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Weighted Ostrowski type inequalities for some classes of convex functions

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Abstract. The main aim of this paper is to study the Ostrowski inequality for convex and $(g, h, \alpha - m)$-convex functions using the weighted Montgomery identity. Through the application of the power mean inequality, we derive results for differentiable functions by analyzing the convexity of the absolute value of their derivatives.

1. Introduction

In 1938, Ostrowski [12] introduced an inequality providing an approximation of the integral $\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t$ to the value of a function at an arbitrary point as follows:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty , \tag{1}$$

for all $x \in [a, b]$, where f is differentiable and $f' \in L_\infty[a, b]$. The above inequality (1) can be proved by using the following identity [4] known as Montgomery identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \int_a^b P(x,t) f'(t)dt, \tag{2}$$

where $P(x, t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a} & \text{for } t \in [a, x], \\ \frac{t-b}{b-a} & \text{for } t \in (x, b]. \end{cases} \tag{3}$$

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The Ostrowski inequality is applied to provide error bounds for numerical quadrature rules of the Riemann type, which further produce the estimations of rectangular, trapezoidal, and midpoint quadrature rules, see [8, 9]. Among others, Milovanović and Pečarić generalized the Ostrowski inequality using the Taylor formula and explored its applications to numerical quadrature rules in [11]. This inequality is also used to establish bounds of relations among special means; for details, we refer readers to [3]. Dragomir and Wang extended the Ostrowski inequality by utilizing the well-known Grüss inequality in [5] and provided applications to special means and numerical quadrature rules. Further extensions and generalizations of the Ostrowski inequality can be found in recent articles, see [3–5] and references therein.
The weighted Montgomery identity [13] is stated as follows:

$$f(x)=\int_a^b w(t)f(t)dt+\int_a^b P_w(x,t)f'(t)dt, \tag{4}$$

where $w:[a,b]\to[0,\infty)$ is some non-negative integrable weight function with $\int_a^b w(s)ds=1$, and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t)=\begin{cases} W(t) & \text{for } t\in[a,x],\\ 1-W(t) & \text{for } t\in(x,b], \end{cases} \tag{5}$$

$W(t)=\int_a^t w(s)ds$, $t\in[a,b]$ with $W(t)=0$, $t<a$ and $W(t)=1$, $t>b$. For weight function $w(t)=\frac{1}{b-a}$, $t\in[a,b]$, the weighted Montgomery identity reduces to (2). We are interested in giving Ostrowski type inequalities by using the weighted Montgomery identity and convexity of a function as well as convexity of the qth power of the absolute value of its derivative. A recently defined convexity, which will be utilized in this paper, is given as follows.

Definition 1.1. *[7] Let h be a non-negative function on $J\subset\mathbb{R}$, $(0,1)\subset J$, $h\neq 0$, and let g be a positive function on $I\subset\mathbb{R}$. A function $f:I\to\mathbb{R}$ is said to be $(g,h,\alpha-m)$-convex if it is non-negative and satisfies the following inequality:*

$$f(\lambda x+m(1-\lambda)y)\leq h(\lambda^\alpha)f(x)g(x)+mh(1-\lambda^\alpha)f(y)g(y), \tag{6}$$

where $\lambda\in[0,1]$, $x,y\in I$.

Remark 1.2. *By setting $\alpha=1$, the definition of (g,h)-convexity defined in [1] is obtained.*

For $m=1$, we consider the case of a $(g,h,\alpha-1)$-convex function, which is commonly known as a (g,h,α)-convex function. All results of convexity will be analyzed for (g,h,α)-convex functions. We rewrite the definition of a (g,h,α)-convex function in the following form:

$$f(z)\leq h\left(\left(\frac{z-y}{x-y}\right)^\alpha\right)f(x)g(x)+h\left(1-\left(\frac{z-y}{x-y}\right)^\alpha\right)f(y)g(y), \tag{7}$$

by substituting $m=1$ and $z=\lambda x+(1-\lambda)y$, $z\in[x,y]$ into inequality (6).

The following well-known power mean inequality is useful for proving the results in this paper.

