frac{L_n \pm F_n\sqrt{5}}{2}$. Moreover, the finite sum identity $\sum_{i=1}^{n} F_i = F_{n+2} - 1$ is derived using the matrix identity:

$$\left(I + Q + Q^2 + \cdots + Q^n\right)(Q - I) = Q^{n+1} - I.$$

Numerous other identities involving Fibonacci numbers arise from the algebraic relations: $Q^mQ^n = Q^{m+n}$, $Q^mQ^nQ^\ell = Q^{m+n+\ell}$, as demonstrated in [13, 27].

Matrix analogues of the binomial and Waring formulas offer yet another powerful tool for constructing identities. Filipponi [21] provided several examples by using a generalized matrix M and its inverse, showing how classical formulas can be adapted to the Fibonacci context.

More generally, many studies have explored generalized Fibonacci sequences through 2×2 matrix representations [3–5, 19, 24]. For example, Wani et al. [1] proposed a generalized Fibonacci sequence defined by a second-order linear recurrence and derived its closed form using matrix methods. This approach enables efficient computation and provides structural insight into the sequence.

Likewise, Jun and Choi [25] studied the generalized Fibonacci sequence $\{q_n\}$ using companion matrix powers to obtain recursive identities and combinatorial interpretations. Cerda-Morales [9] further developed this framework by introducing a new generalization and its associated matrix formulation, thus reinforcing the flexibility and strength of matrix-based approaches in the study of recurrence relations.

In [22], Filipponi examined a special class of symmetric tridiagonal Toeplitz matrices of order 4×4, denoted by $S_4(x, y) = [s_{hk}]_4$, where the main diagonal entries are defined as $s_{hh} = x$ for $1 \leq h \leq 4$, and the sub- and super-diagonal entries are given by $s_{h,h+1} = s_{h+1,h} = y$ for $1 \leq h \leq 3$. This study focused on the powers of these matrices, $S_4^n(x, y)$, under pairs of specific parameters $(x, y) \in \{(F_0, F_1), (F_1, F_2), (F_2, F_3)\}$, where F_n denotes the nth Fibonacci number. The resulting matrix powers, $S_4^n(F_s, F_{s+1})$, have entries expressible in terms of Fibonacci numbers, leading to the derivation of new identities involving these sequences.

Washington [18] further explored the properties of Fibonacci matrices, focusing on 4×4 matrices. The study emphasized their structural connection to the fifth roots of unity, particularly their membership in the unit group, and their relation to the golden ratio α. Several illustrative examples were provided, showcasing the algebraic and number theoretical implications of these matrices.

In [6], the relationship between the Fibonacci and Lucas numbers and the Pascal matrix was investigated in greater depth. The authors introduced the Fibonacci matrix R_L^n and its Lucas counterpart $(R_L - 5I)^n$, demonstrating how their closed-form expressions yield a variety of identities, including novel combinatorial formulas involving Fibonacci and Lucas numbers.

A comprehensive treatment of tridiagonal matrices was given by Ferguson [12], who analyzed their role in the context of the Fibonacci pseudo group. The work presented explicit formulas for characteristic polynomials and eigenvalues, along with applications in fields such as quantum mechanics and magnetohydrodynamics, thereby highlighting the interdisciplinary relevance of Fibonacci-related matrix theory.

Shannon et al. [2] introduced the r-Terraced matrix as a generalization of classical Terraced matrices and proposed its symmetric counterpart. They systematically analyzed these matrices in terms of their spectral and Euclidean norms, characteristic polynomials, and spread upper bounds. By employing Fibonacci sequences in illustrative examples, the authors validated the theoretical results and demonstrated their broader applicability. Their findings emphasize that selecting $r < 1$ and working with lower-dimensional matrices leads to tighter spread bounds and computational efficiency, making these structures particularly suitable for optimization and applied linear algebra.

Peña [14] investigates the eigenvalue localization of symmetric positive Toeplitz matrices by providing inclusion intervals for their spectra. Under certain additional assumptions, the study derives two disjoint subintervals that collectively contain all eigenvalues. Furthermore, the work establishes sufficient conditions for positive definiteness and explores the interplay between total positivity and the Toeplitz structure, culminating in a characterization of symmetric totally positive Circulant matrices.

In [10], Barbarino examines the spectral properties of flipped Toeplitz matrices of the form $H_n(f) = Y_n T_n(f)$. The study identifies an alternating sign pattern in the eigenvalues and provides localization results. These spectral insights are further utilized to evaluate the performance of the MINRES method in solving symmetrized Toeplitz systems, supported by comprehensive numerical experiments.

Recent advancements in public-key cryptography have increasingly aimed to balance computational efficiency with cryptographic robustness by integrating number theoretical structures. One promising direction involves the utilization of generalized Fibonacci matrices in conjunction with classical cryptographic schemes such as the Hill cipher and the ElGamal key exchange protocol.

The Hill cipher, introduced by L. S. Hill in 1929, employs matrix-based linear transformations to encrypt blocks of plaintext, providing notable resistance against frequency analysis attacks. However, its dependency on invertible key matrices poses practical challenges, especially in the context of large-scale or modern cryptographic implementations.

To address these limitations, researchers such as Prasad and Mahato [16] and Zeriouh et al. [20] have proposed schemes that construct key matrices based on Fibonacci-like recurrence relations. These generalized Fibonacci matrices exhibit advantageous algebraic properties such as guaranteed invertibility and predictable structural behavior—that make them particularly well suited for cryptographic applications over finite fields.

Expanding on this foundation, Panchal, Chandra, and Singh [15] introduced a novel public-key encryption and decryption scheme that integrates generalized Fibonacci matrices (under a prime modulus) with the Hill cipher. A distinguishing feature of their approach is the elimination of full key matrix transmission. Instead, the cryptographic key is succinctly represented by a pair of integers (p, θ), thereby reducing both communication overhead and storage requirements.

Complementing these cryptographic innovations, recent studies have also explored the use of Fibonacci matrices in the field of error detection and correction. These efforts aim to enhance the integrity and reliability of data transmission by leveraging the algebraic structure and predictability inherent in Fibonacci-based matrix constructions.

One noteworthy contribution in this area is by Kürüz [7], who introduced new classes of Fibonacci matrices tailored for coding theory applications. The study defines matrix structures such as the Fibonacci X, K, and S matrices, exploring their determinant properties and demonstrating how these can be employed in efficient encoding and decoding schemes. Two main coding methods are proposed: one based on the power of the K_n matrix and the other utilizing the S_n matrix in a double multiplicative framework. Both methods offer significant improvements in simultaneous data transmission volume and demonstrate robust error detection and correction capabilities even with multiple errors by exploiting eigenvalue behavior and asymptotic Fibonacci ratios. The findings illustrate that these new matrix structures can support high-speed, high-accuracy communication without compromising cryptographic integrity.

Motivated by the aforementioned developments in cryptographic applications of Fibonacci matrices particularly those involving the Hill and Affine-Hill ciphers this study advances the theoretical framework by deriving explicit closed-form expressions for the entries of $S_4^n(x, y)$, where x and y are selected from specific Fibonacci number pairs. Based on observations in [22] and our propositions, these results extend previous work by offering both generalizations and new analytical tools.

Beyond the theoretical contributions, we also explore practical implementations of these matrices in classical encryption schemes. Specifically, we demonstrate how the derived expressions for $S_4^n(x, y)$ can be effectively integrated into the Hill cipher and its Affine variant. This application underscores the dual algebraic and computational significance of Fibonacci-based matrix structures, illustrating their interdisciplinary relevance in modern cryptographic systems.

2. The Fibonacci Matrices $S_4^n(x, y)$

In this section, we present a comprehensive analysis of the Fibonacci matrices $S_4^n(x, y)$, covering all relevant cases. Although the relationships among the entries of the matrix $S_4^n(F_s, F_{s+1})$ were previously investigated in [22], explicit closed-form expressions for these entries were not provided. We begin by introducing a lemma that provides closed-form expressions for the elements of the matrix $S_4^n(F_s, F_{s+1})$. The subsequent analysis is divided into three subsections, each devoted to a detailed examination of a specific case.

Following [8, 11, 22], we recall fundamental properties of the symmetric tridiagonal Toeplitz matrix $S_4(x, y)$ of order 4×4. The eigenvalues of this matrix are given by

$$\lambda_j(x, y) = x + 2y \cos\left(\frac{j\pi}{5}\right), \quad j = 1, 2, 3, 4, \tag{8}$$

and the entries of the matrix power $S_4^n(x, y) = \left[s_{hk}^{(n)}(x, y)\right]$ are expressed as

$$s_{hk}^{(n)}(x, y) = \frac{2}{5} \sum_{j=1}^{4} \left(x + 2y \cos\frac{j\pi}{5}\right)^n \sin\left(\frac{jh\pi}{5}\right) \sin\left(\frac{jk\pi}{5}\right), \quad 1 \le h, k \le 4. \tag{9}$$

The symmetry properties of the entries of $S_4^n(x,y)$ are observed as follows:

$$\begin{aligned} s_{11}^{(n)} &= s_{44}^{(n)}, \quad s_{22}^{(n)} = s_{33}^{(n)}, \quad s_{12}^{(n)} = s_{21}^{(n)} = s_{34}^{(n)} = s_{43}^{(n)}, \\ s_{14}^{(n)} &= s_{41}^{(n)}, \quad s_{23}^{(n)} = s_{32}^{(n)}, \quad s_{13}^{(n)} = s_{31}^{(n)} = s_{24}^{(n)} = s_{42}^{(n)}. \end{aligned} \tag{10}$$

Moreover, due to the homogeneity of the matrix with respect to scalar multiplication, the entries satisfy the relation $s_{hk}^{(n)}(px,py) = p^n s_{hk}^{(n)}(x,y)$, so without loss of generality, parameters x and y can be taken such that $\gcd(x,y) = 1$, where $\gcd(x,y)$ denotes the greatest common divisor.

