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Improved soft rough approximations based on soft neighborhoods

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Abstract. This study enhances uncertainty management by integrating soft set theory with neighborhood systems. Existing models use fixed relations, lacking adaptability to varying contexts. We introduce eight soft neighborhood types built from parameterized relations, generalizing existing models like Yao's [17] and Mareay's [12] approaches, enabling flexible approximations. By constructing topologies from these neighborhoods, we develop refined lower and upper approximation operators. This work bridges soft set parameterization with topological structures, offering a foundational framework for uncertainty management in parameter-dependent systems.

1. Introduction

The management of uncertainty and imprecision in knowledge representation has been a central challenge in computational intelligence, driving the development of robust mathematical frameworks such as rough set theory and soft set theory. Introduced by Pawlak in 1982 [14], classical rough set theory revolutionized uncertainty handling by approximating sets through equivalence relations, partitioning the universe into disjoint equivalence classes to define lower and upper approximations. However, this reliance on equivalence relations imposed significant limitations, as real-world data often involves asymmetric or incomplete relationships that cannot be captured by rigid equivalence structures. To address this, the concept of rough neighborhoods emerged as a transformative paradigm, enabling approximations through arbitrary binary relations and bypassing the need for equivalence.

The evolution of rough neighborhoods began with Yao's pioneering work on j-neighborhoods in 1998 [17], which generalized Pawlak's equivalence classes to arbitrary binary relations, thereby expanding rough sets to heterogeneous and incomplete data environments. Building on this foundation, in 2016, R. Mareay [12] introduced a generalized rough set framework that leverages neighborhood systems integrated with topological spaces to extend classical rough set approximations. This approach enhances uncertainty handling by employing neighborhood operators linked to topological properties, providing a more adaptable model for complex data environments. These neighborhoods were called later j-adhesion neighborhoods. In 2021, T. M. Al-shami [4] applied the topological concept of somewhere dense sets to enhance rough set approximations and their accuracy measures. This approach aims to refine the precision of data analysis within rough set theory. In 2022, T. M. Al-shami [6] introduced innovative maximal rough neighborhoods

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by defining new neighborhood operators that extend traditional rough set theory and preserve monotonic accuracy measures. The paper demonstrates their practical utility through a COVID-19 risk classification application. In 2023, M. Hosny et al. [11] developed generalized rough approximation spaces using ideal theory and various maximal neighborhoods. Their model improves boundary region reduction and delivers higher accuracy in uncertain data classification. In 2023, T. M. Al-shami and A. Mhemdi [7] explored approximation spaces derived from subset rough neighborhoods, presenting their theoretical foundations and practical applications. It provides insights into how these spaces can be utilized in various domains, offering a novel perspective on rough set theory. In 2024, T. M. Al-shami and A. Mhemdi [8] introduced overlapping containment rough neighborhoods, offering a refined framework for generalized approximations based on relational properties. Their approach reduces uncertainty and preserves key rough set characteristics, with effective applications in information system analysis. In 2024, T. M. Al-shami and M. Hosny [9] proposed generalized approximation spaces built from $\mathbb{I}_j$-Neighborhoods and ideals, extending classical rough set foundations. Their model improves decision-making accuracy and is validated through an application to Chikungunya disease analysis. In 2025, T. M. Al-shami et al. [10] introduced cardinality rough neighborhoods using ideals to refine rough set approximations based on the count of related elements. This model enhances decision-making in medical applications by better capturing uncertainty through element cardinality. Parallel to these advancements, Molodtsov's soft set theory (1999) [13] introduced a parameterized framework for modeling relational systems, enabling dynamic adaptation to multi-attribute or evolving contexts. Soft sets assign subsets of a universe to parameters, offering a versatile mechanism to handle uncertainty that depends on external factors.

The main objective of this research is to develop an improved framework for soft rough approximations by utilizing refined soft neighborhood systems derived from soft binary relations. Our proposed method leverages the flexibility of soft sets and the adaptability of neighborhood-based structures to handle uncertainty more effectively. This work not only generalizes existing approximation models but also bridges the gap between soft set theory and rough neighborhood systems. The proposed framework is theoretically robust and practically significant, offering valuable improvements for applications where data is imprecise, such as medical diagnosis, decision support systems, and pattern recognition, providing a solid foundation for future research in uncertain and vague data environments.

This article is organized as follows: in section 2, we mention some basic notions needed for this article to be self contained, in section 3, we define neighborhoods based on soft binary relation, in section 4, we construct lower and upper approximations from these neighborhoods and finally, in section 5 we generate new topologies from the new neighborhoods and form new lower and upper approximations based on the generated topologies.

2. Preliminaries

In this section, we recall the concepts and properties that we need to make the manuscript self-contained.

Definition 2.1. *[5] A subset $R \subseteq U \times U$ is called a binary relation, it is said to be reflexive if $(v, v) \in R \quad \forall v \in U$, symmetric if $(u, v) \in R$ whenever $(v, u) \in R$, transitive if $(v, w) \in R$ whenever $(v, u) \in R$ and $(u, w) \in R$ and equivalence if R is reflexive, symmetric and transitive.*

At first, Pawlak associated each set with two crisp sets called lower and upper approximations. These approximations defined with respect to the equivalent classes as follows.

Definition 2.2. *[14] If R is an equivalence relation on a universe U and $[x]_R$ is the equivalence class containing $x \in U$. The lower approximation $\underline{F}(A)$ and upper approximation $\overline{F}(A)$ of a set M of U are given by*

i. $\underline{F}(A) = \{x \in U : [x]_R \subseteq M\}$.

ii. $\overline{F}(A) = \{x \in U : [x]_R \cap M \neq \phi\}$.

Then, the equivalence relation has been replaced by specific (or arbitrary) relation to expand the scope of applications of rough set theory. This leads to deal with the so-called neighborhoods of an element instead of its equivalence classes. As a result, various sorts of rough-set paradigms have been introduced in the published literature. In what follows, we recall a set of these neighborhoods and paradigms that we need to show the importance and robustness of this work.

Definition 2.3. *[2], [3] and [18] Let R be an arbitrary binary relation on a universe set U, then the j-neighborhoods of an element $x \in U$ are defined as follows*
$j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:

i. $N_r(x) = \{y \in U : {}_xR_y\}$

ii. $N_l(x) = \{y \in U : {}_yR_x\}$

iii. $N_i(x) = N_r(x) \cap N_l(x)$

iv. $N_u(x) = N_r(x) \cup N_l(x)$

v. $N_{\langle r\rangle}(x) = \begin{cases} \bigcap_{x \in N_r(y)} N_r(y) & \text{if there exists } x \in N_r(y) \\ \phi & \text{otherwise} \end{cases}$

vi. $N_{\langle l\rangle}(x) = \begin{cases} \bigcap_{x \in N_l(y)} N_l(y) & \text{if there exists } x \in N_l(y) \\ \phi & \text{otherwise} \end{cases}$

vii. $N_{\langle i\rangle}(x) = N_{\langle r\rangle}(x) \cap N_{\langle l\rangle}(x)$

viii. $N_{\langle u\rangle}(x) = N_{\langle r\rangle}(x) \cup N_{\langle l\rangle}(x)$

Definition 2.4. *[1] Let R be an arbitrary binary relation on U and $\psi_j : U \longrightarrow P(U)$ be a mapping which assigns for each z in U its j-neighborhood in $P(U)$. then (U, R, ψ_j) is called a j-neighborhood space (N_jS).*

Definition 2.5. *[12] Let R be an arbitrary binary relation on a universe set U, then the j-adhesion neighborhoods of element $x \in U$ are defined as follows $j \in \{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$:*

i. $P_r(x) = \{y \in U : N_r(x) = N_r(y)\}$

ii. $P_l(x) = \{y \in U : N_l(x) = N_l(y)\}$

iii. $P_i(x) = P_r(x) \cap P_l(x)$

iv. $P_u(x) = P_r(x) \cup P_l(x)$

v. $P_{\langle r\rangle}(x) = \{y \in U : N_{\langle r\rangle}(x) = N_{\langle r\rangle}(y)\}$

vi. $P_{\langle l\rangle}(x) = \{y \in U : N_{\langle l\rangle}(x) = N_{\langle l\rangle}(y)\}$

vii. $P_{\langle i\rangle}(x) = P_{\langle r\rangle}(x) \cap P_{\langle l\rangle}(x)$

viii. $P_{\langle u\rangle}(x) = P_{\langle r\rangle}(x) \cup P_{\langle l\rangle}(x)$

Definition 2.6. *[13] Let U be an initial universe, $P(U)$ be the power set of U, η be a set of parameters, $\eta_1 \subseteq \eta$, then a pair (Ω, η_1) is called a soft set over U, where Ω is a mapping given by:*
$\Omega : \eta_1 \longrightarrow P(U)$.
In other words, a soft set over U is a parametrized family of subsets of the universe U, for every λ_i, $\Omega(\lambda_i)$ is a set of λ_i-approximate elements of the soft set (Ω, η_1). A soft set is not a set.

Definition 2.7. *[16] Let (Ω,η) is a soft set over $U\times V$, that is $\Omega:\eta\longrightarrow P(U\times V)$, then (Ω,η) is a soft binary relation from U to V.*

In other words, a soft binary relation from U to V is a parameterized family of binary relations from U to V. For each $\lambda\in\eta$, define $SS(U\times V,\eta)=\{(\Omega,\eta):(\Omega,\eta)$ is a soft binary relation from U to $V\}$.

Definition 2.8. *[16] If $(\Omega,\eta)\in SS(U\times V,\eta)$, then (Ω,η) is said to be soft reflexive (resp. symmetric, transitive, equivalence) relation over U if $\Omega(\lambda)$ for all $\lambda\in\eta$, is a reflexive (resp. symmetric, transitive, equivalence) relation from U to V.*

Definition 2.9. *[16] Let $M\subseteq V$, then we can define two soft sets over U:*
$\underline{\Omega}^M(\lambda)=\{x\in U:\phi\neq x\Omega(\lambda)\subseteq M\}$
$\overline{\Omega}^M(\lambda)=\{x\in U:x\Omega(\lambda)\cap M\neq\phi\}$
Where $x\Omega(\lambda)=\{y\in V:(x,y)\in\Omega(\lambda)$ for every $\lambda\in\eta\}$

3. Soft Neighborhoods

In this section, we define new types of neighborhoods based on a soft binary relation.

