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Fuzzy $\mathcal{F}$-index of fuzzy zero divisor graphs with MATLAB based algebraic applications

Elif Eryaşar[a,*], **Esra Öztürk Sözen**[a]

[a] *Department of Mathematics, Sinop University, Sinop, Türkiye*

Abstract. This article aims to explore the theoretical properties of topological descriptors used in fuzzy graph theory, which combines elements of graph theory and fuzzy set theory, within algebraic structures. For this purpose, fuzzy $\mathcal{F}$-index (briefly $\mathcal{FF}$-index) has been formulated theoretically for the fuzzy zero divisor graphs of the commutative ring $\mathbb{Z}_n$, where $n = \wp^{\alpha}$, $\wp_1\wp_2$, $\wp_1^2\wp_2$, $\wp_1^2\wp_2^2$, $\wp_1\wp_2\wp_3$ ($\wp_1$, $\wp_2$, $\wp_3$ are primes and $\alpha \geq 3$). In particular, a SageMath-based drawing algorithm that embodies the fuzzy graph structures of the rings is presented for application based convenience. Furthermore, a MATLAB based code was created that directly calculate the fuzzy $\mathcal{F}$-index of the fuzzy zero divisor graphs for every non-prime value of $n > 1$.

1. Introduction

Graph theory is a mathematical discipline that revolves around the analysis of graphs, that is used to depict relations (by edges) between objects (by vertices) via graphical models.

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ and $\mathcal{E}$ representing the set of vertices and edges. The notation $u \sim v$ indicates the relationship between two adjacent vertices $u, v \in \mathcal{V}$. In $\mathcal{G}$, the degree ($\partial(v)$) of a vertex v is determined by counting the number of edges that are connected to it. A weighted graph is a graph whose edges are of numerical values assigned to each edge [7]. A complete graph is a graph where every vertex is connected to every other vertex by an edge. It is represented by K_n, where n represents the number of vertices in the graph, and it has a total of $\frac{n(n-1)}{2}$ edges [23]. A complete bipartite graph is a type of graph where the vertices can be divided into two disjoint sets, and every vertex in one set is connected to every vertex in the other set. In other words, if the sets are denoted as A and B, then every vertex in set A is connected to every vertex in set B. This type of graph is denoted as $K_{m,n}$ where m is the number of vertices in set A and n is the number of vertices in set B [10].

Zero divisor graphs arise from the interaction of graph theory and algebraic structures. In 1988, the zero divisor graph for a commutative ring is proposed in [8], which represents each element of the ring as vertices and focuses on colorings as its main point of interest. The zero divisor graph, handled in [5], is of the vertices which are the nonzero zero divisors and any two of them whose product is zero considered as

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* Corresponding author: Elif Eryaşar
Email addresses: eeryasar@sinop.edu.tr (Elif Eryaşar), esozen@sinop.edu.tr (Esra Öztürk Sözen)
ORCID iDs: https://orcid.org/0000-0002-9852-6662 (Elif Eryaşar), https://orcid.org/0000-0002-2632-2193 (Esra Öztürk Sözen)

adjacent. The zero divisor graph can be used as a tool for studying the algebraic properties of commutative rings via mathematical modeling. Interesting results can be explored for algebraic structures by translating these properties via graph theory language (see [2], [6], [12], [15], [19], [24], [25]).

Chemical graph theory is a popular and functional field of research in modern times. In this field, scientists use graphs, as a tool representing mathematics and other sciences, to explore the structures of compounds. One of the most functional and fundamental concepts here is topological indices. They are numeric values linked to graphs that remain unchanged under graph automorphisms. In particular, they contribute to the investigation and separation of various physical properties of chemicals.

Topological indices play a fundamental role in numerous fields, including molecular chemistry, chemical graph theory, spectral graph theory, and network theory. The Zagreb index, a degree-based topological index, was introduced by Gutman and Trinajstić in 1972 [13] and is used to calculate the π-electron energy of a conjugate system. Furtula and Gutman introduced a new topological index known as the forgotten topological index (F-index) [11], and demonstrated that the F-index is calculated according to the following formula.

$$F(G) = \sum_{vertices} (\partial(i))^3 .$$

In [4], Amin and Nayeem conducted a study on the F-index and F-coindex of subdivision graph and line graph. Abdo et al. [1] in 2017 investigated extremal trees in terms of the F-index. Akhter et al. [3] established the extremal unicyclic and bicyclic graphs with respect to the F-index. De et al. [9] researched the F-index in the context of certain graph operations. In 2021, Islam and Pal [16] introduced and studied the fuzzy $\mathcal{F}$-index (briefly $\mathcal{FF}$- index) for fuzzy graphs. They also presented a number of intriguing findings on the $\mathcal{FF}$-index for a variety of fuzzy graphs, including paths, cycles, stars, complete fuzzy graphs, and etc. [17].

1.1. *Works that are the source of motivation for the study*

In [14], [26], [27], the authors have investigated F-index over graphs for various commutative rings. Also Kuppan and Sankar [20], [21] defined the concept of a fuzzy zero divisor graph for a commutative ring in 2021. The inclusion of fuzzy graphs enables the assignment of fuzzy edge weights, indicating the degree of membership or potential for a connection between vertices. Through the utilization of fuzzy logic and reasoning techniques on fuzzy graphs, it becomes possible to carry out computations and make deductions, even in cases where data is imprecise or incomplete.

1.2. *Significance and objective of the article*

In recent times, a great many researchers have been engaged in the study of topological indices, thanks to the various applications of fuzzy graph theory. In the light of the mentioned sources on motivation of the study and because of the advantages of fuzzy graphs than the traditional ones, we analyzed the fuzzy $\mathcal{F}$-index of fuzzy zero divisor graph of the commutative rings $\mathbb{Z}_n$ for $n = \wp^\alpha,\ \wp_1\wp_2,\ \wp_1^2\wp_2,\ \wp_1^2\wp_2^2,\ \wp_1\wp_2\wp_3$ ($\wp_1,\ \wp_2,\ \wp_3$ are primes and $\alpha \geq 3$) in Section 3. In section 4, we also provide SageMath and MATLAB codes for fuzzy $\mathcal{F}$-index of $\mathcal{G}_f(\mathbb{Z}_n)$. In the last section of the study, the developed codes were used and their applications were included in the rings we were working on. In this way, the $\mathcal{FF}$-index of the fuzzy zero divisor graph can be calculated for every non-prime value of $n > 1$.

