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Equivalence of equicontinuity and distality for real non-autonomous systems

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Abstract. In this paper, we investigate equicontinuity and distality for non-autonomous systems on the interval. We investigate distality of the system using the enveloping cover $E_0(X) = \overline{\{\omega_k : k \in \mathbb{Z}\}}$. We prove that if a sequence of homeomorphisms on the interval (f_n) converges to an injective function, then the sequence (f_n^{-1}) converges to a continuous surjective function. Consequently, we establish equivalence of distality and equicontinuity for non-autonomous systems on the interval.

1. Introduction

Let (X, d) be compact metric space and $\mathbb{F} = \{f_i : i \in \mathbb{N}\}$ be a family of homeomorphisms on X. For any given initial state of the system x_0, any such family generates a *non-autonomous* dynamical system via the relation $x_n = \begin{cases} f_n(x_{n-1}) & : n \geq 1, \\ f_{-n}^{-1}(x_{n+1}) & : n < 0 \end{cases}$. In other words, the non-autonomous dynamical system generated by the family $\mathbb{F}$ can be visualized as an orbit of x_0 under the ordered set $\{\ldots, f_2^{-1}, f_1^{-1}, Id_X, f_1, f_2, \ldots,\}$ (where Id_X is the identity map on X). For a given initial state x_0 of the system, let $\omega_n(x_0)$ denote the state of the system at time n. In particular, $\omega_n(x_0) = \left\{ \begin{array}{ll} f_n \circ f_{n-1} \ldots \circ f_1(x_0) & : n \geq 1, \\ f_{-n}^{-1} \circ f_{-n-1}^{-1} \ldots \circ f_1^{-1}(x_0) & : n < 0 \end{array} \right\}$ and $\omega_0 = Id_X$. The set $O(x) = \{\omega_n(x) : n \in \mathbb{Z}\}$ is called the *orbit* of any point x in X. We refer to the pair $(X, \mathbb{F})$ as a non-autonomous dynamical system. In case the f_i's coincide, the definition coincides with the well-known notion of autonomous dynamical system and is denoted by (X, f).

Dynamical systems have been widely used to approximate the long-term behavior of several natural and physical processes around us. While some good approximations using the autonomous setting have been made, the study using a non-autonomous system provides a better insight into the system and provides a better approximation of the underlying system. Similar complexities and greater precision in comparison to the autonomous system make the study of the non-autonomous systems an interesting prospect, and the area has found applications in many areas of science and engineering [4, 12].

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The study of non-autonomous systems was initially investigated in [7], where the authors investigated topological entropy for a general non-autonomous dynamical system. In [5], the authors derived necessary and sufficient conditions for a non-autonomous system to exhibit periodic solutions. In [10], the authors relate the ω-limit sets of a non-autonomous system generated by a sequence of maps converging uniformly to f on the unit interval I. The authors prove that if the topological entropy of the limiting system (I, f) is zero, then every infinite ω-limit set of (I, f) is an ω-limit set of $(I, \mathbb{F})$. In [9], the authors provide the spectral decomposition of dynamical systems on the interval and investigate the topological entropy in terms of the spectrum of the underlying system. In [2], the authors prove that if f is a continuous interval map such that all wandering intervals converge to periodic orbits, then the family of periodic orbits is dense in ω-limit sets under the Hausdorff metric. In [6], the authors investigate sensitive dependence on initial conditions for a general non-autonomous dynamical system. In [11], the authors study the chaotic behavior of a non-autonomous system generated by a uniform convergent sequence of maps. The authors prove that the chaotic behavior of sequences with the form $(f_n \circ ... \circ f_1)(x)$ is inherited under iterations. In [8], the authors investigate the mixing properties of a non-autonomous system generated by a sequence of linear operators on a topological vector space on certain invariant sets. The authors prove that on such invariant sets, the class of non-autonomous linear dynamical systems, weakly mixing of order n, strictly contains the corresponding class with the weak mixing property of order $n+1$. In [3], the authors relate Li-Yorke chaoticity for the non-autonomous system generated by a uniformly convergent sequence of interval maps with the Li-Yorke chaoticity of the limiting system. In [1], the authors investigate properties such as Li-Yorke chaoticity, topological weak mixing, and topological entropy for a non-autonomous dynamical system. The authors provide examples to establish the existence of non-autonomous systems with positive entropy such that every point is asymptotic to a fixed point and hence prove that the positive entropy of non-autonomous systems need not guarantee Li-Yorke chaoticity of the underlying system. The authors prove that any non-autonomous weak mixing system need not exhibit weak mixing of order 3. Finally, the authors establish that if a non-autonomous system exhibits weak mixing of order 3, then it exhibits weak mixing of all orders.

