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On $\jmath$-flat modules and $\jmath$-von Neumann regular rings

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Abstract. Let R be a commutative ring with a non-zero identity and $\jmath$ be an ideal R. In this paper, we introduce and investigate the concepts of $\jmath$-torsion modules, $\jmath$-torsion free modules, $\jmath$-flat modules and $\jmath$-von Neumann regular rings. Many examples, characterizations, and properties of these notions are given. Moreover, we use them to characterize reduced rings and ZN-rings.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such as a ring. $Nil(R)$, denotes the set of all nilpotent elements of R; and $Z(R)$ denotes the the set of all zero-divisors of R. Recall that a ring R is said to be a ZN-ring if $Z(R) = Nil(R)$. An ideal I of R is said to be a nonnil ideal if $I \not\subseteq Nil(R)$.

Recall from [9, 12] that a prime ideal P of R is called a divided prime if it is comparable to every ideal of R. Set $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. The class of ϕ-rings is a good extension of integral domains to commutative rings with zero-divisors. We recommend [3, 4, 7, 10, 13, 21, 23, 25] for the study of the ring-theoretic characterizations on ϕ-rings.

Let M be an R-module. Set

$$\phi - tor(M) = \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus Nil(R)\}.$$

If ϕ-$tor(M) = M$, then M is called a ϕ-torsion module, and if ϕ-$tor(M) = 0$, then M is said to be a ϕ-torsion free module. Recall from [28] that an R-module F is said to be ϕ-flat, if for every R-monomorphism $f : A \longrightarrow B$ with Coker$f$ is a ϕ-torsion R-module, we have $1_F \otimes_R f : F \otimes_R A \longrightarrow F \otimes_R B$ is an R-monomophism; equivalently, $\mathrm{Tor}_1^R(F, M) = 0$ for every ϕ-torsion R-module M. Suitable background on ϕ-flat modules is [17–20, 26, 27].

The main purpose of this paper is to introduce and investigate the notions of $\jmath$-torsion module, $\jmath$-torsion

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free modules and $\jmath$-flat modules. Let $\jmath$ be an ideal of R, set $R(\jmath) = \{I \mid I$ ideals of R such that $I \not\subseteq \jmath\}$. If $I \in R(\jmath)$, then I is called a $\jmath$-ideal. An R-module M is said to be a $\jmath$-torsion module if $\jmath$-tor$(M) = M$, where $\jmath$-tor$(M) := \{x \in M \mid Ix = 0$ for some $I \in R(\jmath)\}$. On the other hand, M is called a $\jmath$-torsion free module if $\jmath$-tor$(M) = 0$. This note is organized as follows. The second section is dedicated to a number of results concerning $\jmath$-torsion and $\jmath$-torsion free modules. Among many results of this part, we prove in Proposition 2.1 that an ideal $\jmath$ of R is irreducible if and only $\jmath$-tor(M) is a submodule for every (2-generated) R-module M. In addition, we give several characterizations of $\jmath$-torsion and $\jmath$-torsion free modules (see Theorems 2.2 and 2.11). Also, recall from [1] that an R-module M satisfies strong Property A if for any $r_1, \dots r_n \in Z_R(M)$, there exists a nonzero $x \in M$ such that $r_1x = r_2x = \cdots = r_nx = 0$. In this context, D. D. Anderson and S. Chun asked the following question: what R-modules are the homomorphic image of an R-module satisfying strong Property A ? [1, Question 4.4(1)]. We prove in Proposition 2.4 that the rings R in which every module is the homomorphic image of a module satisfying strong Property A are exactly $\jmath$-torsion free rings. The third section deals the notion of $\jmath$-flat modules. Let R be a ring. An R-module M is said to be $\jmath$-flat for some ideal $\jmath$ of R, if for every monomorphism $f : A \to B$ with $\jmath$-torsion coker(f), $f \otimes \mathbf{1} : A \bigotimes_R M \to B \bigotimes_R M$ is monomorphic. In Theorem 3.3, we characterize the $\jmath$-flat modules. Moreover, in Proposition 3.4, we give the relationship between $\jmath$-flat modules and $\jmath$-torsion free modules. In addition, we show that the $\jmath$-flatness of R-modules is a local property (see Theorem 3.10). The last section of this paper is mainly about $\jmath$-von Neumann regular rings. We define a ring R with $\jmath$ as a prime divided ideal of R to be a $\jmath$-von Neumann regular ring if every R-module is $\jmath$-flat. We prove that a ring R is a $\jmath$-von Neumann regular ring for some prime divided ideal $\jmath$ of R if and only if $(R, \jmath)$ is a local ring (see Theorem 4.1).

2. On $\jmath$-torsion modules and $\jmath$-torsion free modules

Let M be an R-module. Set

$$R(\jmath) = \{I \mid I \text{ is an ideal of } R \text{ such that } I \not\subseteq \jmath\}.$$

Also, we define

$$\jmath\text{-tor}(M) := \{x \in M \mid Ix = 0 \text{ for some } I \in R(\jmath)\}.$$

If $\jmath$-tor$(M) = M$, then M is called a $\jmath$-torsion module; and if $\jmath$-tor$(M) = 0$, then M is called a $\jmath$-torsion free module.

We shall begin with the following proposition which allows us to characterize irreducible ideals in terms of the set of $\jmath$-torsion elements.

Proposition 2.1. *Let R be a ring and $\jmath$ be an ideal of R. Then $\jmath$-tor(M) is a submodule for every (2-generated) R-module M if and only if $\jmath$ is an irreducible ideal of R.*

Proof. Suppose that $\jmath$-tor(M) is a submodule for any (2-generated) R-module M and $\jmath$ is not an irreducible ideal of R. So, $(\bar{0})$ is not an irreducible ideal of $R/\jmath$, which implies that there exist nonzero elements $\bar{r_1}, \bar{r_2} \in R/\jmath$ satisfying $(\bar{r_1}) \cap (\bar{r_2}) = (\bar{0})$. Let $M = R/(\jmath + Rr_1) \bigoplus R/(\jmath + Rr_2)$. We have $r_1(\bar{1}, \bar{0}) = (\bar{0}, \bar{0})$ and $r_2(\bar{0}, \bar{1}) = (\bar{0}, \bar{0})$, which gives that $(\bar{1}, \bar{0}), (\bar{0}, \bar{1}) \in \jmath$-tor$(M)$. But $(\bar{1}, \bar{1}) \notin \jmath$-tor$(M)$, a contradiction. For the converse, if $\jmath$ is an irreducible ideal of R and M is an R-module, so $(\bar{0})$ is an irreducible ideal of $R/\jmath$. Let $x_1, x_2 \in \jmath$-tor(M). Then, there exist two elements $r_1, r_2 \in R \setminus \jmath$, $r_ix_i = 0$. By assumption, we can take $0 \neq \bar{r} \in (\bar{r_1}) \cap (\bar{r_2})$. It follows that $r(x_1 + x_2) = 0$, and hence $x_1 + x_2 \in \jmath$-tor(M). □

Let R be a ring and $\jmath$ be an ideal of R. We set

$$\overline{R(\jmath)} = \{I \mid I \text{ is a finitely generated ideal of } R \text{ such that } I \not\subseteq \jmath\}.$$

The following result provides necessary and sufficient conditions for an R-module M to be a $\jmath$-torsion free, for some ideal $\jmath$ of R.