Definition 3.1. *Let (Ω,η) be a soft binary relation over U, $\lambda\in\eta$, then the soft N_j-neighborhoods of $x\in U$ are defined for each j as follows:*

i. $N_r^{\Omega(\lambda)}(x)=\{y\in U:(x,y)\in\Omega(\lambda)\}$ *[16]*

ii. $N_l^{\Omega(\lambda)}(x)=\{y\in U:(y,x)\in\Omega(\lambda)\}$

iii. $N_i^{\Omega(\lambda)}(x)=N_r^{\Omega(\lambda)}(x)\cap N_l^{\Omega(\lambda)}(x)$

iv. $N_u^{\Omega(\lambda)}(x)=N_r^{\Omega(\lambda)}(x)\cup N_l^{\Omega(\lambda)}(x)$

v. $N_{\langle r\rangle}^{\Omega(\lambda)}(x)=\begin{cases}\bigcap\limits_{x\in N_r^{\Omega(\lambda)}(y)} N_r^{\Omega(\lambda)}(y) & \textit{if there exists } x\in N_r^{\Omega(\lambda)}(y)\\ \phi & \textit{Otherwise}\end{cases}$

vi. $N_{\langle l\rangle}^{\Omega(\lambda)}(x)=\begin{cases}\bigcap\limits_{x\in N_l^{\Omega(\lambda)}(y)} N_l^{\Omega(\lambda)}(y) & \textit{if there exists } x\in N_l^{\Omega(\lambda)}(y)\\ \phi & \textit{otherwise}\end{cases}$

vii. $N_{\langle i\rangle}^{\Omega(\lambda)}(x)=N_{\langle r\rangle}^{\Omega(\lambda)}(x)\cap N_{\langle l\rangle}^{\Omega(\lambda)}(x)$

viii. $N_{\langle u\rangle}^{\Omega(\lambda)}(x)=N_{\langle r\rangle}^{\Omega(\lambda)}(x)\cup N_{\langle l\rangle}^{\Omega(\lambda)}(x)$

Definition 3.2. *Let (Ω,η) be a soft binary relation over U and μ_j be a mapping from U to $P(U)$ which associates each $x\in U$ with its soft N_j-neighborhood in $P(U)$, then (U,Ω,η,μ_j) is called a soft N_j-neighborhood space (briefly, S_j-NS).*

Definition 3.3. *Let (U,Ω,η,μ_j) be a soft N_j- neighborhood space over U, $\lambda\in\eta$, then the soft $j-$adhesion neighborhoods of $x\in U$ $P_j^{\Omega(\lambda)}(x)$, $\lambda\in\eta$, $j\in\{r,l,i,u,\langle r\rangle,\langle l\rangle,\langle i\rangle,\langle u\rangle\}$ are defined by:*

i. $P_r^{\Omega(\lambda)}(x)=\{y\in U:N_r^{\Omega(\lambda)}(x)=N_r^{\Omega(\lambda)}(y)\}$.

ii. $P_l^{\Omega(\lambda)}(x)=\{y\in U:N_l^{\Omega(\lambda)}(x)=N_l^{\Omega(\lambda)}(y)\}$.

iii. $P_i^{\Omega(\lambda)}(x)=P_r^{\Omega(\lambda)}(x)\cap P_l^{\Omega(\lambda)}(x)$.

iv. $P_u^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(x) \cup P_l^{\Omega(\lambda)}(x)$.

v. $P_{\langle r\rangle}^{\Omega(\lambda)}(x) = \{y \in U : N_{\langle r\rangle}^{\Omega(\lambda)}(x) = N_{\langle r\rangle}^{\Omega(\lambda)}(y)\}$.

vi. $P_{\langle l\rangle}^{\Omega(\lambda)}(x) = \{y \in U : N_{\langle l\rangle}^{\Omega(\lambda)}(x) = N_{\langle l\rangle}^{\Omega(\lambda)}(y)\}$.

vii. $P_{\langle i\rangle}^{\Omega(\lambda)}(x) = P_{\langle r\rangle}^{\Omega(\lambda)}(x) \cap P_{\langle l\rangle}^{\Omega(\lambda)}(x)$.

viii. $P_{\langle i\rangle}^{\Omega(\lambda)}(x) = P_{\langle r\rangle}^{\Omega(\lambda)}(x) \cup P_{\langle l\rangle}^{\Omega(\lambda)}(x)$.

Proposition 3.4. *Let (Ω, η, μ_j) be a $S_j - NS$ on U. Then for each j:*

i. $N_i^{\Omega(\lambda)}(x) \subseteq N_r^{\Omega(\lambda)}(x) \subseteq N_u^{\Omega(\lambda)}(x)$.

ii. $N_i^{\Omega(\lambda)}(x) \subseteq N_l^{\Omega(\lambda)}(x) \subseteq N_u^{\Omega(\lambda)}(x)$.

iii. $N_{\langle i\rangle}^{\Omega(\lambda)}(x) \subseteq N_{\langle r\rangle}^{\Omega(\lambda)}(x) \subseteq N_{\langle u\rangle}^{\Omega(\lambda)}(x)$.

iv. $N_{\langle i\rangle}^{\Omega(\lambda)}(x) \subseteq N_{\langle l\rangle}^{\Omega(\lambda)}(x) \subseteq N_{\langle u\rangle}^{\Omega(\lambda)}(x)$.

v. *If $\Omega(\lambda)$ is symmetric, then* $N_i^{\Omega(\lambda)}(x) = N_r^{\Omega(\lambda)}(x) = N_l^{\Omega(\lambda)}(x) = N_u^{\Omega(\lambda)}(x)$.

Proof.

i. $N_i^{\Omega(\lambda)}(x) = [N_r^{\Omega(\lambda)}(x) \cap N_l^{\Omega(\lambda)}(x)] \subseteq N_r^{\Omega(\lambda)}(x) \subseteq [N_r^{\Omega(\lambda)}(x) \cup N_l^{\Omega(\lambda)}(x)] = N_u^{\Omega(\lambda)}(x)$.

ii - iv similar to i

v. If $\Omega(\lambda)$ is symmetric, then $x\,\Omega(\lambda)\,y \Leftrightarrow y\,\Omega(\lambda)\,x$. Hence $N_r^{\Omega(\lambda)}(x) = N_l^{\Omega(\lambda)}(x) = N_i^{\Omega(\lambda)}(x) = N_u^{\Omega(\lambda)}(x)$.
□

Proposition 3.5. *Let (U, Ω, η, μ_j) be a $Sj - NS$, then for each j:*

i. $P_i^{\Omega(\lambda)}(x) \subseteq P_r^{\Omega(\lambda)}(x) \subseteq P_u^{\Omega(\lambda)}(x)$.

ii. $P_i^{\Omega(\lambda)}(x) \subseteq P_l^{\Omega(\lambda)}(x) \subseteq P_u^{\Omega(\lambda)}(x)$.

iii. $P_{\langle i\rangle}^{\Omega(\lambda)}(x) \subseteq P_{\langle r\rangle}^{\Omega(\lambda)}(x) \subseteq P_{\langle u\rangle}^{\Omega(\lambda)}(x)$.

iv. $P_{\langle i\rangle}^{\Omega(\lambda)}(x) \subseteq P_{\langle l\rangle}^{\Omega(\lambda)}(x) \subseteq P_{\langle u\rangle}^{\Omega(\lambda)}(x)$.

v. *If μ_j is symmetric, then* $P_i^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(x) = P_l^{\Omega(\lambda)}(x) = P_u^{\Omega(\lambda)}(x)$.

Proof. The proof follows directly from proposition 3.4. □

Proposition 3.6. *Let $(U, \Omega, \eta\mu_j)$ be a $Sj - NS$ on U, $\lambda \in \eta$, $j \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$, then:*

i. *For each $x \in U$, $x \in P_j^{\Omega(\lambda)}(x)$.*

ii. *$y \in P_j^{\Omega(\lambda)}(x)$ if and only if $P_j^{\Omega(\lambda)}(y) = P_j^{\Omega(\lambda)}(x)$.*

Proof.

i. Obvious.

ii. a. At $j = r$. First, if $P_r^{\Omega(\lambda)}(y) = P_r^{\Omega(\lambda)}(x)$, then $y \in P_r^{\Omega(\lambda)}(x)$. Second, let $y \in P_r^{\Omega(\lambda)}(x)$, then $N_r^{\Omega(\lambda)}(y) = N_r^{\Omega(\lambda)}(x)$. Let $z \in P_r^{\Omega(\lambda)}(y)$, then $N_r^{\Omega(\lambda)}(z) = N_r^{\Omega(\lambda)}(y)$, thus $N_r^{\Omega(\lambda)}(z) = N_r^{\Omega(\lambda)}(x)$. So $z \in P_r^{\Omega(\lambda)}(x)$, thus $P_r^{\Omega(\lambda)}(y) \subseteq P_r^{\Omega(\lambda)}(x)$. Similarly $P_r^{\Omega(\lambda)}(x) \subseteq P_r^{\Omega(\lambda)}(y)$.

b. At $j \in \{l, \langle r\rangle, \langle l\rangle\}$. The proof is similar to iia.

c. At $j = i$. First, if $P_i^{\Omega(\lambda)}(y) = P_i^{\Omega(\lambda)}(x)$, thus $y \in P_i^{\Omega(\lambda)}(x)$. Second, let $y \in P_i^{\Omega(\lambda)}(x)$, then $y \in P_r^{\Omega(\lambda)}(x)$ and $y \in P_l^{\Omega(\lambda)}(x)$. Therefore $N_r^{\Omega(\lambda)}(y) = N_r^{\Omega(\lambda)}(x)$ and $N_l^{\Omega(\lambda)}(y) = N_l^{\Omega(\lambda)}(x)$. Let $z \in P_r^{\Omega(\lambda)}(y)$, then $N_r^{\Omega(\lambda)}(z) = N_r^{\Omega(\lambda)}(y) = N_r^{\Omega(\lambda)}(x)$, so $z \in P_r^{\Omega(\lambda)}(x)$. Therefore $P_r^{\Omega(\lambda)}(y) \subseteq P_r^{\Omega(\lambda)}(x)$. Similarly, $P_r^{\Omega(\lambda)}(x) \subseteq P_r^{\Omega(\lambda)}(y)$. Therefore $P_r^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(y)$. Similarly, $P_l^{\Omega(\lambda)}(x) = P_l^{\Omega(\lambda)}(y)$, Hence $P_i^{\Omega(\lambda)}(x) = P_i^{\Omega(\lambda)}(y)$.

d. At $j = \langle i\rangle$ Similar to iic.