It is well known that consecutive Fibonacci numbers (F_s, F_{s+1}) are coprime for all $s \geq 0$, meaning that $\gcd(F_s, F_{s+1}) = 1$. A common choice for the ordered pair (x,y) is therefore (F_{s+1}, F_s) due to this coprimality property. This selection also extends naturally to the negative indices $s < 0$, since the pairs (F_s, F_{s+1}) or (F_{s+1}, F_s) remain coprime in such cases. This broader applicability makes Fibonacci pairs particularly suitable for constructing the matrix $S_4(x,y)$ and exploring its number theoretical properties.

For these computations, Table 2 lists the values of certain sine product expressions that arise in the matrix formulation. These expressions correspond to the eigenvector components associated with $S_4^n(x,y)$, as defined in Eq. (9). Importantly, they remain invariant with respect to the specific choice of (x,y), making them universally applicable across all cases of $S_4^n(x,y)$.

Table 2: Sine product values appearing in the matrix $S_4^n(x,y)$

$\sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5}$	$j=1$	$j=2$	$j=3$	$j=4$
$h=k=1$	$\frac{-\sqrt{5}\beta}{4}$	$\frac{\sqrt{5}\alpha}{4}$	$\frac{\sqrt{5}\alpha}{4}$	$\frac{-\sqrt{5}\beta}{4}$
$h=1,k=2$	$\frac{\sqrt{5}}{4}$	$\frac{\sqrt{5}}{4}$	$\frac{-\sqrt{5}}{4}$	$\frac{-\sqrt{5}}{4}$
$h=1,k=3$	$\frac{\sqrt{5}}{4}$	$\frac{-\sqrt{5}}{4}$	$\frac{-\sqrt{5}}{4}$	$\frac{\sqrt{5}}{4}$
$h=1,k=4$	$\frac{-\sqrt{5}\beta}{4}$	$\frac{-\sqrt{5}\alpha}{4}$	$\frac{\sqrt{5}\alpha}{4}$	$\frac{\sqrt{5}\beta}{4}$
$h=k=2$	$\frac{\sqrt{5}\alpha}{4}$	$\frac{-\sqrt{5}\beta}{4}$	$\frac{-\sqrt{5}\beta}{4}$	$\frac{\sqrt{5}\alpha}{4}$
$h=2,k=3$	$\frac{\sqrt{5}\alpha}{4}$	$\frac{\sqrt{5}\beta}{4}$	$\frac{-\sqrt{5}\beta}{4}$	$\frac{-\sqrt{5}\alpha}{4}$

The entries listed in Table 2 are used in calculating the powers of the matrix $S_4(x,y)$ independently of its specific (x,y) values. These sine product terms are sufficient to determine six distinct matrix entries due to the inherent symmetry property described in Eq. (10).

Lemma 2.1. *Let F_n denote the n^{th} Fibonacci number. Then, the entries of the matrix $S_4^n(F_s, F_{s+1})$ are given by:*

$$s_{11}^{(n)}(F_s,F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n-1} + \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t-1}\right] \tag{11}$$

$$s_{12}^{(n)}(F_s,F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n} - \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t}\right] \tag{12}$$

$$s_{13}^{(n)}(F_s,F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n} + \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t}\right] \tag{13}$$

$$s_{14}^{(n)}(F_s,F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n-1} - \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t-1}\right] \tag{14}$$

$$s_{22}^{(n)}(F_s,F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n+1} + \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t+1}\right] \tag{15}$$

$$s_{23}^{(n)}(F_s, F_{s+1}) = \frac{1}{2}\left[F_{(s+1)n+1} - \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^t F_{s+1}^{n-t}F_{n-t+1}\right] \tag{16}$$

where $\binom{n}{t}$ *is the the binomial coefficient,* $n \geq t \geq 0$.

Proof. For the entries of the matrix $S_4^n(F_s, F_{s+1})$ derived from the general form given in Eq. (9), we obtain

$$s_{hk}^{(n)}(F_s, F_{s+1}) = \frac{2}{5}\sum_{j=1}^{4}\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n \sin\frac{hj\pi}{5}\sin\frac{kj\pi}{5}. \tag{17}$$

Using the equations provided in Eqs. (4)–(6) along with the value from Table 1, we present the following Table 3, which displays the eigenvalues in Eq. (8):

Table 3: Eigenvalues of the matrix $S_4^n(F_s, F_{s+1})$

j	1	2	3	4
$\left(F_s + 2F_{s+1}\cos\frac{j\pi}{5}\right)^n$	$\alpha^{(s+1)n}$	$(F_s - \beta F_{s+1})^n$	$\beta^{(s+1)n}$	$(F_s - \alpha F_{s+1})^n$

The entry $s_{12}^{(n)}(F_s, F_{s+1})$ in Eq. (17) for the case $(h, k) = (1, 2)$ is computed using Tables 2 and 3 as follows:

$$\begin{aligned} s_{12}^{(n)}(F_s, F_{s+1}) &= \frac{1}{2\sqrt{5}}\left[\alpha^{(s+1)n} + (F_s - \beta F_{s+1})^n - \beta^{(s+1)n} - (F_s - \alpha F_{s+1})^n\right] \\ &= \frac{1}{2}\left[F_{(s+1)n} + \frac{1}{\sqrt{5}}\sum_{t=0}^{n}\binom{n}{t}F_s^t F_{s+1}^{n-t}\left((-\beta)^{n-t} - (-\alpha)^{n-t}\right)\right]. \end{aligned}$$

The expression $s_{12}^{(n)}(F_s, F_{s+1})$ is obtained using the Binet formula as given in Eq. (1).

By applying the same methodology, the remaining entries $s_{hk}^{(n)}(F_s, F_{s+1})$ for the relevant (h, k) values in Eq. (17) are calculated using the corresponding values from Tables 2 and 3. This approach establishes the validity of Eqs. (11)–(16). □

Furthermore, the matrix $S_4^n(F_s, F_{s+1})$ is expressed in terms of $S_4^n(F_0, F_1)$ and $S_4^n(F_1, F_2)$ through the following identities:

$$\begin{aligned} S_4^n(F_s, F_{s+1}) &= \left[F_s S_4(1,1) + F_{s-1}S_4(0,1)\right]^n, && (18) \\ &= \left[F_{s+1}S_4(1,1) - F_{s-1}S_4(1,0)\right]^n. && (19) \end{aligned}$$

By applying binomial expansion to Eqs. (18) and (19), respectively, the matrix $S_4^n(F_s, F_{s+1})$ is represented in terms of powers of the matrices $S_4(1,1)$, $S_4(0,1)$, and $S_4(1,0)$ as follows:

$$\begin{aligned} S_4^n(F_s, F_{s+1}) &= \sum_{k=0}^{n}\binom{n}{k}F_s^{n-k}F_{s-1}^k\, S_4^{n-k}(1,1)S_4^k(0,1), && (20) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k F_{s+1}^{n-k}F_{s-1}^k\, S_4^{n-k}(1,1). && (21) \end{aligned}$$

These expansions demonstrate that the matrix $S_4^n(F_s, F_{s+1})$ is expressed as a finite linear combination of matrix powers weighted by products of Fibonacci numbers. Each term in the sum corresponds to a product of the powers of the fundamental building blocks $S_4(1,1)$, $S_4(0,1)$, and $S_4(1,0)$, combined using binomial coefficients and appropriate Fibonacci terms. A more detailed investigation of such results is left to the interested reader.

2.1. The Fibonacci Matrix $S_4^n(F_{s+1}, F_s)$, $s \geq 0$

For the cases $(x, y) = (F_1, F_0)$ and $(x, y) = (F_2, F_1)$, it is observed that $S_4^n(F_1, F_0) = I_4$, and $S_4^n(F_2, F_1)$ coincides with the matrix $S_4^n(F_1, F_2)$ as presented in [22]. Therefore, these cases are not considered further in this study. Instead, we focus on the case $s = 2$, which corresponds to the matrix $S_4^n(2, 1)$, associated with the parameter pair $(x, y) = (F_3, F_2)$.