2. Preliminaries

This part of the article includes basic information that will be needed in the following sections.

Definition 2.1. ***(Fuzzy Zero Divisor Graph)***A fuzzy zero divisor graph $\mathcal{G}_f = (\mathcal{V}, \sigma = \mathcal{V}_f, \mu = \mathcal{E}_f)$ a triple including a non-empty set $\mathcal{V}$, which is the set of nonzero zero divisors of $\mathbb{Z}_n$, with the pair of functions

$$\sigma : \mathcal{V} \longrightarrow (0,1)$$

and

$$\mu : \mathcal{V} \times \mathcal{V} \longrightarrow (0,1]$$

provided that $\sigma(v_i) = \frac{v_i}{n}$ and $\mu(v_i v_j) = \frac{n}{v_i v_j}$ for all $v_i, v_j \in \mathcal{V}$ [21].

Definition 2.2. In a fuzzy graph, the degree ($\partial(v)$) of vertex v is calculated as the sum of the weights of all its corresponding edges [18].

Definition 2.3. Let $\mathcal{G}_f$ be a fuzzy graph. The fuzzy $\mathcal{F}$-index ($\mathcal{FF}(\mathcal{G}_f)$) is defined as follows [17]:

$$\mathcal{FF}(\mathcal{G}_f) = \sum_{u_i \in \sigma} [\sigma(u_i)\partial(u_i)]^3; \ \forall u_i \in \mathcal{V}.$$

3. Main results for fuzzy $\mathcal{F}$-index

Theorem 3.1. *Let $n = \wp^\alpha$ where $\wp > 2$ is a prime integer and $\alpha \in \mathbb{N}$ ($\alpha \geq 3$). Then the $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_n)$ is as follows:*

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp^\alpha})) = \sum_{i=1}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=1}^{|P_i|} \frac{1}{(\wp^{\alpha-i})^3} \left[\sum_{t=\alpha-i}^{\alpha-1} \left(\sum_{x \in P_t^x} \frac{1}{\wp^{t+i-\alpha}} \frac{1}{x} \right) \right]^3 + \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-1} \sum_{j=1}^{|P_i|} \frac{1}{(\wp^{\alpha-i})^3} \left[\sum_{t=\alpha-i}^{\alpha-1} \left(\sum_{x \in P_{t \neq i}^x} \frac{1}{\wp^{t+i-\alpha}} \frac{1}{x} + \sum_{\substack{x \in P_{t=i}^x \\ x \neq j}} \frac{1}{\wp^{t+i-\alpha}} \frac{1}{x} \right) \right]^3.$$

Proof. It is possible to group the zero divisors of $\mathbb{Z}_{\wp^\alpha}$ by considering the following set partitions.

$$P_1 = \{\wp x : x = 1, 2, ..., \wp^{\alpha-1} - 1, \wp \nmid x\},$$
$$P_2 = \{\wp^2 x : x = 1, 2, ..., \wp^{\alpha-2} - 1, \wp \nmid x\},$$
$$\vdots$$
$$P_i = \{\wp^i x : x = 1, 2, ..., \wp^{\alpha-i} - 1, \wp \nmid x\},$$
$$\vdots$$
$$P_{\alpha-1} = \{\wp^{\alpha-1} x : x = 1, 2, ..., \wp - 1, \wp \nmid x\}.$$

This partition is the family of two disjoint subsets of $\mathcal{V}(\mathcal{G}_f(\mathbb{Z}_n))$, whose combination gives the vertex set, and also P_i^x also indicates the set of the elements x providing $\wp^i x \in P_i$. $\forall a \in P_i (1 \leq i \leq \lfloor \frac{\alpha-1}{2} \rfloor)$ is of the form $a = \wp^i x$ ($x = 1, ..., \wp^{\alpha-i} - 1; (\wp, x) = 1$). An edge is established between a and each element b of sets P_t that satisfying the condition $\alpha - i \leq t < \alpha - 1$. According to this,

$$\sigma(a) = \frac{\wp^i x}{\wp^\alpha} = \frac{x}{\wp^{\alpha-i}},$$

$$\partial(a) = \sum_{b \in \mathcal{V}} \mu(ab) = \sum_{t=\alpha-i}^{\alpha-1} \sum_{b \in P_t} \mu(ab) = \sum_{b \in P_{\alpha-i}} \mu(ab) + \sum_{b \in P_{\alpha-i+1}} \mu(ab) + ... + \sum_{b \in P_{\alpha-2}} \mu(ab) + \sum_{b \in P_{\alpha-1}} \mu(ab)$$
$$= \left[\frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i}1)} + \frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i}2)} + ... + \frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i}(\wp^i - 1))} \right] +$$
$$\left[\frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i+1}1)} + \frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i+1}2)} + ... + \frac{\wp^\alpha}{(\wp^i x)(\wp^{\alpha-i+1}(\wp^{i-1} - 1))} \right] + ... +$$

$$\left[\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-2}1)}+\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-2}2)}+...+\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-2}(\wp^{2}-1))}\right]+$$

$$\left[\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-1}1)}+\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-1}2)}+...+\frac{\wp^{\alpha}}{(\wp^{i}x)(\wp^{\alpha-1}(\wp-1))}\right]$$

$$=\left[\frac{1}{x}+\frac{1}{2x}+...+\frac{1}{(\wp^{i}-1)x}\right]+\left[\frac{1}{\wp x}+\frac{1}{2\wp x}+...+\frac{1}{(\wp^{i-1}-1)\wp x}\right]+...+$$

$$\left[\frac{1}{\wp^{i-2}x}+\frac{1}{2\wp^{i-2}x}+...+\frac{1}{(\wp^{2}-1)\wp^{i-2}x}\right]+\left[\frac{1}{\wp^{i-1}x}+\frac{1}{2\wp^{i-1}x}+...+\frac{1}{(\wp-1)\wp^{i-1}x}\right]$$

$$=\frac{1}{x}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{i}-1}\right]+\frac{1}{\wp x}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{i-1}-1}\right]+...+$$