Theorem 2.2. *Let R be a ring, $\jmath$ be an ideal of R and M be an R-module. Then the following statements are equivalent:*

(1) *M is $\jmath$-torsion free.*

(2) $\mathrm{Hom}_R(R/J, M) = 0$ *for any* $J \in R(\jmath)$.

(3) $\mathrm{Hom}_R(R/J, M) = 0$ *for any* $J \in \overline{R(\jmath)}$.

(4) *The natural homomorphism:*

$$\lambda : M \to \mathrm{Hom}_R(J, M) \text{ such that } \lambda(x)(r) = rx,$$

for $x \in M$ and $r \in J$, is a monomorphism for any $J \in R(\jmath)$ (or $J \in \overline{R(\jmath)}$).

(5) $\mathrm{Hom}_R(B, M) = 0$ *for any $J \in R(\jmath)$ (or $J \in \overline{R(\jmath)}$) and any R/J-module B.*

Proof. (1) ⇒ (2) Let M be $\jmath$-torsion free. If $f \in \mathrm{Hom}_R(R/J, M)$, set $x = f(\bar{1})$, then $Jx = 0$, thus $x = 0$. Therefore, $f = 0$ and consequently $\mathrm{Hom}_R(R/J, M) = 0$.

(2) ⇒ (3) Straightforward.

(3) ⇒ (1) Let $x \in M$ such that $Ix = 0$ for some $I \in R(\jmath)$. Then, there is an ideal $J \in \overline{R(\jmath)}$ such that $J \subseteq I$ and $Jx = 0$. Consider the map $f : R/J \to M$, $\bar{r} \mapsto f(\bar{r}) = rx$. Since $\mathrm{Hom}_R(R/J, M) = 0$ for any $J \in \overline{R(\jmath)}$, then $x = 0$.

(2) ⇔ (4) Consider the exact sequence of R-modules

$$0 \to \mathrm{Hom}_R(R/J, M) \to \mathrm{Hom}_R(R, M) = M \to \mathrm{Hom}_R(J, M),$$

λ is a monomorphism if and only if $\mathrm{Hom}_R(R/J, M) = 0$.

(4) ⇒ (5) Let F be a free R/J-module such that $\delta : F \to B$ is an epimorphism. Then there is an exact sequence $0 \to \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(F, M)$. Since $\mathrm{Hom}_R(F, M) \cong \prod \mathrm{Hom}_R(R/J, M) = 0$, so $\mathrm{Hom}_R(B, M) = 0$.

(4) ⇒ (2) It is clear if we set $B = R/J$. □

Let N be an R-module. Then for any family $\{M_i\}_{i\in\Gamma}$ of R-modules, we have the following natural homomorphisms from [22].

$$\theta_1 : \prod_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R\left(N, \prod_{i\in\Gamma} M_i\right),$$
$$\theta_1([f_i])(x) = [f_i(x)] \text{ for } x \in N \text{ and } f_i \in \mathrm{Hom}_R(N, M_i)$$

and

$$\theta_2 : \bigoplus_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \cong \mathrm{Hom}_R\left(N, \bigoplus_{i\in\Gamma} M_i\right),$$
$$\theta_2([f_i])(x) = [f_i(x)] \text{ for } x \in N \text{ and finite non-zero } f_i \in \mathrm{Hom}_R(N, M_i)$$

(1) If N is finitely generated, then θ_1 is an isomorphism, that is,

$$\prod_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \cong \mathrm{Hom}_R\left(N, \prod_{i\in\Gamma} M_i\right).$$

(2) If N is finitely presented, then θ_2 is an isomorphism, that is,

$$\bigoplus_{i\in\Gamma} \mathrm{Hom}_R(N, M_i) \cong \mathrm{Hom}_R\left(N, \bigoplus_{i\in\Gamma} M_i\right).$$

Consider that $N = R/I$ for $I \in \overline{R(\jmath)}$ in above homomorphisms. We have the following result.

Corollary 2.3. *Let R be a ring, $\jmath$ be an ideal of R and $\{M_i \mid i \in \Gamma\}$ be an arbitrary family of R-modules. Then the following assertions are equivalent:*

(1) $\prod_{i\in\Gamma} M_i$ *is $\jmath$-torsion free.*
(2) M_i *is $\jmath$-torsion free for each $i \in \Gamma$.*
(3) $\bigoplus_{i\in\Gamma} M_i$ *is $\jmath$-torsion free for each $i \in \Gamma$.*

Recall from [1] that an R-module M satisfies strong Property A if for any $r_1, \dots r_n \in Z_R(M)$, there exists a nonzero element $x \in M$ such that $r_1x = r_2x = \cdots = r_nx = 0$. In particular, the ring R satisfies Property A if it does as an R-module. Bouchiba et al. [8] characterize the class of these clas of rings. Among other results, they prove that for a ring R, every R-module is the homomorphic image of an R-module satisfying strong Property A if and only if $Z(R) \subseteq J$ for some proper ideal J of R (see [8, Theorem 3.3]. The following result proves that the rings in which every module is the homomorphic image of a module satisfying strong Property A are exactly $\jmath$-torsion free rings.

Proposition 2.4. *Let R be a ring. Then R is a $\jmath$-torsion free ring for some proper ideal $\jmath$ of R if and only if every R-module is the homomorphic image of an R-module satisfying strong Property A.*

Proof. Let $\jmath$ be a proper ideal of R. One can see that R is a $\jmath$-torsion free ring if and only if $Z(R) \subseteq \jmath$. Therefore, an application of [8, Theorem 3.3] completes the proof. □

Let R be a ring and M be an R-module. Then $R \propto M$, the *trivial* (*ring*) *extension of R by M*, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(a_1, m_1)(a_2, m_2) := (a_1a_2, a_1m_2 + a_2m_1)$ for all $a_1, a_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in [2, 5, 6, 14–16].