Theorem 2.2. *Let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. The entries of the matrix $S_4^n(2, 1)$ are given by the following expressions:*

$$s_{21}^{(n)}(2,1) = \frac{1}{2}\left[F_{2n} + 5^{\lfloor n/2 \rfloor}\left(F_n \frac{1+(-1)^n}{2} + L_n \frac{1-(-1)^n}{2}\right)\right] \tag{22}$$

$$s_{31}^{(n)}(2,1) = \frac{1}{2}\left[-F_{2n} + 5^{\lfloor n/2 \rfloor}\left(F_n \frac{1+(-1)^n}{2} + L_n \frac{1-(-1)^n}{2}\right)\right] \tag{23}$$

$$s_{41}^{(n)}(2,1) = \frac{1}{2}\left[-F_{2n+1} + 5^{\lfloor n/2 \rfloor}\left(F_{n-1} \frac{1+(-1)^n}{2} + L_{n-1} \frac{1-(-1)^n}{2}\right)\right] \tag{24}$$

$$s_{44}^{(n)}(2,1) = \frac{1}{2}\left[F_{2n+1} + 5^{\lfloor n/2 \rfloor}\left(F_{n-1} \frac{1+(-1)^n}{2} + L_{n-1} \frac{1-(-1)^n}{2}\right)\right] \tag{25}$$

$$s_{33}^{(n)}(2,1) = \frac{1}{2}\left[F_{2n-1} + 5^{\lfloor n/2 \rfloor}\left(F_{n+1} \frac{1+(-1)^n}{2} + L_{n+1} \frac{1-(-1)^n}{2}\right)\right] \tag{26}$$

$$s_{32}^{(n)}(2,1) = \frac{1}{2}\left[-F_{2n-1} + 5^{\lfloor n/2 \rfloor}\left(F_{n+1} \frac{1+(-1)^n}{2} + L_{n+1} \frac{1-(-1)^n}{2}\right)\right] \tag{27}$$

where symbol $\lfloor x \rfloor$ denotes the floor function, which returns the greatest integer less than or equal to the real number x.

Proof. For any entry of the matrix $S_4^n(2, 1)$, the general expression from Eq. (9) becomes

$$s_{hk}^{(n)}(2,1) = \frac{2}{5}\sum_{j=1}^{4}\left(2 + 2\cos\frac{j\pi}{5}\right)^n \sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5}. \tag{28}$$

Using the identities in Eqs. (4)–(6) along with the values in Table 1, we construct the following table of eigenvalue powers:

Table 4: Eigenvalues of the matrix $S_4^n(2, 1)$

j	1	2	3	4
$\left(2+2\cos\frac{j\pi}{5}\right)^n$	$\left(\sqrt{5}\alpha\right)^n$	α^{2n}	$\left(-\sqrt{5}\beta\right)^n$	β^{2n}

Using the appropriate values from Tables 2 and 4 in Eq. (28) for the specific entry $s_{21}^{(n)}(2, 1)$, we obtain:

$$\begin{aligned} s_{21}^{(n)}(2,1) &= \frac{2}{5}\left[\left(\sqrt{5}\alpha\right)^n \frac{\sqrt{5}}{4} + \alpha^{2n}\frac{\sqrt{5}}{4} + \left(-\sqrt{5}\beta\right)^n\left(\frac{-\sqrt{5}}{4}\right) + \beta^{2n}\left(\frac{-\sqrt{5}}{4}\right)\right] \\ &= \frac{1}{2}\left[\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{5}} + 5^{(n-1)/2}\left(\alpha^n - (-1)^n\beta^n\right)\right]. \end{aligned}$$

By applying Binet formulas given in Eq. (1) and evaluating separately for even and odd n, we find:

$$s_{21}^{(n)}(2,1) = \frac{1}{2}\begin{cases} F_{2n} + 5^{(n-1)/2}L_n, & \text{if } n \text{ is odd,} \\ F_{2n} + 5^{n/2}F_n, & \text{if } n \text{ is even.} \end{cases}$$

By substituting the relevant ordered pairs (h, k) into Eq. (28) and applying the values from Tables 2 and 4, the proofs of Eqs. (22)–(27) are completed. □

The matrix $S_4^n(1,1)$, as studied in [22], has entries expressible in terms of squares and sums of Fibonacci numbers. In contrast, the matrix $S_4^n(2,1)$ has entries, given in Eqs. (22)–(27), that involve both Fibonacci and Lucas numbers. This dual dependence highlights the intrinsic relationship between these two sequences, where the Lucas numbers often regarded a companion to the Fibonacci sequence contribute to the structural and algebraic complexity of $S_4^n(2,1)$.

Motivated by these observations, we now generalize the analysis to the matrix $S_4^n(F_{s+1},F_s)$ and derive closed-form expressions for its entries as follows:

Theorem 2.3. *Let F_n denote the n^{th} Fibonacci number, the entries of the matrix $S_4^n\left(F_{s+1},F_s\right)$ are*

$$s_{21}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[F_{sn}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t}\right], \tag{29}$$

$$s_{31}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[-F_{sn}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t}\right], \tag{30}$$

$$s_{41}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[-F_{sn+1}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t-1}\right], \tag{31}$$

$$s_{44}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[F_{sn+1}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t-1}\right], \tag{32}$$

$$s_{33}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[F_{sn-1}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t+1}\right], \tag{33}$$

$$s_{32}^{(n)}\left(F_{s+1},F_s\right)=\frac{1}{2}\left[-F_{sn-1}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t+1}\right] \tag{34}$$

where $\binom{n}{t}$ is the the binomial coefficient, $n\geq t\geq 0$.

Proof. From Eq. (9), the general term of the matrix $S_4^n\left(F_{s+1},F_s\right)$ is given by

$$s_{hk}^{(n)}\left(F_{s+1},F_s\right)=\frac{2}{5}\sum_{j=1}^{4}\left(F_{s+1}+2F_s\cos\frac{j\pi}{5}\right)^n\sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5}. \tag{35}$$

By applying the identities in Eqs. (4)–(6) and using the values from Table 1, we obtain eigenvalue structure presented in Table 5:

Table 5: Eigenvalues of the matrix $S_4^n(F_{s+1},F_s)$

j	1	2	3	4
$\left(F_{s+1}+2F_s\cos\frac{j\pi}{5}\right)^n$	$(F_{s+1}+\alpha F_s)^n$	α^{sn}	$(F_{s+1}+\beta F_s)^n$	β^{sn}

For the specific entry $s_{44}^{(n)}\left(F_{s+1},F_s\right)$, substituting the appropriate values from Tables 2 and 5 into Eq. (35) yields

$$\begin{aligned}s_{44}^{(n)}\left(F_{s+1},F_s\right)&=\frac{1}{2\sqrt{5}}\left[\left(F_{s+1}+\alpha F_s\right)^n(-\beta)+\alpha^{sn}\alpha+\left(F_{s+1}+\beta F_s\right)^n\alpha+\beta^{sn}(-\beta)\right]\\&=\frac{1}{2}\left[F_{sn+1}+\frac{1}{\sqrt{5}}\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}\left(\alpha^{n-t-1}-\beta^{n-t-1}\right)\right].\end{aligned}$$

The expression $s_{44}^{(n)}(F_{s+1}, F_s)$ is derived by applying Binet formula for Fibonacci numbers, as given in Eq. (1).

Repeating a similar procedure for all entries (h, k) using Eq. (35) and the eigenvalue structures in Tables 2 and 5, we obtain the results stated in Eqs. (29)–(34), completing the proof. □

The matrices $S_4(F_3, F_2)$, $S_4(F_2, F_1)$, and $S_4(F_1, F_0)$ highlight how powers of symmetric tridiagonal Toeplitz matrices reflect the underlying structure of Fibonacci numbers. In particular, the matrix $S_4^n(F_{s+1}, F_s)$ admits the following equivalent formulations:

$$S_4^n(F_{s+1}, F_s) = (F_s\, S_4(F_2, F_1) + F_{s-1}\, S_4(F_1, F_0))^n\,, \quad s > 0, \tag{36}$$

$$= (F_{s+1}\, S_4(F_2, F_1) - F_{s-1}\, S_4(F_0, F_1))^n\,. \tag{37}$$

These forms allow the derivation of finite sum identities involving Fibonacci numbers. By applying the binomial expansion to Eqs. (36) and (37), the matrix $S_4^n(F_{s+1}, F_s)$ can be expressed as:

$$S_4^n(F_{s+1}, F_s) = \sum_{k=0}^{n} \binom{n}{k} F_s^k F_{s-1}^{n-k}\, S_4^k(F_2, F_1), \tag{38}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} F_{s+1}^k F_{s-1}^{n-k} S_4^k(F_2, F_1) S_4^{n-k}(F_0, F_1). \tag{39}$$

These expansions provide a structured approach to expressing the powers of $S_4^n(F_{s+1}, F_s)$ in terms of simpler base matrices and Fibonacci coefficients, allowing for explicit computation of entries, trace, and determinant through recurrence identities and combinatorial formulations.

2.2. *Fibonacci Matrix* $S_4^n\left(F_{-s}, F_{-(s+1)}\right)$*,* $s \geq 0$

For $s = 0$, since $F_0 = 0$ and $F_{-1} = 1$, the matrix $S_4^n(F_0, F_{-1})$ coincides with $S_4^n(F_0, F_1)$, which has already been investigated in detail in [22]. When $s = 1$, taking $F_{-1} = 1$ and $F_{-2} = -1$ from Eq. 2, we arrive at the matrix $S_4^n(1, -1)$. This special case serves as a prototype for matrices of the form $S_4^n(F_{-s}, F_{-(s+1)})$, where negative-indexed Fibonacci numbers introduce alternating signs and reflect the symmetry properties inherent to the Fibonacci sequence. The analysis of such matrices provides additional insight into the behavior of tridiagonal symmetric Toeplitz structures associated with reversed Fibonacci pairs.