$$\frac{1}{\wp^{i-2}x}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{2}-1}\right]+\frac{1}{\wp^{i-1}x}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp-1}\right].$$

$$\sigma(a)\partial(a)=\frac{1}{\wp^{\alpha-i}}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{i}-1}\right]+\frac{1}{\wp^{\alpha-i+1}}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{i-1}-1}\right]$$

$$+...+\frac{1}{\wp^{\alpha-2}}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp^{2}-1}\right]+\frac{1}{\wp^{\alpha-1}}\left[\frac{1}{1}+\frac{1}{2}+...+\frac{1}{\wp-1}\right]$$

$$=\frac{1}{\wp^{\alpha-i}}\sum_{x\in P^{x}_{\alpha-i}}\frac{1}{x}+\frac{1}{\wp^{\alpha-i+1}}\sum_{x\in P^{x}_{\alpha-i+1}}\frac{1}{x}+...+\frac{1}{\wp^{\alpha-2}}\sum_{x\in P^{x}_{\alpha-2}}\frac{1}{x}+\frac{1}{\wp^{\alpha-1}}\sum_{x\in P^{x}_{\alpha-1}}\frac{1}{x}$$

$$=\frac{1}{\wp^{\alpha-i}}\left[\sum_{x\in P^{x}_{\alpha-i}}\frac{1}{x}+\sum_{x\in P^{x}_{\alpha-i+1}}\frac{1}{\wp}\frac{1}{x}+...+\sum_{x\in P^{x}_{\alpha-2}}\frac{1}{\wp^{i-1}}\frac{1}{x}+\sum_{x\in P^{x}_{\alpha-1}}\frac{1}{\wp^{i-2}}\frac{1}{x}\right]$$

$$=\frac{1}{\wp^{\alpha-i}}\left[\sum_{t=\alpha-i}^{\alpha-1}\left(\sum_{x\in P^{x}_{t}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}\right)\right].$$

When a similar analysis is made for each $a\in P_i$ ($\left\lceil\frac{\alpha}{2}\right\rceil\leq i\leq\alpha-1$), each of the set partitions in the lower half will match itself. Meanwhile, since it will establish an edge with all the other elements in the same fragmentation, except of the elements themselves, it is in the form

$$\sigma(a)\partial(a)=\frac{1}{\wp^{\alpha-i}}\left[\sum_{t=\alpha-i}^{\alpha-1}\left(\sum_{x\in P^{x}_{t\neq i}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}+\sum_{x\in P^{x}_{t=i}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}\right)\right].$$

Then, we get

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp^{\alpha}}))=\sum_{i=1}^{\left\lfloor\frac{\alpha-1}{2}\right\rfloor}\sum_{j=1}^{|P_i|}\frac{1}{(\wp^{\alpha-i})^{3}}\left[\sum_{t=\alpha-i}^{\alpha-1}\left(\sum_{x\in P^{x}_{t}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}\right)\right]^{3}+\sum_{i=\left\lceil\frac{\alpha}{2}\right\rceil}^{\alpha-1}\sum_{j=1}^{|P_i|}\frac{1}{(\wp^{\alpha-i})^{3}}\left[\sum_{t=\alpha-i}^{\alpha-1}\left(\sum_{x\in P^{x}_{t\neq i}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}+\sum_{\substack{x\in P^{x}_{t=i}\\ x\neq j}}\frac{1}{\wp^{t+i-\alpha}}\frac{1}{x}\right)\right]^{3}.$$

□

Corollary 3.2. *Let $\wp$ be a prime number, then:*

1. *If $\wp=2$, then $\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp^{2}}))=0$,*

2. *If* $\wp > 2$, *then* $\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp^2})) = \frac{1}{\wp^3}\sum_{j=1}^{\phi(\wp)}\left[\sum_{\substack{i=1\\ i\neq j}}^{\wp-1}\frac{1}{i}\right]^3$.

Proof. 1. Given that the ring $\mathbb{Z}_n$ has only $\overline{2}$ as a zero divisor for $n = 4$, it is obvious from the relationship $\sigma(2) = \frac{1}{2}$ and $\partial(2) = 0$ that $FF(\mathcal{G}_f(\mathbb{Z}_{\wp^2})) = 0$.

2. For each positive prime number $\wp > 2$, all zero divisor elements of the integer set $\mathbb{Z}_{\wp^2}$ are grouped into the set $\mathcal{V}$, which is defined to be $\{\wp, 2\wp, ..., (\wp-1)\wp\}$. For $\forall x \in \mathcal{V}$, it is of the form $x = k\wp$ ($k = \overline{(1, \wp-1)}$) and $\mathcal{V}_f = \{\sigma(x) : x \in \mathcal{V}\} = \left\{\frac{1}{\wp}, \frac{2}{\wp}, \frac{3}{\wp} \cdots \frac{\wp-1}{\wp}\right\}$. Moreover, as $\mathbb{Z}_{\wp^2}$ is a complete graph, then $x \in \mathcal{V}$ will be adjacent to each element of $\mathcal{V} - \{x\}$. Therefore,

$$\partial(x) = \sum_{v\in\mathcal{V}}\mu(xv) = \frac{1}{k} + \frac{1}{2k} + \frac{1}{3k} + ... + \frac{1}{(k-1)k} + \frac{1}{(k+1)k} + ... + \frac{1}{(\wp-1)k}.$$

According to this,

$$\sigma(x)\partial(x) = \frac{k}{\wp}\frac{1}{k}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k-1} + \frac{1}{k+1} + ... + \frac{1}{\wp-1}\right].$$

In conclusion,

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp^2})) = \frac{1}{\wp^3}\sum_{j=1}^{\phi(\wp)}\left[\sum_{\substack{i=1\\ i\neq j}}^{\wp-1}\frac{1}{i}\right]^3$$

where ϕ indicate the Euler's function. □

Theorem 3.3. *Let* $\wp_1$ *and* $\wp_2$ *distinct prime numbers. Then, the fuzzy* $\mathcal{F}$*-index of* $\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2})$ *has the following form:*

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2})) = \frac{\wp_2-1}{\wp_2^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{\wp_1^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3.$$