Proposition 2.5. *Let D be a domain, $\jmath$ be an ideal of D and M be a $\jmath$-torsion free D-module with $\jmath M = 0$. Then $D \propto M$ satisfies strong Property A.*

Proof. Set $R := D \propto M$. Let $I = \sum_{i=1}^{n} R(a_i, m_i)$ be a finitely generated proper ideal of R such that $(a_i, m_i) \in Z(R)$. We show that $a_i \in \jmath$ for each $i = 1, \dots, n$. Deny, there exists $i = 1, \dots, n$ such that $a_i \notin \jmath$ and $(b_i, m_i') \in R \setminus \{(0, 0)\}$ such that $(a_i, m_i)(b_i, m_i') = (0, 0)$. Consequently, $b_i = 0$ (because D is a domain) and so $m_i' = 0$ (since M is a $\jmath$-torsion free module), a desired contradiction. Hence $a_i \in \jmath$ for each $i = 1, \dots, n$. It follows that $(0, m)J \subseteq (0, m)(\jmath \propto M) = (0, 0)$ for each $0 \neq m \in M$ and thus R satisfies strong Property A, as needed. □

Proposition 2.6. *Let R be a ring, $(\jmath_i)_{i\in\Lambda}$ be a family of ideals of R and M be an R-module. Set $\jmath = \cap_{i\in\Lambda}\jmath_i$. Then M is a $\jmath$-torsion free module if and only if M is a $\jmath_i$-torsion free module, for each $i \in \Lambda$.*

Proof. Since $\jmath \subseteq \jmath_i$, we then have the direct implication. Conversely, let $x \in \jmath\text{-}tor(M)$. So, there is $r \in R \setminus \jmath_{i_0}$ such that $rx = 0$ for some $i_0 \in \Lambda$, and hence $x = 0$. Thus $\jmath\text{-}tor(M) = 0$. □

As an immediate consequence of Proposition 2.6, we give a characterization of ϕ-torsion free modules.

Corollary 2.7. *Let R be a ring and M be an R-module. Then the following statements are equivalent:*

(1) *M is a ϕ-torsion free module.*
(2) *M is a $\mathfrak{p}$-torsion free module, for any $\mathfrak{p} \in Min(R)$.*

Proposition 2.8. *Let R be a ring, $\jmath_1$ and $\jmath_2$ be two ideals of R. Then $\jmath_1 \subseteq \jmath_2$ if and only if every $\jmath_2$-torsion R-module is $\jmath_1$-torsion.*

Proof. It suffices to prove the converse. Let $r \in \jmath_1$. Suppose that $r \notin \jmath_2$. Then, R/Rr is a $\jmath_2$-torsion module and hence R/Rr is a $\jmath_1$-torsion module by hypothesis. It follows that there is $a \in R \setminus \jmath_1$ such that $a \in Rr \subseteq \jmath_1$, a desired contradiction. □

The above result allows us to characterize ZN-rings and reduced rings in terms of ϕ-torsion modules.

Corollary 2.9. *Let R be a ring. Then the following statements are satisfied.*

(1) *R is a ZN-ring if and only if every ϕ-torsion R-module is a torsion module.*
(2) *R is a reduced ring if and only if every ϕ-torsion R-module is a (0)-torsion module.*

Remark 2.10. *Let R be a ring, $(\jmath_i)_{i\in\Lambda}$ be a family of ideals of R and M be an R-module. Let $\jmath = \cap_{i\in\Lambda}\jmath_i$. It can be seen that if M is a $\jmath_i$-torsion module, then M is a $\jmath$-torsion module, for each $i \in \Lambda$. However, the converse of the assertion fails. In fact, we consider $R = K[X,Y]$, $\jmath_1 = (X)$, $\jmath_2 = (Y)$ and $M = R/(XY)$. So, M is a $(\jmath_1 \cap \jmath_2)$-torsion module which is not a $\jmath_i$-torsion module, for each i.*

Theorem 2.11. *Let R be a ring and $\jmath$ be a prime ideal of R. Then:*

(1) *An R-module M is $\jmath$-torsion if and only if $\mathrm{Hom}_R(M,N) = 0$ for any $\jmath$-torsion free R-module N.*
(2) *An R-module N is $\jmath$-torsion free if and only if $\mathrm{Hom}_R(M,N) = 0$ for every $\jmath$-torsion R-module M.*
(3) *$\bigoplus_{i\in\Gamma} M_i$ is a $\jmath$-torsion module for any family $\{M_i \mid i \in \Gamma\}$ of $\jmath$-torsion modules.*

Proof. (1) Let M be a $\jmath$-torsion module and $f \in \mathrm{Hom}_R(M,N)$. Then, $\mathrm{Im}(f)$ is a $\jmath$-torsion submodule of N. Since N is $\jmath$-torsion free, we must have $f(M) = 0$, and thus $f = 0$. Conversely, set $T = \jmath\text{-}tor(M)$, since $\jmath$ is a prime ideal of R then T is an R-submodule of M. Set $N = M/T$, so N is $\jmath$-torsion free. It follows that the natural homomorphism $\pi : M \to N$ is the zero homomorphism because $\mathrm{Hom}_R(M,N) = 0$. Therefore $N = 0$, that is, $M = \jmath\text{-}tor(M)$ and hence M is $\jmath$-torsion.

(2) Let N be a $\jmath$-torsion free module. By (1), we obtain that $\mathrm{Hom}_R(M,N) = 0$ for any $\jmath$-torsion module M. For the converse, let $M = \jmath\text{-}tor(N)$. As $\jmath$ is a prime ideal of R then M is an R-submodule of N. Thus, $\mathrm{Hom}_R(M,N) = 0$, which gives that the inclusion homomorphism $M \to N$ is the zero homomorphism. Therefore $M = 0$, and so N is $\jmath$-torsion free.

(3) Follows immediately from (1) by using the following isomorphism

$$\mathrm{Hom}_R\left(\bigoplus_{i\in\Gamma} M_i, N\right) \cong \prod_{i\in\Gamma} \mathrm{Hom}_R\left(M_i, N\right).$$

□

The following examples show that the direct sum of $\jmath$-torsion R-modules is not necessary a $\jmath$-torsion module. Thus the condition that $\jmath$ a prime ideal of R in Theorem 2.11 cannot be removed.