Theorem 2.4. *Let F_n denote the n^{th} Fibonacci numbers. The entries of the matrix $S_4^n(1, -1)$ are given by the following expressions:*

$$s_{11}^{(n)}(1, -1) = \frac{1}{2}\left[F_{2n-1} + F_{n+1}\right] \tag{40}$$

$$s_{12}^{(n)}(1, -1) = \frac{-1}{2}\left[F_{2n} + F_n\right] \tag{41}$$

$$s_{13}^{(n)}(1, -1) = \frac{1}{2}\left[F_{2n} - F_n\right] \tag{42}$$

$$s_{14}^{(n)}(1, -1) = \frac{1}{2}\left[F_{n+1} - F_{2n-1}\right] \tag{43}$$

$$s_{22}^{(n)}(1, -1) = \frac{1}{2}\left[F_{2n+1} + F_{n-1}\right] \tag{44}$$

$$s_{23}^{(n)}(1, -1) = \frac{1}{2}\left[F_{n-1} - F_{2n+1}\right] \tag{45}$$

Proof. From Eq. (9), the closed-form expression for the entries $s_{hk}^{(n)}(1,-1)$ is given by

$$s_{hk}^{(n)}(1,-1) = \frac{2}{5}\sum_{j=1}^{4}\left(1-2\cos\frac{j\pi}{5}\right)^{n}\sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5}. \tag{46}$$

By utilizing the values provided in Table 1 and the identities in Eqs. (4)–(6), we obtain the Table 6 for the eigenvalues of the matrix $S_4^n(1,-1)$:

Table 6: Eigenvalues of the matrix $S_4^n(1,-1)$

j	1	2	3	4
$\left(1-2\cos\frac{j\pi}{5}\right)^{n}$	β^n	β^{2n}	α^n	α^{2n}

Substituting the corresponding values from Tables 2 and 6 into Eq. (46) for the case $s_{11}^{(n)}(1,-1)$, we obtain

$$\begin{aligned} s_{11}^{(n)}(1,-1) &= \frac{2}{5}\left[\beta^n\left(\frac{-\sqrt{5}\beta}{4}\right)+\beta^{2n}\left(\frac{\sqrt{5}\alpha}{4}\right)+\alpha^n\left(\frac{\sqrt{5}\alpha}{4}\right)+\alpha^{2n}\left(\frac{-\sqrt{5}\beta}{4}\right)\right] \\ &= \frac{1}{2\sqrt{5}}\left[-\beta^{n+1}+\beta^{2n}\alpha+\alpha^{n+1}-\alpha^{2n}\beta\right]. \end{aligned}$$

The entry $s_{11}^{(n)}(1,-1)$ is derived using the Binet formula in Eq. (1), along with the identity $\alpha\beta = -1$.

Using similar reasoning and applying the appropriate values from Tables 2 and 6 for each pair (h,k) in Eq. (46), the proofs of Eqs. (41)–(45) follow accordingly. □

Filipponi presented several sum and difference identities involving the entries of the matrix $S_4^n(F_s,F_{s+1})$, as shown in Eqs. (11)–(16) [22]. Analogous identities can also be constructed for the matrices $S_4^n(F_{s+1},F_s)$, which exhibit distinct algebraic structures depending on the specific values of F_{s+1} and F_s. These structural properties influence key matrix invariants such as the trace and determinant, as well as various element-wise summations. In the subsequent analysis, we derive closed-form expressions for the trace, determinant, and selected row and column sums of the matrix $S_4^n(F_{s+1},F_s)$ in terms of Fibonacci and Lucas numbers, thereby uncovering further arithmetic characteristics inherent to this family of matrices.

It has been observed that the entries of the matrix $S_4^n(1,-1)$ are closely related to Fibonacci numbers via explicit formulas. Furthermore, from the expressions in Eqs. (40)–(45), it is observed that for the element wise sums and differences of the matrix $S_4^{(n)}(1,-1)$, the reduction relations involving Fibonacci and Lucas numbers are as in Table 7:

Additionally, the trace of the matrix $S_4^{(n)}(1,-1)$ can be evaluated using the eigenvalue expression given in Table 6 and taking into account the equalities from the identities in Eqs. (40)–(45), the trace, which is the sum of the diagonal elements of the matrix, is given by:

$$\mathrm{Tr}\left(S_4^{(n)}(1,-1)\right) = \sum_{j=1}^{4}\left(1-2\cos\frac{j\pi}{5}\right)^{n} = L_{2n}+L_n.$$

Furthermore, the determinant of the matrix $S_4^{(n)}(1,-1)$ is also evaluated using its eigenvalues. Since the determinant of a matrix is equal to the product of its eigenvalues, it follows that the determinant of the matrix is

$$\det\left(S_4^{(n)}(1,-1)\right) = \prod_{j=1}^{4}\left(1-2\cos\frac{j\pi}{5}\right)^{n} = 5^n.$$

Table 7: Sum and Difference Identities of the Elements in the Matrix $S_4^{(n)}(1,-1)$

$s_{11}^{(n)}(1,-1)+s_{14}^{(n)}(1,-1)=F_{n+1}$	$s_{11}^{(n)}(1,-1)-s_{14}^{(n)}(1,-1)=L_{2n}$
$s_{12}^{(n)}(1,-1)+s_{13}^{(n)}(1,-1)=-F_n$	$s_{12}^{(n)}(1,-1)-s_{13}^{(n)}(1,-1)=-F_{2n}$
$s_{22}^{(n)}(1,-1)+s_{23}^{(n)}(1,-1)=F_{n-1}$	$s_{22}^{(n)}(1,-1)-s_{23}^{(n)}(1,-1)=F_{2n+1}$
$s_{11}^{(n)}(1,-1)+s_{12}^{(n)}(1,-1)=\frac{F_{n-1}-F_{2n-2}}{2}$	$s_{11}^{(n)}(1,-1)-s_{12}^{(n)}(1,-1)=\frac{F_{2n+1}+F_{n+2}}{2}$
$s_{11}^{(n)}(1,-1)+s_{14}^{(n)}(1,-1)=\frac{F_{2n-2}+F_{n-1}}{2}$	$s_{13}^{(n)}(1,-1)-s_{14}^{(n)}(1,-1)=\frac{F_{2n+1}-F_{n+2}}{2}$
$s_{13}^{(n)}(1,-1)+s_{22}^{(n)}(1,-1)=\frac{F_{2n+2}-F_{n-2}}{2}$	$s_{22}^{(n)}(1,-1)-s_{13}^{(n)}(1,-1)=\frac{F_{2n+1}+F_{n+1}}{2}$
$s_{23}^{(n)}(1,-1)+s_{14}^{(n)}(1,-1)=\frac{L_n-L_{2n}}{2}$	$s_{14}^{(n)}(1,-1)-s_{23}^{(n)}(1,-1)=\frac{F+F_n}{2}$
$s_{22}^{(n)}(1,-1)+s_{11}^{(n)}(1,-1)=\frac{L_{2n}+L_n}{2}$	$s_{22}^{(n)}(1,-1)-s_{11}^{(n)}(1,-1)=\frac{F_{2n}+F_n}{2}$

This result highlights a remarkable property of the matrix $S_4^{(n)}(1,-1)$: despite the complexity of its entries, its determinant grows exponentially with n, scaled by a constant base of 5.

Motivated by the connection $S_4^n(F_{-s}, F_{-(s+1)})$, we set $s=2$ and use the identities $F_{-2}=-1$ and $F_{-3}=2$, as provided in Eq. (2). This yields the matrix $S_4(-1,2)$, which leads to the following result.

Theorem 2.5. *Let F_n denote the n^{th} Fibonacci number. Then, the entries of the matrix $S_4^n(F_{-2}, F_{-3})$ are given with*

$$s_{11}^{(n)}(-1,2)=\frac{(-1)^n}{2}\left(F_{3n-1}+5^{\lfloor n/2\rfloor}\right) \tag{47}$$

$$s_{12}^{(n)}(-1,2)=\frac{1}{2}\left[(-1)^{n+1}F_{3n}+5^{(n-1)/2}\left(1-(-1)^n\right)\right] \tag{48}$$

$$s_{13}^{(n)}(-1,2)=\frac{1}{2}\left[(-1)^{n}F_{3n}+5^{(n-1)/2}\left(1-(-1)^n\right)\right] \tag{49}$$

$$s_{14}^{(n)}(-1,2)=\frac{(-1)^{n+1}}{2}\left(F_{3n-1}-5^{\lfloor n/2\rfloor}\right) \tag{50}$$

$$s_{22}^{(n)}(-1,2)=\frac{1}{2}\left((-1)^{n}F_{3n+1}+5^{\lfloor n/2\rfloor}\right) \tag{51}$$

$$s_{23}^{(n)}(-1,2)=\frac{1}{2}\left((-1)^{n+1}F_{3n+1}+5^{\lfloor n/2\rfloor}\right) \tag{52}$$

where symbol $\lfloor x\rfloor$ denotes the floor function, which returns the greatest integer less than or equal to the real number x.