Proof. Since $\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2})$ is a complete bipartite graph, set of the vertices $\mathcal{V}$ can be decomposed into two subsets as:

$$P_1 = \{\wp_1 x : x = 1, 2, ..., \wp_2 - 1\},$$

$$P_2 = \{\wp_2 x : x = 1, 2, ..., \wp_1 - 1\},$$

where, $P_1 \cap P_2 = \emptyset$ and $P_1 \cup P_2 = \mathcal{V}$. For any $a \in P_1$, $\sigma(a)\partial(a) = \frac{1}{\wp_2}\sum_{i=1}^{\phi(\wp_1)}\frac{1}{i}$; similarly for any $a \in P_2$, $\sigma(a)\partial(a) = \frac{1}{\wp_1}\sum_{i=1}^{\phi(\wp_2)}\frac{1}{i}$. As P_1 and P_2 are of the orders $\phi(\wp_2) = \wp_2 - 1$ and $\phi(\wp_1) = \wp_1 - 1$, respectively, then the form of the $\mathcal{FF}$-index is obtained as

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2})) = \sum_{i=1}^{2}\sum_{a\in P_i}[\sigma(a)\partial(a)]^3$$

$$= |P_1|\left[\frac{1}{\wp_2}\left(\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right)\right]^3 + |P_2|\left[\frac{1}{\wp_1}\left(\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right)\right]^3$$

$$= (\wp_2-1)\left[\frac{1}{\wp_2}\left(\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right)\right]^3 + (\wp_1-1)\left[\frac{1}{\wp_1}\left(\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right)\right]^3$$

$$= \frac{\wp_2 - 1}{\wp_2^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{\wp_1 - 1}{\wp_1^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3.$$

□

Theorem 3.4. *Let $\wp_1$ and $\wp_2$ distinct prime numbers. Then, the fuzzy $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2})$ has the following form:*

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2})) = \frac{(\wp_1-1)(\wp_2-1)}{(\wp_1\wp_2)^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{\wp_1^5}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + \frac{\wp_2-1}{\wp_2^3}\left[\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{\wp_1^3}\sum_{j=1}^{\wp_1-1}\left[\sum_{\substack{i=1\\ i\neq j^2\wp_2}}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3.$$

Proof. Let us list the vertex partition sets of $\mathcal{V}$ as follows:

$$P_1 = \{\wp_1 x : x = 1, 2, ..., \wp_1\wp_2 - 1; \wp_1 \nmid x, \wp_2 \nmid x\},$$
$$P_2 = \{\wp_2 x : x = 1, 2, ..., \wp_2^2 - 1; \wp_1 \nmid x\},$$
$$P_3 = \{\wp_1^2 x : x = 1, 2, ..., \wp_2 - 1\},$$
$$P_4 = \{\wp_1\wp_2 x : x = 1, 2, ..., \wp_1 - 1\},$$

where, $\cup_{i=1}^4 P_i = \mathcal{V}$ and $P_i \cap P_j = \emptyset$ for any distinct $i, j \in \{1, 2, 3, 4\}$. Here, note that $|P_1| = (\wp_1-1)(\wp_2-1)$, $|P_2| = \wp_1(\wp_1 - 1)$, $|P_3| = \wp_2 - 1$, $|P_4| = \wp_1 - 1$.

Morever, for every element $a \in \mathcal{V}$ we have,

$$\sigma(a)\partial(a) = \frac{1}{\wp_1\wp_2}\sum_{i=1}^{\wp_1-1}\frac{1}{i},\ a \in P_1,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1^2}\sum_{i=1}^{\wp_2-1}\frac{1}{i},\ a \in P_2,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_2}\sum_{i=1}^{\wp_1^2-1}\frac{1}{i},\ a \in P_3,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1}\sum_{i=1}^{\wp_1\wp_2-1}\frac{1}{i},\ a \in P_4,$$

for any $\{P_i\}_{i\in I=\{1,2,3,4\}}$. Hence,

$$F(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2})) = \sum_{a\in\mathcal{V}}[\sigma(a)\partial(a)]^3$$

$$= \sum_{a\in\mathcal{V}_1}[\sigma(a)\partial(a)]^3 + \sum_{a\in\mathcal{V}_2}[\sigma(a)\partial(a)]^3 + \sum_{a\in\mathcal{V}_3}[\sigma(a)\partial(a)]^3 + \sum_{a\in\mathcal{V}_4}[\sigma(a)\partial(a)]^3$$

$$= |P_1|\left[\frac{1}{\wp_1\wp_2}\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + |P_2|\left[\frac{1}{\wp_1^2}\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + |P_3|\left[\frac{1}{\wp_2}\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3 + |P_4|\left[\frac{1}{\wp_1}\sum_{\substack{i=1\\ i\neq\wp_2}}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3$$

$$= (\wp_1-1)(\wp_2-1)\left[\frac{1}{\wp_1\wp_2}\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \wp_1(\wp_1-1)\left[\frac{1}{\wp_1^2}\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + (\wp_2-1)\left[\frac{1}{\wp_2}\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3 + (\wp_1-1)\left[\frac{1}{\wp_1}\sum_{i=1}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3$$

Then, we get

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2})) = \frac{(\wp_1-1)(\wp_2-1)}{(\wp_1\wp_2)^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{\wp_1^5}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + \frac{\wp_2-1}{\wp_2^3}\left[\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{\wp_1^3}\sum_{j=1}^{\wp_1-1}\left[\sum_{\substack{i=1\\ i\neq j^2\wp_2}}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3.$$

□

Theorem 3.5. *Let $\wp_1$ and $\wp_2$ distinct prime numbers. Then, the $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2^2})$ has the following form:*

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2^2})) = \frac{(\wp_1-1)(\wp_2-1)}{\wp_1^3\wp_2^5}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{(\wp_1-1)(\wp_2-1)}{\wp_1^5\wp_2^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + \frac{\wp_2-1}{\wp_2^5}\left[\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3$$

$$+\frac{\wp_1-1}{\wp_1^5}\left[\sum_{i=1}^{\wp_2^2-1}\frac{1}{i}\right]^3 + \frac{1}{\wp_1^3\wp_2^3}\sum_{j=1}^{(\wp_1-1)(\wp_2-1)}\left[\sum_{\substack{i_j=1\\ i_j\neq j}}^{\wp_1\wp_2-1}\frac{1}{i_j}\right]^3 + \frac{1}{\wp_2^3}\sum_{j=1}^{\wp_2-1}\left[\sum_{\substack{i=1\\ i\neq j^2\wp_1^2}}^{\wp_1^2\wp_2-1}\frac{1}{i}\right]^3 + \frac{1}{\wp_1^3}\sum_{j=1}^{\wp_1-1}\left[\sum_{\substack{i=1\\ i\neq j^2\wp_2^2}}^{\wp_1\wp_2^2-1}\frac{1}{i}\right]^3.$$

Proof. The set of vertex of the fuzzy graph $\mathcal{V}$, $\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2^2})$ is divided as follows, since the integer $\wp_1^2\wp_2^2$ has 7-proper divisor. The family of the sets $\{P_i\}_{i\in I=\{1,2,3,4,5,6,7\}}$ is a decomposition of $\mathcal{V}$.