Let $(X, \mathbb{F})$ be a non-autonomous dynamical system generated by a family $\mathbb{F}$ of continuous self-maps on X. The system $(X, \mathbb{F})$ is said to be distal if for every pair of distinct points $x, y \in X$, $\liminf_k d(\omega_k(x), \omega_k(y)) > 0$. A non-autonomous system $(X, \mathbb{F})$ is said to be *equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(\omega_n(x), \omega_n(y)) < \epsilon$ for all $n \in \mathbb{Z}$, $x, y \in X$. It is known that if the system $(X, \mathbb{F})$ is autonomous, every equicontinuous system is distal. However, the map $f : I \times S^1 \to I \times S^1$ defined as $f(r, \theta) = (r, r + \theta)$ is an example of a distal system that fails to be equicontinuous and hence the distinction between the two notions is well known for autonomous systems. It is known that the action of an abelian group T is equicontinuous if and only if its closure in the pointwise topology ($E(X)$, also known as the enveloping semigroup) forms a group (under composition) of homeomorphisms.

In the present paper, we investigate equicontinuity and distality for non-autonomous systems on the interval. We define the enveloping cover of the system $(X, \mathbb{F})$ as $E_0(X) = \overline{\{\omega_k : k \in \mathbb{Z}\}}$, and investigate the dynamics of the system using the enveloping cover. We prove that if a sequence of homeomorphisms (f_n) on the interval converges to an injective function then the sequence (f_n^{-1}) converges to a continuous surjective function. Consequently, we establish the equivalence of distality and equicontinuity for non-autonomous systems on the interval.

2. Main results

Proposition 2.1. *Let $(X, \mathbb{F})$ be a non-autonomous system generated by a commutative family of homeomorphisms. If $(X, \mathbb{F})$ is equicontinuous then, $(X, \mathbb{F})$ is distal.*

Proof. Let $(X, \mathbb{F})$ be equicontinuous and x and y be proximal. Then there exists sequence (n_k) of integers such that $\lim_{k\to\infty} d(\omega_{n_k}(x), \omega_{n_k}(y)) = 0$. For any $\epsilon > 0$, there exists $\delta > 0$ such that $d(a, b) < \delta$ ensures $d(\omega_n(a), \omega_n(b)) < \epsilon$ for all $n \in \mathbb{Z}$. As $d(\omega_{n_r}(x), \omega_{n_r}(y)) < \delta$ (for some n_r), we have $d(x, y) < \epsilon$. As the argument holds for any $\epsilon > 0$, we have $x = y$ and hence the system $(X, \mathbb{F})$ is distal. □

Proposition 2.2. *For any non-autonomous system $(X, \mathbb{F})$ generated by a commutative family, $(X, \mathbb{F})$ is distal if and only if every element in $E_0(X)$ is injective.*

Proof. Let $(X, \mathbb{F})$ be distal and let $p \in E_0(X)$. As $p \in E_0(X)$ and x_1, x_2 be pair of distinct elements in X. Then, there exists a sequence (n_k) in $\mathbb{Z}$ such that $(\omega_{n_k}(x_i))$ converges to $p(x_i)$ for $i = 1, 2$. Thus, if $p(x_1) = p(x_2)$ then $(\omega_{n_k}(x_1))$ and $(\omega_{n_k}(x_2))$ converge to the same point and hence x_1 and x_2 are proximal. As $(X, \mathbb{F})$ is distal, we have $x_1 = x_2$ and hence p is injective. Conversely, if x and y are proximal for $(X, \mathbb{F})$, along a subsequence (ω_{m_k}) (say), then the pointwise limit of (ω_{m_k}) (its subsequence) fails to be injective. Thus, if every element in $E_0(X)$ is injective, then $(X, \mathbb{F})$ is distal, and the proof is complete. □

Remark 2.3. The above result characterizes the distality of any non-autonomous system $(X, \mathbb{F})$. As the results hold good when the generating maps f_n coincide, the results hold good for any autonomous system (X, f). Thus, for any autonomous system (X, f), elements in $E(X)$ are invertible if and only if they are injective.