□

Proposition 3.7. *Let (U, Ω, η, μ_j) be a $Sj-NS$, then for each j, the class $\xi_{P_j}^{\Omega(\lambda)}(U) = \{P_j^{\Omega(\lambda)}(x) : x \in U, \lambda \in \eta$, $j \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$ forms a partition on U.*

Proposition 3.8. *Let (U, Ω, η, μ_j) be a $Sj-NS$ and $\lambda \in \eta$, then for each $x \in U$:*

i. $P_{\langle r\rangle}^{\Omega(\lambda)}(x) = P_l^{\Omega(\lambda)}(x)$.

ii. $P_{\langle l\rangle}^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(x)$.

Proof.

i. Let $y \in P_{\langle r\rangle}^{\Omega(\lambda)}(x)$, then $P_{\langle r\rangle}^{\Omega(\lambda)}(y) = P_{\langle r\rangle}^{\Omega(\lambda)}(x)$ and $N_{\langle r\rangle}^{\Omega(\lambda)}(y) = N_{\langle r\rangle}^{\Omega(\lambda)}(x)$. If $y \in P_l^{\Omega(\lambda)}(x)$ then the proof is complete. Assume that $y \notin P_l^{\Omega(\lambda)}(x)$, then $x \notin P_l^{\Omega(\lambda)}(y)$, thus $N_l^{\Omega(\lambda)}(x) \neq N_l^{\Omega(\lambda)}(y)$ and so there exists $z \in N_l^{\Omega(\lambda)}(y)$ such that $z \notin N_l^{\Omega(\lambda)}(x)$. Therefore $y \in N_r^{\Omega(\lambda)}(z)$ and $x \notin N_r^{\Omega(\lambda)}(z)$. So $N_{\langle r\rangle}^{\Omega(\lambda)}(x) \neq N_{\langle r\rangle}^{\Omega(\lambda)}(y)$ contradiction. Thus $y \in P_l^{\Omega(\lambda)}(x)$, hence $P_{\langle r\rangle}^{\Omega(\lambda)}(x) \subseteq P_l^{\Omega(\lambda)}(x)$. Similarly, $P_l^{\Omega(\lambda)}(x) \subseteq P_{\langle r\rangle}^{\Omega(\lambda)}(x)$.

ii. Similar.

□

Corollary 3.9. *Let (U, Ω, η, μ_j) be $Sj-NS$ and $\lambda \in \eta$, then for each $x \in U$:*

i. $P_{\langle i\rangle}^{\Omega(\lambda)}(x) = P_i^{\Omega(\lambda)}(x)$.

ii. $P_{\langle u\rangle}^{\Omega(\lambda)}(x) = P_u^{\Omega(\lambda)}(x)$.

Proof.

i. $P_i^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(x) \cap P_l^{\Omega(\lambda)}(x) = P_{\langle l\rangle}^{\Omega(\lambda)}(x) \cap P_{\langle r\rangle}^{\Omega(\lambda)}(x) = P_{\langle i\rangle}^{\Omega(\lambda)}(x)$.

ii. $P_u^{\Omega(\lambda)}(x) = P_r^{\Omega(\lambda)}(x) \cup P_l^{\Omega(\lambda)}(x) = P_{\langle l\rangle}^{\Omega(\lambda)}(x) \cup P_{\langle r\rangle}^{\Omega(\lambda)} = P_{\langle u\rangle}^{\Omega(\lambda)}$.

□

Proposition 3.10. *Let (U, Ω, η, μ_j) be $Sj-NS$ and Ω is a soft reflexive relation on U, then for each j, $P_j^{\Omega(\lambda)}(x) \subseteq N_j^{\Omega(\lambda)}(x)$ for each $x \in U$ and all $\lambda \in \eta$.*

Proof. We prove the case $j = r$ and the rest is similar. Let $y \in P_r^{\Omega(\lambda)}(x)$, then $N_r^{\Omega(\lambda)}(y) = N_r^{\Omega(\lambda)}(x)$. Since $\Omega(\lambda)$ is a soft reflexive relation on U, then $y \in N_r^{\Omega(\lambda)}(y)$, thus $y \in N_r^{\Omega(\lambda)}(x)$. Hence $P_r^{\Omega(\lambda)}(x) \subseteq N_r^{\Omega(\lambda)}(x)$ for each $x \in U$ and each $\lambda \in \eta$. □

In the following example, we evaluate the soft N_j and soft P_j-neighborhoods of each element in a universe U based on a soft binary relation (Ω).

Example 3.11. *Let $U = \{u_1, u_2, u_3\}$ and $\eta = \{\lambda_1, \lambda_2\}$. Define $\Omega : \eta \longrightarrow 2^{U\times U}$ by:*
$\Omega(\lambda_1) = \{(u_1, u_2), (u_2, u_1), (u_2, u_3), (u_3, u_2)\}$
$\Omega(\lambda_2) = \{(u_1, u_3), (u_2, u_2), (u_2, u_3), (u_3, u_3)\}$
Then the soft N_j-neighborhoods of each element in U are:

Table 1: Soft N_j-Neighborhoods

x	$N_r^{\Omega(\lambda_1)}(x)$	$N_l^{\Omega(\lambda_1)}(x)$	$N_i^{\Omega(\lambda_1)}(x)$	$N_u^{\Omega(\lambda_1)}(x)$	$N_{\langle r\rangle}^{\Omega(\lambda_1)}(x)$	$N_{\langle l\rangle}^{\Omega(\lambda_1)}(x)$	$N_{\langle i\rangle}^{\Omega(\lambda_1)}(x)$	$N_{\langle u\rangle}^{\Omega(\lambda_1)}(x)$
u_1	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$
u_2	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
u_3	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$
x	$N_r^{\Omega(\lambda_2)}(x)$	$N_l^{\Omega(\lambda_2)}(x)$	$N_i^{\Omega(\lambda_2)}(x)$	$N_u^{\Omega(\lambda_2)}(x)$	$N_{\langle r\rangle}^{\Omega(\lambda_2)}(x)$	$N_{\langle l\rangle}^{\Omega(\lambda_2)}(x)$	$N_{\langle i\rangle}^{\Omega(\lambda_2)}(x)$	$N_{\langle u\rangle}^{\Omega(\lambda_2)}(x)$
u_1	$\{u_3\}$	ϕ	ϕ	$\{u_3\}$	ϕ	U	ϕ	U
u_2	$\{u_2, u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2, u_3\}$	$\{u_2, u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2, u_3\}$
u_3	$\{u_3\}$	U	$\{u_3\}$	U	$\{u_3\}$	U	$\{u_3\}$	U

The soft P_j-neighborhoods of each element in U are:

Table 2: Soft P_j-Neighborhoods

x	$P_r^{\Omega(\lambda_1)}(x)$	$P_l^{\Omega(\lambda_1)}(x)$	$P_i^{\Omega(\lambda_1)}(x)$	$P_u^{\Omega(\lambda_1)}(x)$	$P_{\langle r\rangle}^{\Omega(\lambda_1)}(x)$	$P_{\langle l\rangle}^{\Omega(\lambda_1)}(x)$	$P_{\langle i\rangle}^{\Omega(\lambda_1)}(x)$	$P_{\langle u\rangle}^{\Omega(\lambda_1)}(x)$
u_1	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$
u_2	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
u_3	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$
x	$P_r^{\Omega(\lambda_2)}(x)$	$P_l^{\Omega(\lambda_2)}(x)$	$P_i^{\Omega(\lambda_2)}(x)$	$P_u^{\Omega(\lambda_2)}(x)$	$P_{\langle r\rangle}^{\Omega(\lambda_2)}(x)$	$P_{\langle l\rangle}^{\Omega(\lambda_2)}(x)$	$P_{\langle i\rangle}^{\Omega(\lambda_2)}(x)$	$P_{\langle u\rangle}^{\Omega(\lambda_2)}(x)$
u_1	$\{u_1, u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1, u_3\}$
u_2	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
u_3	$\{u_1, u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_1, u_3\}$	$\{u_1, u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_1, u_3\}$

4. Novel types of soft rough sets based on soft neighborhoods

In this section we construct new types of lower and upper approximations based on the neighborhoods defined in the previous section.