$P_1 = \{\wp_1 x : x = 1,2,...,\wp_1\wp_2^2-1; \wp_1 \nmid x, \wp_2 \nmid x\}$,

$P_2 = \{\wp_2 x : x = 1,2,...,\wp_1^2\wp_2-1; \wp_1 \nmid x, \wp_2 \nmid x\}$,

$P_3 = \{\wp_1^2 x : x = 1,2,...,\wp_2^2-1; \wp_2 \nmid x\}$,

$P_4 = \{\wp_2^2 x : x = 1,2,...,\wp_1^2-1; \wp_1 \nmid x\}$,

$P_5 = \{\wp_1\wp_2 x : x = 1,2,...,\wp_1\wp_2-1; \wp_1 \nmid x, \wp_2 \nmid x\}$,

$P_6 = \{\wp_1^2\wp_2 x : x = 1,2,...,\wp_2-1\}$,

$P_7 = \{\wp_1\wp_2^2 x : x = 1,2,...,\wp_1-1\}$,

where, $|P_1| = (\wp_1-1)(\wp_2^2-\wp_2)$, $|P_2| = (\wp_1^2-\wp_1)(\wp_2-1)$, $|P_3| = \wp_2(\wp_2-1)$, $|P_4| = \wp_1(\wp_1-1)$, $|P_5| = (\wp_1-1)(\wp_2-1)$, $|P_6| = (\wp_2-1)$, $|P_7| = (\wp_1-1)$.

Morever, for every element $a \in \mathcal{V}$,

$$\sigma(a)\partial(a) = \frac{1}{\wp_1\wp_2^2}\sum_{i=1}^{|P_7|}\frac{1}{i},\ a \in P_1,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1^2\wp_2}\sum_{i=1}^{|P_6|}\frac{1}{i},\ a \in P_2,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_2^2}\sum_{i=1}^{|P_4|+|P_7|}\frac{1}{i},\ a \in P_3,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1^2}\sum_{i=1}^{|P_3|+|P_6|}\frac{1}{i},\ a \in P_4,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1\wp_2}\sum_{\substack{i_j=1\\(i_j\neq j)}}^{|P_5|+|P_6|+|P_7|-1}\frac{1}{i_j},\ a\in P_5\ (1\le j\le \wp_1\wp_2-1),$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_2}\sum_{\substack{i=1\\(i\neq j^2\wp_1^2)}}^{|P_2|+|P_4|+|P_5|+|P_6|+|P_7|-1}\frac{1}{i},\ a\in P_6\ (1\le j\le \wp_2-1),$$

$$\sigma(a)\partial(a) = \sum_{\substack{i=1\\(i\neq j^2\wp_2^2)}}^{|P_1|+|P_3|+|P_5|+|P_6|+|P_7|-1}\frac{1}{i},\ a\in P_7\ (1\le j\le \wp_1-1)$$

can be determined as the forms which are given above. Then, by from the $\sigma(a)\partial(a)$, we get

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2^2})) = \sum_{a\in V}[\sigma(a)\partial(a)]^3$$

$$= \sum_{i=1}^{7}\sum_{a\in P_i}[\sigma(a)\partial(a)]^3$$

$$= \sum_{a\in P_1}[\sigma(a)\partial(a)]^3+\sum_{a\in P_2}[\sigma(a)\partial(a)]^3+\sum_{a\in P_3}[\sigma(a)\partial(a)]^3+\sum_{a\in P_4}[\sigma(a)\partial(a)]^3+\sum_{a\in P_5}[\sigma(a)\partial(a)]^3+\sum_{a\in P_6}[\sigma(a)\partial(a)]^3+\sum_{a\in P_7}[\sigma(a)\partial(a)]^3$$

$$= |P_1|\,[\sigma(a)\partial(a)]^3+|P_2|\,[\sigma(a)\partial(a)]^3+|P_3|\,[\sigma(a)\partial(a)]^3+|P_4|\,[\sigma(a)\partial(a)]^3\ |P_5|\,[\sigma(a)\partial(a)]^3+|P_6|\,[\sigma(a)\partial(a)]^3+|P_7|\,[\sigma(a)\partial(a)]^3$$

$$= (\wp_1-1)\wp_2(\wp_2-1)\left[\frac{1}{\wp_1\wp_2^2}\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3+\wp_1(\wp_1-1)(\wp_2-1)\left[\frac{1}{\wp_1^2\wp_2}\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3$$

$$+\wp_2(\wp_2-1)\left[\frac{1}{\wp_1^2}\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3+\wp_1(\wp_1-1)\left[\frac{1}{\wp_1^2}\sum_{i=1}^{\wp_2^2-1}\frac{1}{i}\right]^3+\sum_{j=1}^{(\wp_1-1)(\wp_2-1)}\left[\frac{1}{\wp_1\wp_2}\sum_{i_j=1}^{\wp_1\wp_2-1}\frac{1}{i_j}\right]^3+\sum_{j=1}^{\wp_2-1}\left[\frac{1}{\wp_2}\sum_{\substack{i=1\\(i\neq j^2\wp_1^2)}}^{\wp_1^2\wp_2-1}\frac{1}{i}\right]^3+\sum_{j=1}^{\wp_1-1}\left[\frac{1}{\wp_1}\sum_{\substack{i=1\\(i\neq j^2\wp_2^2)}}^{\wp_1\wp_2^2-1}\frac{1}{i}\right]^3.$$