Proof. Using Eq. (9), the closed-form expression for the entries $s_{hk}^{(n)}(-1,2)$ is given by

$$s_{hk}^{(n)}(-1,2)=\frac{2}{5}\sum_{j=1}^{4}\left(-1+4\cos\frac{j\pi}{5}\right)^n\sin\frac{jh\pi}{5}\sin\frac{jk\pi}{5}. \tag{53}$$

Based on the identities in Eqs. (4)–(6) and the values in Table 1, the corresponding powers of the eigenvalues are listed in Table 8:

Table 8: Eigenvalues of the matrix $S_4^n(-1,2)$

j	1	2	3	4
$\left(-1+4\cos\frac{j\pi}{5}\right)^n$	$5^{n/2}$	$(-1)^n\beta^{3n}$	$(-1)^n5^{n/2}$	$(-1)^n\alpha^{3n}$

Substituting the relevant values from Tables 2 and 8 into Eq. (53) for the entry $(h,k)=(1,3)$ yields

$$s_{13}^{(n)}(-1,2)=\frac{1}{2\sqrt{5}}\left[(-1)^{n}\left(\alpha^{3n}-\beta^{3n}\right)+5^{n/2}\left(1-(-1)^{n}\right)\right].$$

This result follows directly from the Binet formula given in Eq. (1). The remaining entries in Eqs. (47)–(52) can be derived analogously by applying the appropriate eigenvalue identities and sine function products corresponding to each (h,k) pair in Eq. (53). □

The entries of the matrix $S_4^n(F_{-2},F_{-3})$ are expressed in terms of powers of 5 and the Fibonacci numbers F_{3n} and $F_{3n\pm1}$. To generalize these results, using the definition $F_{-s}=(-1)^{s+1}F_s$ in Eq. (9), we study on the matrix $S_4^n\left(F_{-s},F_{-(s+1)}\right)$ as follows:

Theorem 2.6. *Let F_n denote the n^{th} Fibonacci number. The entries of the matrix $S_4^n\left(F_{-s},F_{-(s+1)}\right)$ hold:*

$$s_{11}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(F_{(s+1)n-1}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t-1}\right), \tag{54}$$

$$s_{12}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(-F_{(s+1)n}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t}\right), \tag{55}$$

$$s_{13}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(F_{(s+1)n}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t}\right), \tag{56}$$

$$s_{14}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(-F_{(s+1)n-1}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t-1}\right), \tag{57}$$

$$s_{22}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(F_{(s+1)n+1}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t+1}\right), \tag{58}$$

$$s_{23}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{(-1)^{(s+1)n}}{2}\left(-F_{(s+1)n+1}+\sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^tF_{s+1}^{n-t}F_{n-t+1}\right), \tag{59}$$

where $\binom{n}{t}$ is the the binomial coefficient, $n\geq t\geq 0$.

Proof. From Eq. (9), the entries $s_{hk}^{(n)}\left(F_{-s},F_{-(s+1)}\right)$ are expressed as

$$s_{hk}^{(n)}\left(F_{-s},F_{-(s+1)}\right)=\frac{2}{5}\sum_{j=1}^{4}\left(F_{-s}+2F_{-(s+1)}\cos\frac{j\pi}{5}\right)^n\sin\frac{hj\pi}{5}\sin\frac{kj\pi}{5}. \tag{60}$$

By applying the identities in Eqs. (5)–(6) along with the values in Table 1, we obtain the eigenvalues listed in Table 9.

Table 9: Eigenvalues of the matrix $S_4^n\left(F_{-s},F_{-(s+1)}\right)$

j	1	2	3	4
$\left(F_{-s}+2F_{-(s+1)}\cos\frac{j\pi}{5}\right)^n$	$\left(F_{-s}+\alpha F_{-(s+1)}\right)^n$	$(-\beta)^{(s+1)n}$	$\left(F_{-s}+\beta F_{-(s+1)}\right)^n$	$(-\alpha)^{(s+1)n}$

Substituting the appropriate values from Tables 2 and 9 into Eq. (60) for the case $(h,k) = (2,3)$ yields

$$\begin{aligned} s_{23}^{(n)}\left(F_{-s}, F_{-(s+1)}\right) &= \frac{(-1)^{(s+1)n}}{2\sqrt{5}}\left[\left(F_s - \alpha F_{s+1}\right)^n \alpha + \beta^{(s+1)n}\beta + \left(F_s - \beta F_{s+1}\right)^n(-\beta) + \alpha^{(s+1)n}(-\alpha)\right] \\ &= \frac{(-1)^{(s+1)n}}{2}\left[-F_{(s+1)n+1} + \sum_{t=0}^{n}\binom{n}{t}(-1)^{n-t}F_s^t F_{s+1}^{n-t}\frac{\alpha^{n-t+1}-\beta^{n-t+1}}{\sqrt{5}}\right]. \end{aligned}$$

The desired expression is thus obtained using the Binet formula from Eq. (1). Applying the same reasoning and substitutions from Tables 2 and 9 for the remaining index pairs in Eqs. (54)–(59), the proof is completed. □

Extending our analysis to matrices of the form $S_4^n\left(F_{-s}, F_{-(s+1)}\right)$, constructed from negatively indexed Fibonacci numbers, we derive closed-form expressions involving the matrices $S_4(F_0, F_{-1})$ and $S_4(F_{-1}, F_{-2})$ (or equivalently, $S_4(F_{-2}, F_{-1})$). More precisely, by applying the binomial theorem to the right-hand sides of Eqs. (61) and (63), we arrive at the expanded forms given in Eqs. (62) and (64).

$$S_4^n\left(F_{-s}, F_{-(s+1)}\right) = (-1)^{(s+1)n}\left[F_s S_4(F_{-1}, F_{-2}) - F_{s-1} S_4(F_0, F_{-1})\right]^n, \quad s > 0, \tag{61}$$

$$= (-1)^{(s+1)n}\sum_{k=0}^{n}\binom{n}{k}(F_s)^k(-F_{s-1})^{n-k} S_4^k(F_{-1}, F_{-2}) S_4^{n-k}(F_0, F_{-1}), \tag{62}$$

$$= (-1)^{sn}\left[F_s S_4(F_{-2}, F_{-1}) + F_{s-1} S_4(F_0, F_{-1})\right]^n, \tag{63}$$

$$= (-1)^{sn}\sum_{k=0}^{n}\binom{n}{k}(F_s)^k(F_{s-1})^{n-k} S_4^k(F_{-2}, F_{-1}) S_4^{n-k}(F_0, F_{-1}). \tag{64}$$

These identities provide novel representations for $S_4^n\left(F_{-s}, F_{-(s+1)}\right)$, thereby enriching our understanding of the algebraic structure and combinatorial properties inherent in such Fibonacci matrices.

2.3. *The Fibonacci Matrix $S_4^n(F_{-(s+1)}, F_{-s})$, $s \geq 0$*

For the base case $s = 0$, we have $S_4^n(F_{-1}, F_0) = I_4$, the 4×4 identity matrix, which requires no further analysis. When $s = 1$, it is observed that $S_4(F_{-2}, F_{-1})$ is equivalent to $(-1)S_4(F_{-1}, F_{-2})$, as established in Theorem 2.4. Accordingly, the entries of the matrix $S_4^n(-1,1)$ satisfy the relation $S_4^n(-1,1) = (-1)^n S_4^n(1,-1)$, with explicit formulas for the entries provided in Eqs. (40)–(45).

For the case $s = 2$, we consider the ordered pair $(x, y) = (F_{-3}, F_{-2})$ and investigate the matrix $S_4^n(2,-1)$:

Theorem 2.7. *Let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. Then, the entries of the matrix $S_4^n(2,-1)$ are given by:*

$$s_{41}^{(n)}(2,-1) = \frac{1}{2}\left[F_{2n+1} - 5^{\lfloor n/2\rfloor}\left(F_{n-1}\left(\tfrac{1+(-1)^n}{2}\right) + L_{n-1}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{65}$$

$$s_{43}^{(n)}(2,-1) = \frac{1}{2}\left[-F_{2n} - 5^{\lfloor n/2\rfloor}\left(F_{n}\left(\tfrac{1+(-1)^n}{2}\right) + L_{n}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{66}$$

$$s_{44}^{(n)}(2,-1) = \frac{1}{2}\left[F_{2n+1} + 5^{\lfloor n/2\rfloor}\left(F_{n-1}\left(\tfrac{1+(-1)^n}{2}\right) + L_{n-1}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{67}$$

$$s_{31}^{(n)}(2,-1) = \frac{1}{2}\left[-F_{2n} + 5^{\lfloor n/2\rfloor}\left(F_{n}\left(\tfrac{1+(-1)^n}{2}\right) + L_{n}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{68}$$

$$s_{32}^{(n)}(2,-1) = \frac{1}{2}\left[F_{2n-1} - 5^{\lfloor n/2\rfloor}\left(F_{n+1}\left(\tfrac{1+(-1)^n}{2}\right) + L_{n+1}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{69}$$

$$s_{33}^{(n)}(2,-1)=\frac{1}{2}\left[F_{2n-1}+5^{\lfloor n/2\rfloor}\left(F_{n+1}\left(\tfrac{1+(-1)^n}{2}\right)+L_{n+1}\left(\tfrac{1-(-1)^n}{2}\right)\right)\right] \tag{70}$$

where symbol $\lfloor x\rfloor$ denotes the greatest integer less than or equal to a real number x, commonly referred to as the floor function.

Proof. For the matrix $S_4^n(2,-1)$, by using the general expression for matrix entries from Eq. (9), we rewrite the terms as follows:

$$s_{hk}^{(n)}(2,-1)=\frac{2}{5}\sum_{j=1}^{4}\left(2-2\cos\frac{j\pi}{5}\right)^n\sin\frac{hj\pi}{5}\sin\frac{kj\pi}{5}. \tag{71}$$

By employing the identities in Eqs. (4)–(6) along with the values from Table 1, we obtain the eigenvalue powers presented in Table 10.