Definition 4.1. *Let (Ω, η, μ_j) be a $S_j - NS$ on U and $A \subseteq U$, then:*

i. $\underline{\sigma}_{N_j}^{A}(\lambda) = \{x \in U : N_j^{\Omega(\lambda)}(x) \subseteq A\}$ *(called soft N_j-lower approximation of A).*

ii. $\overline{\sigma}_{N_j}^{A}(\lambda) = \{x \in U : N_j^{\Omega(\lambda)}(x) \cap A \neq \phi\}$ *(called soft N_j-upper approximation of A).*

iii. $B_{N_j}^{A}(\lambda) = \overline{\sigma}_{N_j}^{A}(\lambda) - \underline{\sigma}_{N_j}^{A}(\lambda)$ *(called soft N_j-boundary region of A).*

iv. $POS_{N_j}^{A}(\lambda) = \underline{\sigma}_{N_j}^{A}(\lambda)$ *(called soft N_j-positive region of A).*

v. $NEG_{N_j}^{A}(\lambda) = U - \overline{\sigma}_{N_j}^{A}(\lambda)$ *(called soft N_j-negative region of A).*

vi. $M_{N_j}^{A}(\lambda) = \frac{|\underline{\sigma}_{N_j}^{A}(\lambda)\cap A|}{|\overline{\sigma}_{N_j}^{A}(\lambda)\cup A|}$ *(called soft N_j-accuracy measure of A).*

Let (U, Ω, η, μ_j) be a $Sj - NS$ on U and $A \subseteq U$, then:

i. $\underline{\sigma}^A_{P_j}(\lambda) = \{x \in U : P_j^{\Omega(\lambda)}(x) \subseteq A\}$ (called soft P_j-lower approximation of A).

ii. $\overline{\sigma}^A_{P_j}(\lambda) = \{x \in U : P_j^{\Omega(\lambda)}(x) \cap A \neq \phi\}$ (called soft P_j-upper approximation of A).

iii. $B^A_{P_j}(\lambda) = \overline{\sigma}^A_{P_j}(\lambda) - \underline{\sigma}^A_{P_j}(\lambda)$ (called soft P_j-boundary region of A).

iv. $POS^A_{P_j}(\lambda) = \underline{\sigma}^A_{P_j}(\lambda)$ (called soft P_j-positive region of A).

v. $NEG^A_{P_j}(\lambda) = U - \overline{\sigma}^A_{P_j}(\lambda)$ (called soft P_j-negative region of A).

vi. $M^A_{P_j}(\lambda) = \frac{|\underline{\sigma}^A_{P_j}(\lambda)|}{|\overline{\sigma}^A_{P_j}(\lambda)|}$ (called soft P_j-accuracy measure of A).

Proposition 4.2. *Let (Ω, η, μ_j) be a $S_j - NS$ on U and $A, B \subseteq U$, then the following properties hold for each j:*

i. $\underline{\sigma}^U_{N_j}(\lambda) = U$.

ii. $\overline{\sigma}^\phi_{N_j}(\lambda) = \phi$.

iii. If $A \subseteq B$, then $\underline{\sigma}^A_{N_j}(\lambda) \subseteq \underline{\sigma}^B_{N_j}(\lambda)$.

iv. $\underline{\sigma}^{A\cap B}_{N_j}(\lambda) = \underline{\sigma}^A_{N_j}(\lambda) \cap \underline{\sigma}^B_{N_j}(\lambda)$.

v. $\underline{\sigma}^{A^c}_{N_j}(\lambda) = (\overline{\sigma}^A_{N_j}(\lambda))^c$.

vi. If $A \subseteq B$, then $\overline{\sigma}^A_{N_j}(\lambda) \subseteq \overline{\sigma}^B_{N_j}(\lambda)$.

vii. $\overline{\sigma}^{A\cup B}_{N_j}(\lambda) = \overline{\sigma}^A_{N_j}(\lambda) \cup \overline{\sigma}^B_{N_j}(\lambda)$.

viii. $\overline{\sigma}^{A^c}_{N_j}(\lambda) = (\underline{\sigma}^A_{N_j}(\lambda))^c$.

Proof. i, ii Straightforward.

iii. Since $A \subseteq B$, then $\underline{\sigma}^A_{N_j}(\lambda) = \{x \in U : N_j^{\Omega(\lambda)}(x) \subseteq A\} \subseteq \{x \in U : N_j^{\Omega(\lambda)}(x) \subseteq B\} = \underline{\sigma}^B_{N_j}(\lambda)$.

iv. $(A \cap B) \subseteq A$, thus $\underline{\sigma}^{A\cap B}_{N_j}(\lambda) \subseteq \underline{\sigma}^A_{N_j}(\lambda)$. Similarly, $\underline{\sigma}^{A\cap B}_{N_j}(\lambda) \subseteq \underline{\sigma}^B_{N_j}(\lambda)$. Then, $\underline{\sigma}^{A\cap B}_{N_j}(\lambda) \subseteq \underline{\sigma}^A_{N_j}(\lambda) \cap \underline{\sigma}^B_{N_j}(\lambda)$. Conversely, let $x \in \underline{\sigma}^A_{N_j}(\lambda) \cap \underline{\sigma}^B_{N_j}(\lambda)$, then $x \in \underline{\sigma}^A_{N_j}(\lambda)$ and $x \in \underline{\sigma}^B_{N_j}(\lambda)$, thus $N_j^{\Omega(\lambda)}(x) \subseteq A$ and $N_j^{\Omega(\lambda)}(x) \subseteq B$, therefore $N_j^{\Omega(\lambda)}(x) \subseteq (A \cap B)$. Consequently $x \in \underline{\sigma}^{A\cap B}_{N_j}(\lambda)$, therefore $\underline{\sigma}^A_{N_j}(\lambda) \cap \underline{\sigma}^B_{N_j}(\lambda) \subseteq \underline{\sigma}^{A\cap B}_{N_j}(\lambda)$. Hence $\underline{\sigma}^A_{N_j}(\lambda) \cap \underline{\sigma}^B_{N_j}(\lambda) = \underline{\sigma}^{A\cap B}_{N_j}(\lambda)$.

v. $x \in \underline{\sigma}^{A^c}_{N_j}(\lambda) \Longleftrightarrow N_j^{\Omega(\lambda)}(x) \subseteq A^c \Longleftrightarrow N_j^{\Omega(\lambda)}(x) \cap A = \phi \Longleftrightarrow x \notin \overline{\sigma}^A_{N_j}(\lambda) \Longleftrightarrow x \in (\overline{\sigma}^A_{N_j}(\lambda))^c$.

vi. Since $A \subseteq B$, then $\overline{\sigma}^A_{N_j}(\lambda) = \{x \in U : N_j^{\Omega(\lambda)}(x) \cap A \neq \phi\} \subseteq \{x \in U : N_j^{\Omega(\lambda)}(x) \cap B \neq \phi\} = \overline{\sigma}^B_{N_j}(\lambda)$.

vii. $A \subseteq (A \cup B)$, therefore $\overline{\sigma}^A_{N_j}(\lambda) \subseteq \overline{\sigma}^{A\cup B}_{N_j}(\lambda)$. Similarly, $\overline{\sigma}^B_{N_j}(\lambda) \subseteq \overline{\sigma}^{A\cup B}_{N_j}(\lambda)$. Thus, $\overline{\sigma}^A_{N_j}(\lambda) \cup \overline{\sigma}^B_{N_j}(\lambda) \subseteq \overline{\sigma}^{A\cup B}_{N_j}(\lambda)$. Conversely, let $x \in \overline{\sigma}^{A\cup B}_{N_j}(\lambda)$, thus $N_j^{\Omega(\lambda)}(x) \cap (A \cup B) \neq \phi$, therefore $N_j^{\Omega(\lambda)}(x) \cap A \neq \phi$ or $N_j^{\Omega(\lambda)}(x) \cap B \neq \phi$. Consequently $x \in \overline{\sigma}^A_{N_j}(\lambda)$ or $x \in \overline{\sigma}^B_{N_j}(\lambda)$, then $x \in \overline{\sigma}^A_{N_j}(\lambda) \cup \overline{\sigma}^B_{N_j}(\lambda)$, therefore $\overline{\sigma}^{A\cup B}_{N_j}(\lambda) \subseteq \overline{\sigma}^A_{N_j}(\lambda) \cup \overline{\sigma}^B_{N_j}(\lambda)$. Hence $\overline{\sigma}^{A\cup B}_{N_j}(\lambda) = \overline{\sigma}^A_{N_j}(\lambda) \cup \overline{\sigma}^B_{N_j}(\lambda)$.

viii. It follows directly from v.

□

Let (U, Ω, η, μ_j) be a $Sj - NS$, $\lambda \in \eta$ and $A, B \subseteq U$. Then for each j, the following properties hold:

i. $\underline{\sigma}^{U}_{P_j}(\lambda) = \overline{\sigma}^{U}_{P_j}(\lambda) = U$.

ii. $\underline{\sigma}^{\phi}_{P_j}(\lambda) = \overline{\sigma}^{\phi}_{P_j}(\lambda) = \phi$.

iii. If $A \subseteq B$, then $\underline{\sigma}^{A}_{P_j}(\lambda) \subseteq \underline{\sigma}^{B}_{P_j}(\lambda)$.

iv. $\underline{\sigma}^{A \cap B}_{P_j}(\lambda) = \underline{\sigma}^{A}_{P_j} \cap \underline{\sigma}^{B}_{P_j}(\lambda)$.

v. $\underline{\sigma}^{A^c}_{P_j}(\lambda) = (\overline{\sigma}^{A}_{P_j}(\lambda))^c$.

vi. If $A \subseteq B$, then $\overline{\sigma}^{A}_{P_j}(\lambda) \subseteq \overline{\sigma}^{B}_{P_j}(\lambda)$.

vii. $\overline{\sigma}^{A \cup B}_{P_j}(\lambda) = \overline{\sigma}^{A}_{P_j}(\lambda) \cup \overline{\sigma}^{B}_{P_j}(\lambda)$.

viii. $\overline{\sigma}^{A^c}_{P_j}(\lambda) = (\underline{\sigma}^{A}_{P_j}(\lambda))^c$.