Definition 1.3. *Let f, g be real valued functions defined on $[a,b]$. If $|f|, |f||g|^q\in L[a,b]$; then for $q\geq 1$ the following inequality holds:*

$$\int_a^b |f(t)||g(t)|dt\leq\left(\int_a^b |f(t)|dt\right)^{1-\frac{1}{q}}\left(\int_a^b |f(t)||g(t)|^q dt\right)^{\frac{1}{q}}. \tag{8}$$

2. Main result

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), $|f'|^q \in L[a,b]$ and let $\left|f'\right|^q$ be (g,h,α)-convex, for $q \geq 1$. Further, let $w : [a,b] \to [0,+\infty)$ be some non-negative integrable weight function. Then we have:*

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x W(t)dt\right)^{\frac{1}{p}} \left[|f'(a)|^q.g(a)\int_a^x W(t)h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt\right. \tag{9}$$
$$\left.+|f'(x)|^q.g(x)\int_a^x W(t)h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt\right]^{\frac{1}{q}} + \left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}}\left[|f'(x)|^q.g(x)\right.$$
$$\left.\int_x^b (1-W(t))h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt + |f'(b)|^q.g(b)\int_x^b (1-W(t))h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt\right]^{\frac{1}{q}}$$
$$\leq \left(\int_a^x W(t)dt + \int_x^b (1-W(t))dt\right)^{\frac{1}{p}}\left[|f'(a)|^q.g(a)\int_a^x W(t)h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt + |f'(x)|^q\right.$$
$$\times g(x)\int_a^x W(t)h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt + |f'(x)|^q.g(x)\int_x^b (1-W(t))h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt$$
$$\left.+\,|f'(b)|^q.g(b)\int_x^b (1-W(t))h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt\right]^{\frac{1}{q}},$$

where $1 - \frac{1}{q} = \frac{1}{p}$.

Proof. The weighted Montgomery identity (4) gives the following modulus inequality:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left|\int_a^x W(t)f'(t)dt\right| + \left|\int_x^b (1-W(t))f'(t)dt\right| \tag{10}$$
$$\leq \int_a^x W(t)\left|f'(t)\right|dt + \int_x^b (1-W(t))\left|f'(t)\right|dt.$$

Using the power mean inequality for two integrals appearing on the right-hand side of the above inequality, one can have the following inequality:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x |W(t)|dt\right)^{1-\frac{1}{q}}\left(\int_a^x |W(t)||f'(t)|^q dt\right)^{\frac{1}{q}} \tag{11}$$
$$+\left(\int_x^b |1-W(t)|dt\right)^{1-\frac{1}{q}}\left(\int_x^b |1-W(t)||f'(t)|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is (g,h,α)-convex, by applying (7) we get

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x W(t)dt\right)^{\frac{1}{p}}\left(\int_a^x W(t)\left(h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)|f'(a)|^q.g(a)\right.\right. \tag{12}$$
$$\left.\left.+h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)|f'(x)|^q.g(x)dt\right)\right)^{\frac{1}{q}} + \left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}}$$
$$\left(\int_x^b (1-W(t))\left(h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)|f'(x)|^q.g(x) + h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)|f'(b)|^q.g(b)\right)dt\right)^{\frac{1}{q}}.$$

By rearranging the terms of the above inequality, one can get the following inequality:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x W(t)dt\right)^{\frac{1}{p}}\left[|f'(a)|^q.g(a)\int_a^x W(t)h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt\right. \tag{13}$$

$$+|f'(x)|^q.g(x)\int_a^x W(t)h\left(1-\left(\frac{t-x}{a-x}\right)^\alpha\right)dt\Bigg]^{\frac{1}{q}}+\left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}}\Bigg[|f'(x)|^q.g(x)$$

$$\int_x^b (1-W(t))h\left(\left(\frac{t-b}{x-b}\right)^\alpha\right)dt+|f'(b)|^q.g(b)\int_x^b (1-W(t))h\left(1-\left(\frac{t-b}{x-b}\right)^\alpha\right)dt\Bigg]^{\frac{1}{q}}.$$

The required inequality can be obtained by using the Hölder inequality in discrete form. □

Some special cases are given in the forthcoming theorems. First, we give the following result for (g, h)-convex functions defined in [1].