Then, we get

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1^2\wp_2^2})) = \frac{(\wp_1-1)(\wp_2-1)}{\wp_1^3\wp_2^5}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3+\frac{(\wp_1-1)(\wp_2-1)}{\wp_1^5\wp_2^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3+\frac{\wp_2-1}{\wp_2^5}\left[\sum_{i=1}^{\wp_1^2-1}\frac{1}{i}\right]^3$$

$$+\frac{\wp_1-1}{\wp_1^5}\left[\sum_{i=1}^{\wp_2^2-1}\frac{1}{i}\right]^3+\frac{1}{\wp_1^3\wp_2^3}\sum_{j=1}^{(\wp_1-1)(\wp_2-1)}\left[\sum_{\substack{i_j=1\\i_j\neq j}}^{\wp_1\wp_2-1}\frac{1}{i_j}\right]^3+\frac{1}{\wp_2^3}\sum_{j=1}^{\wp_2-1}\left[\sum_{\substack{i=1\\i\neq j^2\wp_1^2}}^{\wp_1^2\wp_2-1}\frac{1}{i}\right]^3+\frac{1}{\wp_1^3}\sum_{j=1}^{\wp_1-1}\left[\sum_{\substack{i=1\\i\neq j^2\wp_2^2}}^{\wp_1\wp_2^2-1}\frac{1}{i}\right]^3.$$

□

Theorem 3.6. *Let $\wp_1,\wp_2$ and $\wp_3$ distinct prime numbers. Then, the $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2\wp_3})$ has the following form:*

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2\wp_3})) = \frac{(\wp_1-1)(\wp_3-1)}{(\wp_2\wp_3)^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3+\frac{(\wp_1-1)(\wp_3-1)}{(\wp_1\wp_3)^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3+\frac{(\wp_1-1)(\wp_2-1)}{(\wp_1\wp_2)^3}\left[\sum_{i=1}^{\wp_3-1}\frac{1}{i}\right]^3+$$

$$\frac{\wp_3-1}{\wp_3^3}\left[\sum_{i=1}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3+\frac{\wp_2-1}{(\wp_2)^3}\left[\sum_{i=1}^{\wp_1\wp_3-1}\frac{1}{i}\right]^3+\frac{\wp_1-1}{(\wp_1)^3}\left[\sum_{i=1}^{\wp_2\wp_3-1}\frac{1}{i}\right]^3.$$

Proof. Let $\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2\wp_3})$ be the fuzzy zero divisor graph of $\mathbb{Z}_{\wp_1\wp_2\wp_3}$. By vertex partition technique, family of sets $\{P_i\}_{i\in I=\{1,2,3,4,5,6\}}$ builds a disjoint decomposition of $\mathcal{V}$ as follows.

$P_1 = \{\wp_1 x : x = 1, 2, ..., \wp_2\wp_3 - 1; \wp_2 \nmid x, \wp_3 \nmid x\}$,

$P_2 = \{\wp_2 x : x = 1, 2, ..., \wp_1\wp_3 - 1; \wp_1 \nmid x, \wp_3 \nmid x\}$,

$P_3 = \{\wp_3 x : x = 1, 2, ..., \wp_1\wp_2 - 1; \wp_1 \nmid x, \wp_2 \nmid x\}$,

$P_4 = \{\wp_1\wp_2 x : x = 1, 2, ..., \wp_3 - 1\}$,

$P_5 = \{\wp_1\wp_3 x : x = 1, 2, ..., \wp_2 - 1\}$,

$P_6 = \{\wp_2\wp_3 x : x = 1, 2, ..., \wp_1 - 1\}$,

where, $|P_1| = (\wp_1-1)(\wp_3-1)$, $|P_2| = (\wp_1-1)(\wp_3-1)$, $|P_3| = (\wp_1-1)(\wp_2-1)$, $|P_4| = \wp_3-1$, $|P_5| = \wp_2-1$, $|P_6| = \wp_1 - 1$.

Morever, for every element $a \in \mathcal{V}$ we have,

$$\sigma(a)\partial(a) = \frac{1}{\wp_2\wp_3}\sum_{i=1}^{|P_6|}\frac{1}{i},\ a \in P_1,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1\wp_3}\sum_{i=1}^{|P_5|}\frac{1}{i}, a \in\ P_2,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1\wp_2}\sum_{i=1}^{|P_4|}\frac{1}{i}, a \in\ P_3,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_3}\sum_{i=1}^{|P_3|+|P_5|+|P_6|}\frac{1}{i}, a \in\ P_4,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_2}\sum_{i=1}^{|P_2|+|P_4|+|P_6|}\frac{1}{i}, a \in\ P_5,$$

$$\sigma(a)\partial(a) = \frac{1}{\wp_1}\sum_{i=1_{(i\neq j^2\wp_1^2)}}^{|P1|+|P_4|+|P_5|}\frac{1}{i}, a \in\ P_6.$$

Hence,

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2\wp_3})) = \sum_{a\in\mathcal{V}}[\sigma(a)\partial(a)]^3$$

$$= \sum_{i=1}^{6}\sum_{a\in P_i}[\sigma(a)\partial(a)]^3$$

$$= \sum_{a\in P_1}[\sigma(a)\partial(a)]^3 + \sum_{a\in P_2}[\sigma(a)\partial(a)]^3 + \sum_{a\in P_3}[\sigma(a)\partial(a)]^3 + \sum_{a\in P_4}[\sigma(a)\partial(a)]^3 + \sum_{a\in P_5}[\sigma(a)\partial(a)]^3 + \sum_{a\in P_6}[\sigma(a)\partial(a)]^3$$

$$= |P_1|\,[\sigma(a)\partial(a)]^3 + |P_2|\,[\sigma(a)\partial(a)]^3 + |P_3|\,[\sigma(a)\partial(a)]^3 + |P_4|\,[\sigma(a)\partial(a)]^3 + |P_5|\,[\sigma(a)\partial(a)]^3 + |P_6|\,[\sigma(a)\partial(a)]^3.$$