Examples 2.12. (1) *Let $R = K[X,Y]$ with K is a field and set $\jmath = (XY)$. Let $M_1 = R/(X)$ and $M_2 = R/(Y)$ be R-modules. Then M_1 and M_2 are $\jmath$-torsion modules, however $M_1 \bigoplus M_2$ is not a $\jmath$-torsion module.*
(2) *Let $p \neq q$ be two prime numbers. Consider $R = \mathbb{Z}/p^2q^2\mathbb{Z}$ and set $\jmath = Nil(R) = pqR$. Take $M_1 = R/pR$ and $M_2 = R/qR$. Then M_1 and M_2 are ϕ-torsion modules, but $M_1 \bigoplus M_2$ is not a ϕ-torsion module.*

Proposition 2.13. *Let $R \subseteq T$ be an extension of rings and $\jmath$ be a prime ideal of R. If M is a $\jmath$-torsion R-module, then $M \bigotimes_R T$ is a $\jmath T$-torsion T-module. In particular, if M is a $\jmath$-torsion R-module, then $M[x]$ is a $\jmath[x]$-torsion $R[x]$-module.*

Proof. Let $x = \sum_{i=1}^{n} x_i \bigotimes t_i \in M \bigotimes_R T$. Since M is a $\jmath$-torsion R-module, for every index i there exists $r_i \in R \setminus \jmath$ such that $r_i x_i = 0$. Thus $rx = 0$ with $r = r_1 \cdots r_n \in R \setminus \jmath$, which gives that $r \in T \setminus \jmath T$. □

Proposition 2.14. *Let $f : R \to T$ be an epimorphism of rings. If M is an $f(\jmath)$-torsion T-module, then M is a $\jmath$-torsion R-module.*
In particular, if $I \subseteq \jmath$ are two ideals of R and M an R-module such that M/IM is a $(\jmath/I)$-torsion R/I-module, then M is a $\jmath$-torsion R-module.

Proof. One can see that if J is an $f(\jmath)$-ideal of T, then $f^{-1}(J)$ is a $\jmath$-ideal of R. □

3. On $\jmath$-flat modules

Definition 3.1. *Let $\jmath$ be an ideal of a ring R. An R-module M is said to be $\jmath$-flat, if for every monomorphism $f : A \to B$ with $\jmath$-torsion* $\operatorname{coker}(f)$, $f \otimes \mathbf{1} : A \bigotimes_R M \to B \bigotimes_R M$ *is also monomorphic; equivalently, if $0 \to A \to R \to C \to 0$ is an exact R-sequence where C is $\jmath$-torsion, then $0 \to A \bigotimes_R M \to B \bigotimes_R M \to C \bigotimes_R M \to 0$ is exact.*

Remark 3.2. *Let R be a ring and M be an R-module.*

(1) *Assume that $\jmath = Nil(R)$. Then M is a $\jmath$-flat module if and only if M is a ϕ-flat module.*
(2) *If $\jmath_1 \subseteq \jmath_2$ are two ideals of R and M is a $\jmath_1$-flat module, then M is a $\jmath_2$-flat module.*

In the following theorem, we give several characterizations of $\jmath$-flat modules.

Theorem 3.3. *Let R be a ring, $\jmath$ be an ideal of R and M be an R-module. Then the following conditions are equivalent:*

(1) *M is a $\jmath$-flat module.*
(2) $\operatorname{Tor}_1^R(P, M) = 0$ *for all $\jmath$-torsion R-modules P.*
(3) $\operatorname{Tor}_1^R(R/I, M) = 0$ *for all $\jmath$-ideals I of R.*
(4) $0 \to I \bigotimes_R M \to R \bigotimes_R M$ *is an exact sequence for all $\jmath$-ideals I of R.*
(5) $I \bigotimes_R M \cong IM$ *for all $\jmath$-ideals I of R.*
(6) $- \bigotimes_R M$ *is exact for every exact R-sequence $0 \to N \to F \to C \to 0$, where N, F, C are finitely generated, C is a $\jmath$-torsion R-module and F is free.*
(7) $- \bigotimes_R M$ *is exact for every exact R-sequence $0 \to N \to F \to C \to 0$, where C is a $\jmath$-torsion R-module and F is free.*
(8) $\operatorname{Tor}_1^R(R/I, M) = 0$ *for all finitely generated $\jmath$-ideals I of R.*
(9) $0 \to I \bigotimes_R M \to R \otimes_R M$ *is an exact sequence for all finitely generated $\jmath$-ideals I of R,*
(10) $I \bigotimes_R M \cong IM$ *for all finitely generated $\jmath$-ideals I of R.*
(11) $\operatorname{Ext}_R^1(I, M^+) = 0$ *for any $\jmath$-ideal I of R, where M^+ denote by the character module* $\operatorname{Hom}_Z(M, Q/Z)$.
(12) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \cap FI = IK$ for all $\jmath$-ideals I of R.*
(13) *Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \bigcap FI = IK$ for all finitely generated $\jmath$-ideals I of R.*

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (9) ⇒ (10), (12) ⇒ (13) ⇒ (10) ⇒ (8) and (3) ⇔ (11) are similar to those of flat modules (see for example [22, Theorems 2.5.6 and 2.5.7].

(10) ⇒ (5) Let $\sigma\left(\sum_{i=1}^n a_i \otimes x_i\right) = \sum_{i=1}^n a_i x_i = 0, a_i \in I, x_i \in M$. Since I is a $\jmath$-ideal, there exists $a_0 \in I \setminus \jmath$, Set $I_0 = Ra_0 + Ra_1 + \cdots + Ra_n$. Then $I_0 \subseteq I$ and I_0 is a $\jmath$-ideal. Consider the following commutative diagram:

$$\begin{array}{ccc} I_0 \otimes_R M & \longrightarrow & I \otimes_R M \\ \big\downarrow{\scriptstyle \sigma_I} & & \big\downarrow{\scriptstyle \sigma_R} \\ I_0 M & \longrightarrow & IM. \end{array}$$

It is clear that σ_I is an epimorphism and σ_R is an isomorphism. So σ_I is a monomorphism, which yields that σ_I is an isomorphism.

(5) ⇒ (12) Define $g_0 : IF \to IM$ by $g_0\left(\sum_i a_i x_i\right) = \sum_i a_i g(x_i), a_i \in I, x_i \in F$. Then $\operatorname{Ker}(g_0) = K \cap IF$. By [22, p. 103], we obtain that $0 \to IK \to IF \xrightarrow{g_0} IM \to 0$ is exact if and only if $K \cap IF = IK$. Now, let $\sigma_X : I \otimes_R X \to IX$ be the natural homomorphism for $X = K, F, M$. By hypotheses, we get σ_F and σ_M are isomorphisms since F is free. Set $N = \operatorname{Ker}(I \otimes_R K \to I \otimes_R F)$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} N & \longrightarrow & I \otimes_R K & \longrightarrow & I \otimes_R F & \longrightarrow & I \otimes_R M & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \sigma_K} & & \big\downarrow{\scriptstyle \sigma_F} & & {\scriptstyle \sigma_M}\big\downarrow & & \\ 0 & \longrightarrow & K \cap IF & \longrightarrow & IF & \longrightarrow & IM & \longrightarrow & 0. \end{array}$$

Then σ_K is an epimorphism by Five Lemma. Hence $K \cap IF = \operatorname{Im}(\sigma_K) = IK$.

