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Fuzzification of the category of (L, M)-fuzzy convex spaces

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Abstract. In this paper, a fuzzy approach to the category of (L, M)-fuzzy convex spaces (denoted by **LM-FCon**) is introduced. To be specific, the objects and morphisms of **LM-FCon** are given some degree value, denoted by $\omega(X, \mathscr{C})$ and $\mu(f)$ respectively, which extends the **LM-FCon** to the fuzzy case. What is more, we show that the set $\mathfrak{C}_\alpha(L, M, X) = \{\mathscr{C} : L^X \to M \mid \omega(X, \mathscr{C}) \geqslant \alpha\}$ is a bounded complete lattice and discuss the relationships among $\mathfrak{C}_\alpha(L, M, X)$, L-convex structures and (L, M)-fuzzy convex structures. Finally, subspace, product space, join space and quotient space of $\mathfrak{C}_\alpha(L, M, X)$ are studied and related properties are obtained.

1. Introduction

Originally influenced by geometry, the concept of convex sets was introduced in the last century and now has become an important research direction of mathematics [4, 22]. Inspired by axiomatic methods, some scholars abstractly generalized convex sets to convex structures (or called abstract convexities in [2, 11]). A convex structure is a subset of powerset $\mathcal{P}(X)$ which includes both X and $\emptyset$, and which is closed for arbitrary intersections and up-directed unions (see [28]).

The development of fuzzy mathematics has opened up the scope of mathematical research and lots of fuzzy mathematical branches have been studied, such as fuzzy category [16, 17, 23, 32], fuzzy metric spaces and fuzzy norm spaces [25, 37–40, 42, 43] fuzzy topological structures [5, 24, 30, 41] fuzzy posets and fuzzy convergence structures [9, 15, 34]. Naturally, the abstract convex structure has been generalized to the fuzzy case. Many researchers did various works from different directions in fuzzy convex structures [6, 7, 10, 14, 20, 21, 26, 31, 33, 35, 36].

In [12, 18], Rosa and Maruyama presented the concept of L-fuzzy convex structures (now called L-convex structures) in the scope of the $[0, 1]$ interval and the completely distributive lattice L, respectively. Under the preliminary work of L-convex spaces, Jin et al. [3] and Pang et al. [16] introduced several kinds of subcategories of the category of L-convex spaces and studied the categorical interrelationships between stratified L-convex spaces and convex spaces. From a totally different perspective, Shi and Xiu [20] introduced the concept of M-fuzzifying convex structures and the category of M-fuzzifying convex spaces.

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Subsequently, Shi and Xiu [21] presented the notion of (L, M)-convex structures, which further generalized the fuzzification method for convex structures by containing L-convex structures and M-fuzzifying convex structures as their special cases.

What's more, the relationship between the category of M-fuzzifying convex spaces and the category of (L, M)-fuzzy convex spaces (denoted by **LM-FCon**) has been discussed in [32]. It shows that there exists a coreflectively embedding functor from the category of M-fuzzyfying convex spaces to **LM-FCon**. And, Wu and Li [32] discussed different subcategories of (L, M)-fuzzy convex spaces. Li [8] expounded a subcategory of the category of (L, M)-fuzzy convex spaces, called the category of enriched (L, M)-fuzzy convex spaces and studied its properties. Up to now, there are three kinds of categories of fuzzy convex spaces, including the category of L-convex spaces, the category of M-fuzzifying convex spaces, and the category of (L, M)-fuzzy convex spaces. The category of (L, M)-fuzzy convex space is the most general one, as it includes the others. Although the objects and morphisms in **LM-FCon** are fuzzy, the category is crisp. A natural question arise: whether can we generalize the crisp **LM-FCon** to the fuzzy case?

Inspired by the ideas of fuzzy category [5, 23, 24], the main purpose of this paper is to endow the objects and morphisms of **LM-FCon** with some degrees. And try to give a fuzzy approach to the category of (L, M)-convex spaces.

The paper is structured as follows. In Section 2, some essential concepts and notations of a frame, an (L, M)-fuzzy convex space, and an L-fuzzy category are recalled. In Section 3, the degree to which a mapping ω is an (L, M)-fuzzy convex space and the degree to which a mapping μ is an (L, M)-fuzzy convexity-preserving mapping are defined. Based on these, we give a fuzzification method of **LM-FCon** and obtain a corresponding M-fuzzy category. In Section 4, it is show that the set $\mathfrak{C}_\alpha(L, M, X) = \{\mathscr{C} : L^X \to M \mid \omega(X, \mathscr{C}) \geqslant \alpha\}$ is a bounded complete lattice and the relationship among $\mathfrak{C}_\alpha(L, M, X)$, L-convex structures and (L, M)-fuzzy convex structures are discussed. In Section 5, by means of fuzzy powerset operators on M^{L^X} and M^{L^Y}, the concept of subspace, join space, quotient space and product space of $\mathfrak{C}_\alpha(L, M, X)$ are proposed and their related properties are studied.

2. Preliminaries

Throughout this paper, we assume the underlying lattice is a frame. Firstly, we recall the definition of a frame.

Definition 2.1 ([27]). A *frame* (or called a *complete Heyting algebra*) is a complete lattice satisfying $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I}(x \wedge y_i)$.

Unless otherwise noted, L and M denote frames. The smallest elements and the largest elements in L and M are denoted by $\perp_L$, $\top_L$ and $\perp_M$, $\top_M$, respectively.

For a frame L, the related implication operator $\rightarrowtail: L \times L \longrightarrow L$ can be defined by $\forall x, y \in L$, $x \rightarrowtail y = \bigvee \{z \in L | x \wedge z \leq y\}$.

Some properties of the implication are listed below.

Lemma 2.2 ([5]). *Let L be a frame and $\rightarrowtail$ be the related implication operator. Then for any $x, y, z \in L$, $\{x_i\}_{i \in I}, \{y_j\}_{j \in J} \subseteq L$, the following properties hold.*

(G1) $x \wedge y \leqslant z \Leftrightarrow x \leqslant y \rightarrowtail z$.

(G2) $x \rightarrowtail y = \top_L \Leftrightarrow x \leqslant y$.

(G3) $\top_L \rightarrowtail x = x$.

(G4) $x \rightarrowtail (\bigwedge_{j \in J} y_j) = \bigwedge_{j \in J}(x \rightarrowtail y_j)$, *in particular*, $y \leqslant z \Rightarrow x \rightarrowtail y \leqslant x \rightarrowtail z$.

(G5) $(\bigvee_{i \in I} x_i) \rightarrowtail y = \bigwedge_{i \in I}(x_i \rightarrowtail y)$, *in particular*, $x \leqslant z \Rightarrow x \rightarrowtail y \geqslant z \rightarrowtail y$.

(G6) $x \rightarrowtail y \geqslant (x \rightarrowtail z) \wedge (z \rightarrowtail y)$.

Suppose X is a non-empty set. Denote L^X be the mappings from X to L. As all algebraic operators on L could be point-wisely extended to L^X , so L^X is also a frame and a residual implication operator $\rightarrowtail: L^X \times L^X \longrightarrow L^X$ can be defined by $(A \rightarrowtail B)(x) = A(x) \rightarrowtail B(x)$ for any $A, B \in L^X$ and $x \in X$.

In [23, 24], the authors proposed the concept of a fuzzy category. A fuzzy category is that the potential objects and potential morphisms are to some certain degree. Next, let us recall the definition of an L-valued or an L-fuzzy category.

Definition 2.3 ([23, 24]). Let $\mathfrak{C} = (Ob_{\mathfrak{C}}, Mor_{\mathfrak{C}}, dom, cod, \circ)$ be an ordinary (or a classical) category. Let $\omega : Ob_{\mathfrak{C}} \longrightarrow L$ and $\mu : Mor_{\mathfrak{C}} \longrightarrow L$ be L-fuzzy subclasses of the classes of its objects and morphisms, respectively. Then we call the triple $(\mathfrak{C}, \omega, \mu)$ is *an L-valued (or an L-fuzzy category) if it satisfies the following conditions:*

(**LCA1**) $\forall X, Y \in Ob_{\mathfrak{C}}, \forall f \in Mor_{\mathfrak{C}}(X, Y), \mu(f) \leqslant \omega(X) \wedge \omega(Y)$;

(**LCA2**) $\mu(f) \wedge \mu(g) \leqslant \mu(g \circ f)$, *where* $\circ$ *is the composite operation;*

(**LCA3**) $\mu(e_X) = \omega(X)$, *where* $e_X : X \longrightarrow X$ *denotes the identity morphism.*

Remark 2.4. In [23, 24], the corresponding lattice of the original definition is a GL-monoid and the condition (LCA2) is replaced by $\mu(f) * \mu(g) \leqslant \mu(g \circ f)$.

The authors had discussed the method of how to fuzzify the category of (L, M)-fuzzy topological spaces and presented a notion of M-fuzzy category $FTOP(L, M)$.

Now, we should naturally consider the question how to fuzzify the category of (L, M)-fuzzy convex spaces? Before solving the problem, we need to recall the definitions of convex spaces and fuzzy convex spaces in the following.

In the standard case, a set in an n-dimensional Euclidean space is convex if and only if it contains the whole segments joining each two of its points. By axiomatizing the properties of convex sets in Euclidean spaces, the concept of a convex structure was introduced in the following definition.

Definition 2.5 ([2, 4, 28]). A subset C of 2^X is called a convex structure (or called a convexity), if it satisfies the following conditions.