Proof. Similar to proposition 4.2. □

Proposition 4.3. *Let (Ω, η, μ_j) be a $S_j - NS$ on U and $A \subseteq U$, then:*

i. $\underline{\sigma}^{A}_{N_u}(\lambda) \subseteq \underline{\sigma}^{A}_{N_r}(\lambda) \subseteq \underline{\sigma}^{A}_{N_i}(\lambda)$.

ii. $\underline{\sigma}^{A}_{N_u}(\lambda) \subseteq \underline{\sigma}^{A}_{N_l}(\lambda) \subseteq \underline{\sigma}^{A}_{N_i}(\lambda)$.

iii. $\underline{\sigma}^{A}_{N_{\langle u \rangle}}(\lambda) \subseteq \underline{\sigma}^{A}_{N_{\langle r \rangle}}(\lambda) \subseteq \underline{\sigma}^{A}_{N_{\langle i \rangle}}(\lambda)$.

iv. $\underline{\sigma}^{A}_{N_{\langle u \rangle}}(\lambda) \subseteq \underline{\sigma}^{A}_{N_{\langle l \rangle}}(\lambda) \subseteq \underline{\sigma}^{A}_{N_{\langle i \rangle}}(\lambda)$.

v. $\overline{\sigma}^{A}_{N_i}(\lambda) \subseteq \overline{\sigma}^{A}_{N_r}(\lambda) \subseteq \overline{\sigma}^{A}_{N_u}(\lambda)$.

vi. $\overline{\sigma}^{A}_{N_i}(\lambda) \subseteq \overline{\sigma}^{A}_{N_l}(\lambda) \subseteq \overline{\sigma}^{A}_{N_u}(\lambda)$.

vii. $\overline{\sigma}^{A}_{N_{\langle i \rangle}}(\lambda) \subseteq \overline{\sigma}^{A}_{N_{\langle r \rangle}}(\lambda) \subseteq \overline{\sigma}^{A}_{N_{\langle u \rangle}}(\lambda)$.

viii. $\overline{\sigma}^{A}_{N_{\langle i \rangle}}(\lambda) \subseteq \overline{\sigma}^{A}_{N_{\langle l \rangle}}(\lambda) \subset \overline{\sigma}^{A}_{N_{\langle u \rangle}}(\lambda)$.

Proof. The proof follows directly from proposition 3.4. □

Corollary 4.4. *Let (Ω, η, μ_j) be a $S_j - NS$ and $X \subseteq U$. If η is reflexive, then for each j:*

i. $M^{X}_{N_u}(\lambda) \subseteq M^{X}_{N_r}(\lambda) \subseteq M^{X}_{N_i}(\lambda)$.

ii. $M^{X}_{N_u}(\lambda) \subseteq M^{X}_{N_l}(\lambda) \subseteq M^{X}_{N_i}(\lambda)$.

iii. $M^{X}_{N_{\langle u \rangle}}(\lambda) \subseteq M^{X}_{N_{\langle r \rangle}}(\lambda) \subseteq M^{X}_{N_{\langle i \rangle}}(\lambda)$.

iv. $M^{X}_{N_{\langle u \rangle}}(\lambda) \subseteq M^{X}_{N_{\langle l \rangle}}(\lambda) \subseteq M^{X}_{N_{\langle i \rangle}}(\lambda)$.

Proposition 4.5. *Let (U, Ω, η, μ_j) be a $Sj - NS$, $\lambda \in \eta$ and $A \subseteq U$, then the following properties hold for each j:*

i. $\underline{\sigma}^A_{P_u}(\lambda) \subseteq \underline{\sigma}^A_{P_r}(\lambda) \subseteq \underline{\sigma}^A_{P_i}(\lambda)$.

ii. $\underline{\sigma}^A_{P_u}(\lambda) \subseteq \underline{\sigma}^A_{P_l}(\lambda) \subseteq \underline{\sigma}^A_{P_i}(\lambda)$.

iii. $\underline{\sigma}^A_{P_{\langle u\rangle}}(\lambda) \subseteq \underline{\sigma}^A_{P_{\langle r\rangle}}(\lambda) \subseteq \underline{\sigma}^A_{P_{\langle i\rangle}}(\lambda)$.

iv. $\underline{\sigma}^A_{P_{\langle u\rangle}}(\lambda) \subseteq \underline{\sigma}^A_{P_{\langle l\rangle}}(\lambda) \subseteq \underline{\sigma}^A_{P_{\langle i\rangle}}(\lambda)$.

v. $\overline{\sigma}^A_{P_i}(\lambda) \subset \overline{\sigma}^A_{P_r}(\lambda) \subseteq \overline{\sigma}^A_{P_u}(\lambda)$.

vi. $\overline{\sigma}^A_{P_i}(\lambda) \subseteq \overline{\sigma}^A_{P_l}(\lambda) \subseteq \overline{\sigma}^A_{P_u}(\lambda)$.

vii. $\overline{\sigma}^A_{P_{\langle i\rangle}}(\lambda) \subseteq \overline{\sigma}^A_{P_{\langle r\rangle}}(\lambda) \subseteq \overline{\sigma}^A_{P_{\langle u\rangle}}(\lambda)$.

viii. $\overline{\sigma}^A_{P_{\langle i\rangle}}(\lambda) \subseteq \overline{\sigma}^A_{P_{\langle l\rangle}}(\lambda) \subseteq \overline{\sigma}^A_{P_{\langle u\rangle}}(\lambda)$.

Proof. The proof follows directly from proposition 3.5. □

Corollary 4.6. *i.* $M^A_{P_u}(\lambda) \subset M^A_{P_r}(\lambda) \subseteq M^A_{P_i}(\lambda)$.

ii. $M^A_{P_u}(\lambda) \subseteq M^A_{P_l}(\lambda) \subseteq M^A_{P_i}(\lambda)$.

iii. $M^A_{P_{\langle u\rangle}}(\lambda) \subseteq M^A_{P_{\langle r\rangle}}(\lambda) \subseteq M^A_{P_{\langle i\rangle}}$.

iv. $M^A_{P_{\langle u\rangle}}(\lambda) \subseteq M^A_{P_{\langle l\rangle}}(\lambda) \subseteq M^A_{P_{\langle i\rangle}}(\lambda)$.

In the following example, we continue example 3.11. We evaluate the lower and upper approximations of each set in the power set of U.

Example 4.7. *Continued from example (3.11). The soft N_j-lower approximations of each set in $P(U)$ are:*

Table 3: Soft N_j-Lower Approximations

A	$\underline{\sigma}^A_{N_r}(\lambda_1)$	$\underline{\sigma}^A_{N_l}(\lambda_1)$	$\underline{\sigma}^A_{N_i}(\lambda_1)$	$\underline{\sigma}^A_{N_u}(\lambda_1)$	$\underline{\sigma}^A_{N_{\langle r\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{N_{\langle l\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{N_{\langle i\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{N_{\langle u\rangle}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1,u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U
A	$\underline{\sigma}^A_{N_r}(\lambda_2)$	$\underline{\sigma}^A_{N_l}(\lambda_2)$	$\underline{\sigma}^A_{N_i}(\lambda_2)$	$\underline{\sigma}^A_{N_u}(\lambda_2)$	$\underline{\sigma}^A_{N_{\langle r\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{N_{\langle l\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{N_{\langle i\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{N_{\langle u\rangle}}(\lambda_2)$
ϕ	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ
$\{u_1\}$	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ
$\{u_2\}$	ϕ	$\{u_1,u_2\}$	$\{u_1,u_2\}$	ϕ	$\{u_1\}$	$\{u_2\}$	$\{u_1,u_2\}$	ϕ
$\{u_3\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_3\}$	ϕ	$\{u_1,u_3\}$	ϕ
$\{u_1,u_2\}$	ϕ	$\{u_1,u_2\}$	$\{u_1,u_2\}$	ϕ	$\{u_1\}$	$\{u_2\}$	$\{u_1,u_2\}$	ϕ
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_3\}$	ϕ	$\{u_1,u_3\}$	ϕ
$\{u_2,u_3\}$	U	$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	U	$\{u_2\}$	U	$\{u_2\}$
U	U	U	U	U	U	U	U	U

The soft P_j-lower approximations of each set in $P(U)$ are:

Table 4: Soft P_j-Lower Approximations

A	$\underline{\sigma}^A_{P_r}(\lambda_1)$	$\underline{\sigma}^A_{P_l}(\lambda_1)$	$\underline{\sigma}^A_{P_i}(\lambda_1)$	$\underline{\sigma}^A_{P_u}(\lambda_1)$	$\underline{\sigma}^A_{P_{\langle r\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{P_{\langle l\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{P_{\langle i\rangle}}(\lambda_1)$	$\underline{\sigma}^A_{P_{\langle u\rangle}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1,u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U
A	$\underline{\sigma}^A_{P_r}(\lambda_2)$	$\underline{\sigma}^A_{P_l}(\lambda_2)$	$\underline{\sigma}^A_{P_i}(\lambda_2)$	$\underline{\sigma}^A_{P_u}(\lambda_2)$	$\underline{\sigma}^A_{P_{\langle r\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{P_{\langle l\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{P_{\langle i\rangle}}(\lambda_2)$	$\underline{\sigma}^A_{P_{\langle u\rangle}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	$\{u_3\}$	$\{u_3\}$	ϕ	ϕ	$\{u_3\}$	$\{u_3\}$	ϕ
$\{u_1,u_2\}$	$\{u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1,u_2\}$	$\{u,u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U

The soft N_j-upper approximations of each set in $P(U)$ are:

Table 5: Soft N_j-Upper Approximations

A	$\overline{\sigma}^A_{N_r}(\lambda_1)$	$\overline{\sigma}^A_{N_l}(\lambda_1)$	$\overline{\sigma}^A_{N_i}(\lambda_1)$	$\overline{\sigma}^A_{N_u}(\lambda_1)$	$\overline{\sigma}^A_{N_{\langle r\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{N_{\langle l\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{N_{\langle i\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{N_{\langle u\rangle}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_1,u_2\}$	U	U	U	U	U	U	U	U
$\{u_1,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U
A	$\overline{\sigma}^A_{N_r}(\lambda_2)$	$\overline{\sigma}^A_{N_l}(\lambda_2)$	$\overline{\sigma}^A_{N_i}(\lambda_2)$	$\overline{\sigma}^A_{N_u}(\lambda_2)$	$\overline{\sigma}^A_{N_{\langle r\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{N_{\langle l\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{N_{\langle i\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{N_{\langle u\rangle}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	$\{u_3\}$	ϕ	$\{u_3\}$	ϕ	$\{u_1,u_3\}$	ϕ	$\{u_1,u_3\}$
$\{u_2\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2\}$	U	$\{u_2\}$	U
$\{u_3\}$	U	$\{u_3\}$	$\{u_3\}$	U	$\{u_2,u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	U
$\{u_1,u_2\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2\}$	U	$\{u_2\}$	U
$\{u_1,u_3\}$	U	$\{u_3\}$	$\{u_3\}$	U	$\{u_2,u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	U
$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U
U	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U