(8) ⇒ (3) Let I be a $\jmath$-ideal of R, then I is the direct limit of all finitely generated $\jmath$-subideals I_i of I, that is, $I = \varinjlim I_i$. Hence $\operatorname{Tor}_2^R(I_i, M) \cong \operatorname{Tor}_1^R(R/I_i, M) = 0$, so

$$\operatorname{Tor}_2^R(\varinjlim I_i, M) \cong \varinjlim \operatorname{Tor}_2^R(I_i, M) \cong \varinjlim \operatorname{Tor}_2^R(I_i, M) = 0$$

by [22, Theorem 3.4.14]. Therefore

$$\operatorname{Tor}_1^R(R/I, M) \cong \operatorname{Tor}_2^R(I, M) \cong \operatorname{Tor}_2^R(\varinjlim I_i, M) = 0.$$

(4) ⇒ (6) Let $X = \{e_i\}_{i=1}^n$ be a basis of F. The case for $n = 1$ is true by hypothesis and the following result. If $0 \to I \to R \to R/I \to 0$ is exact, and R/I is a $\jmath$-torsion R-module, then $I = \operatorname{Ann}_R(\overline{1}) \nsubseteq \jmath$. Therefore, I is a $\jmath$-ideal of R. Suppose that $n > 1$. Set $F_1 = Re_2 \bigoplus \cdots \bigoplus Re_n$ and $A = N \cap Re_1$. Let $I = \{r \in R \mid re_1 \subseteq A\}$. Then $A = Ie_1 \cong I$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
D & \longrightarrow & A & \longrightarrow & N & \xrightarrow{\pi} & N/A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Re_1 & \longrightarrow & F & \xrightarrow{p} & F_1 & \longrightarrow & 0.
\end{array}$$

where π is the natural homomorphism, p is the projection and f is the homomorphism induced by the left square. If $u \in N$ with $f(\bar{u}) = p(u) = 0$, we must have $u \in Re_1$. Thus $u \in A$, whence f is monomorphic. Now, we consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & N & \longrightarrow & A/N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Re_1 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which all columns and rows are exact. The fact that C is a $\jmath$-torsion R-module ensures that C', C'' are $\jmath$-torsion R-modules.

Set $N' = \ker\left(A \bigotimes_R M \to N \bigotimes_R M\right)$. Tensoring by M, we get the following commutative diagram with the top row exact

$$\begin{array}{ccccccccc}
N' & \longrightarrow & A \bigotimes_R M & \longrightarrow & N \bigotimes_R M & \longrightarrow & N/A \bigotimes_R M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Re_1 \bigotimes_R M & \longrightarrow & F \bigotimes_R M & \longrightarrow & F_1 \bigotimes_R M & \longrightarrow & 0
\end{array}$$

Since

$$F \bigotimes_R M \cong \left(Re_1 \bigoplus F_1\right) \otimes M \cong (Re_1 \otimes M) \bigoplus (F_1 \otimes M),$$

the bottom row is also exact. Notice that $A \bigotimes_R M = Ie_1 \bigotimes_R M \to Re_1 \bigotimes_R M$ is monomorphic by hypothesis and $N/A \bigotimes_R M \to F_1 \bigotimes_R M$ is monomorphic by induction. Hence, we obtain that $N \bigotimes_R M \to F \bigotimes_R M$ is monomorphic by Five Lemma.

(6) ⇒ (7) Let $u_i \in N$ and $x_i \in M$ such that $\sum_{i=1}^{m} u_i \otimes x_i = 0$ in $F \bigotimes_R M$. We will prove that $\sum_{i=1}^{m} u_i \otimes x_i = 0$ in $N \bigotimes_R M$. Set $N_0 = Ru_1 + \cdots + Ru_m$. Then, there are a finitely generated free submodule F_0 and a free submodule F_1 of F such that $F = F_0 \bigoplus F_1$ and $N_0 \subseteq F_0$. In the following commutative diagram

$$\begin{array}{ccccccccc}
D & \longrightarrow & N_0 & \longrightarrow & F_0 & \xrightarrow{\pi} & F_0/N_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{p} & C & \longrightarrow & 0
\end{array}$$

The fact that f is a monomorphic by Five Lemma and C is a $\jmath$-torsion R-module implies that F_0/N_0 is a $\jmath$-torsion R-module. Thus $N_0 \bigotimes_R M \to F_0 \bigotimes_R M$ is monomorphic by assumption. Consider the following commutative diagram

$$\begin{array}{ccc}
N_0 \bigotimes_R M & \longrightarrow & N \bigotimes_R M \\
\downarrow & & \downarrow \\
F_0 \bigotimes_R M & \longrightarrow & F \bigotimes_R M
\end{array}$$

Since $F_0 \bigotimes_R M \to F \bigotimes_R M$ is monomorphic and $\sum_{i=1}^{m} u_i \otimes x_i = 0$ in $F_0 \bigotimes_R M$, we have $\sum_{i=1}^{m} u_i \otimes x_i = 0$ in $N_0 \bigotimes_R M$ by hypothesis. Thus, we conclude that $\sum_{i=1}^{m} u_i \otimes x_i = 0$ in $N \bigotimes_R M$ from this diagram.

(7) ⇒ (1) Let A be a submodule of a module B. Pick a free module F and an epimorphism $g : F \to B$. Set $N = g^{-1}(A)$ and $K = \ker(g)$. Then, we have the following commutative diagram (a pullback diagram) with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & & C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Tensoring by M, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K \bigotimes_R M & \longrightarrow & N \bigotimes_R M & \longrightarrow & A \bigotimes_R M & \longrightarrow & 0 \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\
K \bigotimes_R M & \longrightarrow & F \bigotimes_R M & \longrightarrow & B \bigotimes_R M & \longrightarrow & 0
\end{array}$$

Since $N \bigotimes_R M \to F \bigotimes_R M$ is monomorphic, we get $A \bigotimes_R M \to B \bigotimes_R M$ is monomorphic by Five Lemma. □

Proposition 3.4. *Let R be a ring. If $\jmath$ is a prime ideal of R satisfying $Z(R) \subseteq \jmath$, then every $\jmath$-flat R-module is a $\jmath$-torsion free module.*

Proof. Let M be a $\jmath$-flat R-module for some prime ideal $\jmath$ of R satisfying $Z(R) \subseteq \jmath$. Then $R_\jmath/R$ is a $\jmath$-torsion R-module. It follows that the natural exact sequence $0 \to R \to R_\jmath \to R_\jmath/R \longrightarrow 0$ implies that $0 \to M = R \bigotimes_R M \to R_\jmath \bigotimes_R M \to R_\jmath/R \bigotimes_R M \longrightarrow 0$ is also exact. In particular, $0 \to M \to M_\jmath$ is exact. Now, if $I \in R(\jmath)$ and $x \in M$ such that $Ix = 0$, then there is an element $s \in R \setminus \jmath$ such that $sx = 0$. This implies that $x = \frac{x}{1} = \frac{sx}{s} = 0$. Hence M is a $\jmath$-torsion free module, as required. □