(C1) $\emptyset, X \in C$;

(C2) If $\{A_i \mid i \in I\} \subseteq C$ is non-empty, then $\bigcap_{i \in I} A_i \in C$;

(C3) If $\{A_j \mid j \in J\} \subseteq C$ is non-empty and up-directed, then $\bigcup_{j \in J}^{dir} A_j \in C$.

And the pair (X, C) is called a convex space. The members of $A \in C$ are called convex sets.

Convex structures exists extensively, including vector spaces, partially ordered sets, metric spaces, lattices, graphs, matroids and so on.

Example 2.6. (Standard convex structure [28]) Let $(X, +, \cdot)$ be a vector space and $A \subseteq X$. We say A is convex if $x, y \in A$ implies $tx + (1 - t)y \in A$ for each $t \in [0, 1]$.

Example 2.7. (Order convex structure [28]) Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$. We say A is convex if $\forall z \in X, \forall x, y \in A, x \leq z \leq y$ implies $z \in A$.

Example 2.8. (Geodesic convex structure [28]) Let (X, d) be a metric space and $A \subseteq X$. We say A is convex if $\forall z \in X, \forall x, y \in A, d(x, y) = d(x, z) + d(z, y)$ implies $z \in A$.

The notion of a fuzzy convex structure is introduced by M.V.Rosa in [18]. In 2009, Y. Maruyama generalized it to the L-fuzzy setting in [12] and gave the definition of an L-convex structure as follows.

Definition 2.9 ([12]). A subset $\mathscr{C}$ of L^X is called an *L-convex structure* (or called an L-convexity) if it satisfies the following conditions.

(**LC1**) $\perp_{L^X}, \top_{L^X} \in \mathscr{C}$;

(**LC2**) If $\{A_i : i \in I\} \subseteq \mathscr{C}$ is non-empty, then $\bigwedge_{i \in I} A_i \in \mathscr{C}$;

(**LC3**) If $\{A_j : j \in J\} \subseteq \mathscr{C}$ is non-empty and up-directed, then $\bigvee_{j \in J}^{dir} A_j \in \mathscr{C}$.

And the pair $(X, \mathscr{C})$ is called an L-convex space.

In 2014, the definition of an M-fuzzifying convex structure (or an M-fuzzifying convexity) is presented in [20]. Recently, F.G. Shi and Z.Y. Xiu [21] further extended it to an M-fuzzy subset of L^X (called an (L, M)-fuzzy convex structure) as follows.

Definition 2.10 ([21]). An (L, M)-*fuzzy convex structure* (or called an (L, M)-fuzzy convexity) is a mapping $\mathscr{C} : L^X \to M$ which satisfies the following conditions.

(**LMFC1**) $\mathscr{C}(\perp_{L^X}) = \mathscr{C}(\top_{L^X}) = \top_M$;

(**LMFC2**) If $\{A_i : i \in I\} \subseteq L^X$ is non-empty, then $\mathscr{C}\left(\bigwedge_{i\in I} A_i\right) \geqslant \bigwedge_{i\in I} \mathscr{C}(A_i)$;

(**LMFC3**) If $\{A_j : j \in J\} \subseteq L^X$ is non-empty and up-directed, then $\mathscr{C}\left(\bigvee^{dir}_{j\in J} A_j\right) \geqslant \bigwedge_{j\in J} \mathscr{C}(A_j)$.

And the pair $(X, \mathscr{C})$ is called an (L, M)-fuzzy convex space.

Remark 2.11. If we denote $\{0, 1\} = \mathbf{2}$ and denote $L/M = [0, 1]$, then a convexity in [28] is a $(\mathbf{2},\mathbf{2})$-fuzzy convexity and an L-convexity in [12] is an $(L, \mathbf{2})$-fuzzy convexity, an M-fuzzifying convexity in [20] is a $(\mathbf{2}, M)$-fuzzy convexity.

The concave structure is closely related to the convex structure and also plays an important role in mathematics. In 2021, I. Alshammari, A. M. Alghamdi and A. Ghareeb [1] generalized it to the (L, M)-fuzzy situation as follows.

Definition 2.12 ([1]). An (L, M)-*fuzzy concave structure* on X is a mapping $\mathscr{C} : L^X \longrightarrow M$ which satisfies the following conditions.

(**LMFCA1**) $\mathscr{C}(\perp_{L^X}) = \mathscr{C}(\top_{L^X}) = \top_M$;

(**LMFCA2**) If $\{A_i : i \in I\} \subseteq L^X$ is non-empty, then $\mathscr{C}\left(\bigvee_{i\in I} A_i\right) \geqslant \bigwedge_{i\in I} \mathscr{C}(A_i)$;

(**LMFCA3**) If $\{A_j : j \in J\} \subseteq L^X$ is co-directed, then $\mathscr{C}\left(\bigwedge^{cdir}_{j\in J} A_j\right) \geqslant \bigwedge_{j\in J} \mathscr{C}(A_j)$. And the pair $(X, \mathscr{C})$ is called an (L, M)-fuzzy concave space.

For a mapping $f : X \longrightarrow Y$, define $f^{\rightarrow}_L : L^X \longrightarrow L^Y$ and $f^{\leftarrow}_L : L^Y \longrightarrow L^X$ by $f^{\rightarrow}_L(A)(y) = \bigvee_{f(x)=y} A(x)$, $f^{\leftarrow}_L(B)(x) = B(f(x))$ for any $A \in L^X$ and $B \in L^Y$. For a mapping $g : Y \longrightarrow Z$, we get $(g \circ f)^{\leftarrow}_L(C)(x) = f^{\leftarrow}_L(g^{\leftarrow}_L(C))(x) = C(g(f(x)))$ for any $C \in L^Z$ in [19].

Definition 2.13 ([21]). A mapping $f : X \longrightarrow Y$ between (L, M)-fuzzy convex spaces $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ is called (L, M)-fuzzy convexity-preserving if $\mathscr{C}_Y(B) \leqslant \mathscr{C}_X(f^{\leftarrow}_L(B))$ for any $B \in L^Y$.

It is easy to check that all (L, M)-fuzzy convex spaces as objects and all (L, M)-fuzzy convexity-preserving mappings as morphisms constitute a category. It is named as a category of (L, M)-fuzzy convex spaces, denoted by **LM-FCon**.

3. *M*-fuzzy category LM-FCon

Although the objects and morphisms of **LM-FCon** are fuzzy, the category itself is a classic category. A natural question arises, can we generalize this crisp category to the fuzzy case?

In this section, we will give a fuzzy method for **LM-FCon**. When we attempt to generalize **LM-FCon** to fuzzy cases, the first and important problem is that how to define the degree to objects and morphisms of a classical category.

Definition 3.1. let $\mathfrak{C}$ be a classical category whose objects are pairs $(X, \mathscr{C})$, where X is a non-empty set and $\mathscr{C} : L^X \longrightarrow M$ is a mapping, and whose morphisms are arbitrary mappings $f : X \longrightarrow Y$. Then the mapping $\omega : Ob_{\mathfrak{C}} \longrightarrow M$ defined by

$$\begin{aligned}\omega(X,\mathscr{C}) &= \mathscr{C}(\perp_{L^X}) \wedge \mathscr{C}(\top_{L^X}) \wedge \left(\bigwedge_{\{A_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \mathscr{C}(A_i) \rightarrowtail \mathscr{C}\left(\bigwedge_{i\in I} A_i\right)\right)\right) \\ &\wedge \left(\bigwedge_{\{A_j\}^{dir}_{j\in J}\subseteq L^X} \left(\bigwedge_{j\in J} \mathscr{C}(A_j) \rightarrowtail \mathscr{C}\left(\bigvee^{dir}_{j\in J} A_j\right)\right)\right),\end{aligned}$$

is called the *degree to which* $(X, \mathscr{C})$ *is an* (L, M)*-fuzzy convex space. And the mapping* $\mu : Mor_{\mathfrak{C}} \longrightarrow M$ *defined by*

$$\mu(f) = \left(\bigwedge_{B \in L^Y} (\mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B))) \right) \wedge \omega(X, \mathscr{C}_X) \wedge \omega(Y, \mathscr{C}_Y),$$

is called the degree to which f *is an* (L, M)*-fuzzy convexity preserving mapping.*

For convenience, we denote

$$\omega_1(X, \mathscr{C}) = \mathscr{C}(\bot_{L^X}) \wedge \mathscr{C}(\top_{L^X}),$$

$$\omega_2(X, \mathscr{C}) = \left(\bigwedge_{\{A_i\}_{i \in I} \subseteq L^X} \left(\bigwedge_{i \in I} \mathscr{C}(A_i) \rightarrowtail \mathscr{C}\left(\bigwedge_{i \in I} A_i \right) \right) \right),$$

$$\omega_3(X, \mathscr{C}) = \left(\bigwedge_{\{A_j\}_{j \in J}^{dir} \subseteq L^X} \left(\bigwedge_{j \in J} \mathscr{C}(A_j) \rightarrowtail \mathscr{C}\left(\bigvee_{j \in J}^{dir} A_j \right) \right) \right),$$

$$\nu(f) = \bigwedge_{B \in L^Y} (\mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B))).$$

Then

$$\omega = \omega_1 \wedge \omega_2 \wedge \omega_3$$

$$\mu(f) = \nu(f) \wedge \omega(X, \mathscr{C}_X) \wedge \omega(Y, \mathscr{C}_Y).$$

When the degree of objects and morphisms are given, we will obtain the important conclusion in the following theorem.