Then, we get

$$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{\wp_1\wp_2\wp_3})) = \frac{(\wp_1-1)(\wp_3-1)}{(\wp_2\wp_3)^3}\left[\sum_{i=1}^{\wp_1-1}\frac{1}{i}\right]^3 + \frac{(\wp_1-1)(\wp_3-1)}{(\wp_1\wp_3)^3}\left[\sum_{i=1}^{\wp_2-1}\frac{1}{i}\right]^3 + \frac{(\wp_1-1)(\wp_2-1)}{(\wp_1\wp_2)^3}\left[\sum_{i=1}^{\wp_3-1}\frac{1}{i}\right]^3 +$$

$$\frac{\wp_3-1}{\wp_3^3}\left[\sum_{i=1}^{\wp_1\wp_2-1}\frac{1}{i}\right]^3 + \frac{\wp_2-1}{(\wp_2)^3}\left[\sum_{i=1}^{\wp_1\wp_3-1}\frac{1}{i}\right]^3 + \frac{\wp_1-1}{(\wp_1)^3}\left[\sum_{i=1}^{\wp_2\wp_3-1}\frac{1}{i}\right]^3.$$

□

4. SageMath and MATLAB code

This section contains SageMath code for drawing fuzzy zero divisor graphs and a MATLAB algorithm that calculates the $\mathcal{FF}$-index using the adjacency matrix [22]. Simply it is enought to input n to utilize it.

4.1. SageMath code

```
n=input ("Type in the value of n")
V=[]
for u in [(2/n),(3/n),..,((n-1)/n)]:
   if gcd(n*u,n)!=1:
    V.append(u)

E=[]
for n1 in V:
    for n2 in V:
        if ((n1*n)*(n2*n))%n==0 and n1!=n2:
            E.append((n1,n2))
G=Graph()

G.add_edges(E)
for m1, m2 in G.edges(sort=True, labels=False):
    G.set_edge_label(m1,m2,1/(n*(m1*m2)))

G.show(edge_labels=True, color_by_label=True, edge_style='solid', vertex_size
   =500, figsize=[12,12])
```

4.2. MATLAB code

```
n= input
Vert = strings (1,n -2) ;
Adj= zeros (n -2) ;
Deg= zeros (1,n -2) ;
for i=2:n -1
Vert (i -1) = int2str (i) ;
for j =2:n -1
if (i==j) , continue , end
if mod (i*j,n)==0
Adj(i -1,j -1) =n/(i*j);
Deg(i -1)=i/n;
end
end
end
for i= size (Deg ,2):- 1:1
if (Deg (i) ==0)
Adj(i ,:) =[];
Adj (:,i) =[];
Vert (i) =[];
Deg(i) =[];
end
end
ffi =0;
```

```
for i =1: size (Deg , 2)
ffi =ffi +(sum(Adj(i ,:))*Deg(i))^3
end
fprintf ( '%d %d %d\n' , ffi)
```

5. Applications

By utilizing the codes from the previous section, the $\mathbb{Z}_n$ fuzzy graph structures were graphically displayed and their $\mathcal{FF}$-index was calculated for values of $n = \wp^{\alpha},\ \wp_1\wp_2,\ \wp_1^2\wp_2,\ \wp_1^2\wp_2^2,\ \wp_1\wp_2\wp_3$.

Example 5.1. The visual representations and $\mathcal{FF}$-index values of $\mathbb{Z}_n$ for $n = 9,\ 15,\ 20,\ 30,\ 36$ are displayed below. (See Table 1, Table 2, Table 3, Table 4, Table 5.)

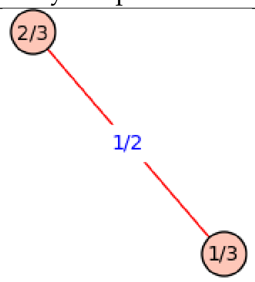
Fuzzy Graph Structure	$\mathcal{FF}$-index values
	$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_9)) = 0.042$

Table 1: Fuzzy graph structure and $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_9)$

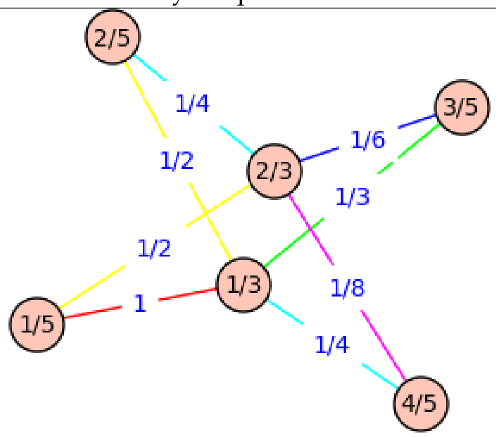
Fuzzy Graph Structure	$\mathcal{FF}$-index values
	$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{15})) = 0.778$

Table 2: Fuzzy graph structure and $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{15})$

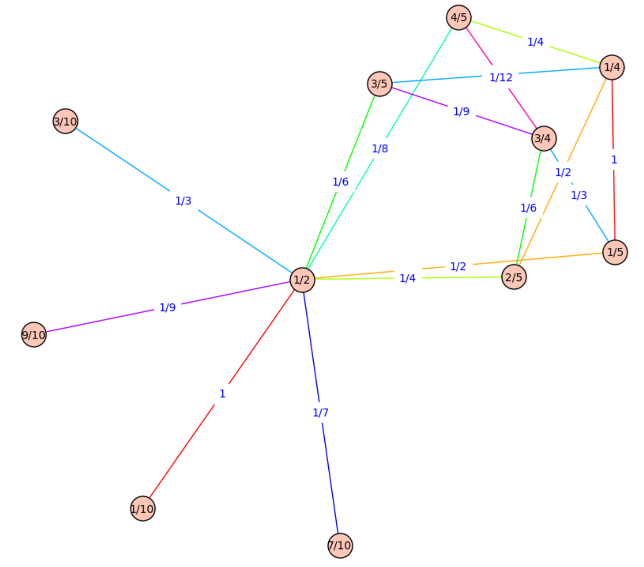
Fuzzy Graph Structure	$\mathcal{FF}$-index values
	$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{20})) = 2.755$

Table 3: Fuzzy graph structure and $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{20})$