Example 3.5. *Every flat R-module is $\jmath$-flat. If $\jmath = 0$, then every $\jmath$-flat R-module is flat.*

Proposition 3.6. *Let R be a ring, $(\jmath_i)_{i\in\Gamma}$ be a family of ideals of R, $\jmath = \cap_{i\in\Gamma}\jmath_i$ and let M be an R-module. Then the following assertions are equivalent:*

(1) *M is a $\jmath$-flat module,*
(2) *M is a $\jmath_i$-flat module, for all $i \in \Gamma$.*

Proof. If M is a $\jmath$-flat module, then M is clearly a $\jmath_i$-flat module since $\jmath \subseteq \jmath_i$. Conversely, let J be a $\jmath$-ideal, so there exists $x \in J$ such that $x \notin \jmath$. Since $\jmath = \cap_{i\in\Gamma} \jmath_i$, $x \notin \jmath_{i_0}$ for some $i_0 \in \Gamma$. By assumption, we get M is a $\jmath_{i_0}$-flat module, and so $\mathrm{Tor}_1^R(R/I, M) = 0$ by Theorem 3.3. Consequently $\mathrm{Tor}_1^R(R/I, M) = 0$ for all $\jmath$-ideals J of R, which implies that M is a $\jmath$-flat module. □

In the light of the above proposition, we give a new characterization of ϕ-flat module.

Corollary 3.7. *Let R be a ring and M be an R-module. Then the following conditions are equivalent:*

(1) *M is a ϕ-flat module,*
(2) *M is a $\mathfrak{p}$-flat module for each $\mathfrak{p} \in Min(R)$.*

Remark 3.8. *Note that if M is a ϕ-flat R-module, then M is an $\mathfrak{m}$-flat module for each $\mathfrak{m} \in Max(R)$. It is interesting to see that the converse of the above assertion would fail. In fact, let $(D, \mathfrak{m})$ be a local ϕ-ring which is not a ϕ-von Neumann regular ring (i.e., $\mathfrak{m} \neq Nil(R)$). Since the only $\mathfrak{m}$-ideal of R is R, we get that every R-module is $\mathfrak{m}$-flat. However, by [28, Theorem 4.1], there exists an R-module M that is not ϕ-flat.*

Proposition 3.9. *Let M be a $\jmath$-flat R-module and let S be a multiplicative subset of R. Then $S^{-1}M$ is a $\jmath$-flat R-module.*

Proof. The proof is analogous to that of [22, Theorem 2.5.10].. □

We next prove that the $\jmath$-flatness of R-modules is a local property.

Theorem 3.10. *Let R be a ring and let M be an R-module, then the following conditions are equivalent:*

(1) *M is a $\jmath$-flat R-module.*
(2) *$M_\mathfrak{p}$ is a $\jmath_\mathfrak{p}$-flat $R_\mathfrak{p}$-module, for each prime ideal $\mathfrak{p}$ of R.*
(3) *$M_\mathfrak{m}$ is a $\jmath_\mathfrak{m}$-flat $R_\mathfrak{m}$-module, for each maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) ⇒ (2) Let $\mathfrak{p}$ prime ideal of R, and let J be a $\jmath_\mathfrak{p}$-ideal of $S^{-1}R$, then $J = I_\mathfrak{p}$ with I is a $\jmath$-ideal of R. Then, we have

$$\begin{aligned}\mathrm{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}/J) &\cong \mathrm{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, (R/I)_\mathfrak{p}) \\ &\cong \mathrm{Tor}_1^R(M, R/I)_\mathfrak{p} = 0.\end{aligned}$$

Then $M_\mathfrak{p}$ is $\jmath_\mathfrak{p}$-flat $R_\mathfrak{p}$-module.

(2) ⇒ (3) This is straightforward.

(3) ⇒ (1) Assume that $M_\mathfrak{m}$ is a ϕ-P-flat $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ of R. We must show that the morphism $f : M \bigotimes_R J \to M \bigotimes_R R$ is monomorphic for every $\jmath$-ideal J of R. As $M_\mathfrak{m}$ is a $\jmath_\mathfrak{m}$-flat $R_\mathfrak{m}$-module, we get that $f_\mathfrak{m} : M_\mathfrak{m} \bigotimes_{R_\mathfrak{m}} (Ra)_\mathfrak{m} \to M_\mathfrak{m} \bigotimes_{R_\mathfrak{m}} R_\mathfrak{m}$ is a monomorphic for each maximal ideal $\mathfrak{m}$ of R. As a result of [22, Theorem 1.5.21], f is monomorphic. This completes the proof. □

Theorem 3.11. *Let $f : R \to T$ be an epimorphism of rings. If M is a $\jmath$-flat R-module, then $M \bigotimes_R T$ is a $f(\jmath)$-flat T-module.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of T-modules, where C is a $f(\jmath)$-torsion module. By Proposition 2.14, $0 \to A \to B \to C \to 0$ is also an exact R-sequence, and C is a $f(\jmath)$-torsion module. Now, we consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \bigotimes_R M & \longrightarrow & B \bigotimes_R M & \longrightarrow & C \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & A \bigotimes_T T \bigotimes_R M & \longrightarrow & B \bigotimes_T T \bigotimes_R M & \longrightarrow & C \bigotimes_T T \bigotimes_R M & \longrightarrow & 0
\end{array}$$

The above row exact implies the below row exact, which gives that $M \bigotimes_R T$ is a $f(\jmath)$-flat T-module. □

Corollary 3.12. *Let M be a $\jmath$-flat R-module and I be an ideal of R such that $I \subseteq \jmath$. Then M/IM is a $\jmath/I$-flat R/I-module.*

Theorem 3.13. *Let R be a ring, $\jmath$ be a prime divided ideal of R, M be an R-module and I be an ideal of R. Assume that $I \subseteq \jmath$ and $I \bigotimes_R M \cong IM$. Then M is a $\jmath$-flat R-module if and only if M/IM is a $\jmath/I$-flat R/I-module.*

Proof. We suppose M/IM is a $\jmath/I$-flat R/I-module. For any $\jmath$-ideal J of R, consider the following commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & J/I \bigotimes_{R/I} R/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_{R/I} R/I \bigotimes_R M \\
 & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & J/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_R M
\end{array}$$

The above row exact implies the below row exact, thus consider the following commutative diagram with rows exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J/I \bigotimes_R M & \longrightarrow & R/I \bigotimes_R M & \longrightarrow & R/J \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & JM/IM & \longrightarrow & M/IM & \longrightarrow & M/JM & \longrightarrow & 0
\end{array}$$

So, $J/I \bigotimes_R M \cong JM/IM$ according to the Five lemma. Consider the following commutative diagram with rows exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I \bigotimes_R M & \longrightarrow & J \bigotimes_R M & \longrightarrow & I/J \bigotimes_R M & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & IM & \longrightarrow & JM & \longrightarrow & JM/IM & \longrightarrow & 0
\end{array}$$