Theorem 3.2. *The triple* $(\mathfrak{C}, \omega, \mu)$ *is an M-valued (or an M-fuzzy) category.*

Proof. (**LCA1**) and (**LCA3**) are easily to be proved. We only need to check the condition (**LCA2**). By Lemma 2.2, we know

$$\begin{aligned} \nu(g \circ f) &= \bigwedge_{D \in L^Z} \left(\mathscr{C}_Z(D) \rightarrowtail \mathscr{C}_X\left((g \circ f)_L^{\leftarrow}(D) \right) \right) \\ &= \bigwedge_{D \in L^Z} \left(\mathscr{C}_Z(D) \rightarrowtail \mathscr{C}_X\left(f_L^{\leftarrow}(g_L^{\leftarrow}(D)) \right) \right) \\ &\geqslant \bigwedge_{D \in L^Z} \left(\mathscr{C}_Z(D) \rightarrowtail \mathscr{C}_Y(g_L^{\leftarrow}(D)) \right) \wedge \left(\mathscr{C}_Y(g_L^{\leftarrow}(D)) \rightarrowtail \mathscr{C}_X\left(f_L^{\leftarrow}(g_L^{\leftarrow}(D)) \right) \right) \\ &\geqslant \left(\bigwedge_{D \in L^Z} \left(\mathscr{C}_Z(D) \rightarrowtail \mathscr{C}_Y(g_L^{\leftarrow}(D)) \right) \right) \wedge \left(\bigwedge_{B \in L^Y} \left(\mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B)) \right) \right) \\ &= \nu(g) \wedge \nu(f) \end{aligned}$$

This shows $\nu(g \circ f) \geqslant \nu(g) \wedge \nu(f)$. Further $\mu(g \circ f) \geqslant \mu(f) \wedge \mu(g)$, which means (**LCA2**) holds. Hence $(\mathfrak{C}, \omega, \mu)$ is an M-fuzzy category. □

Remark 3.3. (1) If $\omega(X, \mathscr{C}) = \top_M$, then $\mathscr{C}(\bot_{L^X}) = \mathscr{C}(\top_{L^X}) = \top_M$; $\bigwedge_{i \in I} \mathscr{C}(A_i) \leqslant \mathscr{C}\left(\bigwedge_{i \in I} A_i\right)$ for all $\{A_i\}_{i \in I} \subseteq L^X$; $\bigwedge_{j \in J} \mathscr{C}(A_j) \leqslant \mathscr{C}\left(\bigvee_{j \in J}^{dir} A_j\right)$ for all $\{A_j\}_{j \in J}^{dir} \subseteq L^X$, which means the pair of $(X, \mathscr{C})$ is precisely an (L, M)-fuzzy convex space.

(2) If $\mu(f) = \top_M$, then $\omega(X, \mathscr{C}_X) = \top_M$, $\omega(Y, \mathscr{C}_Y) = \top_M$, $\nu(f) = \top_M$, namely, $\mathscr{C}_Y(B) \leqslant \mathscr{C}_X(f_L^{\leftarrow}(B)))$ for any $B \in L^Y$. It shows f is precisely an (L, M)-fuzzy convexity-preserving mappings from $(X, \mathscr{C}_X)$ to $(Y, \mathscr{C}_Y)$.

In addition, we can easily get the following conclusions.

Proposition 3.4. *For each mapping $\mathscr{C} : L^X \longrightarrow M$. Let $\neg\mathscr{C} : L^X \longrightarrow M$ be the mapping defined by $\forall A \in L^X$,*

$$\neg\mathscr{C}(A) = \mathscr{C}(A \rightarrowtail \bot_{L^X}).$$

Then we have

(1) $\delta_1(X, \neg\mathscr{C}) \triangleq \neg\mathscr{C}(\bot_{L^X}) \wedge \neg\mathscr{C}(\top_{L^X}) = \omega_1(X, \mathscr{C})$;

(2) $\delta_2(X, \neg\mathscr{C}) \triangleq \bigwedge_{\{U_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \neg\mathscr{C}(U_i) \rightarrowtail \neg\mathscr{C}\left(\bigvee_{i\in I} U_i\right)\right) \geqslant \omega_2(X, \mathscr{C})$;

(3) $\delta_3(X, \neg\mathscr{C}) \triangleq \bigwedge_{\{U_j\}_{j\in J}^{cdir}\subseteq L^X} \left(\bigwedge_{j\in J} \neg\mathscr{C}(U_j) \rightarrowtail \neg\mathscr{C}\left(\bigwedge_{j\in J}^{cdir} U_j\right)\right) \geqslant \omega_3(X, \mathscr{C})$.

Proof. (1) By the definition of $\neg\mathscr{C}$, we have $\delta_1(X, \neg\mathscr{C}) = \neg\mathscr{C}(\bot_{L^X}) \wedge \neg\mathscr{C}(\top_{L^X}) = \mathscr{C}(\bot_{L^X} \rightarrowtail \bot_{L^X}) \wedge \mathscr{C}(\top_{L^X} \rightarrowtail \bot_{L^X}) = \mathscr{C}(\top_{L^X}) \wedge \mathscr{C}(\bot_{L^X}) = \omega_1(X, \mathscr{C})$.

(2) Depend on the definition of $\neg\mathscr{C}$, we have

$$\begin{aligned}
\delta_2(X, \neg\mathscr{C}) &= \bigwedge_{\{U_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \neg\mathscr{C}(U_i) \rightarrowtail \neg\mathscr{C}\left(\bigvee_{i\in I} U_i\right)\right) \\
&= \bigwedge_{\{U_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \mathscr{C}(U_i \rightarrowtail \bot_{L^X}) \rightarrowtail \mathscr{C}\left(\left(\bigvee_{i\in I} U_i\right) \rightarrowtail \bot_{L^X}\right)\right) \\
&= \bigwedge_{\{U_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \mathscr{C}(U_i \rightarrowtail \bot_{L^X}) \rightarrowtail \mathscr{C}\left(\bigwedge_{i\in I} (U_i \rightarrowtail \bot_{L^X})\right)\right) \\
&= \bigwedge_{\{\widetilde{U}_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \mathscr{C}(\widetilde{U}_i) \rightarrowtail \mathscr{C}\left(\bigwedge_{i\in I} \widetilde{U}_i\right)\right) \\
&\geqslant \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X} \left(\bigwedge_{i\in I} \mathscr{C}(A_i) \rightarrowtail \mathscr{C}\left(\bigwedge_{i\in I} A_i\right)\right) = \omega_2(X, \mathscr{C}),
\end{aligned}$$

where $\widetilde{U}_i = U_i \rightarrowtail \bot_{L^X}$.

(3) Similar to (2), we have

$$\begin{aligned}
\delta_3(X, \neg\mathscr{C}) &= \bigwedge_{\{U_j\}_{j\in J}^{cdir}\subseteq L^X} \left(\bigwedge_{j\in J} \neg\mathscr{C}(U_j) \rightarrowtail \neg\mathscr{C}\left(\bigwedge_{j\in J}^{cdir} U_j\right)\right) \\
&= \bigwedge_{\{U_j\}_{j\in J}^{cdir}\subseteq L^X} \left(\bigwedge_{j\in J} \mathscr{C}(U_j \rightarrowtail \bot_{L^X}) \rightarrowtail \mathscr{C}\left(\left(\bigwedge_{j\in J}^{cdir} U_j\right) \rightarrowtail \bot_{L^X}\right)\right) \\
&= \bigwedge_{\{U_j\}_{j\in J}^{cdir}\subseteq L^X} \left(\bigwedge_{j\in J} \mathscr{C}(U_j \rightarrowtail \bot_{L^X}) \rightarrowtail \mathscr{C}\left(\bigvee_{j\in J}^{dir} (U_j \rightarrowtail \bot_{L^X})\right)\right) \\
&= \bigwedge_{\{\widetilde{U}_j\}_{j\in J}^{dir}\subseteq L^X} \left(\bigwedge_{j\in J} \mathscr{C}(\widetilde{U}_j) \rightarrowtail \mathscr{C}\left(\bigvee_{j\in J}^{dir} \widetilde{U}_j\right)\right) \\
&\geqslant \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X} \left(\bigwedge_{j\in J} \mathscr{C}(A_j) \rightarrowtail \mathscr{C}\left(\bigvee_{j\in J}^{dir} A_j\right)\right) = \omega_3(X, \mathscr{C}),
\end{aligned}$$

where $\widetilde{U}_j = U_j \rightarrowtail \bot_{L^X}$ and $\{U_j\}_{j\in J}$ is co-directed implies $\{\widetilde{U}_j\}_{j\in J}$ is up-directed. □

Remark 3.5. In fact, $\delta_1(X,\neg\mathscr{C}) \wedge \delta_2(X,\neg\mathscr{C}) \wedge \delta_3(X,\neg\mathscr{C})$ could be used to define the degree to which $(X,\neg\mathscr{C})$ is an (L,M)-fuzzy concave space. If $\delta_1(X,\neg\mathscr{C}) \wedge \delta_2(X,\neg\mathscr{C}) \wedge \delta_3(X,\neg\mathscr{C}) = \top_M$, then $\neg\mathscr{C}(\bot_{L^X}) = \neg\mathscr{C}(\top_{L^X}) = \top_M$; $\bigwedge_{i\in I}\mathscr{C}(U_i) \leqslant \mathscr{C}\left(\bigvee_{i\in I} U_i\right)$, $\forall \{U_i\}_{i\in I} \subseteq L^X$; $\bigwedge_{j\in J}\mathscr{C}(U_j) \leqslant \mathscr{C}\left(\bigwedge^{cdir}_{j\in J} U_j\right)$, $\forall \{U_j\}^{cdir}_{j\in J} \subseteq L^X$, which means the pair of $(X,\neg\mathscr{C})$ is precisely an (L,M)-fuzzy concave space.