Table 10: Eigenvalues of the matrix $S_4^n(2,-1)$

j	1	2	3	4
$\left(2-2\cos\frac{j\pi}{5}\right)^n$	β^{2n}	$\left(-\sqrt{5}\beta\right)^n$	α^{2n}	$\left(\sqrt{5}\alpha\right)^n$

Substituting the appropriate values from Tables 2 and 10 into Eq. (71) for the ordered pair $(h,k)=(4,3)$, we obtain:

$$\begin{aligned}s_{43}^{(n)}(2,-1)&=\frac{2}{5}\left[\beta^{2n}\tfrac{\sqrt{5}}{4}+\left(-\sqrt{5}\beta\right)^n\tfrac{\sqrt{5}}{4}+\alpha^{2n}\left(-\tfrac{\sqrt{5}}{4}\right)+\left(\sqrt{5}\alpha\right)^n\left(-\tfrac{\sqrt{5}}{4}\right)\right]\\&=\frac{1}{2}\left[-\frac{\alpha^{2n}-\beta^{2n}}{\sqrt{5}}-5^{(n-1)/2}\left(\alpha^n-(-1)^n\beta^n\right)\right].\end{aligned}$$

Using Binet formulas in Eq. (1), and noting the parity of n, we derive:

$$s_{43}^{(n)}(2,-1)=\begin{cases}\frac{1}{2}\left(-F_{2n}-5^{\frac{n-1}{2}}L_n\right), & \text{if } n \text{ is odd},\\ \frac{1}{2}\left(-F_{2n}-5^{\frac{n}{2}}F_n\right), & \text{if } n \text{ is even}.\end{cases}$$

Finally, by applying the corresponding values for other ordered pairs (h,k) from Tables 2 and 10 into Eq. (71), the remaining results in Eqs. (65)–(70) are verified, thus completing the proof. □

The entries of the matrix $S_4^n(F_{-3},F_{-2})$ involve not only powers of 5 and the Fibonacci numbers F_{2n} and $F_{2n\pm1}$, but also elements from both the Fibonacci sequence F_n and the Lucas sequence L_n. Motivated by this rich structure, we now generalize the formulation to a broader class of matrices defined by negatively indexed Fibonacci pairs.

Utilizing the identity $F_{-s}=(-1)^{s+1}F_s$, and applying Eq. (9) to the ordered pair $(x,y)=\left(F_{-(s+1)},F_{-s}\right)$ for $s>0$, the matrix $S_4^n\left(F_{-(s+1)},F_{-s}\right)$ can be expressed as follows:

Theorem 2.8. *Let F_n denote the n^{th} Fibonacci number. Then, for any integer $s>0$, the entries of the matrix $S_4^n\left(F_{-(s+1)},F_{-s}\right)$ are given by:*

$$s_{21}^{(n)}\left(F_{-(s+1)},F_{-s}\right)=\frac{(-1)^{sn}}{2}\left[-F_{sn}-\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t}\right] \tag{72}$$

$$s_{22}^{(n)}\left(F_{-(s+1)},F_{-s}\right)=\frac{(-1)^{sn}}{2}\left[F_{sn-1}+\sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_s^{n-t}F_{n-t+1}\right] \tag{73}$$

$$s_{23}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) = \frac{(-1)^{sn}}{2}\left[F_{sn-1} - \sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_{s}^{n-t}F_{n-t+1}\right] \tag{74}$$

$$s_{41}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) = \frac{(-1)^{sn}}{2}\left[F_{sn+1} - \sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_{s}^{n-t}F_{n-t-1}\right] \tag{75}$$

$$s_{42}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) = \frac{(-1)^{sn}}{2}\left[-F_{sn} + \sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_{s}^{n-t}F_{n-t}\right] \tag{76}$$

$$s_{44}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) = \frac{(-1)^{sn}}{2}\left[F_{sn+1} + \sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_{s}^{n-t}F_{n-t-1}\right] \tag{77}$$

where $\binom{n}{t}$ *is the the binomial coefficient,* $n \geq t \geq 0$.

Proof. For $s > 0$, the entries $s_{hk}^{(n)}\left(F_{-(s+1)}, F_{-s}\right)$ from Eq. (9) are written as

$$s_{hk}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) = \frac{2}{5}\sum_{j=1}^{4}\left(F_{-(s+1)} + 2F_{-s}\cos\frac{j\pi}{5}\right)^{n}\sin\frac{hj\pi}{5}\sin\frac{kj\pi}{5}. \tag{78}$$

Using the identities from Eqs. (4)–(6) and the eigenvalue representations in Table 1, the eigenvalues corresponding to $S_4^n(F_{-(s+1)}, F_{-s})$ are arranged as shown in Table 11.

Table 11: Eigenvalues of the matrix $S_4^n\left(F_{-(s+1)}, F_{-s}\right)$ for $s > 0$

j	1	2	3	4
$\left(F_{-(s+1)} + 2F_{-s}\cos\frac{j\pi}{5}\right)^n$	$(-\beta)^{sn}$	$\left(F_{-(s+1)} - \beta F_{-s}\right)^n$	$(-\alpha)^{sn}$	$\left(F_{-(s+1)} - \alpha F_{-s}\right)^n$

Substituting the appropriate values from Tables 2 and 11 into Eq. (78) for the case $(h, k) = (2, 2)$, and applying Binet formula from Eq. (1), we obtain the following:

$$\begin{aligned} s_{22}^{(n)}\left(F_{-(s+1)}, F_{-s}\right) &= \frac{(-1)^{sn}}{2\sqrt{5}}\left[\beta^{sn}\alpha + (F_{s+1} + \beta F_s)^n(-\beta) + \alpha^{sn}(-\beta) + (F_{s+1} + \alpha F_s)^n\alpha\right] \\ &= \frac{(-1)^{sn}}{2\sqrt{5}}\left[\alpha^{sn-1} - \beta^{sn-1} + \sum_{t=0}^{n}\binom{n}{t}F_{s+1}^{t}F_{s}^{n-t}\left(\alpha^{n-t+1} - \beta^{n-t+1}\right)\right]. \end{aligned}$$

Finally, by applying the appropriate (h, k) values from Tables 2 and 11 into Eq. (78), the proofs of Eqs. (72)–(77) are established. □

Furthermore, the matrix $S_4^n\left(F_{-(s+1)}, F_{-s}\right)$ for $s > 0$ can be represented in two distinct forms:

$$\begin{aligned} S_4^n\left(F_{-(s+1)}, F_{-s}\right) &= (-1)^{(s+1)n}\left[F_s S_4(F_{-2}, F_{-1}) - F_{s-1}I_4\right]^n \\ &= (-1)^{(s+1)n}\sum_{t=0}^{n}\binom{n}{t}(-F_{s-1})^{n-t}F_s^t S_4^t(F_{-2}, F_{-1}), \\ &= (-1)^{sn}\left[F_s S_4(F_{-1}, F_{-2}) + F_{s-1}I_4\right]^n \\ &= (-1)^{sn}\sum_{t=0}^{n}\binom{n}{t}F_{s-1}^{n-t}F_s^t S_4^t(F_{-1}, F_{-2}). \end{aligned}$$

These binomial expansions reveal that the powers of the matrix $S_4^n\left(F_{-(s+1)}, F_{-s}\right)$ are expressible as linear combinations of powers of fixed matrices, where the coefficients involve Fibonacci numbers. This formulation encapsulates the recursive structure inherent in the matrix and underscores the role of Fibonacci and Lucas numbers in determining the evolution of its entries across powers.

3. Numerical Illustrations

In this section, we present numerical results that illustrate Theorems 2.2 and 2.6. Specifically, we demonstrate the computation of the matrix powers $S_4^n(F_3, F_2)$ for $n = 5$ and $n = 6$, as well as $S_4^n(F_{-s}, F_{-(s+1)})$ for $s = 2$ and $n = 4$. Using standard matrix multiplication, we also compare the number of arithmetic operations—namely, multiplications and additions—performed between matrix elements in each case.

3.1. Numerical Evaluation of Theorem 2.2 for $n = 5$ and $n = 6$

To illustrate the applicability of Theorem 2.2, we evaluate the explicit expressions for the entries of the matrix $S_4^n(2, 1)$ for two specific values: $n = 5$ and $n = 6$. The necessary Fibonacci and Lucas numbers used in these computations are as follows:

- For $n = 5$: $F_5 = 5$, $F_{10} = 55$, $F_4 = 3$, $F_6 = 8$, $F_{11} = 89$, $F_9 = 34$; $L_5 = 11$, $L_4 = 7$, $L_6 = 18$, $\left\lfloor \frac{5}{2} \right\rfloor = 2$, and $5^2 = 25$.

$$s_{21}^{(5)}(2,1) = \frac{1}{2}\left[F_{10} + 25 \cdot (F_5 \cdot 0 + L_5 \cdot 1)\right] = \frac{1}{2}\left[55 + 25 \cdot 11\right] = \frac{1}{2}\left[55 + 275\right] = 165$$
$$s_{31}^{(5)}(2,1) = \frac{1}{2}\left[-F_{10} + 25 \cdot L_5\right] = \frac{1}{2}\left[-55 + 275\right] = 110$$

$$s_{41}^{(5)}(2,1) = \frac{1}{2}\left[-F_{11} + 25 \cdot L_4\right] = 43 \qquad s_{44}^{(5)}(2,1) = \frac{1}{2}\left[F_{11} + 25 \cdot L_4\right] = 132$$
$$s_{33}^{(5)}(2,1) = \frac{1}{2}\left[F_9 + 25 \cdot L_6\right] = 242 \qquad s_{32}^{(5)}(2,1) = \frac{1}{2}\left[-F_9 + 25 \cdot L_6\right] = 208$$