We conclude that $J \bigotimes_R M \cong JM$ and thus M is a $\jmath$-flat R-module. □

Proposition 3.14. *Let R be a ring, $\jmath$ be a prime divided ideal of R and I be a $\jmath$-ideal of R. Then I is a $\jmath$-flat R-module if and only if $I/\jmath$ is a flat $R/\jmath$-module.*

Proof. Assume that I is a $\jmath$-flat R-module and let $K/\jmath$ be a nonzero ideal of $R/\jmath$. Then K is a $\jmath$-ideal of R. This gives that R/K is $\jmath$-torsion and so is $R/K \otimes_R R/\jmath$. Consider the following exact sequence $0 \to K \to R \to R/K \to 0$. Note that $R/\jmath$ is $\jmath$-flat, so $0 \to K \otimes_R R/\jmath \to R \otimes_R R/\jmath \to R/K \otimes_R R/\jmath \to 0$ is exact. Since I is $\jmath$-flat, we then have the following exact sequence

$$0 \to I \otimes_R K \otimes_R R/\jmath \to I \otimes_R R \otimes_R R\jmath \to I \otimes_R R/K \otimes_R R/\jmath \to 0.$$

Now, let $x \in \jmath$. Since I is a $\jmath$-ideal, then there exists $r \in I$ such that $\jmath \subseteq Rr$, whence $x = ra \in I\jmath$ as $I\jmath$ is prime. Therefore $I \otimes_R R/\jmath = I/I\jmath = I/\jmath$ and likewise we find that $K \otimes_R R/\jmath = K/K\jmath = K/\jmath\ K$ is $\jmath$-ideal. Consequently, we have the following exact sequence

$$\begin{aligned} 0 &\rightarrow (I \otimes_R R/\jmath) \otimes_{R/\jmath} (K \otimes_R R/\jmath) \\ &\rightarrow (I \otimes_R R/\jmath) \otimes_{R/\jmath} (R \otimes_R R/\jmath) \\ &\rightarrow (I \otimes_R R/\jmath) \otimes_{R/\jmath} (R/K \otimes_R R/\jmath) \rightarrow 0. \end{aligned}$$

That is,

$$0 \rightarrow I/\jmath \otimes_{R/\jmath} K/\jmath \rightarrow I/\jmath \otimes_{R/\jmath} R/\jmath \rightarrow I/\jmath) \otimes_{R/\jmath} R/K \rightarrow 0$$

is exact. Therefore $I/\jmath$ is flat over $R/\jmath$. The converse follows immediately from Theorem 3.13. This completes the proof. □

Theorem 3.15. *Let R be a ring, $\jmath$ be an ideal of R and M be a $\jmath$-flat module and $0 \rightarrow A \rightarrow B \rightarrow M \rightarrow 0$ be an exact sequence. If A is $\jmath$-flat R-module, then so is B.*

Proof. Assume that A is $\jmath$-flat. Let $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ be an exact sequence with I is $\jmath$-ideal of R. Consider the commutative diagram (with exact lines)

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & ker(i \bigotimes 1_B) & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & I \bigotimes A & \longrightarrow & I \bigotimes B & \longrightarrow & I \bigotimes M & \longrightarrow & 0 \\ & & {\scriptstyle i \bigotimes 1_A}\downarrow & & {\scriptstyle i \bigotimes 1_B}\downarrow & & {\scriptstyle i \bigotimes 1_M}\downarrow & & \\ 0 & \longrightarrow & R \bigotimes A & \xrightarrow{1_R \bigotimes u} & R \bigotimes B & \longrightarrow & R \bigotimes M & \longrightarrow & 0 \end{array}$$

where $i \bigotimes 1_A$, $1_R \bigotimes u$ and $i \bigotimes 1_M$ are monomorphisms since R, A and M are $\jmath$-flat. By the Snake Lemma [22, Theorem 1.9.10], the sequence $0 \rightarrow ker(i \bigotimes 1_B) \rightarrow 0$ is exact, that is, $i \bigotimes 1_B$ is a monomorphism. Hence, B is $\jmath$-flat by Theorem 3.3, as needed. □

Definition 3.16. *Let R be a ring and M be an R-module. Then M is called a strongly $\jmath$-flat module if* $\mathrm{Tor}_n^R(T, M) = 0$ *for any $\jmath$-torsion module T and any $n \geq 1$.*

Lemma 3.17. *Let R be a ring and M be an R-module. Then M is strongly $\jmath$-flat if and only if* $\mathrm{Tor}_n^R(R/I, M) = 0$ *for any (finitely generated) $\jmath$-ideal I of R and any $n \geq 1$.*

Proof. It follows from that an R-module M is strongly $\jmath$-flat if and only if each syzygies $\Omega^n(M)$ of M is $\jmath$-flat, and that each $\Omega^n(M)$ is $\jmath$-flat if and only if $\mathrm{Tor}_1^R(R/I, \Omega^n(M)) = 0$ for any $\jmath$-ideal I of R for any (finitely generated) $\jmath$-ideal I of R). □

Proposition 3.18. *Let R be a ring and $\jmath$ be an ideal of R. Then the following statements hold.*

(1) *The class of strongly $\jmath$-flat modules is closed under direct limits, direct summands and extensions.*
(2) *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of R-modules. If B and C are strongly $\jmath$-flat modules, then so is A.*

Proof. (1) It is similar to that of flat modules (see for example [22, Theorems 2.5.2 and 2.5.34]).
(2) Let T be a $\jmath$-torsion module. Then we have an exact sequence $\cdots \rightarrow \mathrm{Tor}_{n+1}^R(T, C) \rightarrow \mathrm{Tor}_n^R(T, A) \rightarrow \mathrm{Tor}_n^R(T, B) \rightarrow \cdots \rightarrow \mathrm{Tor}_2^R(T, C) \rightarrow \mathrm{Tor}_1^R(T, A) \rightarrow \mathrm{Tor}_1^R(T, B) \rightarrow \mathrm{Tor}_1^R(T, C)$. Since B and C are strongly $\jmath$-flat modules, $\mathrm{Tor}_n^R(T, B) = \mathrm{Tor}_n^R(T, C) = 0$ for any $n \geq 1$. Hence $\mathrm{Tor}_n^R(T, A) = 0$ for any $n \geq 1$, whence A is strongly $\jmath$-flat. □

Obviously, every strongly $\jmath$-flat module is $\jmath$-flat, and if $\jmath = Nil(R)$ then the notion of strongly $\jmath$-flat is identical with strongly ϕ-flat introduced by Zhang in [24], it follows by [24, Example 1.1] that ϕ-flat modules are not always strongly ϕ-flat, and consequently $\jmath$-flat modules are not always strongly $\jmath$-flat. But the following result exhibits that over a rings ring R with $Z(R) \subseteq \jmath$, $\jmath$-flat modules are exactly strongly $\jmath$-flat .