Based on Proposition 3.5, it is easy to get the following conclusions.

Proposition 3.6. *Let $\neg\neg\mathscr{C} : L^X \longrightarrow M$ be the mapping defined by $\forall A \in L^X$,*

$$\neg\neg\mathscr{C}(A) = \neg\mathscr{C}(A \rightarrowtail \bot_{L^X}) = \mathscr{C}((A \rightarrowtail \bot_{L^X}) \rightarrowtail \bot_{L^X}).$$

Then we have

(1) $\omega_1(X,\neg\neg\mathscr{C}) \triangleq \neg\neg\mathscr{C}(\bot_{L^X}) \wedge \neg\neg\mathscr{C}(\top_{L^X}) = \delta_1(X,\neg\mathscr{C}) = \omega_1(X,\mathscr{C})$.

(2) $$\omega_2(X,\neg\neg\mathscr{C}) \triangleq \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}\neg\neg\mathscr{C}(A_i) \rightarrowtail \neg\neg\mathscr{C}\left(\bigwedge_{i\in I} A_i\right)\right) \geqslant \delta_2(X,\neg\mathscr{C}) \geqslant \omega_2(X,\mathscr{C}).$$

(3) $$\omega_3(X,\neg\neg\mathscr{C}) \triangleq \bigwedge_{\{A_j\}^{dir}_{j\in J}\subseteq L^X}\left(\bigwedge_{j\in J}\neg\neg\mathscr{C}(A_j) \rightarrowtail \neg\neg\mathscr{C}\left(\bigvee^{dir}_{j\in J} A_j\right)\right) \geqslant \delta_3(X,\neg\mathscr{C}) \geqslant \omega_3(X,\mathscr{C}).$$

4. The relationships among $\mathfrak{C}_\alpha(L,M,X)$, L-convex structures and (L,M)-fuzzy convex structures

In this section, $\mathfrak{C}_\alpha(L,M,X)$ denote the set of all mappings $\mathscr{C} : L^X \to M$ satisfying $\omega(X,\mathscr{C}) \geqslant \alpha$, it is meaningful to further study the interrelation among $\mathfrak{C}_\alpha(L,M,X)$, L-convex structures and (L,M)-fuzzy convex structures.

For any $\alpha \in M$. Denote $\mathfrak{C}_\alpha(L,M,X) = \{\mathscr{C} : L^X \to M \mid \omega(X,\mathscr{C}) \geqslant \alpha\}$.

If $\omega(X,\mathscr{C}) \geqslant \alpha$, then $\omega_1(X,\mathscr{C}) \geqslant \alpha$, $\omega_2(X,\mathscr{C}) \geqslant \alpha$ and $\omega_3(X,\mathscr{C}) \geqslant \alpha$. Further,

(i) $\omega_1(X,\mathscr{C}) \geqslant \alpha$, this shows $\mathscr{C}(\bot_{L^X}) \geqslant \alpha$, $\mathscr{C}(\top_{L^X}) \geqslant \alpha$;

(ii) $\omega_2(X,\mathscr{C}) \geqslant \alpha$, this shows $\forall \{A_i\}_{i\in I} \subseteq L^X$, $\bigwedge_{i\in I}\mathscr{C}(A_i) \rightarrowtail \mathscr{C}(\bigwedge_{i\in I} A_i) \geqslant \alpha$, i.e., $(\bigwedge_{i\in I}\mathscr{C}(A_i)) \wedge \alpha \leqslant \mathscr{C}(\bigwedge_{i\in I} A_i)$;

(iii) $\omega_3(X,\mathscr{C}) \geqslant \alpha$, this shows $\forall \{A_j\}^{dir}_{j\in J} \subseteq L^X$,$\bigwedge_{j\in J}\mathscr{C}(A_j) \rightarrowtail \mathscr{C}(\bigvee^{dir}_{j\in J} A_j) \geqslant \alpha$, i.e., $(\bigwedge_{i\in I}\mathscr{C}(A_i)) \wedge \alpha \leqslant \mathscr{C}(\bigvee^{dir}_{j\in J} A_j)$.

Definition 4.1. If $\mathscr{C}_1 \leqslant \mathscr{C}_2$ for any $\mathscr{C}_1, \mathscr{C}_2 \in \mathfrak{C}_\alpha(L,M,X)$, then $\mathscr{C}_1$ is called coarser than $\mathscr{C}_2$ (or $\mathscr{C}_2$ is finer than $\mathscr{C}_1$).

Based on the order relation in Definition 4.1, we get the following conclusion.

Theorem 4.2. *$(\mathfrak{C}_\alpha(L,M,X),\leq)$ is a bounded complete lattice.*

Proof. (1) Let $\mathscr{C}_{dis} : L^X \longrightarrow M$ defined by $\forall A \in L^X$, $\mathscr{C}_{dis}(A) = \top_M$. It is obvious that $\mathscr{C}_{dis}$ is the largest element of $\mathfrak{C}_\alpha(L,M,X)$. On the other hand, let $\mathscr{C}_{ind} : L^X \longrightarrow M$ defined by $\forall A \in L^X$,

$$\mathscr{C}_{ind}(A) = \begin{cases} \alpha, & A = \bot_{L^X} \text{ or } A = \top_{L^X}; \\ \bot_M, & A \in L^X \setminus \{\bot_{L^X}, \top_{L^X}\}. \end{cases}$$

Then $\mathscr{C}_{ind}$ is the smallest element of $\mathfrak{C}_\alpha(L,M,X)$.

(2) Next, we verify that $\mathfrak{C}_\alpha(L,M,X)$ is closed for arbitrary intersection. For any $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathfrak{C}_\alpha(L,M,X)$, define the mapping $\mathscr{C}_0 : L^X \longrightarrow M$ by $\forall A \in L^X$, $\mathscr{C}_0(A) = \bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda(A)$. Then

$$\omega_1(X,\mathscr{C}_0) = \bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda(\bot_{L^X}) \wedge \bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda(\top_{L^X}) = \bigwedge_{\lambda\in\Lambda}\omega_1(X,\mathscr{C}_\lambda) \geqslant \alpha.$$

$$
\begin{aligned}
\omega_2(X,\mathscr{C}_0) &= \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}\mathscr{C}_0(A_i)\rightarrowtail\mathscr{C}_0\left(\bigwedge_{i\in I}A_i\right)\right)\\
&= \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}\bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda(A_i)\rightarrowtail\bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda\left(\bigwedge_{i\in I}A_i\right)\right)\\
&\geqslant \bigwedge_{\lambda\in\Lambda}\ \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}\mathscr{C}_\lambda(A_i)\rightarrowtail\mathscr{C}_\lambda\left(\bigwedge_{i\in I}A_i\right)\right)=\bigwedge_{\lambda\in\Lambda}\omega_2((X,\mathscr{C}_\lambda))\geqslant\alpha.
\end{aligned}
$$

$$
\begin{aligned}
\omega_3(X,\mathscr{C}_0) &= \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}\mathscr{C}_0(A_j)\rightarrowtail\mathscr{C}_0\left(\bigvee_{j\in J}^{dir}A_j\right)\right)\\
&= \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}\bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda(A_j)\rightarrowtail\bigwedge_{\lambda\in\Lambda}\mathscr{C}_\lambda\left(\bigvee_{j\in J}^{dir}A_j\right)\right)\\
&\geqslant \bigwedge_{\lambda\in\Lambda}\ \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}\mathscr{C}_\lambda(A_j)\rightarrowtail\mathscr{C}_\lambda\left(\bigvee_{j\in J}^{dir}A_j\right)\right)=\bigwedge_{\lambda\in\Lambda}\omega_3(X,\mathscr{C}_\lambda)\geqslant\alpha.
\end{aligned}
$$

Hence $\omega(X,\mathscr{C}_0)=\omega_1(X,\mathscr{C}_0)\wedge\omega_2(X,\mathscr{C}_0)\wedge\omega_3(X,\mathscr{C}_0)\geqslant\alpha$. This shows $\mathscr{C}_0\in\mathfrak{C}_\alpha(L,M,X)$ and $\mathscr{C}_0$ is the smallest element of $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}$.

(3) For each $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}\subseteq\mathscr{C}_\alpha(L,M,X)$. By the conclusion of (2), we can establish the upper bound of $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}$ is

$$\sup\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}=\bigwedge\{\mathscr{C}\in\mathfrak{C}_\alpha(L,M,X)|\forall\mathscr{C}_\lambda\in\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda},\mathscr{C}_\lambda\leq\mathscr{C}\}.$$

Therefore $(\mathfrak{C}_\alpha(L,M,X),\leq)$ is a bounded complete lattice. □

In what follows, we will discuss the relationships among L-convex structures, (L,M)-fuzzy convex structures and $\mathfrak{C}_\alpha(L,M,X)$.

Theorem 4.3. *If $\mathscr{C}\in\mathfrak{C}_\alpha(L,M,X)$ and $\gamma\leqslant\alpha$ for any $\gamma\in M$, then $\mathscr{C}_{[\gamma]}=\{A\in L^X\mid\mathscr{C}(A)\geqslant\gamma\}$ is an L-convex structure. Conversely, if $\mathscr{C}_{[\gamma]}$ is an L-convex structure for any $\gamma\leqslant\alpha$, then $\mathscr{C}\in\mathfrak{C}_\alpha(L,M,X)$.*

Proof. Firstly, we show $\mathscr{C}_{[\gamma]}$ is an L-convex structure.