Theorem 2.2. *Under the assumptions of Theorem 2.1, we have the following Ostrowski type inequality for the (g, h)-convex function:*

$$\left|f(x)-\int_a^b w(t)f(t)dt\right|\le\left(\int_a^x W(t)dt\right)^{\frac{1}{p}}\Bigg[|f'(a)|^q.g(a)\int_a^x W(t)h\left(\frac{t-x}{a-x}\right)dt \tag{14}$$

$$+|f'(x)|^q.g(x)\int_a^x W(t)h\left(\frac{a-t}{a-x}\right)dt\Bigg]^{\frac{1}{q}}+\left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}}\Bigg[|f'(x)|^q.g(x)$$

$$\int_x^b (1-W(t))h\left(\frac{t-b}{x-b}\right)dt+|f'(b)|^q.g(b)\int_x^b (1-W(t))h\left(\frac{x-t}{x-b}\right)dt\Bigg]^{\frac{1}{q}}$$

$$\le\left(\int_a^x W(t)dt+\int_x^b (1-W(t))dt\right)^{\frac{1}{p}}\Bigg[|f'(a)|^q.g(a)\int_a^x W(t)h\left(\frac{t-x}{a-x}\right)dt$$

$$+|f'(x)|^q g(x)\int_a^x W(t)h\left(\frac{a-t}{a-x}\right)dt+|f'(x)|^q.g(x)\int_x^b (1-W(t))h\left(\frac{t-b}{x-b}\right)dt$$

$$+|f'(b)|^q.g(b)\int_x^b (1-W(t))h\left(\frac{x-t}{x-b}\right)dt\Bigg]^{\frac{1}{q}},$$

where $1-\frac{1}{q}=\frac{1}{p}$.

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b), $|f'|^q\in L[a,b]$ and let $\left|f'\right|^q$ be convex, for $q\ge 1$. Further, let $w:[a,b]\to[0,+\infty)$ be some non-negative integrable weight function. Then we have:*

$$\left|f(x)-\int_a^b w(t)f(t)dt\right| \tag{15}$$

$$\le\Omega_1(x)^{\frac{1}{p}}\left[\frac{x\left|f'(a)\right|^q-a\left|f'(x)\right|^q}{x-a}\Omega_1(x)+\frac{\left|f'(x)\right|^q-\left|f'(a)\right|^q}{2(x-a)}\int_a^x (x^2-s^2)w(s)ds\right]^{\frac{1}{q}}$$

$$+\Omega_2(x)^{\frac{1}{p}}\left[\frac{b\left|f'(x)\right|^q-x\left|f'(b)\right|^q}{b-x}\Omega_2(x)+\frac{\left|f'(b)\right|^q-\left|f'(x)\right|^q}{2(b-x)}\int_x^b (s^2-x^2)w(s)ds\right]^{\frac{1}{q}}$$

$$\le(\Omega_1(x)+\Omega_2(x))^{\frac{1}{p}}\Bigg[\frac{x\left|f'(a)\right|^q-a\left|f'(x)\right|^q}{x-a}\Omega_1(x)+\frac{b\left|f'(x)\right|^q-x\left|f'(b)\right|^q}{b-x}\Omega_2(x)$$

$$+\frac{\left|f'(x)\right|^q-\left|f'(a)\right|^q}{2(x-a)}\int_a^x (x^2-s^2)w(s)ds+\frac{\left|f'(b)\right|^q-\left|f'(x)\right|^q}{2(b-x)}\int_x^b (s^2-x^2)w(s)ds\Bigg]^{\frac{1}{q}},$$

where

$$\Omega_1(x)=\int_a^x (x-s)w(s)ds,\quad \Omega_2(x)=\int_x^b (s-x)w(s)ds. \tag{16}$$