The soft P_j-upper approximations of each set in $P(U)$ are:

Table 6: Soft P_j-Upper Approximations

A	$\overline{\sigma}^A_{P_r}(\lambda_1)$	$\overline{\sigma}^A_{P_l}(\lambda_1)$	$\overline{\sigma}^A_{P_i}(\lambda_1)$	$\overline{\sigma}^A_{P_u}(\lambda_1)$	$\overline{\sigma}^A_{P_{\langle r\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{P_{\langle l\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{P_{\langle i\rangle}}(\lambda_1)$	$\overline{\sigma}^A_{P_{\langle u\rangle}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_1,u_2\}$	U	U	U	U	U	U	U	U
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U
A	$\overline{\sigma}^A_{P_r}(\lambda_2)$	$\overline{\sigma}^A_{P_l}(\lambda_2)$	$\overline{\sigma}^A_{P_i}(\lambda_2)$	$\overline{\sigma}^A_{P_u}(\lambda_2)$	$\overline{\sigma}^A_{P_{\langle r\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{P_{\langle l\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{P_{\langle i\rangle}}(\lambda_2)$	$\overline{\sigma}^A_{P_{\langle u\rangle}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1\}$
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_3\}$
$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$
U	U	U	U	U	U	U	U	U

The soft N_j-accuracy measures of each set in $P(U)$ are:

Table 7: Soft N_j-Accuracy Measures

A	$M^A_{N_r}(\lambda_1)$	$M^A_{N_l}(\lambda_1)$	$M^A_{N_i}(\lambda_1)$	$M^A_{N_u}(\lambda_1)$	$M^A_{N_{\langle r\rangle}}(\lambda_1)$	$M^A_{N_{\langle l\rangle}}(\lambda_1)$	$M^A_{N_{\langle i\rangle}}(\lambda_1)$	$M^A_{N_{\langle u\rangle}}(\lambda_1)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	0	0	0	0	0	0	0
$\{u_2\}$	0	0	0	0	1	1	1	1
$\{u_3\}$	0	0	0	0	0	0	0	0
$\{u_1,u_2\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{u_1,u_3\}$	0	0	0	0	1	1	1	1
$\{u_2,u_3\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1
A	$M^A_{N_r}(\lambda_2)$	$M^A_{N_l}(\lambda_2)$	$M^A_{N_i}(\lambda_2)$	$M^A_{N_u}(\lambda_2)$	$M^A_{N_{\langle r\rangle}}(\lambda_2)$	$M^A_{N_{\langle l\rangle}}(\lambda_2)$	$M^A_{N_{\langle i\rangle}}(\lambda_2)$	$M^A_{N_{\langle u\rangle}}(\lambda_2)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	$\frac{1}{2}$	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$
$\{u_2\}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{3}$	1	$\frac{1}{3}$
$\{u_3\}$	0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{3}$
$\{u_1,u_2\}$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{1}{3}$	1	$\frac{2}{3}$	1	$\frac{2}{3}$
$\{u_1,u_3\}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$
$\{u_2,u_3\}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	1	$\frac{2}{3}$	1	$\frac{2}{3}$
U	1	1	1	1	1	1	1	1

The soft P_j-accuracy measures of each set in $P(U)$ are:

Table 8: Soft P_j-Accuracy Measures

A	$M^A_{P_r}(\lambda_1)$	$M^A_{P_l}(\lambda_1)$	$M^A_{P_i}(\lambda_1)$	$M^A_{P_u}(\lambda_1)$	$M^A_{P_{\langle r\rangle}}(\lambda_1)$	$M^A_{P_{\langle l\rangle}}(\lambda_1)$	$M^A_{P_{\langle i\rangle}}(\lambda_1)$	$M^A_{P_{\langle u\rangle}}(\lambda_1)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	0	0	0	0	0	0	0
$\{u_2\}$	1	1	1	1	1	1	1	1
$\{u_3\}$	0	0	0	0	0	0	0	0
$\{u_1, u_2\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{u_1, u_3\}$	1	1	1	1	1	1	1	1
$\{u_2, u_3\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1
A	$M^A_{P_r}(\lambda_2)$	$M^A_{P_l}(\lambda_2)$	$M^A_{P_i}(\lambda_2)$	$M^A_{P_u}(\lambda_2)$	$M^A_{P_{\langle r\rangle}}(\lambda_2)$	$M^A_{P_{\langle l\rangle}}(\lambda_2)$	$M^A_{P_{\langle i\rangle}}(\lambda_2)$	$M^A_{P_{\langle u\rangle}}(\lambda_2)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	1	1	0	0	1	1	0
$\{u_2\}$	1	1	1	1	1	1	1	1
$\{u_3\}$	0	1	1	0	0	1	1	0
$\{u_1, u_2\}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$
$\{u_1, u_3\}$	1	1	1	1	1	1	1	1
$\{u_2, u_3\}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1

Remark 4.8. *Example 4.7 the converse of the following results doesn't hold in general. Let $A, B \subseteq U$, then:*

I. *If $A \subseteq B$, then $\underline{\sigma}^A_{N_j}(\lambda) \subseteq \underline{\sigma}^B_{N_j}(\lambda)$.*

Let $A = \{u_1\}$ and $B = \{u_1, u_3\}$

- *i.* $\underline{\sigma}^A_{N_i}(\lambda_2) = \{u_1\}$
- *ii.* $\underline{\sigma}^B_{N_i}(\lambda_2) = \{u_1, u_3\}$

II. *If $A \subseteq B$, then $\overline{\sigma}^A_{N_j}(\lambda) \subseteq \overline{\sigma}^B_{N_j}(\lambda)$*

Let $A = \{u_1\}$ and $B = \{u_1, u_3\}$

- *i.* $\overline{\sigma}^A_{N_u}(\lambda_2) = \{u_2\}$
- *ii.* $\overline{\sigma}^B_{N_u}(\lambda_2) = U$

III. *If $A \subseteq B$, then $\underline{\sigma}^A_{P_j}(\lambda) \subseteq \underline{\sigma}^B_{P_j}(\lambda)$*

Let $A = \{u_2\}$ and $B = \{u_2, u_3\}$

- *i.* $\underline{\sigma}^A_{P_l}(\lambda_2) = \{u_2\}$
- *ii.* $\underline{\sigma}^B_{P_l}(\lambda_2) = \{u_2, u_3\}$

IV. If $A \subseteq B$, *then* $\overline{\sigma}^{A}_{P_j}(\lambda) \subseteq \overline{\sigma}^{B}_{P_j}(\lambda)$

Let $A = \{u_2\}$ *and* $B = \{u_2, u_3\}$

i. $\overline{\sigma}^{A}_{P_r}(\lambda_2) = \{u_2\}$

ii. $\overline{\sigma}^{B}_{P_r}(\lambda_2) = U$

V. $\underline{\sigma}^{A}_{N_u}(\lambda) \subseteq \underline{\sigma}^{A}_{N_l}(\lambda) \subseteq \underline{\sigma}^{A}_{N_i}(\lambda)$.

Let $A = \{u_1\}$ *and* $B = \{u_3\}$

i. $\underline{\sigma}^{A}_{N_u}(\lambda_2) = \phi$

ii. $\underline{\sigma}^{A}_{N_l}(\lambda_2) = \{u_1\}$

iii. $\underline{\sigma}^{B}_{N_u}(\lambda_2) = \{u_1\}$

iv. $\underline{\sigma}^{B}_{N_i}(\lambda_2) = \{u_1, u_3\}$

VI. $\overline{\sigma}^{A}_{N_i}(\lambda) \subseteq \overline{\sigma}^{A}_{N_r}(\lambda) \subseteq \overline{\sigma}^{A}_{N_u}(\lambda)$

Let $A = \{u_3\}$ *and* $B = \{u_2\}$

i. $\overline{\sigma}^{A}_{N_i}(\lambda_2) = \{u_3\}$

ii. $\overline{\sigma}^{A}_{N_r}(\lambda_2) = U$

iii. $\overline{\sigma}^{B}_{N_r}(\lambda_2) = \{u_2\}$

iv. $\overline{\sigma}^{B}_{N_u}(\lambda_2) = \{u_2, u_3\}$

VII. $\underline{\sigma}^{A}_{P_u}(\lambda) \subseteq \underline{\sigma}^{A}_{P_l}(\lambda) \subseteq \underline{\sigma}^{A}_{P_i}(\lambda)$.

Let $A = \{u_1\}$

i. $\underline{\sigma}^{A}_{P_u}(\lambda_2) = \phi$

ii. $\underline{\sigma}^{A}_{P_l}(\lambda_2) = \{u_1\}$

VIII. $\overline{\sigma}^{A}_{P_i}(\lambda) \subseteq \overline{\sigma}^{A}_{P_l}(\lambda) \subseteq \overline{\sigma}^{A}_{P_u}(\lambda)$

Let $A = \{u_1\}$

i. $\overline{\sigma}^{A}_{P_i}(\lambda_2) = \{u_1\}$

ii. $\overline{\sigma}^{A}_{P_r}(\lambda_2) = \{u_1, u_3\}$

5. Topologies generated by soft N_j-neighborhoods and soft P_j-neighborhoods

In this section we generate new topologies from the neighborhoods that we define in section 3 and we also form new lower and upper approximations from these topologies.

Theorem 5.1. *Let* (U, Ω, η, μ_j) *be a soft* N_j- *neighborhood space over* U. *Then* $\tau_{N_j}(\lambda) = \{A \subseteq U : N_j^{\Omega(\lambda)}(x) \subseteq A,$ *for all* $x \in A,$ *all* $\lambda \in \eta$ *and all* $j\}$ *is a topology on* U.