Proposition 2.4. *For any non-autonomous system generated by a commutative family $\mathbb{F} = (f_n)$, if $(X, \mathbb{F})$ is equicontinuous, then every element in $E_0(X)$ is a homeomorphism.*

Proof. Since $(X, \mathbb{F})$ is equicontinuous, $(X, \mathbb{F})$ is distal which implies every element in $E_0(X)$ is injective. Further, as the system $(X, \mathbb{F})$ is equicontinuous, the topologies of uniform convergence and pointwise convergence coincide on $E_0(X)$ and hence every element in $E_0(X)$ is surjective and continuous and hence a homeomorphism. □

Example 2.5. Let $X = \{(r, \theta) : r \in \{2 - \frac{1}{2^n} : n \in N\} \cup \{1, 2\}, 0 \leq \theta \leq 1\}$ and $f : X \to X$ is defined as $f(r, \theta) = (r, \theta + r)$. Let $f_1 = f^2$ and define $f_n : X \to X$ as $f_n = f^{2^{n-1}}$ (for $n \geq 2$). Then, for non-autonomous system $(X, \mathbb{F})$ generated by family $\mathbb{F} = (f_n)$, $\omega_k = f^{2^k}$ and converges pointwise to the identity map. Thus, $E_0(X) = \{\omega_k : k \in \mathbb{Z}\}$. However, as the map is sensitive (on points lying on $r = 2$), the system need not be equicontinuous when every element in $E_0(X)$ is a homeomorphism.

Remark 2.6. The above result establishes that if the non-autonomous system is equicontinuous, then the elements in the enveloping cover are necessarily homomorphisms. However, the above example provides an instance when the converse of the same fails to hold good. Further, as the above result holds good when the generating maps f_n coincide, the result holds good for any autonomous system (X, f) (in fact forms a group of homeomorphisms). We now try to establish similar results for non-autonomous systems on an interval. Before we move further, we establish some of the results required.

Lemma 2.7. *If (f_n) is a sequence of monotonically increasing (decreasing) functions converging pointwise to a continuous map f, then (f_n) converges uniformly to f on compact subsets of $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{R}$ $(a < b)$, $\epsilon > 0$ be given and $I = [a, b]$. As f is continuous on I, there exists $\delta > 0$ such that $d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\epsilon}{9}$ $\forall x, y \in I$. Let $\{x_1, x_2, \ldots, x_t\}$ be a $\frac{\delta}{2}$-cover of I. Also, as (f_n) converges pointwise to f, there exists $k \in \mathbb{N}$ such that $d(f_n(x_i), f(x_i)) < \frac{\epsilon}{9}$, for all $n \geq k, i \in \{1, 2, ..., t\}$. As any $x \in I$ lies in $[x_i, x_{i+1}]$ for some $i \in \{1, 2, \ldots, t-1\}$, we have, $d(f_n(x_i), f_n(x)) \leq d(f_n(x_i), f_n(x_{i+1})) \leq d(f_n(x_i), f(x_i)) + d(f(x_i), f(x_{i+1})) + d(f(x_{i+1}), f_n(x_{i+1})) < \frac{\epsilon}{3}$, $\forall n > k$. Hence we have $d(f_n(x), f(x)) \leq d(f_n(x), f_n(x_i)) + d(f_n(x_i), f(x_i)) + d(f(x_i), f(x)) < \epsilon$ $\forall n > k$ and $\forall\, x \in I$. As any compact subset of $\mathbb{R}$ is contained in some compact interval of $\mathbb{R}$, (f_n) converges uniformly to f on compact subsets of $\mathbb{R}$. Further, as the above arguments hold when (f_n) is a sequence of monotonically decreasing functions, the result holds when (f_n) is a sequence of monotonically increasing (decreasing) functions, and the proof is complete. □

Proposition 2.8. *Let I be a compact interval and let f be a homeomorphism on I. If (f^n) converges pointwise to g then the following are equivalent:*