Proof. By setting h as an identity function, $\alpha = 1$ and $g \equiv 1$ in (13), the inequality for the convex function $|f'|^q$ holds in the following form:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x W(t)dt\right)^{\frac{1}{p}} \left[|f'(a)|^q \int_a^x W(t)\left(\frac{t-x}{a-x}\right)dt + |f'(x)|^q \right. \tag{17}$$

$$\left. \times \int_a^x W(t)\left(\frac{a-t}{a-x}\right)dt\right]^{\frac{1}{q}} + \left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}} \left[|f'(x)|^q \int_x^b (1-W(t))\left(\frac{t-b}{x-b}\right)dt\right.$$

$$\left. + |f'(b)|^q \int_x^b (1-W(t))\left(\frac{x-t}{x-b}\right)dt\right]^{\frac{1}{q}}.$$

This further leads to the following inequality:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left(\int_a^x W(t)dt\right)^{\frac{1}{p}} \left[\frac{x\left|f'(a)\right|^q - a\left|f'(x)\right|^q}{(x-a)} \int_a^x W(t)dt\right. \tag{18}$$

$$\left. + \frac{\left|f'(x)\right|^q - \left|f'(a)\right|^q}{(x-a)} \int_a^x tW(t)dt\right]^{\frac{1}{q}} + \left(\int_x^b (1-W(t))dt\right)^{\frac{1}{p}} \left[\frac{b\left|f'(x)\right|^q - x\left|f'(b)\right|^q}{(b-x)}\right.$$

$$\left. \int_x^b (1-W(t))dt + \frac{\left|f'(b)\right|^q - \left|f'(x)\right|^q}{(b-x)} \int_x^b t(1-W(t))dt\right]^{\frac{1}{q}}.$$

By applying the definition of $W(t)$ and using a change of order of integration we get

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \tag{19}$$

$$\leq \Omega_1(x)^{\frac{1}{p}} \left[\frac{x\left|f'(a)\right|^q - a\left|f'(x)\right|^q}{x-a}\Omega_1(x) + \frac{\left|f'(x)\right|^q - \left|f'(a)\right|^q}{2(x-a)} \int_a^x (x^2-s^2)w(s)ds\right]^{\frac{1}{q}}$$

$$+ \Omega_2(x)^{\frac{1}{p}} \left[\frac{b\left|f'(x)\right|^q - x\left|f'(b)\right|^q}{b-x}\Omega_2(x) + \frac{\left|f'(b)\right|^q - \left|f'(x)\right|^q}{2(b-x)} \int_x^b (s^2-x^2)w(s)ds\right]^{\frac{1}{q}}.$$

By applying the Hölder inequality one can get the required inequality. □

Remark 2.4. *By fixing different values of the function g, one can obtain Ostrowski type inequalities for several classes of functions related to convex functions. For instance, by setting $h(x) = x$, $\alpha = 1$ and $g(x) = \exp(-\eta x)$ in* (23)*, the Ostrowski type inequality for exponentially convex functions is obtained.*

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), $|f'| \in L_q[a,b]$ and $\left|f'\right|^q$ the (g,h,α)-convex function. Further, let $w : [a,b] \to [0,+\infty)$ be some non-negative integrable weight function. Then we have:*

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \leq \left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}} \left[|f'(a)|^q.g(a) \int_a^x \left(h\left(\left(\frac{t-x}{a-x}\right)^\alpha\right)dt\right.\right. \tag{20}$$

$$\left. + |f'(x)|^q.g(x) \int_a^x h\left(1-\left(\frac{t-x}{a-x}\right)^\alpha\right)dt\right]^{\frac{1}{q}} + \left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}$$

$$\left[|f'(x)|^q.g(x) \int_x^b h\left(\left(\frac{t-b}{x-b}\right)^\alpha\right)dt + |f'(b)|^q.g(b) \int_x^b h\left(1-\left(\frac{t-b}{x-b}\right)^\alpha\right)dt\right]^{\frac{1}{q}}$$

$$\leq \left(\int_a^x W^p(t)dt + \int_x^b (1-W(t))^p dt\right)^{\frac{1}{p}} \left[|f'(a)|^q.g(a) \int_a^x h\left(\left(\frac{t-x}{a-x}\right)^\alpha\right)dt\right.$$

$$+|f'(x)|^q g(x)\int_a^x h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt+|f'(x)|^q.g(x)\int_x^b h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt$$
$$+|f'(b)|^q.g(b)\int_x^b h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt\Bigg]^{\frac{1}{q}}.$$