Proof.

i. Obviously, $U, \phi \in \tau_{N_j}(\lambda)$.

ii. Let $A_i \in \tau_{N_j}(\lambda)$, $i \in I$, then $N_j^{\Omega(\lambda)}(x) \subseteq A_i$ for all $x \in A_i$ and for each $i \in I$. Let $x \in \bigcup_{i\in I} A_i$, thus there exists $i_0 \in I$ such that $x \in A_{i_0}$, therefore $N_j^{\Omega(\lambda)}(x) \subseteq A_{i_0} \subseteq \bigcup_{i\in I} A_i$. Hence $\bigcup_{i\in I} A_i \in \tau_{N_j}(\lambda)$.

iii. Let $A_1, A_2 \in \tau_{N_j}(\lambda)$, then $N_j^{\Omega(\lambda)}(x) \subseteq A_1$ for each $x \in A_1$ and $N_j^{\Omega(\lambda)}(x) \subseteq A_2$ for each $x \in A_2$. Let $x \in (A_1 \cap A_2)$, thus $x \in A_1$ and $x \in A_2$, therefore $N_j^{\Omega(\lambda)}(x) \subseteq A_1$ and $N_j^{\Omega(\lambda)}(x) \subseteq A_2$, thus $N_j^{\Omega(\lambda)}(x) \subseteq (A_1 \cap A_2)$. Hence $(A_1 \cap A_2) \in \tau_{N_j}(\lambda)$.
□

Theorem 5.2. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U. Then $\tau_{P_j}(\lambda) = \{A \subseteq U : P_j^{\Omega(\lambda)}(x) \subseteq A,\ for\ all\ x \in A,\ all\ \lambda \in \eta\ and\ all\ j\}$ is a topology on U.*

Proof. Similar to theorem 5.1. □

Definition 5.3. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U, $\tau_{N_j}(\lambda)$ be a topology on U and $A \subseteq U$, then for each j:*

i. $\underline{\sigma}^A_{\tau_{N_j}}(\lambda) = A^0_{N_j}(\lambda)$. *(Called soft τ_{N_j}-lower approximation of A, where $A^0_{N_j}(\lambda)$ is the interior of A)*

ii. $\overline{\sigma}^A_{\tau_{N_j}}(\lambda) = \overline{A}_{\tau_{N_j}}(\lambda)$. *(Called soft τ_{N_j}-upper approximation of A, where $\overline{A}_{\tau_{N_j}}(\lambda)$ is the closure of A)*

iii. $B^A_{\tau_{N_j}}(\lambda) = \overline{\sigma}^A_{\tau_{N_j}}(\lambda) - \underline{\sigma}^A_{\tau_{N_j}}(\lambda)$. *(Called soft τ_{N_j}-boundary of A)*

iv. $POS^A_{\tau_{N_j}}(\lambda) = \underline{\sigma}^A_{\tau_{N_j}}(\lambda)$. *(Called soft τ_{N_j}-positive region of A)*

v. $NEG^A_{\tau_{N_j}}(\lambda) = U - \overline{\sigma}^A_{\tau_{N_j}}(\lambda)$. *(Called soft τ_{N_j}-negative region of A)*

vi. $M^A_{\tau_{N_j}}(\lambda) = \frac{|\underline{\sigma}^A_{\tau_{N_j}}(\lambda)|}{|\overline{\sigma}^A_{\tau_{N_j}}(\lambda)|}$. *(Called soft τ_{N_j}-accuracy measure of A)*

Definition 5.4. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U, $\tau_{P_j}(\lambda)$ be a topology on U and $A \subseteq U$, then for each j:*

i. $\underline{\sigma}^A_{\tau_{P_j}}(\lambda) = A^o_{P_j}(\lambda)$. *(Called soft τ_{P_j}-lower approximation of A, where $A^0_{P_j}(\lambda)$ is the interior of A).*

ii. $\overline{\sigma}^A_{\tau_{P_j}}(\lambda) = \overline{A}_{\tau_{P_j}}(\lambda)$. *(Called soft τ_{P_j}-upper approximation of A, where $\overline{A}_{\tau_{P_j}}(\lambda)$ is the closure of A).*

iii. $B^A_{\tau_{P_j}}(\lambda) = \overline{\sigma}^A_{\tau_{P_j}}(\lambda) - \underline{\sigma}^A_{\tau_{P_j}}(\lambda)$. *(Called soft τ_{P_j}-boundary of A)*

iv. $POS^A_{\tau_{P_j}}(\lambda) = \underline{\sigma}^A_{\tau_{P_j}}(\lambda)$. *(Called soft τ_{P_j}-positive region of A)*

v. $NEG^A_{\tau_{P_j}}(\lambda) = U - \overline{\sigma}^A_{\tau_{P_j}}(\lambda)$. *(Called soft τ_{P_j}-negative region of A)*

vi. $M^A_{\tau_{P_j}}(\lambda) = \frac{|\underline{\sigma}^A_{\tau_{P_j}}(\lambda)|}{|\overline{\sigma}^A_{\tau_{P_j}}(\lambda)|}$. *(Called soft τ_{P_j}-accuracy measure of A)*

In the following propositions we will relate the approximations we construct in this section via topology with that we form in the previous section without topology.

Proposition 5.5. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U, $\tau_{P_j}(\lambda)$ be a topology on U and $A \subseteq U$, then for each j: $\underline{\sigma}^A_{\tau_{N_j}}(\lambda) \subseteq \underline{\sigma}^A_{N_j}(\lambda)$ and $\overline{\sigma}^A_{N_j}(\lambda) \subseteq \overline{\sigma}^A_{\tau_{N_j}}(\lambda)$.*

Proof. First, Let $z \in \underline{\sigma}^A_{\tau_{N_j}}(\lambda)$, then $z \in A^o_{N_j}(\lambda)$. Thus there exists an open set G such that $z \in G \subseteq A$. Therefore $N_j^{\Omega(\lambda)}(z) \subseteq G \subseteq A$. Hence $\underline{\sigma}^A_{\tau_{N_j}}(\lambda) \subseteq \underline{\sigma}^A_{N_j}(\lambda)$. Second, Let $z \notin \overline{\sigma}^A_{\tau_{N_j}}(\lambda)$, then $z \notin \overline{A}_{N_j}(\lambda)$. So there exists a closed set $F \supseteq A$ such that $z \notin F$. Therefore $z \in F^c$ which is an open set and $F^c \cap A = \phi$. Since $z \in F^c$ which is open, thus $N_j^{\Omega(\lambda)}(z) \subseteq F^c$ and $N_j^{\Omega(\lambda)}(z) \cap A = \phi$. Therefore $z \notin \overline{\sigma}^A_{N_j}(\lambda)$. Hence $\overline{\sigma}^A_{N_j}(\lambda) \subseteq \overline{\sigma}^A_{\tau_{N_j}}(\lambda)$. □

Corollary 5.6. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U, $\tau_{P_j}(\lambda)$ be a topology on U and $A \subseteq U$, then for each j: $M^A_{\tau_{N_j}}(\lambda) \subset M^A_{N_j}(\lambda)$.*

Proposition 5.7. *Let (U, Ω, η, μ_j) be a soft N_j- neighborhood space over U, $\tau_{P_j}(\lambda)$ be a topology on U and $A \subseteq U$, then for each j:*

i. $\underline{\sigma}^A_{\tau_{P_j}}(\lambda) \subseteq \underline{\sigma}^A_{P_j}(\lambda)$

ii. $\overline{\sigma}^A_{P_j}(\lambda) \subseteq \overline{\sigma}^A_{\tau_{P_j}}(\lambda)$.

iii. $M^A_{\tau_{P_j}}(\lambda) \subset M^A_{P_j}(\lambda)$.

Proof. The proof is similar to proposition 5.5. □

In the following example, we continue example 4.7. We generate new topologies based on the new approximations that we construct in the last section and also we evaluate the new lower and upper approximations that we form from the generated topologies for each set.

Example 5.8. *Continued from example 4.7.*
The generated N_j-topologies are:
$\tau_{N_j}(\lambda_1) = \{U, \phi\}$ *for* $j = \{r, l, i, u\}$
$\tau_{N_j}(\lambda_1) = \{U, \phi, \{u_2\}, \{u_1, u_3\}\}$ *for* $j = \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$
$\tau_{N_r}(\lambda_2) = \{U, \phi, \{u_3\}, \{u_1, u_3\}, \{u_2, u_3\}\}$
$\tau_{N_l}(\lambda_2) = \{U, \phi, \{u_1\}, \{u_2\}, \{u_1, u_2\}\}$
$\tau_{N_j}(\lambda_2) = \{U, \phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}\}$ *for* $j = \{i, \langle i\rangle\}$
$\tau_{N_j}(\lambda_2) = \{U, \phi\}$ *for* $j = \{u, \langle u\rangle\}$
$\tau_{N_{\langle r\rangle}}(\lambda_2) = \{U, \phi, \{u_1\}, \{u_3\}, \{u_1, u_3\}, \{u_2, u_3\}\}$
$\tau_{N_{\langle l\rangle}}(\lambda_2) = \{U, \phi, \{u_2\}\}$

The generated P_j-topologies are:
$\tau_{P_j}(\lambda_1) = \{U, \phi, \{u_2\}, \{u_1, u_3\}\}$ *for all j.*
$\tau_{P_j}(\lambda_2) = \{U, \phi, \{u_2\}, \{u_1, u_3\}\}$ *for* $j = \{r, u, \langle r\rangle, \langle u\rangle\}$
$\tau_{P_j}(\lambda_2) = \{U, \phi, \{u_1\}, \{u_2\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2, u_3\}\}$ *for* $j = \{l, i, \langle l\rangle, \langle i\rangle\}$

The soft τ_{N_j}-lower approximations of each set in $P(U)$ are:

Table 9: Soft τ_{N_j}-Lower Approximations

A	$\underline{\sigma}^A_{\tau_{N_r}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_l}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_i}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_u}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_{\langle r\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_{\langle l\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_{\langle i\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{N_{\langle u\rangle}}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_2\}$	ϕ	ϕ	ϕ	ϕ	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1,u_2\}$	ϕ	ϕ	ϕ	ϕ	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	ϕ	ϕ	ϕ	ϕ	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	ϕ	ϕ	ϕ	ϕ	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U
A	$\underline{\sigma}^A_{\tau_{N_r}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_l}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_i}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_u}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_{\langle r\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_{\langle l\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_{\langle i\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{N_{\langle u\rangle}}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ	$\{u_1\}$	ϕ
$\{u_2\}$	ϕ	$\{u_2\}$	$\{u_2\}$	ϕ	ϕ	$\{u_2\}$	$\{u_2\}$	ϕ
$\{u_3\}$	$\{u_3\}$	ϕ	$\{u_3\}$	ϕ	$\{u_3\}$	ϕ	$\{u_3\}$	ϕ
$\{u_1,u_2\}$	ϕ	$\{u_1,u_2\}$	$\{u_1,u_2\}$	ϕ	$\{u_1\}$	$\{u_2\}$	$\{u_1,u_2\}$	ϕ
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1,u_3\}$	ϕ	$\{u_1,u_3\}$	ϕ	$\{u_1,u_3\}$	ϕ
$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	ϕ	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	ϕ
U	U	U	U	U	U	U	U	U

The soft τ_{P_j}-lower approximations of each set in $P(U)$ are:

Table 10: Soft τ_{P_j}-Lower Approximations

A	$\underline{\sigma}^A_{\tau_{P_r}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_l}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_i}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_u}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_{\langle r\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_{\langle l\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_{\langle i\rangle}}}(\lambda_1)$	$\underline{\sigma}^A_{\tau_{P_{\langle u\rangle}}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1,u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U
A	$\underline{\sigma}^A_{\tau_{P_r}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_l}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_i}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_u}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_{\langle r\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_{\langle l\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_{\langle i\rangle}}}(\lambda_2)$	$\underline{\sigma}^A_{\tau_{P_{\langle u\rangle}}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	ϕ	$\{u_1\}$	$\{u_1\}$	ϕ	$\{u_1\}$	$\{u_1\}$	$\{u_1\}$	ϕ
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	ϕ	$\{u_3\}$	$\{u_3\}$	ϕ	$\{u_3\}$	$\{u_3\}$	$\{u_3\}$	ϕ
$\{u_1,u_2\}$	$\{u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_1,u_2\}$	$\{u_2\}$
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2,u_3\}$	$\{u_2\}$
U	U	U	U	U	U	U	U	U

The soft τ_{N_j}-upper approximations of each set in $P(U)$ are:

Table 11: Soft τ_{N_j}-Upper Approximations

A	$\overline{\sigma}^A_{\tau_{N_r}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_l}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_i}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_u}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_{\langle r\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_{\langle l\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_{\langle i\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{N_{\langle u\rangle}}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	U	U	U	U	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2\}$	U	U	U	U	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	U	U	U	U	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_1,u_2\}$	U	U	U	U	U	U	U	U
$\{u_1,u_3\}$	U	U	U	U	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U
A	$\overline{\sigma}^A_{\tau_{N_r}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_l}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_i}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_u}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_{\langle r\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_{\langle l\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_{\langle i\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{N_{\langle u\rangle}}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	U	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	U
$\{u_2\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_2\}$	U	$\{u_2\}$	U	$\{u_2\}$	U
$\{u_3\}$	U	$\{u_3\}$	$\{u_3\}$	U	$\{u_2,u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	U
$\{u_1,u_2\}$	$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	U
$\{u_1,u_3\}$	U	$\{u_1,u_3\}$	$\{u_1,u_3\}$	U	U	$\{u_1,u_3\}$	$\{u_1,u_3\}$	U
$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	U
U	U	U	U	U	U	U	U	U

The soft τ_{P_j}-upper approximations of each set in $P(U)$ are:

Table 12: Soft τ_{P_j}-Upper Approximations

A	$\overline{\sigma}^A_{\tau_{P_r}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_l}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_i}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_u}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_{\langle r\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_{\langle l\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_{\langle i\rangle}}}(\lambda_1)$	$\overline{\sigma}^A_{\tau_{P_{\langle u\rangle}}}(\lambda_1)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_1,u_2\}$	U	U	U	U	U	U	U	U
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	U	U	U	U	U	U	U
U	U	U	U	U	U	U	U	U
A	$\overline{\sigma}^A_{\tau_{P_r}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_l}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_i}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_u}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_{\langle r\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_{\langle l\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_{\langle i\rangle}}}(\lambda_2)$	$\overline{\sigma}^A_{\tau_{P_{\langle u\rangle}}}(\lambda_2)$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1\}$	$\{u_1\}$	$\{u_1,u_3\}$
$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$	$\{u_2\}$
$\{u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_3\}$	$\{u_3\}$	$\{u_1,u_3\}$
$\{u_1,u_2\}$	U	$\{u_1,u_2\}$	$\{u_1,u_2\}$	U	U	$\{u_1,u_2\}$	$\{u_1,u_2\}$	U
$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$	$\{u_1,u_3\}$
$\{u_2,u_3\}$	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U	U	$\{u_2,u_3\}$	$\{u_2,u_3\}$	U
U	U	U	U	U	U	U	U	U

The soft τ_{N_j}-accuracy measures of each set in $P(U)$ are:

Table 13: Soft τ_{N_j}-Accuracy Measures

A	$M^A_{\tau_{N_r}}(\lambda_1)$	$M^A_{\tau_{N_l}}(\lambda_1)$	$M^A_{\tau_{N_i}}(\lambda_1)$	$M^A_{\tau_{N_u}}(\lambda_1)$	$M^A_{\tau_{N_{\langle r\rangle}}}(\lambda_1)$	$M^A_{\tau_{N_{\langle l\rangle}}}(\lambda_1)$	$M^A_{\tau_{N_{\langle i\rangle}}}(\lambda_1)$	$M^A_{\tau_{N_{\langle u\rangle}}}(\lambda_1)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	0	0	0	0	0	0	0
$\{u_2\}$	0	0	0	0	1	1	1	1
$\{u_3\}$	0	0	0	0	0	0	0	0
$\{u_1,u_2\}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{u_1,u_3\}$	0	0	0	0	1	1	1	1
$\{u_2,u_3\}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1
A	$M^A_{\tau_{N_r}}(\lambda_2)$	$M^A_{\tau_{N_l}}(\lambda_2)$	$M^A_{\tau_{N_i}}(\lambda_2)$	$M^A_{\tau_{N_u}}(\lambda_2)$	$M^A_{\tau_{N_{\langle r\rangle}}}(\lambda_2)$	$M^A_{\tau_{N_{\langle l\rangle}}}(\lambda_2)$	$M^A_{\tau_{N_{\langle i\rangle}}}(\lambda_2)$	$M^A_{\tau_{N_{\langle u\rangle}}}(\lambda_2)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	$\frac{1}{2}$	1	0	1	0	1	0
$\{u_2\}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{3}$	1	0
$\{u_3\}$	$\frac{1}{3}$	0	1	0	$\frac{1}{2}$	0	1	0
$\{u_1,u_2\}$	0	$\frac{2}{3}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	1	0
$\{u_1,u_3\}$	$\frac{2}{3}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	0	1	0
$\{u_2,u_3\}$	$\frac{2}{3}$	$\frac{1}{2}$	1	0	1	$\frac{1}{3}$	1	0
U	1	1	1	1	1	1	1	1

The soft τ_{P_j}-accuracy measures of each set in $P(U)$ are:

Table 14: Soft τ_{P_j}-Accuracy Measures

A	$M^A_{\tau_{P_r}}(\lambda_1)$	$M^A_{\tau_{P_l}}(\lambda_1)$	$M^A_{\tau_{P_i}}(\lambda_1)$	$M^A_{\tau_{P_u}}(\lambda_1)$	$M^A_{\tau_{P_{\langle r\rangle}}}(\lambda_1)$	$M^A_{\tau_{P_{\langle l\rangle}}}(\lambda_1)$	$M^A_{\tau_{P_{\langle i\rangle}}}(\lambda_1)$	$M^A_{\tau_{P_{\langle u\rangle}}}(\lambda_1)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	0	0	0	0	0	0	0
$\{u_2\}$	1	1	1	1	1	1	1	1
$\{u_3\}$	0	0	0	0	0	0	0	0
$\{u_1,u_2\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\{u_1,u_3\}$	1	1	1	1	1	1	1	1
$\{u_2,u_3\}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1
A	$M^A_{\tau_{P_r}}(\lambda_2)$	$M^A_{\tau_{P_l}}(\lambda_2)$	$M^A_{\tau_{P_i}}(\lambda_2)$	$M^A_{\tau_{P_u}}(\lambda_2)$	$M^A_{\tau_{P_{\langle r\rangle}}}(\lambda_2)$	$M^A_{\tau_{P_{\langle l\rangle}}}(\lambda_2)$	$M^A_{\tau_{P_{\langle i\rangle}}}(\lambda_2)$	$M^A_{\tau_{P_{\langle u\rangle}}}(\lambda_2)$
ϕ	0	0	0	0	0	0	0	0
$\{u_1\}$	0	1	1	0	0	1	1	0
$\{u_2\}$	1	1	1	1	1	1	1	1
$\{u_3\}$	0	1	1	0	0	1	1	0
$\{u_1,u_2\}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$
$\{u_1,u_3\}$	1	1	1	1	1	1	1	1
$\{u_2,u_3\}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$	$\frac{1}{3}$	1	1	$\frac{1}{3}$
U	1	1	1	1	1	1	1	1

6. Summary and conclusion

This study advances the theoretical foundations of rough set theory by integrating soft set parameterization with generalized neighborhood systems, addressing critical gaps in modeling parameter-dependent uncertainty. Building on the foundational work of Yao's j-neighborhoods [17] and Mareay's j-adhesion neighborhoods [12], we introduce eight novel soft neighborhood types derived from parameterized soft binary relations

The unification of soft sets with neighborhood systems offers a robust framework for uncertainty management in systems governed by variable relational parameters. By bridging soft set parameterization with precise topological frameworks, this refined approach offers a promising direction for future research in uncertain data analysis, machine learning, and decision support systems.

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