1. *g is continuous*
2. *g is surjective*

3. (f^n) *converges to g uniformly*

Proof. As the uniform limit of a sequence of continuous surjective maps is continuous and surjective, (3) $\implies$ (1) and (3) $\implies$ (2) hold. Note that if f is monotonically decreasing, (f^n) fails to converge to any g (as the endpoints swap values at every iteration), and thus, if (f^n) converges pointwise, then f must be monotonically increasing. Consequently, f^n is also a monotonically increasing function for each $n \in \mathbb{N}$ and (1) $\implies$ (3) holds (by Lemma 2.7). Also, as the pointwise limit of monotonically increasing functions is increasing, any point of discontinuity x_0 forces g to skip values in the jump interval $[g(x_0^-), g(x_0^+))] \setminus \{g(x_0)\}$ and thus ensures that g cannot be surjective. Thus, (2) $\implies$ (1) also holds, and the proof is complete. □

Remark 2.9. The above result establishes the equivalence of continuity and surjectivity for any limit of iterates of a homeomorphism. As similar arguments establish the result for any limit point of the sequence (f^n), the continuity of members of $E(X)$ can be concluded using the surjectivity of the element under consideration. Further, as the arguments above can be applied to any sequence (f_n) of homeomorphisms, we get the following result.

Proposition 2.10. *Let I be a compact interval and let (f_n) be a sequence of homeomorphisms on I. If (f_n) converges pointwise to g then the following are equivalent:*

1. *g is continuous*
2. *g is surjective*
3. *(f_n) converges to g uniformly*

Proof. As any sequence (f_n) converging pointwise must be eventually a sequence of increasing (or decreasing) maps, the result follows from discussions in Remark 2.9 and Lemma 2.7. □

Remark 2.11. The above result establishes that any pointwise limit (say g) of homeomorphisms is surjective if and only if it is continuous. Thus, if (f_n^{-1}) converges pointwise (to say h), then $g \circ h(x) = x$ and hence h is one-one (follows directly from that fact that if (f_n) converges uniformly to f and (g_n) converges pointwise to g the $(f_n \circ g_n)$ converges pointwise to $f \circ g$).

Lemma 2.12. *For any monotone function $g : [a, b] \to [c, d]$, if Range (g) is dense in $[c, d]$ then g is continuous and surjective.*

Proof. As any point of discontinuity x_0 forces g to skip values in the jump interval $[g(x_0^-), g(x_0^+))] \setminus \{g(x_0)\}$, denseness of range of g forces g to be continuous. Thus, $g([a, b]) = [c, d]$ (as g is continuous and *Range*(g) is dense in $[c, d]$) and the proof is complete. □

Lemma 2.13. *For any monotonically strictly increasing (decreasing) self map g on $[a, b]$, there exists a sequence of strictly increasing (decreasing) piecewise linear functions (g_n) such that (g_n) converges pointwise to g (on $[a, b]$).*

Proof. $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of $[a, b]$ containing the points of discontinuities for g and Let $D_k = \{d_1, d_2, \ldots, d_k\} \cup \{d_0 = a, d_\infty = b\}$. Without loss of generality, let $d_0 \le d_1 < d_2 < \ldots < d_k \le d_\infty\}$ and let g_k be k^{th}-order piecewise linear approximation of g obtained by linear joining of d_i and d_{i+1} (and linearly joining d_k with d_∞).

In particular,

$$g_k(x) = \begin{cases} g(d_i)\frac{d_{i+1}-x}{d_{i+1}-d_i} + g(d_{i+1})\frac{x-d_i}{d_{i+1}-d_i} & : x \in [d_i, d_{i+1}],\ \ 0 \le i \le k, \\ g(d_\infty)\frac{x-d_k}{d_\infty-d_k} + g(d_k)\frac{d_\infty-x}{d_\infty-d_k} & : x \in [d_k, d_\infty] \end{cases}$$

As $g_k(x) = g(x)$ for all $x \in D$, $(g_k(x))$ converges to $g(x)$ for all $x \in D$. Further, as points of D^c are points of continuity of g, $(g_k(x))$ converges to $g(x)$ for all $x \in D^c$ and hence $(g_k(x))$ converges to $g(x)$ for all $x \in X$. □

Lemma 2.14. *Let g be an injective monotone self map on $[a, b]$ satisfying $g(a) = a$ and $g(b) = b$. If h is a monotone self-map on $[a, b]$ such that $h \circ g(x) = x$, for all $x \in [a, b]$, then h is unique.*