- For $n = 6$: $F_6 = 8$, $F_{12} = 144$, $F_5 = 5$, $F_7 = 13$, $F_{13} = 233$, $F_{11} = 89$; $L_6 = 18$, $L_5 = 11$, $L_7 = 29$, $\left\lfloor \frac{6}{2} \right\rfloor = 3$, and $5^3 = 125$.

$$s_{21}^{(6)}(2,1) = \frac{1}{2}\left[F_{12} + 125 \cdot F_6\right] = 572 \qquad s_{31}^{(6)}(2,1) = \frac{1}{2}\left[-F_{12} + 125 \cdot F_6\right] = 428$$
$$s_{41}^{(6)}(2,1) = \frac{1}{2}\left[-F_{13} + 125 \cdot F_5\right] = 196 \qquad s_{44}^{(6)}(2,1) = \frac{1}{2}\left[F_{13} + 125 \cdot F_5\right] = 429$$
$$s_{33}^{(6)}(2,1) = \frac{1}{2}\left[F_{11} + 125 \cdot F_7\right] = 857 \qquad s_{32}^{(6)}(2,1) = \frac{1}{2}\left[-F_{11} + 125 \cdot F_7\right] = 768$$

These numerical results confirm both the correctness and the practical computability of the closed-form expressions presented in Theorem 2.2. Furthermore, they demonstrate that the structure of the entries in $S_4^n(2, 1)$ is highly sensitive to the parity of n, as reflected in the alternating appearance of the Fibonacci and Lucas numbers.

3.2. Numerical Evaluation of Theorem 2.6 for $s = 2$ and $n = 4$

We now illustrate Theorem 2.6 by evaluating the matrix $S_4^n(F_{-s}, F_{-(s+1)})$ for $s = 2$ and $n = 4$. Using the identity $F_{-n} = (-1)^{n+1}F_n$, we obtain $F_{-2} = (-1)^3F_2 = -1, \quad F_{-3} = (-1)^4F_3 = 2$. We substitute these values into the closed-form expression given in Theorem 2.6. The individual entries of the resulting matrix $S_4^4(F_{-2}, F_{-3})$ are computed as follows:

$$s_{11}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(F_{11} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t-1}\right) = 57,$$

$$s_{12}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(-F_{12} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t}\right) = -72,$$

$$s_{13}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(F_{12} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t}\right) = 72,$$

$$s_{14}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(-F_{11} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t-1}\right) = -32,$$

$$s_{22}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(F_{13} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t+1}\right) = 129,$$

$$s_{23}^{(4)}(-1,2) = \frac{(-1)^{12}}{2}\left(-F_{13} + \sum_{t=0}^{4}\binom{4}{t}(-1)^{4-t}(-1)^t 2^{4-t}F_{4-t+1}\right) = -104.$$

This example demonstrates the practical application of Theorem 2.6 for computing specific entries of the matrix $S_4^n(x, y)$, where $x = F_{-2}$ and $y = F_{-3}$ are negative-indexed Fibonacci numbers. These values are considered within the framework of Theorem 2.5.

3.3. Computational Aspects of omputing $S_4^n(2, 1)$

The Fibonacci and Lucas numbers required in the closed-form expressions can be computed efficiently using a variety of methods. The most straightforward technique is the recursive definition, though it incurs exponential time complexity if not optimized with memoization or dynamic programming. A significantly faster alternative is the iterative approach, which computes all terms up to F_n or L_n in linear time using a simple loop. Specifically, the n-th Fibonacci number can be calculated iteratively via the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. This method requires exactly $n - 1$ addition operations for $n \geq 2$, since each term from F_2 to F_n is obtained through a single addition.

For improved efficiency—particularly when dealing with large values of n—one can employ the Binet formulas provided in Eq. (1). These closed-form expressions enable constant-time approximations, although attention must be paid to numerical stability and floating-point precision. In practical computational settings, it is often advantageous to precompute and store Fibonacci and Lucas numbers in lookup tables, especially when repeated access is required.

To evaluate the computational cost of raising the matrix $S_4(2, 1)$ to a power, we consider three distinct methods:

Closed-form expressions: This method circumvents full matrix multiplication and is particularly efficient when only specific entries of the matrix are needed. It leverages the symmetry of the six independent elements of the symmetric matrix $S_4^n(2, 1)$, thereby reducing computational redundancy. The evaluation relies solely on the closed-form expressions of Fibonacci and Lucas numbers, which further simplifies arithmetic operations. Instead of computing the entire matrix, only the required entries of $S_4^n(2, 1)$ are calculated using the formulas provided in Theorem 2.2.

Standard matrix multiplication: The matrix power $S_4^n(2, 1)$ is computed iteratively by performing $n - 1$ successive multiplications of the base 4×4 matrix. Each multiplication requires 64 scalar multiplications and

48 scalar additions, amounting to 112 arithmetic operations per step. As a result, the total computational cost increases linearly with n, requiring $112(n-1)$ arithmetic operations in total.

Optimized multiplication: Matrix powers are computed more efficiently by exploiting the associativity of matrix multiplication. Instead of performing $n-1$ successive multiplications, the power $S_4^n(2,1)$ is evaluated using techniques such as repeated squaring or grouped products—for instance, $S_4^6(2,1) = (S_4^2(2,1))^3$ or $S_4^{10}(2,1) = (S_4^5(2,1))^2$. This approach significantly reduces the total number of required multiplications. For example:

- When $n = 5$, the matrix power is computed as $S^2 \cdot S^2 \cdot S$, requiring three matrix multiplications.
- When $n = 6$, we first compute S^2 and then raise it to the third power: $(S^2)^3$, which involves only two additional multiplications after computing S^2.
- When $n = 10$, we compute S^5 and then square it: $(S^5)^2$, significantly reducing the number of operations compared to performing nine consecutive multiplications.

By strategically structuring the computation based on the value of n, this method reduces the total number of matrix multiplications, offering improved computational efficiency over direct repeated multiplication. These examples illustrate how the use of closed-form expressions and optimized multiplication techniques facilitates fast and efficient evaluation of matrix powers.

Table 12 presents the computational cost associated with three different methods for computing powers of the matrix $S_4(2,1)$ across various values of n: closed-form expressions, standard matrix multiplication, and optimized multiplication.

Table 12: Asymptotic operation counts for computing $S_4^n(2,1)$

Power n	**Closed-form expressions**	**Standard multiplication**	**Optimized multiplication**
5	72	448	336
6	72	560	224
8	72	784	336
10	72	1008	448
15	72	1568	672
20	72	2240	784
25	72	2800	896

The closed-form approach, based on Theorem 2.2, exhibits the highest efficiency by computing only the required entries of the matrix using identities involving Fibonacci and Lucas numbers. In scenarios where only selected matrix elements are needed—such as in cryptographic or numerical applications—this method significantly minimizes arithmetic complexity, maintaining a constant number of operations regardless of the matrix exponent n. In contrast, the standard approach computes the full matrix power through repeated 4×4 matrix multiplications. This method incurs a linearly increasing number of arithmetic operations, requiring 448 and 560 operations for $n = 5$ and $n = 6$, respectively. Since each multiplication involves 64 scalar multiplications and 48 additions, the method becomes increasingly inefficient as n grows.

The third method improves performance by exploiting the associativity of matrix multiplication to reorganize the computation. For instance, computing $(S^2)^3$ instead of performing six consecutive multiplications reduces the number of matrix products. As a result, the operation count drops to 336 for $n = 5$ and 224 for $n = 6$. This optimization reflects common strategies used in numerical linear algebra libraries (e.g., BLAS, LAPACK), where efficient operation ordering and reuse are essential to minimizing computational complexity and enhancing performance.

In summary, the closed-form method provides the most efficient solution when only partial matrix information is required, whereas optimized multiplication significantly reduces computational cost in full matrix power evaluations, particularly for large values of n.

4. Dynamic Matrix-Based Key Generation Based on $S_4^n(x, y)$ for Hill and Affine Hill Ciphers

This section presents a novel dynamic key generation scheme for both the classical Hill cipher and its affine variant, the Affine Hill Cipher, by leveraging parameterized 4×4 symmetric matrices denoted as $S_4^n(x, y)$. These matrices are dynamically adapted based on plaintext characteristics and temporal context, thereby enhancing both the cryptographic strength and the adaptability of the cipher system. For a comprehensive treatment of this topic, see [26].

4.1. Parameter Definition for (x, y, s, n)

The temporal variable n is computed as a function of the current date, for example: $n = ((\text{day} + \text{month}) \times \text{year}) \bmod 29$, and is used as the exponent in the key matrix to introduce day-based variability into the encryption process.