Proof. By using the Hölder inequality in integral form, from inequality (10) one can have

$$\left|f(x)-\int_a^b w(t)f(t)dt\right|\leq\left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}}\left[\int_a^x |f'(t)|^q dt\right]^{\frac{1}{q}} \tag{21}$$
$$+\left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}\cdot\left[\int_x^b |f'(t)|^q\right]^{\frac{1}{q}}.$$

By using (g,h,α)-convexity of $|f'(t)|^q$ over $[a,x]$ and $[b,x]$, respectively, we get

$$\left|f(x)-\int_a^b w(t)f(t)dt\right| \tag{22}$$
$$\leq\left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}}\cdot\Bigg[\int_a^x\Bigg(h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)|f'(a)|^q.g(a)+h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)$$
$$|f'(x)|^q.g(x)\Bigg)dt\Bigg]^{\frac{1}{q}}+\left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}\cdot\Bigg[\int_x^b\Bigg(h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)|f'(x)|^q.g(x)$$
$$+h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)|f'(b)|^q.g(b)\Bigg)dt\Bigg]^{\frac{1}{q}}=\left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}}\Bigg[|f'(a)|^q.g(a)$$
$$\int_a^x h\left(\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt+|f'(x)|^q.g(x)\int_a^x h\left(1-\left(\frac{t-x}{a-x}\right)^{\alpha}\right)dt\Bigg]^{\frac{1}{q}}+\left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}$$
$$\Bigg[|f'(x)|^q.g(x)\int_x^b h\left(\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt+|f'(b)|^q.g(b)\int_x^b h\left(1-\left(\frac{t-b}{x-b}\right)^{\alpha}\right)dt\Bigg]^{\frac{1}{q}}.$$

Using the Hölder inequality in discrete form, from inequality (22) we get the required inequality (20). □

Theorem 2.6. *Under the assumptions of Theorem 2.5, we have the following Ostrowski type inequality for the (g,h)-convex function:*

$$\left|f(x)-\int_a^b w(t)f(t)dt\right|\leq\left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}}\Bigg[|f'(a)|^q.g(a)\int_a^x\Bigg(h\left(\frac{t-x}{a-x}\right)dt \tag{23}$$
$$+|f'(x)|^q.g(x)\int_a^x h\left(\frac{a-t}{a-x}\right)dt\Bigg]^{\frac{1}{q}}+\left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}$$
$$\Bigg[|f'(x)|^q.g(x)\int_x^b\Bigg(h\left(\frac{t-b}{x-b}\right)dt+|f'(b)|^q.g(b)\int_x^b h\left(\frac{x-t}{x-b}\right)\Bigg)dt\Bigg]^{\frac{1}{q}}$$
$$\leq\left(\int_a^x W^p(t)dt+\int_x^b (1-W(t))^p dt\right)^{\frac{1}{p}}\Bigg[|f'(a)|^q.g(a)\int_a^x h\left(\frac{t-x}{a-x}\right)dt+|f'(x)|^q g(x)$$
$$\int_a^x h\left(\frac{a-t}{a-x}\right)dt+|f'(x)|^q.g(x)\int_x^b h\left(\frac{t-b}{x-b}\right)dt+|f'(b)|^q.g(b)\int_x^b h\left(\frac{x-t}{x-b}\right)dt\Bigg]^{\frac{1}{q}},$$

where $1-\frac{1}{q}=\frac{1}{p}$.

Theorem 2.7. *Under the assumptions of Theorem 2.3, the following inequality holds:*

$$\left|f(x)-\int_a^b w(t)f(t)dt\right| \tag{24}$$

$$\leq \left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}} \left[\frac{(x-a)(|f'(a)|^q + |f'(x)|^q)}{2}\right]^{\frac{1}{q}} + \left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}}$$

$$\left[\frac{(b-x)(|f'(x)|^q + |f'(b)|^q)}{2}\right]^{\frac{1}{q}} \leq \left(\int_a^x W^p(t)dt + \int_x^b (1-W(t))^p dt\right)^{\frac{1}{p}}$$

$$\left[\frac{(x-a)(|f'(a)|^q + |f'(x)|^q) + (b-x)(|f'(x)|^q + |f'(b)|^q)}{2}\right]^{\frac{1}{q}}.$$