Proof. Let h_1 and h_2 be two monotonic functions such that $h_1 \circ g(x) = h_2 \circ g(x) = x$ for all $x \in [a, b]$. Firstly note that as $h_1(a) = h_2(a) = a$ and $h_1(b) = h_2(b) = b$, h_1 and h_2 are monotonically increasing functions. Also note that for any element $g(z)$ in $Range(g)$, $h_1(g(z)) = h_2(g(z)) = z$ and hence h_1 and h_2 coincide on $Range(g)$. Finally for any element y not in $Range(g)$ there exists $x_0 \in [a, b]$ such that $y \in [g(x_0^-), g(x_0^+)]$ and hence for any $k \in \mathbb{N}$ we have $h_1(g(x_0 - \frac{1}{k})) \leq h_1(y), h_2(y) \leq h_2(g(x_0 + \frac{1}{k}))$. Thus, $x_0 - \frac{1}{k} \leq h_1(y), h_2(y) \leq x_0 + \frac{1}{k}$ for any $k \in \mathbb{N}$ (as $h \circ g = I$) and hence $h_1(y) = h_2(y)$ and the proof is complete. □

Lemma 2.15. *If g is an injective monotonically increasing self map on $[a, b]$ such that $g(a) = a$ and $g(b) = b$ then (g_n^{-1}) (g_n defined as in Lemma 2.13) converges uniformly to a surjective monotonically increasing self map (on $[a, b]$).*

Proof. Firstly note that as g_n are piecewise linear, g_n^{-1} are piecewise linear and exhibit a convergent subsequence (by Helly's selection principle). Thus, there exists a subsequence $(g_{n_k}^{-1})$ and a monotonically increasing function h such that $(g_{n_k}^{-1})$ converges pointwise to h. Also, as $(g_{n_k}^{-1} \circ g(x))$ converges to $(h \circ g(x))$ (for any $x \in X$) and $(g_{n_k}^{-1} \circ g(d))$ converges to d for all $d \in D$, we have $h \circ g(d) = d$ for all $d \in D$. Thus, $Range(h)$ is dense in $[a, b]$ and hence h is a continuous surjection (by Lemma 2.12) and $(g_{n_k}^{-1})$ converges uniformly to h (Proposition 2.10). As $(g_{n_k}^{-1})$ converges uniformly to h and (g_{n_k}) converges to g (pointwise), $h \circ g(x) = x$ for all $x \in X$. Finally, as h is a monotone function satisfying $h \circ g(x) = x$ for all $x \in X$, h is unique (Lemma 2.14). As the argument holds for any subsequence of (g_n), (g_n^{-1}) itself converges (uniformly) to h and the proof is complete. □

Proposition 2.16. *If (f_n) is a sequence of homeomorphisms on $[a, b]$ converging pointwise to an injective self map g then (f_n^{-1}) converges pointwise to h, where h is a continuous surjective self map on $[a, b]$.*

Proof. Firstly note that as g is injective the sequence (g_n) (as constructed in Lemma 2.13) converges to g (pointwise), (g_n^{-1}) converges pointwise to h and satisfies $h \circ g(x) = x$ (Lemma 2.15). Let $y \in [a, b]$. As each f_n is surjective, there exists $x_n \in [a, b]$ such that $f_n(x_n) = y$. Also, for any convergent subsequence (x_{n_k}) of (x_n), $(f_{n_k}^{-1} - g_{n_k}^{-1})(y) = (f_{n_k}^{-1} - g_{n_k}^{-1})(f_{n_k}(x_{n_k})) = x_{n_k} - g_{n_k}^{-1}(f_{n_k}(x_{n_k}))$. As g_n^{-1} converges uniformly to h (and (f_n) converges pointwise to g), $(g_{n_k}^{-1} \circ f_{n_k})$ converges pointwise to identity map (and hence uniformly by Lemma 2.10). Thus, we have $(x_{n_k} - g_{n_k}^{-1} \circ f_{n_k}(x_{n_k}))$ converges to 0 and hence $(f_{n_k}^{-1} - g_{n_k}^{-1})(y)$ converges to 0. As the argument holds for any subsequence of $((f_n^{-1} - g_n^{-1})(y))$, $((f_n^{-1} - g_n^{-1})(y))$ converges to 0. As the argument holds for any y, $(f_n^{-1} - g_n^{-1})$ converges to 0 (pointwise) and hence (f_n^{-1}) converges to a continuous surjective map h. □