The key generation mechanism incorporates two primary variables: s and n. The variable s serves a dual purpose. First, it is used to map a designated plaintext character (typically the first character) to a numerical value based on the extended English alphabet: $A = s$, $B = s + 1$, ..., $Z = s + 25$, space $= s + 26$, comma $= s + 27$, period $= s + 28$. Second, s is defined as the number of plaintext blocks obtained by partitioning the message into 4×1 vectors. That is, if the plaintext is divided into m such blocks, then: $s = m$. This assignment ensures that s reflects the structural length of the input, thereby dynamically influencing the Fibonacci-based parameter selection used in constructing the key matrix.

$$s = \begin{cases} m, & \text{if } m \leq 29 \\ m \bmod 29, & \text{if } m > 29 \end{cases}$$

This conditional assignment ensures that s remains within a computationally manageable range, particularly in the context of Fibonacci-based parameterization, where excessively large indices may result in computational inefficiencies or potential overflow. Moreover, the alphabetic characters used in the encryption scheme are numerically mapped using the modular expression $(s + k) \bmod 29$, where $k \in \{1, 2, \ldots, 28\}$. This mapping introduces a dynamic offset that depends on the structure of the plaintext, thereby effectively shifting the alphabet. As a result, it enhances the variability of the numeric representation of characters across different encryption sessions, contributing to the overall robustness of the encryption process.

The matrix $S_4^n(x, y)$ maintains a limited number of independent parameters while preserving structural flexibility through the dynamic variables x, y, and n. In this construction, the parameters x and y are not selected arbitrarily; rather, they are specifically derived from Fibonacci numbers based on the plaintext-dependent index s. This mapping introduces nonlinear growth and structural diversity into the key space, thereby enhancing the cryptographic complexity of the system.

4.2. Matrix Type Selection Based on Calendar Modulo Mapping

To further enhance diversity in key construction, four structurally distinct key matrices K_0, K_1, K_2, and K_3 are generated. Each matrix is defined as a specific power n of the base matrix $S_4(x, y)$, where the parameters x and y are derived using both positive and negative Fibonacci indices:

$$\begin{aligned} K_0 &= S_4^n(F_s, F_{s+1}), & K_1 &= S_4^n(F_{s+1}, F_s), \\ K_2 &= S_4^n(F_{-s}, F_{-(s+1)}), & K_3 &= S_4^n(F_{-(s+1)}, F_{-s}) \end{aligned} \tag{79}$$

where F_i denotes the i^{th} Fibonacci number. This bidirectional indexing introduces structural inversion and numerical asymmetry across the matrix types. This approach introduces asymmetry and a broader range of variability into the key generation process, thereby strengthening the resistance of the cryptosystem against structural cryptanalysis.

The matrix used for encryption on a given day is selected based on the calendar day modulo 4:

$$K_{\text{type}} = K_{\text{day mod } 4}. \tag{80}$$

This cyclic yet non-repetitive selection mechanism ensures temporal variability in key usage, thereby reducing susceptibility to statistical and pattern-based attacks. Each matrix K_j, where $j \in \{0, 1, 2, 3\}$, is constructed through a unique mapping $f_j(s, n)$, embedding both plaintext-contextual and temporal entropy into the encryption process.

By integrating Fibonacci-based parameterization with day-dependent exponentiation, the proposed method adheres to the principles of dynamic symmetric encryption—enhancing unpredictability, expanding the key space complexity, and providing increased resilience against classical cryptanalytic techniques.

4.3. Encryption and Decryption

Based on the dynamically generated key matrices K_j, defined as powers of symmetric matrices $S_4(x, y)$, we formulate the encryption and decryption processes for both Hill and Affine Hill ciphers. The plaintext is first partitioned into 4-dimensional column vectors P_i, $i = \{1, 2, ..., m\}$, over $\mathbb{Z}_{29}$, where each element corresponds to the index of a character in the English alphabet. For further details on this topic, refer to [26].

Hill Cipher Let $P_i \in \mathbb{Z}_{29}^{4\times1}$, $i = \{1, 2, ..., m\}$ be a plaintext vector, and let $K_j \in \mathbb{Z}_{29}^{4\times4}$, $j = \{0, 1, 2, 3\}$ be the encryption matrix selected as described in Eqs. (79) and (80). The ciphertext vector $C_i \in \mathbb{Z}_{29}^{4\times1}$ is computed via standard Hill cipher encryption as follows:

$$C_i = K_j \cdot P_i \bmod 29.$$

Hill Decryption

Assuming that K_j is invertible over $\mathbb{Z}_{29}$, the original plaintext vector P_i can be recovered by computing the modular inverse K_j^{-1} as follows:

$$P_i = K_j^{-1} \cdot C_i \bmod 29.$$

Affine Hill Cipher We propose a modified Affine Hill cipher scheme, in which each plaintext vector $P_i \in \mathbb{Z}_{29}^{4\times1}$ is encrypted using a single column vector selected from the matrix $K_j = S_4^n(x, y) \in \mathbb{Z}_{29}^{4\times4}$, according to the rule:

$$t = \begin{cases} n \bmod 4, & \text{if } n \bmod 4 \neq 0 \\ 4, & \text{if } n \bmod 4 = 0 \end{cases} \tag{81}$$

Let the selected t^{th} column vector from K_j be denoted as $k_j^t \in \mathbb{Z}_{29}^{4\times1}$, where t is determined in Eq. (81). The encryption process uses this column vector in an affine transformation, incorporating it as a dynamically changing scalar weight vector.

Affine Hill Encryption Given a plaintext vector $P_i \in \mathbb{Z}_{29}^{4\times1}$, the corresponding ciphertext vector $C_i \in \mathbb{Z}_{29}^{4\times1}$ is computed as:

$$C_i = K_j \cdot P_i + k_j^t \bmod 29.$$

Here, K_j is the selected matrix as defined in the matrix selection mechanism based on the day, and k_j^t is the t-th column of K_j where $t = n \bmod 4$ (with $t = 4$ if $n \equiv 0 \bmod 4$).

Affine Hill Decryption To recover the plaintext vector P_i from the ciphertext C_i, the inverse matrix $K_j^{-1} \in \mathbb{Z}_{29}^{4\times4}$ is used (assuming K_j is invertible), and the corresponding column vector k_j^t is subtracted prior to matrix inversion:

$$P_i = K_j^{-1} \cdot (C_i - k_j^t) \bmod 29.$$

This approach introduces a hybrid encryption structure in which the affine component is dynamically generated for each plaintext block by selecting a specific column from an exponentiated symmetric key matrix, based on modular arithmetic. By combining a fixed linear transformation (via the key matrix) with a variable, column-indexed offset determined by the temporal parameter n, the system achieves both structural and temporal entropy.

This design ensures that each plaintext block is associated with a unique affine shift, thereby enhancing ciphertext variability even in the presence of repeated plaintext inputs. As a result, the scheme strengthens resistance against classical linear cryptanalysis while maintaining computational efficiency, thus offering a robust and adaptable cryptographic framework.

4.4. Discussion of Results

The integration of matrix exponentiation, $K_j = S_4^n(x, y)$, in conjunction with Fibonacci-based parameterization and calendar-driven matrix rotation, significantly enhances the cryptographic strength of both Hill and Affine Hill ciphers. By dynamically adjusting the key matrix on a daily basis and embedding plaintext-dependent variables, the proposed approach introduces multidimensional entropy into the system.

A notable contribution of this scheme is the dynamic affine vector k_j^t, selected from the matrix K_j using a modular column selection strategy. This vector serves as an additional layer of complexity, effectively obscuring the linear relationship between plaintext and ciphertext. Consequently, the system demonstrates improved resistance against both linear and differential cryptanalysis. The cryptographic advantages of the proposed design can be summarized as follows:

Temporal variability: Achieved through the exponentiation index n, which is tied to the current date.

Plaintext-adaptive behavior: Introduced via the parameter s, which reflects the structure and length of the input message.

Structural diversification: Maintained by alternating among four matrix types using a modular, day-based selection mechanism.

These features collectively contribute to increased unpredictability and reduced susceptibility to known-plaintext and ciphertext-only attacks. The parametrized symmetric matrix design substantially enlarges the key space while preserving computational efficiency, ensuring the method's practicality in real-world cryptographic applications.

In conclusion, the proposed method offers a robust and scalable enhancement to classical Hill-type encryption schemes by embedding adaptive, temporal, and structural variability at the core of the key generation and encryption process.

5. Conclusions and Recommendations

This study proposes a method for identifying values of x and y for which the matrix $S_4^n(x, y)$ is regarded as a generalized Fibonacci matrix. In Chapter 2, closed-form expressions were derived for specific and generalized entries of $S_4^n(x, y)$, particularly for the ordered pairs (F_{s+1}, F_s), $(F_{-s}, F_{-(s+1)})$, and $(F_{-(s+1)}, F_{-s})$. Examples following each theorem were included to support the analysis of sum and difference relationships between matrix entries. When examined through matrix norms or algebraic operations, these expressions offer deeper insight into the structural properties of the matrices. Moreover, the Fibonacci-based constructions introduced in this work possess the potential to yield identities analogous to those of Fibonacci and Lucas numbers, thereby enriching the algebraic theory of recursive matrix families.

In addition to the theoretical developments, the proposed framework enhances classical Hill-type ciphers by incorporating dynamic key evolution, Fibonacci-based parameterization, and calendar-driven matrix modulation. These features collectively introduce temporal and structural entropy into the encryption process, increasing resilience against conventional attacks while maintaining computational efficiency.

Future work may extend this research by exploring generalized matrix constructions based on higher-order recurrence sequences or negative-indexed Lucas numbers. It would also be valuable to adapt the cryptographic component to lightweight protocols suitable for constrained environments, and to perform formal security analyses under a variety of threat models. Such investigations may further validate the robustness and broaden the applicability of Fibonacci matrices in modern cryptographic systems.

Acknowledgment. We wish to thank the blinded reviewers in advance for their valuable feedback and constructive comments, which we believe will contribute significantly to enhancing the quality of this manuscript.

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