Proof. By setting h as an identity function, $\alpha = 1$ and $g \equiv 1$ in (22), the inequality for the convex function $|f'|^q$, holds in the following form:

$$\begin{aligned} &\left|f(x) - \int_a^b w(t)f(t)dt\right| \\ &\leq \left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}} \left[|f'(a)|^q \int_a^x \frac{t-x}{a-x}dt + |f'(x)|^q \int_a^x \frac{a-t}{a-x}dt\right]^{\frac{1}{q}} \\ &+ \left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}} \left[|f'(x)|^q \int_x^b \frac{t-b}{x-b}dt + |f'(b)|^q \int_x^b \frac{x-t}{x-b}dt\right]^{\frac{1}{q}} \\ &= \left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}} \left[\frac{|f'(a)|^q}{x-a} \int_a^x (x-t)dt + \frac{|f'(x)|^q}{x-a} \int_a^x (t-a)dt\right]^{\frac{1}{q}} \\ &+ \left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}} \left[\frac{|f'(x)|^q}{b-x} \int_x^b (b-t)dt + \frac{|f'(b)|^q}{b-x} \int_x^b (t-x)dt\right]^{\frac{1}{q}}. \end{aligned} \tag{25}$$

After integrating and carrying out some computations, we obtain

$$\begin{aligned} &\left|f(x) - \int_a^b w(t)f(t)dt\right| \\ &\leq \left[\int_a^x W^p(t)dt\right]^{\frac{1}{p}} \left[\frac{(x-a)(|f'(a)|^q + |f'(x)|^q)}{2}\right]^{\frac{1}{q}} \\ &+ \left[\int_x^b (1-W(t))^p dt\right]^{\frac{1}{p}} \left[\frac{(b-x)(|f'(x)|^q + |f'(b)|^q)}{2}\right]^{\frac{1}{q}}. \end{aligned} \tag{26}$$

Applying the Hölder inequality in discrete form we get the required inequality. □

Without loss of generality, inequality (24) can be formulated on the symmetric interval $[a, b] = [-1, 1]$ as

$$\left|f(x) - \int_{-1}^1 w(t)f(t)dt\right| \leq C_p(w, x) \left[\frac{1+x}{2}|f'(-1)|^q + |f'(x)|^q + \frac{1-x}{2}|f'(1)|^q\right]^{\frac{1}{q}}, \tag{27}$$

where

$$C_p(w, x) = \left(\int_{-1}^x W^p(t)dt + \int_x^1 (1-W(t))^p dt\right)^{\frac{1}{p}}, \tag{28}$$

$1/p + 1/q = 1$, $q \geq 1$.

Now, we will calculate $C_p(w, x)$ for certain weight functions w on $(-1, 1)$.

For Chebyshev weight of the first kind $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, where $t \in (-1,1)$, we easily obtain

$$W(t) = \frac{1}{2} + \frac{1}{\pi}\arcsin t.$$

So by applying (28) we can calculate the values of $C_p\left(\frac{1}{\pi\sqrt{1-t^2}}, x\right)$ for certain values of x and p:

$$\begin{aligned}
&C_2\left(\frac{1}{\pi\sqrt{1-t^2}}, x\right) = \frac{\sqrt{2\pi\sqrt{1-x^2} + 2\pi x \arcsin x - 4}}{\pi},\\
&C_2\left(\frac{1}{\pi\sqrt{1-t^2}}, 0\right) = 0.480973, C_2\left(\frac{1}{\pi\sqrt{1-t^2}}, -\frac{1}{2}\right) = C_2\left(\frac{1}{\pi\sqrt{1-t^2}}, \frac{1}{2}\right) = 0.559206,\\
&C_{10}\left(\frac{1}{\pi\sqrt{1-t^2}}, 0\right) = 0.440414, C_{10}\left(\frac{1}{\pi\sqrt{1-t^2}}, -\frac{1}{2}\right) = C_{10}\left(\frac{1}{\pi\sqrt{1-t^2}}, \frac{1}{2}\right) = 0.560557.
\end{aligned}$$