Proposition 2.17. *Let I be the unit interval and $\mathcal{F} = \{f_\lambda : \lambda \in \Lambda\}$ be a family of homeomorphisms on I. Then for every $g \in \overline{\mathcal{F}}$ there exists a sequence (f_n) in $\mathcal{F}$ such that (f_n) converges pointwise to g.*

Proof. Firstly, as $g \in \overline{\mathcal{F}}$, there exists a subnet $\{f_\alpha\}_{\alpha\in\Lambda}$ (of $\mathcal{F}$) which converges pointwise to g. As each f_α is a homeomorphism, $f_\alpha(0)$ is either 0 or 1. As (f_α) converges, without loss of generality, assume $f_\alpha(0) = 0$ (for infinitely many α) and hence $g(0) = 0$. As f_α's are homeomorphisms, f_α's are monotonically increasing (for all $\alpha \geq \beta$). As the limit of monotonically increasing functions is increasing, g is monotonically increasing (not necessarily strict). As g is monotonically increasing, the set of discontinuities for g is countable (say D_g) and hence the set $D = (\mathbb{Q} \cup D_g) \cap [0, 1]$ is countable. As I^D is metrizable, the set $\mathcal{G} = \overline{\{f_\alpha|_D : \alpha \in \Lambda \text{ and } \alpha \geq \beta\}}$ is metrizable and hence first countable (under product topology). Since f_α converging pointwise to g in I^I, $(f_\alpha|_D)$ converges pointwise to $f|_D$ in I^D and consequently there exists a sequence (f_n) in $\{f_\alpha : \alpha \in \Lambda \text{ and } \alpha \geq \beta\}$ such that $(f_n|_D)$ converges pointwise to $f|_D$ (as X^D is first countable). Further, for any x in D^c, there exists an increasing sequence (q_n) in D and a decreasing sequence (r_n) in D such that (q_n) converges to x and (r_n) converges to x. Then, $f_n(q_k) < f_n(x) < f_n(r_k)$ (as f_n's are monotonically increasing). Thus, we have $\lim_n f_n(q_k) \leq \lim_n f_n(x) \leq \lim_n f_n(r_k)$ or $g(q_k) \leq \liminf_n f_n(x) \leq \limsup_n f_n(x) \leq g(r_k)$. As $x \in D^c$ is a point of continuity for g, we have $\lim_n f_n(x) = f(x)$, and the proof is complete. □

Proposition 2.18. *For a non-autonomous system generated by a commutative family $\mathbb{F} = (f_n)$, $([a, b], \mathbb{F})$ is distal if and only if every element in $E_0(X) = \overline{\{\omega_n : n \in \mathbb{Z}\}}$ is a homeomorphism.*

Proof. Let $([a,b],\mathbb{F})$ be distal. By Proposition 2.17, for any element g in $E_0(X)$ there exists a sequence (n_k) of integers such that (ω_{n_k}) converges pointwise to g and hence $(\omega_{n_k}^{-1})$ converges pointwise to a continuous surjective map h (say) (Proposition 2.16). As $(X,\mathbb{F})$ is distal, h is injective. Again as the limit of $\omega_{n_k}^{-1}$ is injective map h, $(\omega_{n_k}^{-1})^{-1} = \omega_{n_k}$ converges to a continuous surjective map (Proposition 2.16) and hence g is a homeomorphism. Conversely, if every element of $E_0(X) = \overline{\{\omega_n : n \in \mathbb{Z}\}}$ is a homeomorphism then by Proposition 2.2, $([a,b],\mathbb{F})$ is distal and the proof is complete. □

Proposition 2.19. *For a non-autonomous system generated by a commutative family* $\mathbb{F} = (f_n)$, $([a,b],\mathbb{F})$ *is distal if and only if* $([a,b],\mathbb{F})$ *is equicontinuous.*

Proof. Firstly note that every equicontinuous system is distal (Proposition 2.1). Also, if $([a,b],\mathbb{F})$ is distal then every element g of $E_0(X)$ is a homeomorphism (Proposition 2.18) and hence there exists a sequence (ω_{n_k}) converging pointwise to g (Proposition 2.17). As pointwise convergence in such a case ensures uniform convergence (Proposition 2.10), $([a,b],\mathbb{F})$ is equicontinuous. □

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