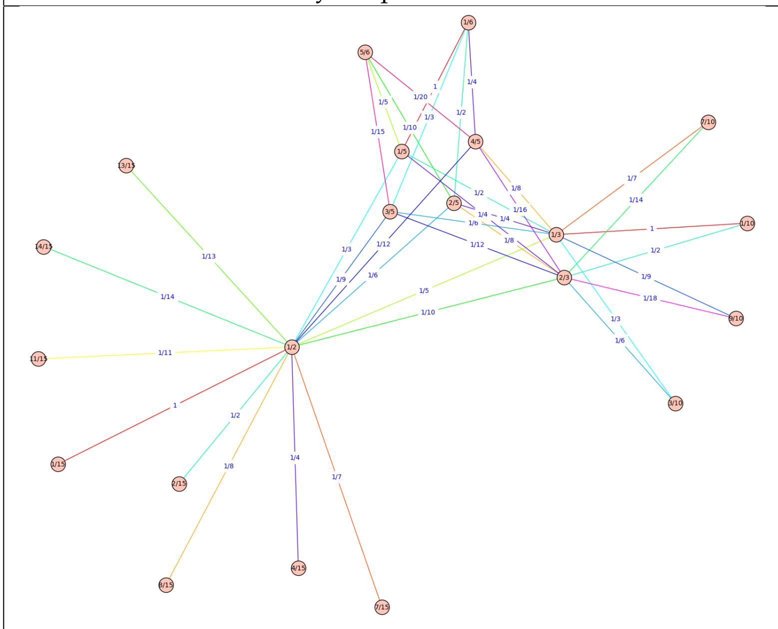
Fuzzy Graph Structure	$\mathcal{FF}$-index values
	$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{30})) = 6.455$

Table 4: Fuzzy graph structure and $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{30})$

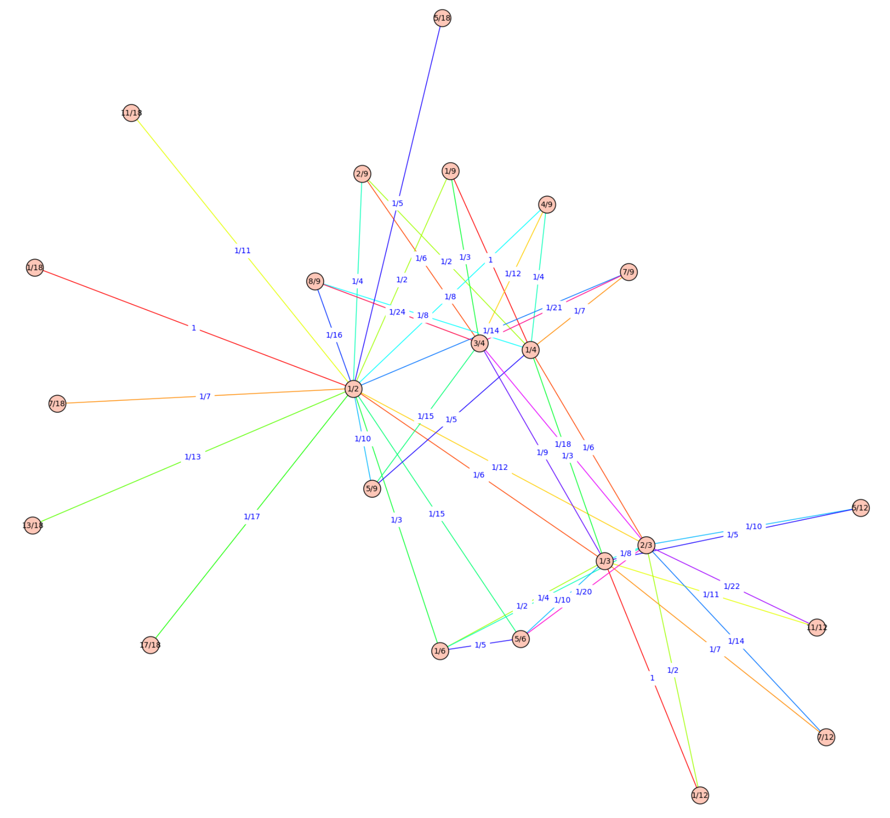
Fuzzy Graph Structure	$\mathcal{FF}$-index values
	$\mathcal{FF}(\mathcal{G}_f(\mathbb{Z}_{36})) = 7.034$

Table 5: Fuzzy graph structure and $\mathcal{FF}$-index of $\mathcal{G}_f(\mathbb{Z}_{36})$

6. Conclusion

Graph theory, used to model the relationship between the elements of a set, visualizes the problem through drawing in basic sciences. This concept is used in theoretical sciences as well as in applied sciences. Graph theoretical models of the algebraic relations between the elements of algebraic structures or the relations between substructures containing certain algebraic elements have attracted the attention of many researchers. In this way, it has become possible to model algebraic structures concretely, which, until the very recent past, could only be defined as an abstract concept.

The problem of uncertainty is especially important for scientists in the field of artificial intelligence (prediction development, risk analysis). The first information to understand uncertainty and derive appropriate solutions came from Zadeh [28] with fuzzy set theory. According to fuzzy logic, life is not just about right and wrong, black and white. A fuzzy set is a concept created with a graduated membership function, unlike the classical set expressed with a characteristic function. Following the work of Euler, who laid the foundations of a wide scientific field in the field of graphs, Rosenfeld pioneered the idea of fuzzy graphs and made assessments of thought processes.

Based on this combination, in the 3th section of this study, the graph parametric topological characters of fuzzy zero divisor graphs defined on commutative rings by Kuppan and Sankar (2021) [20] were investigated.

The 4th section of the study includes a SageMath code that illustrates the fuzzy zero divisor model of the ring $\mathbb{Z}_n$. This allows for a concrete demonstration of the weighted form of fuzzy zero divisor graphs, which are a more general structure than the zero divisor graph, on the ring $\mathbb{Z}_n$. Furthermore, software has been developed that allows for the direct calculation of the fuzzy topological graph parameters by providing a MATLAB code that constructs the degree and adjacency fuzzy matrices of the graphs in question. This is in accordance with the theoretical form given in the previous sections.

In the last section of the study, the developed codes were used and their applications were included in the rings we were working on. In this way, the fuzzy $\mathcal{FF}$-index of the fuzzy zero divisor graph can be calculated for every non-prime value of $n > 1$ and of the ring $\mathbb{Z}_n$.

As a whole, this study provides sustainable and valuable insights into general fuzzy graphs, special forms of n, and algebraic structures with varying fuzzy topological indices.

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