Further, for Chebyshev weight of the second kind $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, where $t \in (-1,1)$, we derive

$$W(t) = \frac{1}{2} + \frac{1}{\pi}\left(t\sqrt{1-t^2} + \arcsin t\right).$$

By using (28) we can calculate the values of $C_p\left(\frac{2}{\pi}\sqrt{1-t^2}, x\right)$, for the selected values of x and p:

$$\begin{aligned}
&C_2\left(\frac{2}{\pi}\sqrt{1-t^2}, x\right) = \frac{1}{\pi}\sqrt{\frac{-128 + 30\pi\sqrt{1-x^2}(2+x^2) + 90\pi x \arcsin x}{45}},\\
&C_2\left(\frac{2}{\pi}\sqrt{1-t^2}, 0\right) = 0.369067, C_2\left(\frac{2}{\pi}\sqrt{1-t^2}, -\frac{1}{2}\right) = C_2\left(\frac{2}{\pi}\sqrt{1-t^2}, \frac{1}{2}\right) = 0.540334,\\
&C_{10}\left(\frac{2}{\pi}\sqrt{1-t^2}, 0\right) = 0.411738, C_{10}\left(\frac{2}{\pi}\sqrt{1-t^2}, -\frac{1}{2}\right) = C_{10}\left(\frac{2}{\pi}\sqrt{1-t^2}, \frac{1}{2}\right) = 0.653723.
\end{aligned}$$

Moreover, for uniform weight $w(t) = \frac{1}{2}$, where $t \in (-1,1)$, we derive

$$W(t) = \frac{1}{2}(t+1)$$

and

$$C_p\left(\frac{1}{2}, x\right) = \frac{1}{2}\left(\frac{(1-x)^{p+1} + (x+1)^{p+1}}{p+1}\right)^{1/p}.$$

So we calculate the values of $C_p\left(\frac{1}{2}, x\right)$, for some x and p:

$$\begin{aligned}
&C_p\left(\frac{1}{2}, 0\right) = \frac{1}{2}\left(\frac{2}{p+1}\right)^{1/p},\\
&C_p\left(\frac{1}{2}, \frac{1}{2}\right) = C_p\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1+3^{p+1}}{2^{p+1}(p+1)}\right)^{1/p},\\
&C_{1.5}\left(\frac{1}{2}, 0\right) = 0.430887, C_{1.5}\left(\frac{1}{2}, -\frac{1}{2}\right) = C_{1.5}\left(\frac{1}{2}, \frac{1}{2}\right) = 0.556114,\\
&C_2\left(\frac{1}{2}, 0\right) = 0.408248, C_2\left(\frac{1}{2}, -\frac{1}{2}\right) = C_2\left(\frac{1}{2}, \frac{1}{2}\right) = 0.540062.
\end{aligned}$$

Finally, in the next three figures, we present the graphs of the function $x \mapsto C_p(w, x)$, for the selected values of p.

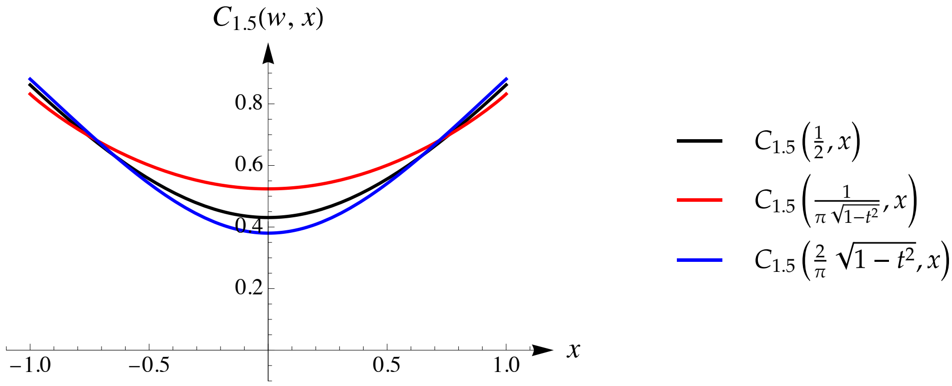

Figure 1: The bound function $x \mapsto C_{1.5}(w, x)$ for uniform weight, and Chebyshev weights of the first and second kind