(**LC1**) Since $\mathscr{C}(\bot_{L^X})\geqslant\alpha\geqslant\gamma$ and $\mathscr{C}(\top_{L^X})\geqslant\alpha\geqslant\gamma$, we have $\bot_{L^X},\top_{L^X}\in\mathscr{C}_{[\gamma]}$.

(**LC2**) Take any $\{A_i\}_{i\in I}\subseteq\mathscr{C}_{[\gamma]}$. Then $\mathscr{C}(A_i)\geqslant\gamma$, $\forall i\in I$. Since $\alpha\leqslant\bigwedge_{i\in I}\mathscr{C}(A_i)\rightarrowtail\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)$, we get $\alpha\wedge\left(\bigwedge_{i\in I}\mathscr{C}(A_i)\right)\leqslant\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)$. Hence $\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)\geqslant\alpha\wedge\gamma=\gamma$. It shows $\bigwedge_{i\in I}A_i\in\mathscr{C}_{[\gamma]}$.

(**LC3**) The proof is similar to that of (ii) and omitted here.

Conversely, if we want to show that $\mathscr{C}\in\mathfrak{C}_\alpha(L,M,X)$, we need to show $\omega_1(X,\mathscr{C})\geqslant\alpha$, $\omega_2(X,\mathscr{C})\geqslant\alpha$ and $\omega_3(X,\mathscr{C})\geqslant\alpha$.

(i) Take $\gamma=\alpha$. Since $\bot_{L^X},\top_{L^X}\in\mathscr{C}_\alpha$, we have $\mathscr{C}(\bot_{L^X})\wedge\mathscr{C}(\top_{L^X})\geqslant\alpha$, which means $\omega_1(X,\mathscr{C})\geqslant\alpha$.

(ii) For any $\{A_i\}_{i\in I}\subseteq L^X$, take any $\gamma\in M$ such that $\gamma\leqslant\alpha\wedge(\bigwedge_{i\in I}\mathscr{C}(A_i))$. Then $\gamma\leqslant\alpha$ and $\gamma\leqslant\mathscr{C}(A_i)$, i.e., $A_i\in\mathscr{C}_{[\gamma]}$. This means $\bigwedge_{i\in I}A_i\in\mathscr{C}_{[\gamma]}$, i.e., $\gamma\leq\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)$. By the arbitrary of γ, we have $\alpha\wedge\left(\bigwedge_{i\in I}\mathscr{C}(A_i)\right)\leq\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)$. So $\alpha\leqslant\bigwedge_{i\in I}\mathscr{C}(A_i)\rightarrowtail\mathscr{C}\left(\bigwedge_{i\in I}A_i\right)$. Hence $\omega_2(X,\mathscr{C})\geqslant\alpha$.

(iii) The proof of $\omega_3(X,\mathscr{C})\geqslant\alpha$ is similar to $\omega_2(X,\mathscr{C})\geqslant\alpha$ and omitted here. □

Corollary 4.4. *If $\mathscr{C}\in\mathfrak{C}_\alpha(L,M,X)$, then $\mathscr{C}_{[\alpha]}=\{A\in L^X\mid\mathscr{C}(A)\geqslant\alpha\}$ is an L-convex structure.*

Theorem 4.5. *Let $\mathcal{D} \in \mathfrak{C}_\beta(L, M, X)$ and $\alpha \leqslant \beta$. Define the mapping $\mathscr{C} : L^X \longrightarrow M$ by $\forall A \in L^X$,*

$$\mathscr{C}(A) = \alpha \rightarrowtail \mathcal{D}(A).$$

Then $\mathscr{C}$ is an (L, M)-fuzzy convex structure.

Proof. We need to show that $\mathscr{C}$ satisfies (**LMFC1**)-(**LMFC3**).

(**LMFC1**) Since $\alpha \leqslant \beta$, it follows from Lemma 2.2 that $\mathscr{C}(\perp_{L^X}) = \alpha \rightarrowtail \mathcal{D}(\perp_{L^X}) \geqslant \alpha \rightarrowtail \beta \geqslant \alpha \rightarrowtail \alpha = \top_M$, $\mathscr{C}(\top_{L^X}) = \alpha \rightarrowtail \mathcal{D}(\top_{L^X}) \geqslant \alpha \rightarrowtail \beta \geqslant \alpha \rightarrowtail \alpha = \top_M$. Hence $\mathscr{C}(\perp_{L^X}) = \mathscr{C}(\top_{L^X}) = \top_M$.

(**LMFC2**) For any $r \in M$, we have $\alpha \rightarrowtail \gamma \wedge \beta = \alpha \rightarrowtail \gamma$.
In fact, $\alpha \rightarrowtail \gamma = \bigvee\{\lambda | \lambda \wedge \alpha \leqslant \gamma\} \leqslant \bigvee\{\lambda | \lambda \wedge \alpha \wedge \beta \leqslant \gamma \wedge \beta\} = \bigvee\{\lambda | \lambda \wedge \alpha \leqslant \gamma \wedge \beta\} = \alpha \rightarrowtail \gamma \wedge \beta$. On the other hand, since $\gamma \wedge \beta \leqslant \gamma$, we have $\alpha \rightarrowtail \gamma \wedge \beta \leq \alpha \rightarrowtail \gamma$.

Next, take any $\{A_i\}_{i \in I} \subseteq L^X$. Since $\beta \leqslant \bigwedge_{i \in I} \mathcal{D}(A_i) \rightarrowtail \mathcal{D}\left(\bigwedge_{i \in I} A_i\right)$, i.e., $\beta \wedge \left(\bigwedge_{i \in I} \mathcal{D}(A_i)\right) \leqslant \mathcal{D}\left(\bigwedge_{i \in I} A_i\right)$, it follows that

$$\bigwedge_{i \in I} \mathscr{C}(A_i) = \bigwedge_{i \in I} (\alpha \rightarrowtail \mathcal{D}(A_i)) = \alpha \rightarrowtail \bigwedge_{i \in I} \mathcal{D}(A_i) = \alpha \rightarrowtail \beta \wedge \left(\bigwedge_{i \in I} \mathcal{D}(A_i)\right) \leqslant \alpha \rightarrowtail \mathcal{D}\left(\bigwedge_{i \in I} A_i\right) = \mathscr{C}\left(\bigwedge_{i \in I} A_i\right).$$

(**LMFC3**) The proof is similar to (LMFC2), and is omitted here. □

From Theorem 4.5, it is not difficult to get the following corollary.

Corollary 4.6. *If $\mathcal{D} \in \mathfrak{C}_\alpha(L, M, X)$, then the mapping $\mathscr{C} : L^X \longrightarrow M$ defined by $\forall A \in L^X, \mathscr{C}(A) = \alpha \rightarrowtail \mathcal{D}(A)$ is an (L, M)-fuzzy convex structures. Particularly, if $\mathcal{D}$ is an (L, M)-fuzzy convex structure, then the mapping $\mathscr{C}$ is also an (L, M)-fuzzy convex structure.*

The relationships among $\mathfrak{C}_\alpha(L, M, X)$, L-convex structures and (L, M)-fuzzy convex structures can be summarized in the following figure 1.

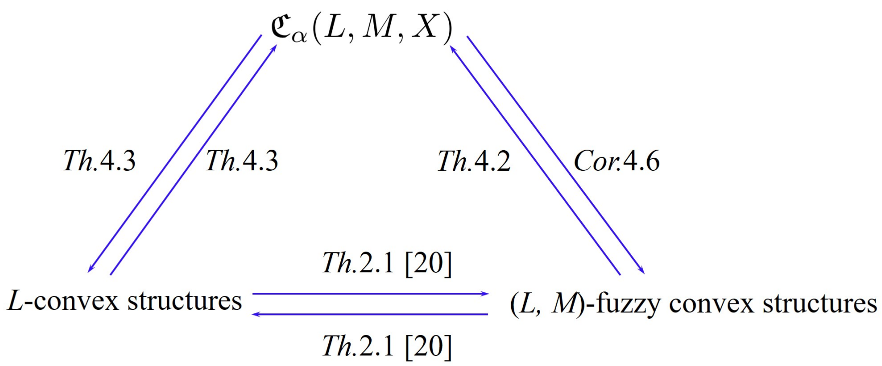

Figure 1: Relationships among $\mathfrak{C}_\alpha(L, M, X)$, L-convex structures, (L, M)-fuzzy convex structures

In what follows, we shall propose the concept of the subbase of $\mathfrak{C}_\alpha(L, M, X)$, and discuss its properties.

Definition 4.7. Let $\mathcal{S} : L^X \longrightarrow M$ be a mapping and let the mapping $\mathscr{C}_\mathcal{S} : L^X \longrightarrow M$ defined by

$$\mathscr{C}_\mathcal{S} = \bigwedge\{\mathscr{C} \in \mathfrak{C}_\alpha(L, M, X) \mid \mathcal{S} \leqslant \mathscr{C}\}.$$

By Theorem 4.2, we get $\mathscr{C}_\mathcal{S} \in \mathfrak{C}_\alpha(L, M, X)$. Then $\mathcal{S}$ is called a subbase of $\mathscr{C}_\mathcal{S}$.