Theorem 3.19. *Let R be a ring R with $Z(R) \subseteq \jmath$. Then an R-module M is $\jmath$-flat if and only if M is strongly $\jmath$-flat.*

Proof. Suppose M is a $\jmath$-flat R-module. Let J be a $\jmath$-ideal of R, so J contains a non-zero-divisor a of R. Hence $\mathrm{Tor}_n^R(R/aR, M) = 0$ for any positive integer n. It follows by [11, Proposition 4.1.1] that

$$\mathrm{Tor}_1^{R/aR}(R/J, M/aM) \cong \mathrm{Tor}_1^{R/aR}(R/J, M \otimes_R R/aR) \cong \mathrm{Tor}_1^R(R/J, M) = 0.$$

Hence M/Ma is a flat R/aR-module. Consequently, for any $n \geq 1$ we have

$$\mathrm{Tor}_n^R(R/J, M) \cong \mathrm{Tor}_n^{R/aR}(R/J, M \otimes_R R/aR) \cong \mathrm{Tor}_n^{R/aR}(R/J, M/aM) = 0.$$

This yields that M is a strongly $\jmath$-flat R-module according to Lemma 3.17. □

4. On $\jmath$-von Neumann regular rings

We define a ring R with $\jmath$ is a prime divided ideal of R to be a $\jmath$-von Neumann regular ring if every R-module is $\jmath$-flat.

Theorem 4.1. *Let R be a ring with $\jmath$ is a prime divided ideal of R. The following conditions are equivalent:*

1. *R is a $\jmath$-von Neumann regular ring.*
2. *For any element $a \in R \setminus \jmath$, we have $Ra = Ra^2$.*
3. *Every principal $\jmath$-ideal I of R is generated by an idempotent element $e \in R$.*
4. *Every finitely generated $\jmath$-ideal I of R is generated by an idempotent element $e \in R$.*
5. *R is a local ring with maximal ideal ideal $\jmath$.*

Proof. (1) ⇒ (2) For each $a \in R \setminus \jmath$, $0 \to Ra \to R \to R/Ra \to 0$ is exact. Since R/Ra is $\jmath$-flat, $Ra = Ra \bigcap Ra = Ra^2$ by Theorem 3.3. Then there is an element $x \in R \setminus \jmath$ such that $a = xa^2$.
(2) ⇒ (3) Let $a \in R \setminus \jmath$, $0 \to Ra \to R \to R/Ra \to 0$ is exact. Since R/Ra is $\jmath$-flat, by Theorem 3.3, $Ra = Ra \cap Ra = Ra^2$. Therefore there exists $x \in R$ such that $a = xa^2$.
(3) ⇒ (4) Let $I = Ra_1 + \cdots + Ra_n$ be a $\jmath$-ideal of R. Since $\jmath$ is a prime divided ideal of R, we may assume that each $a_i \in R \setminus \jmath$, and so $Ra_i = Re_i$ for some idempotent elements e_i. Consequently $I = Re_1 + \cdots + Re_n$. For any $x \in I$, $x = r_1e_1 + \cdots + r_ne_n = r_1e_1^2 + \cdots + r_ne_n^2 \in I^2$. Thus $I^2 = I$, and therefore I is generated by an idempotent element.
(4) ⇒ (1) Let M be an R-module and $0 \to A \to F \to M \to 0$ be exact, where F is free. Let I be a finitely generated $\jmath$-ideal of R. Then, by hypothesis, $I = Re$ for some idempotent $e \in R$. For each $x \in A \cap IF$, we have $x = ey = e^2y = ex \in IA$ (where $y \in F$). This implies that $A \bigcap IF = IA$ and thus M is $\jmath$-flat by Theorem 3.3.
(4) ⇒ (5) Let J be a non-zero principal ideal of $R/\jmath$. Then $J = I/\jmath$ with I is a $\jmath$-ideal of R. So $I = Re$ with e is idempotent, which gives that J is generated by an idempotent element $\overline{e} \in R/\jmath$. Hence $R/\jmath$ is a von Neumann reqular ring, so $R/\jmath$ is a field. It follows that $\jmath$ is a maximal ideal of R. Since $\jmath$ is a divided ideal of R, we can easily conclude that R is a local ring with maximal ideal $\jmath$.
(5) ⇒ (4) This is straightforward, since in this case R is the only $\jmath$-ideal of R. □

The following corollary shows the relationship between the concepts of von Neumann regular rings and $\jmath$-von Neumann regular rings.

Corollary 4.2. *Let R be a ring with a prime divided ideal $\jmath$.*

1. *Assume that R is a $\jmath$-von Neumann regular ring. Then R is a von Neumann regular ring if and only if it is a field.*

(2) *Suppose that R is a von Neumann regular ring. Then R is a $\jmath$-von Neumann regular ring if and only if it is a field.*

Armed with Corollary 4.2, we can easily construct a $\jmath$-von Neumann regular ring which is not a von Neumann regular ring and a von Neumann regular ring that is not a $\jmath$-von Neumann regular ring.

Example 4.3. *Let p be a prime number and $n > 1$. Then:*

(1) $\mathbb{Z}/p^n\mathbb{Z}$ *is a* $p\mathbb{Z}/p^n\mathbb{Z}$*-von Neumann regular ring which is not a von Neumann regular ring .*
(2) $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ *is a von Neumann regular ring which is not a $\jmath$-von Neumann regular ring for every ideal $\jmath$ of R.*

Proposition 4.4. *Let R be a ring with a prime divided ideal $\jmath$. Then R is a $\jmath$-von Neumann regular ring if and only if every descending chain of $\jmath$-ideals is stationary.*

Proof. Assume that R is a $\jmath$-von Neumann regular ring, then every descending chain of $\jmath$-ideals is stationary. Conversely, let $(J_n)_{n\in\mathbb{N}}$ be a descending chain of non-zero ideal of $R/\jmath$. For each $n \in \mathbb{N}$, we set $J_n = I_n + \jmath$ where I_n is a $\jmath$-ideal of R. Therefore $(I_n)_{n\in\mathbb{N}}$ is stationary which implies that $(J_n)_{n\in\mathbb{N}}$ is stationary. Hence $R/\jmath$ is an artinian domain and thus $R/\jmath$ is a field. It follows that $\jmath$ is a maximal ideal of R. As $\jmath$ is divided, we conclude that $\jmath$ is the only maximal ideal of R. Therefore R is a $\jmath$-von Neumann regular ring. □

We end with the following question.

Question 4.5. *When $\jmath$-flat (resp., $\jmath$-torsion free) modules are all flat (resp., torsion free) modules ?*

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