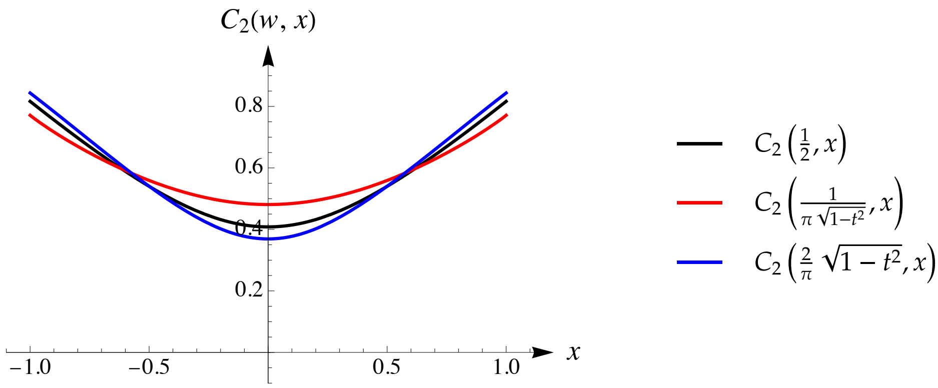

Figure 2: The bound function $x \mapsto C_2(w, x)$ for uniform weight, and Chebyshev weights of the first and second kind

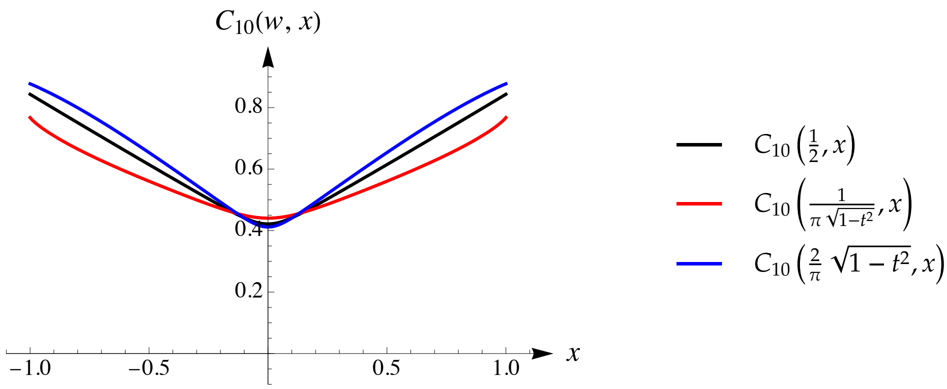

Figure 3: The function $x \mapsto C_{10}(w, x)$ for uniform weight, and Chebyshev weights of the first and second kind

In Figures 1-3, the red curves represent the function $x \mapsto C_p\left(\frac{1}{\pi\sqrt{1-t^2}}, x\right)$, the blue curves represent the function $x \mapsto C_p\left(\frac{2}{\pi}\sqrt{1-t^2}, x\right)$, and the black curves represent the function $x \mapsto C_p\left(\frac{1}{2}, x\right)$, all for $p = 1.5, p = 2$, and $p = 10$. All three figures show that for $x = 0$, the Chebyshev weight function of the first kind has the maximum value of the constant $C_p(w, 0)$, while the Chebyshev weight function of the second kind has the smallest value, for every selection of p.

For graphical representations and numerical computations, we used Wolfram Mathematica and the

Mathematica package "OrthogonalPolynomials" (see [2] and [10]).

3. Conclusion

In this study, we examined the Ostrowski inequality pertaining to $(h, g; \alpha - m)$-convex functions through the application of the weighted Montgomery identity. Utilizing the power mean inequality, we produced novel outcomes for differentiable functions by investigating various types of convexity of the absolute values of their derivatives. Our findings extend and generalize classical results, offering a broader perspective on the role of convexity in integral inequalities. These results provide deeper insights into the behavior of differentiable functions and their applications in mathematical analysis and related fields.

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