Theorem 4.8. *Let $f : (X, \mathscr{C}_X) \longrightarrow (Y, \mathscr{C}_Y)$ be a mapping. Let $\mathscr{C}_X \in \mathfrak{C}_\beta(L, M, X)$ and $\mathscr{C}_Y \in \mathfrak{C}_\alpha(L, M, Y)$ and $\alpha \leqslant \beta$. If the mapping $\mathcal{S}_Y : L^Y \longrightarrow M$ is a subbase of $\mathscr{C}_Y$, then the following conditions are equivalent.*

(1) $\forall B \in L^Y, \mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B)) \geqslant \alpha$;

(2) $\forall B \in L^Y, \mathcal{S}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B)) \geqslant \alpha$.

Proof. (1) ⇒ (2) It follows from $\mathscr{C}_Y(B) \geqslant \mathcal{S}_Y(B)$ that

$$\alpha \leqslant \mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B)) \leqslant \mathcal{S}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B)).$$

(2) ⇒ (1) Firstly, define the mapping $\mathscr{C}'_Y : L^Y \longrightarrow M$ by $\forall B \in L^Y$, $\mathscr{C}'_Y(B) = \mathscr{C}_X(f_L^{\leftarrow}(B))$. Then $\omega(Y, \mathscr{C}'_Y) \geqslant \alpha$. In fact, $\mathscr{C}_X \in \mathfrak{C}_\beta(L, M, X)$, i.e., $\omega(X, \mathscr{C}_X) \geqslant \beta$, we know $\omega_1(X, \mathscr{C}_X) \geqslant \beta$, $\omega_2(X, \mathscr{C}_X) \geqslant \beta$ and $\omega_3(X, \mathscr{C}_X) \geqslant \beta$. Then $\mathscr{C}_X(\bot_{L^X}) \geqslant \beta$, $\mathscr{C}_X(\top_{L^X}) \geqslant \beta$, and $(\bigwedge_{i\in I} \mathscr{C}_X(A_i)) \wedge \beta \leqslant \mathscr{C}_X(\bigwedge_{i\in I} A_i)$ and $\left(\bigwedge_{j\in J} \mathscr{C}_X(A_j)\right) \wedge \beta \leqslant \mathscr{C}_X(\bigvee^{dir}_{j\in J} A_j)$. Note that $f_L^{\leftarrow}(\bot_{L^Y}) = \bot_{L^X}$, $f_L^{\leftarrow}(\top_{L^Y}) = \top_{L^X}$. Then $\mathscr{C}'_Y(\bot_{L^Y}) = \mathscr{C}_X(f_L^{\leftarrow}(\bot_{L^Y})) = \mathscr{C}_X(\bot_{L^X}) \geqslant \beta$ and $\mathscr{C}'_Y(\top_{L^Y}) = \mathscr{C}_X(f_L^{\leftarrow}(\top_{L^Y})) = \mathscr{C}_X(\top_{L^X}) \geqslant \beta$. Since

$$\mathscr{C}'_Y(\bigwedge_{i\in I} B_i) = \mathscr{C}_X\left(f_L^{\leftarrow}(\bigwedge_{i\in I} B_i)\right) = \mathscr{C}_X\left(\bigwedge_{i\in I} f_L^{\leftarrow}(B_i)\right) \geqslant \left(\bigwedge_{i\in I} \mathscr{C}_X(f_L^{\leftarrow}(B_i))\right) \wedge \beta,$$

$$\mathscr{C}'_Y(\bigvee^{dir}_{j\in J} B_j) = \mathscr{C}_X\left(f_L^{\leftarrow}(\bigvee^{dir}_{j\in J} B_j)\right) = \mathscr{C}_X\left(\bigvee^{dir}_{j\in J} f_L^{\leftarrow}(B_j)\right) \geqslant \left(\bigwedge_{j\in J} \mathscr{C}_X(f_L^{\leftarrow}(B_i))\right) \wedge \beta.$$

This means $\omega_1(Y, \mathscr{C}'_Y) \geqslant \beta$, $\omega_2(Y, \mathscr{C}'_Y) \geqslant \beta$, $\omega_3(Y, \mathscr{C}'_Y) \geqslant \beta$. Hence $\omega(Y, \mathscr{C}'_Y) = \omega_1(Y, \mathscr{C}'_Y) \wedge \omega_2(Y, \mathscr{C}'_Y) \wedge \omega_3(Y, \mathscr{C}'_Y) \geqslant \beta \geqslant \alpha$.

Secondly, define the mapping $\mathscr{C}''_Y : L^Y \longrightarrow M$ by $\forall B \in L^Y$, $\mathscr{C}''_Y(B) = \alpha \rightarrowtail \mathscr{C}'_Y(B)$. By Theorem 4.5, we have $\mathscr{C}''_Y$ is an (L, M)-fuzzy convex structures on Y. Since $\mathcal{S}_Y(B) \leqslant \alpha \rightarrowtail \mathscr{C}'_Y(B) = \mathscr{C}''_Y(B)$ and $\mathcal{S}_Y(B)$ is the subbase of $\mathscr{C}_Y$, it follows that $\mathcal{S}_Y(B) \leqslant \mathscr{C}_Y(B) \leqslant \mathscr{C}''_Y(B) = \alpha \rightarrowtail \mathscr{C}'_Y(B) = \alpha \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B))$. Hence $\alpha \leqslant \mathscr{C}_Y(B) \rightarrowtail \mathscr{C}_X(f_L^{\leftarrow}(B))$. □

5. Subspace, Join space, Quotient space and Product space of $\mathfrak{C}_\alpha(L, M, X)$

In this section, we will use fuzzy powerset operators on M^{L^X} and M^{L^Y} to study the subspace, join space, quotient space and product space of $\mathfrak{C}_\alpha(L, M, X)$.

Let X and Y be non-empty sets and let $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ be a mapping satisfying $f_L^{\leftarrow}(\bigvee_{i\in I} A) = \bigvee_{i\in I} f_L^{\leftarrow}(A)$, $f_L^{\leftarrow}(\bigwedge_{i\in I} B) = \bigwedge_{i\in I} f_L^{\leftarrow}(B)$. And $f_L^{\leftarrow}(\bot_{L^Y}) = \bot_{L^X}$, $f_L^{\leftarrow}(\top_{L^Y}) = \top_{L^X}$.

Now we shall use $f_L^{\leftarrow}$ to define the fuzzy powerset operator on M^{L^Y} and M^{L^X}.

Definition 5.1 ([19])**.** The fuzzy powerset operator $(f_L^{\leftarrow})^{\rightarrow} : M^{L^Y} \longrightarrow M^{L^X}$ is defined by $\forall \mathscr{C}_Y \in M^{L^Y}$, $\forall A \in L^X$, $\forall B \in L^Y$,

$$(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A) = \bigvee\{\mathscr{C}_Y(B) | f_L^{\leftarrow}(B) = A\}.$$

Definition 5.2 ([19])**.** The fuzzy powerset operator $(f_L^{\leftarrow})^{\leftarrow} : M^{L^X} \longrightarrow M^{L^Y}$ is defined by $\forall \mathscr{C}_X \in M^{L^X}$, $\forall B \in L^Y$,

$$(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(B) = \mathscr{C}_X(f_L^{\leftarrow}(B)).$$

We show that the fuzzy powerset operators $(f_L^{\leftarrow})^{\rightarrow}$ and $(f_L^{\leftarrow})^{\leftarrow}$ do not decrease the degree of $\omega(Y, \mathscr{C}_Y)$ and $\omega(X, \mathscr{C}_X)$.

Theorem 5.3. *Let $\mathscr{C}_Y \in M^{L^Y}$. If M is a completely distributive lattice, then $\omega(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega(Y, \mathscr{C}_Y)$.*

Proof. In order to prove $\omega(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega(Y, \mathscr{C}_Y)$, we need to check that $\omega_1(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega_1(Y, \mathscr{C}_Y)$, $\omega_2(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega_2(Y, \mathscr{C}_Y)$ and $\omega_3(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega_3(Y, \mathscr{C}_Y)$.

(i) Since

$$(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(\top_{L^X}) = \bigvee\{\mathscr{C}_Y(B) | f_L^{\leftarrow}(B) = \top_{L^X}\} \geq \mathscr{C}_Y(\top_{L^Y}),$$

$$(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(\bot_{L^X}) = \bigvee\{\mathscr{C}_Y(B) | f_L^{\leftarrow}(B) = \bot_{L^X}\} \geq \mathscr{C}_Y(\bot_{L^Y}),$$

it implies that $\omega_1(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega_1(Y, \mathscr{C}_Y)$.

(ii) Take any $\{A_i\}_{i\in I} \subseteq L^X$. Then

$$\bigwedge_{i\in I}(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A_i) \rightarrowtail (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)\left(\bigwedge_{i\in I} A_i\right) = \bigwedge_{i\in I} \bigvee_{f_L^{\leftarrow}(B_i)=A_i} \mathscr{C}_Y(B_i) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigwedge_{i\in I} A_i} \mathscr{C}_Y(B).$$

If there exists $i \in I$, such that $f_L^{\leftarrow}(B) \neq A_i$ for any $B \in L^Y$, then $\bigwedge_{i\in I}(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A_i) = \bot_M$. This implies $\omega_2(X,(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) = \top_M \geq \omega_2(Y,\mathscr{C}_Y)$.

Assume that for any $i \in I$, there exists $B_i \in L^Y$ such that $f_L^{\leftarrow}(B_i) = A_i$. Then $f_L^{\leftarrow}\left(\bigwedge_{i\in I} B_i\right) = \bigwedge_{i\in I} f_L^{\leftarrow}(B_i) = \bigwedge_{i\in I} A_i$. This shows

$$\begin{aligned}
\omega_2(X,(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) &= \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A_i) \rightarrowtail (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)\left(\bigwedge_{i\in I} A_i\right)\right)\\
&= \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\left(\bigwedge_{i\in I}\bigvee_{f_L^{\leftarrow}(B_i)=A_i}\mathscr{C}_Y(B_i) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigwedge_{i\in I} A_i}\mathscr{C}_Y(B)\right)\\
&\overset{By (G5)}{=\!=} \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\bigwedge_{f_L^{\leftarrow}(B_i)=A_i}\left(\bigwedge_{i\in I}\mathscr{C}_Y(B_i) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigwedge_{i\in I} A_i}\mathscr{C}_Y(B)\right)\\
&\geqslant \bigwedge_{\{A_i\}_{i\in I}\subseteq L^X}\bigwedge_{f_L^{\leftarrow}(B_i)=A_i}\left(\bigwedge_{i\in I}\mathscr{C}_Y(B_i) \rightarrowtail \mathscr{C}_Y\left(\bigwedge_{i\in I} B_i\right)\right)\\
&\geqslant \bigwedge_{\{B_i\}_{i\in I}\subseteq L^Y}\left(\bigwedge_{i\in I}\mathscr{C}_Y(B_i) \rightarrowtail \mathscr{C}_Y\left(\bigwedge_{i\in I} B_i\right)\right) = \omega_2(Y,\mathscr{C}_Y).
\end{aligned}$$

(iii) Take any $\{A_j\}_{j\in J}^{dir} \subseteq L^X$. Then

$$\bigwedge_{j\in J}(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A_j) \rightarrowtail (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)\left(\bigvee_{j\in J}^{dir} A_j\right) = \bigwedge_{j\in J} \bigvee_{f_L^{\leftarrow}(B_j)=A_j} \mathscr{C}_Y(B_j) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigvee_{j\in J}^{dir} A_j} \mathscr{C}_Y(B).$$

As the previous process of proof (ii), it is sufficient to assume that for any $j \in J$, there exists $B_j \in L^Y$ such that $f_L^{\leftarrow}(B_j) = A_j$. Then

$$f_L^{\leftarrow}\left(\bigvee_{j\in J} B_j\right) = \bigvee_{j\in J} f_L^{\leftarrow}(B_j) = \bigvee_{j\in J} A_j.$$

Since $\{A_j\}_{j\in J} \subseteq L^X$ is up-directed, we know $\{B_j\}_{j\in J} \subseteq L^Y$ is also up-directed. This shows

$$\omega_3(X,(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) = \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A_j) \rightarrowtail (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)\left(\bigvee_{j\in J}^{dir} A_j\right)\right)$$

$$= \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}\bigvee_{f_L^{\leftarrow}(B_j)=A_j}\mathscr{C}_Y(B_j) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigvee_{j\in J}^{dir}A_j}\mathscr{C}_Y(B)\right)$$

$$\overset{By (G5)}{=\!=} \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\bigwedge_{f_L^{\leftarrow}(B_j)=A_j}\left(\bigwedge_{j\in J}\mathscr{C}_Y(B_j) \rightarrowtail \bigvee_{f_L^{\leftarrow}(B)=\bigvee_{j\in J}^{dir}A_j}\mathscr{C}_Y(B)\right)$$

$$\geqslant \bigwedge_{\{A_j\}_{j\in J}^{dir}\subseteq L^X}\bigwedge_{f_L^{\leftarrow}(B_j)=A_j}\left(\bigwedge_{j\in J}\mathscr{C}_Y(B_j) \rightarrowtail \mathscr{C}_Y\left(\bigvee_{j\in J}^{dir} B_j\right)\right)$$

$$\geqslant \bigwedge_{\{B_j\}_{j\in J}^{dir}\subseteq L^X}\left(\bigwedge_{j\in J}\mathscr{C}_Y(B_j) \rightarrowtail \mathscr{C}_Y\left(\bigvee_{j\in J}^{dir} B_j\right)\right) = \omega_3(Y,\mathscr{C}_Y).$$

Combining (i),(ii),(iii), we get $\omega(X,(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega(Y,\mathscr{C}_Y)$. □

Theorem 5.4. *Let $\mathscr{C}_X \in M^{L^X}$. Then $\omega(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega(X,\mathscr{C}_X)$.*

Proof. In order to prove $\omega(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega(X,\mathscr{C}_X)$, it suffices to check $\omega_1(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega_1(X,\mathscr{C}_X)$, $\omega_2(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega_2(X,\mathscr{C}_X)$ and $\omega_3(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega_3(X,\mathscr{C}_X)$.

(i) $\omega_1(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) = (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(\bot_{L^Y})\wedge(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(\top_{L^Y}) = \mathscr{C}_X(f_L^{\leftarrow}(\bot_{L^Y}))\wedge\mathscr{C}_X(f_L^{\leftarrow}(\top_{L^Y})) = \mathscr{C}_X(\bot_{L^X})\wedge\mathscr{C}_X(\top_{L^X}) = \omega_1(X,\mathscr{C}_X)$.

(ii) Take any $\{B_i\}_{i\in I} \subseteq L^Y$. Then $f_L^{\leftarrow}\left(\bigwedge_{i\in I} B_i\right) = \bigwedge_{i\in I} f_L^{\leftarrow}(B_i)$ and

$$\omega_2(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) = \bigwedge_{\{B_i\}_{i\in I}\subseteq L^Y}\left(\bigwedge_{i\in I}(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(B_i) \rightarrowtail (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)\left(\bigwedge_{i\in I} B_i\right)\right)$$

$$= \bigwedge_{\{B_i\}_{i\in I}\subseteq L^Y}\left(\bigwedge_{i\in I}\mathscr{C}_X(f_L^{\leftarrow}(B_i)) \rightarrowtail \mathscr{C}_X\left(f_L^{\leftarrow}\left(\bigwedge_{i\in I} B_i\right)\right)\right)$$

$$= \bigwedge_{\{B_i\}_{i\in I}\subseteq L^Y}\left(\bigwedge_{i\in I}\mathscr{C}_X(f_L^{\leftarrow}(B_i)) \rightarrowtail \mathscr{C}_X\left(\bigwedge_{i\in I} f_L^{\leftarrow}(B_i)\right)\right) \geqslant \omega_2(X,\mathscr{C}_X).$$

(iii) Take any $\{B_j\}_{j\in J}^{dir} \subseteq L^Y$. Then $f_L^{\leftarrow}\left(\bigvee_{j\in J} B_j\right) = \bigvee_{j\in J} f_L^{\leftarrow}(B_j)$ and $\{f_L^{\leftarrow}(B_j)\}_{j\in J}$ is also up-directed. This shows

$$\omega_3(Y,(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) = \bigwedge_{\{B_j\}_{j\in J}^{dir}\subseteq L^Y}\left(\bigwedge_{j\in J}(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(B_j) \rightarrowtail (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)\left(\bigvee_{j\in J}^{dir} B_j\right)\right)$$

$$= \bigwedge_{\{B_j\}_{j\in J}^{dir}\subseteq L^Y}\left(\bigwedge_{j\in J}\mathscr{C}_X(f_L^{\leftarrow}(B_j)) \rightarrowtail \mathscr{C}_X\left(f_L^{\leftarrow}\left(\bigvee_{j\in J}^{dir} B_j\right)\right)\right)$$

$$= \bigwedge_{\{B_j\}_{j\in J}^{dir}\subseteq L^Y}\left(\bigwedge_{j\in J}\mathscr{C}_X(f_L^{\leftarrow}(B_j)) \rightarrowtail \mathscr{C}_X\left(\bigvee_{j\in J}^{dir} f_L^{\leftarrow}(B_j)\right)\right) \geqslant \omega_3(X,\mathscr{C}_X).$$

Therefore $\omega(Y, (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega(X, \mathscr{C}_X)$. $\square$

Corollary 5.5. *Assume $\mathscr{C}_Y \in \mathfrak{C}_\alpha(L, M, Y)$. If M is completely distributive, then $(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y) \in \mathfrak{C}_\alpha(L, M, X)$ and it is the coarsest one making $\nu(f) = \top_M$.*

Proof. By Theorem 5.3, we have $\omega(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \geqslant \omega(Y, \mathscr{C}_Y) \geqslant \alpha$. This implies $(X, (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)) \in \mathfrak{C}_\alpha(L, M, X)$. Since

$$(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(f_L^{\leftarrow}(B)) = \bigvee\{\mathscr{C}_Y(\widetilde{B}) | f_L^{\leftarrow}(\widetilde{B}) = f_L^{\leftarrow}(B)\} \geqslant \mathscr{C}_Y(B),$$

we have $\nu(f) = \bigwedge_{B \in L^Y}(\mathscr{C}_Y(B) \rightarrow (f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(f_L^{\leftarrow}(B)) = \top_M$. Assume that $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$ making $\nu(f) = \top_M$, we need to prove that $(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y) \leqslant \mathscr{C}_X$. In fact, $\forall A \in L^X$, $(f_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_Y)(A) = \bigvee\{\mathscr{C}_Y(B) | f_L^{\leftarrow}(B) = A\} \leqslant \mathscr{C}_X(f_L^{\leftarrow}(B)) = \mathscr{C}_X(A)$. $\square$

Corollary 5.6. *Suppose $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$. Then $(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X) \in \mathfrak{C}_\alpha(L, M, Y)$ and it is the finest one making $\nu(f) = \top_M$.*

Proof. By Theorem 5.4, we have $\omega(Y, (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \geqslant \omega(X, \mathscr{C}_X) \geqslant \alpha$. This means $(Y, (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)) \in \mathscr{C}_\alpha(L, M, Y)$. Since $(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(B) = \mathscr{C}_X(f_L^{\leftarrow}(B))$, we have $\nu(f) = \bigwedge_{B \in L^Y}(((f_L^{\leftarrow})^{\leftarrow}\mathscr{C}_X)(B) \rightarrow \mathscr{C}_X(f_L^{\leftarrow}(B))) = \top_M$. Suppose $\mathscr{C}_Y \in \mathscr{C}(L, M, Y)$ making $\nu(f) = \top_M$, we need to prove that $\mathscr{C}_Y \leqslant (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)$. In fact, for each $B \in L^Y$, $(f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)(B) = \mathscr{C}_Y(B)$. $\square$

Finally, we will use fuzzy powerset operators $(f_L^{\leftarrow})^{\rightarrow}$, $(f_L^{\leftarrow})^{\leftarrow}$, Corollary 5.5 and Corollary 5.6 to give definitions of the subspace, join space, quotient space and product space of $\mathscr{C}_\alpha(L, M, X)$ and get some conclusions.

Definition 5.7. ***(Subspace)*** Let M be a completely distribute lattice and let $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$. For each $X_0 \subseteq X$, let $e : X_0 \longrightarrow X$ be the embedding mapping. Define

$$\mathscr{C}_0 = (e_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_X) \in M^{L^{X_0}}.$$

Then we call the pair $(X_0, \mathscr{C}_0)$ as the subspace of $(X, \mathscr{C}_X)$.

Proposition 5.8. *Let M be a completely distributive lattice and $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$, $X_0 \subseteq X$ and $\mathscr{C}_0 = (e_L^{\leftarrow})^{\rightarrow}(\mathscr{C}_X)$. Then the subspace $(X_0, \mathscr{C}_0)$ is the coarsest one that makes $\nu(e) = \top_M$.*

Proof. It is easy to see from Corollary 5.5 and omitted here. $\square$

Definition 5.9. ***(Join space)*** Let $\mathscr{C}_{X_i} \in \mathfrak{C}_\alpha(L, M, X_i)$ for any $i \in I$ and $X = \oplus_{i \in I} X_i$ be the direct sum of the corresponding sets. Suppose $e_i : X_i \longrightarrow X$ is the embedding mapping. Define

$$\mathcal{S}_i = ((e_i)_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_{X_i}) \in M^{L^X}, \ \ \mathscr{C}_X = \bigwedge_{i \in I} \mathcal{S}_i.$$

Then we call the pair $(X, \mathscr{C}_X)$ as the join space of $\{(X_i, \mathscr{C}_{X_i})\}_{i \in I}$.

Proposition 5.10. *Let $\mathscr{C}_{X_i} \in \mathfrak{C}_\alpha(L, M, X_i)$ for any $i \in I$, and $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$, where $X = \oplus_{i \in I} X_i$, $\mathscr{C}_X = \bigwedge_{i \in I} \mathcal{S}_i$. Then the join space $(X, \mathscr{C}_X)$ is the finest one that makes $\nu(e_i) = \top_M$ for any $i \in I$.*

Proof. It is easy to see from Corollary 5.6 and omitted here. $\square$

Definition 5.11. ***(Quotient space)*** Let $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$. Assume $f : X \longrightarrow Y$ to be a surjective mapping. Define

$$\mathscr{C}_Y = (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X) \in M^{L^Y}.$$

Then we call the pair $(Y, \mathscr{C}_Y)$ as the quotient space of $(X, \mathscr{C}_X)$.

Proposition 5.12. *Let $\mathscr{C}_X \in \mathfrak{C}_\alpha(L, M, X)$ and $\mathscr{C}_Y \in \mathfrak{C}_\alpha(L, M, Y)$, where $\mathscr{C}_Y = (f_L^{\leftarrow})^{\leftarrow}(\mathscr{C}_X)$.Then the quotient space $(Y, \mathscr{C}_Y)$ is the finest one making $\nu(f) = \top_M$*

As an important concept for further study, the conclusion about the product space is not self-explanatory, so we give a detailed proof of it.

Definition 5.13. (Product space) Let M be a completely distributive lattice and $\mathscr{C}_{X_i} \in \mathfrak{C}_\alpha(L, M, X_i)$ for any $i \in I$. Let $X = \Pi_{i\in I} X_i$ be the product of the corresponding sets and $p_i : X \to X_i$ be the projections. Define

$$\widehat{\mathscr{C}i} = \left((p_i)_L^{\leftarrow}\right)^{\rightarrow}(\mathscr{C}_{X_i}) \in M^{L^X},\ \ \mathcal{S} = \bigvee_{i\in I} \widehat{\mathscr{C}_i}\,.$$

$\mathscr{C}_X$ is generated by the subbase $\mathcal{S}$. Then we call the pair $(X, \mathscr{C}_X)$ as the product space of $\{(X_i, \mathscr{C}_{X_i})\}_{i\in I}$.

Proposition 5.14. *Let M be a completely distributive lattice and $\mathscr{C}_{X_i} \in \mathfrak{C}_\alpha(L, M, X_i)$ for any $i \in I$. $\mathscr{C}_X$ is generated by the subbase $\mathcal{S}$, $\mathcal{S} = \bigvee_{i\in I} \widehat{\mathscr{C}_i}$, $X = \Pi_{i\in I} X_i$. Then the product space $(X, \mathscr{C}_X)$ is the coarsest one that makes $\nu(p_i) = \top_M$ for any $i \in I$.*

Proof. In order to verify $\nu(p_i) = \top_M$ for any $i \in I$, we need to prove $\mathscr{C}_{X_i}(A_i) \leqslant \mathscr{C}_X\left((p_i)_L^{\leftarrow}(A_i)\right)$ for any $i \in I$ and $A_i \in L^{X_i}$. According to Corollary 5.5, $\widehat{\mathscr{C}_i} \in \mathfrak{C}_\alpha(L, M, X)$ and $\nu(p_i) = \top_M$ for any $i \in I$. This implies $\mathscr{C}_{X_i}(A_i) \leqslant \widehat{\mathscr{C}_i}\left((p_i)_L^{\leftarrow}(A_i)\right)$. Then

$$\mathscr{C}_X\left((p_i)_L^{\leftarrow}(A_i)\right) = \left(\bigwedge_{\mathscr{C}\geqslant\mathcal{S}} \mathscr{C}\right)\left((p_i)_L^{\leftarrow}(A_i)\right) \geqslant \mathcal{S}\left((p_i)_L^{\leftarrow}(A_i)\right) = \left(\bigvee_{i\in I} \widehat{\mathscr{C}_i}\right)\left((p_i)_L^{\leftarrow}(A_i)\right) \geqslant \mathscr{C}_{X_i}(A_i).$$

Next, we prove $(X, \mathscr{C}_X)$ is the coarsest one making $\nu(p_i) = \top_M$ for any $i \in I$. Assume $\widehat{\mathscr{C}_X} \in \mathfrak{C}_\alpha(L, M, X)$ which makes $\nu(p_i) = \top_M$, for any $i \in I$, we need to prove $\mathscr{C}_X \leqslant \widehat{\mathscr{C}_X}$. Since $\forall i \in I, \forall A \in L^X$,

$$\widehat{\mathscr{C}_i}(A) = \left((p_i)_L^{\leftarrow}\right)^{\rightarrow}(\mathscr{C}_{X_i})(A) = \bigvee_{(p_i)_L^{\leftarrow}(A_i)=A} \mathscr{C}_{X_i}(A_i) \leqslant \widehat{\mathscr{C}_X}\left((p_i)_L^{\leftarrow}\right)(A_i) = \widehat{\mathscr{C}_X}(A).$$

It follows that $\widehat{\mathscr{C}_i} \leqslant \widehat{\mathscr{C}_X}$ for any $i \in I$. Further $\mathcal{S} = \bigvee_{i\in I} \widehat{\mathscr{C}_i} \leqslant \widehat{\mathscr{C}_X}$. Hence $\widehat{\mathscr{C}_X} \in \mathfrak{C}_\alpha(L, M, X)$ containing $\mathcal{S}$. Therefore $\mathscr{C}_X \leqslant \widehat{\mathscr{C}_X}$. □

6. Conclusions

In this paper, the degree of the mapping $\omega : Ob_{\mathfrak{C}} \to M$ to be an (L, M)-fuzzy convex space and the degree of the mapping $\mu : Mor_{\mathfrak{C}} \to M$ to be an (L, M)-fuzzy convexity preserving mapping were defined. Afterwards, we gave a fuzzification method of the ordinary category **LM-FCon** and prove that the triple $(\mathfrak{C}, \omega, \mu)$ is an M-fuzzy category. Besides, we introduced the notion of $\mathfrak{C}_\alpha(L, M, X)$ and discussed its subspace, join space, quotient space and especially product space of $\mathfrak{C}_\alpha(L, M, X)$.

Driven by the above work, there are plenty of other research in the framework of M-fuzzy category **LM-FCon** can be considered, such as restricted hull operators, convex hull operators, the JHC and CUP property, etc., which will be the directions of our future works.

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