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Proposed results concerning Codazzi and torsion coupleds on pure metallic metric geometries

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Abstract. The paper explores the intricacies of metallic pseudo-Riemannian manifolds represented as (M, J, g), where M is a smooth manifold with a metallic structure denoted as J, a pseudo-Riemannian metric denoted as g and a linear connection with torsion denoted as ∇. The focus is on linear connections with torsion, introducing novel conditions and classifications. The paper introduces a new integrability condition for the metallic structure J, including the torsion-coupling condition, and provides specific outcomes for Codazzi-coupled and torsion-coupled scenarios. It explores Codazzi-couplings involving a torsion tensor and investigates properties of associated tensor fields. Additionally, it discusses conditions for the purity of connections and their operators, deriving significant results through torsion or Codazzi coupling. It establishes conditions for a metallic pseudo-Riemannian manifold to become locally metallic pseudo-Riemannian manifold and quasi-metallic pseudo-Riemannian manifold. Overall, the paper contributes to understanding integrable structures with torsion in metallic pseudo-Riemannian manifolds.

1. Introduction

The concept of tensor structures on differentiable manifolds is fundamental in modern differential geometry. Among these tensor structures, one of the most well-known is the almost complex structure, denoted as F. An almost complex structure is characterized by being a $(1, 1)$-tensor field, which means it maps tangent vectors to tangent vectors. Furthermore, at each point on the manifold, the square of this tensor field is minus the identity, i.e., $F^2 = -id$, where id represents the identity transformation. Importantly, this structure is defined on an even-dimensional manifold, typically of dimension $2n$.

In many cases, the almost complex structure F is associated with a Hermitian metric g. This Hermitian metric can be either Riemannian or pseudo-Riemannian in nature, and it has the property of preserving the almost complex structure F. In other words, F acts as an isometry with respect to the (pseudo-)Riemannian metric. When you have this combination of an almost complex structure and a Hermitian metric, it is

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referred to as an almost Hermitian structure. The associated $(0, 2)$-tensor of the Hermitian metric g is a 2-form denoted as ω, which is defined as $\omega = g(FX, Y)$, where X and Y are vector fields on M. This 2-form is closely related to symplectic geometry, which is a branch of mathematics dealing with symplectic structures and their associated properties. A Kähler manifold is a well-known concept defined by the conditions that the almost complex structure F is integrable which means that either the Nijenhuis tensor, defined as $N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y]$ for all vector fields X and Y on M, equals zero or when the almost complex structure F is covariantly constant with respect to a torsion-free linear connection ∇ and the Kähler form ω is closed, indicating that $d\omega = 0$. Regarding the Levi-Civita connection ∇^g associated with the metric g, the triple (M, F, g) is a Kähler manifold if and only if the almost complex structure F is covariantly constant concerning the Levi-Civita connection ∇^g, meaning that $\nabla^g F = 0$. It can be readily observed that the conditions $N_F = 0$ and $d\omega = 0$ are equivalent to the condition $\nabla^g F = 0$.

Zhang and his coauthors have created two paper [9] and [15] by offering an alternative perspective on the definition of (para-)Kähler geometry, departing from the conventional approach. In [9], Fei and Zhang investigate the interplay between Codazzi couplings and (para-)Kähler geometry. They have successfully derived significant structural outcomes, revealing how the Kleinian group interacts with an arbitrary affine connection through g-conjugation, ω-conjugation, and L-gauge transformation. Here, g represents the pseudo-Riemannian metric, ω signifies a non-degenerate 2-form, and L denotes the tangent bundle isomorphism on smooth manifolds. Moreover, they establish a connection between Codazzi couplings of a torsion-free connection and a compatible triple. They also demonstrate the compatibility of a pair of connections with Kähler and para-Kähler structures. This generalizes the concept from special Kähler geometry, where the connection must be curvature-free, to Codazzi-Kähler geometry, where such a constraint is not necessary. In [15], Grigorian and Zhang delve into the interaction between an affine connection ∇, which typically exhibits torsion, and both the Riemannian metric g and the almost (para-)Hermitian structure L on a manifold (M, L, g). Here, L can represent either an almost complex structure F with $F^2 = -id$ or an almost para-complex structure K with $K^2 = id$. Their research reveals that ∇ becomes (para-)holomorphic, and L becomes integrable if and only if the pair (∇, L) adheres to a torsion coupling condition. They explore (para-)Hermitian manifolds M in which this torsion coupling condition is met by four distinct connections, all of which may carry torsion: ∇, ∇^L, ∇^*, and $\nabla^\dagger$, where ∇^L is the L-conjugate transformation of ∇, and ∇^* is the g-conjugate transformation of ∇, and is defined as $\nabla^\dagger = (\nabla^*)^L = (\nabla^L)^*$. This investigation leads to the following noteworthy cases, where T signifies torsion: *(i)* In the scenario where $T = T^*$, $T^L = T^\dagger$, all four connections are Codazzi-coupled to g, but $d\omega \neq 0$. Consequently, the manifold M is referred to as Codazzi-(para-)Hermitian; *(ii)* In the case where $T = -T^\dagger$, $T^L = -T^*$, $d\omega = 0$, which implies that the manifold M becomes (para-)Kähler. In this latter case, quadruples of (para-)holomorphic connections, all possessing non-vanishing torsions, can exist in (para-)Kähler manifolds. This complements the findings of Fei and Zhang [9], which demonstrated the existence of pairs of torsion-free connections, each Codazzi-coupled to both g and L, in Codazzi-(para-)Kähler manifolds.

In contrast to the almost Hermitian structure, there is another case where the almost complex structure F acts as an anti-isometry with respect to a pseudo-Riemannian metric g. This pseudo-Riemannian metric is referred to as an anti-Hermitian metric or a Norden metric, which was first studied by and named after Norden [20]. The Norden metric is necessarily pseudo-Riemannian with a neutral signature. Thus, the pair (F, g) is known as an almost anti-Hermitian structure, almost Norden structure, or almost B-structure. The associated $(0, 2)$-tensor of any Norden metric, denoted as G, is defined as $G = g(FX, Y)$. This tensor is also a Norden metric. In this context, we have a pair of mutually associated Norden metrics, often referred to as twin Norden metrics. The classification of anti-Kähler manifolds is a topic in differential geometry, specifically related to almost anti-Hermitian manifolds. An anti-Kähler manifold is defined as an almost anti-Hermitian manifold (M, F, g) with the property that $\nabla^g F = 0$. This condition is a generalization of the concept of Kähler manifolds. Ganchev and Borisov [10] , following a method similar to the one used by Gray and Hervella [14] for almost Hermitian manifolds introduced a classification scheme for almost complex manifolds equipped with a Norden metric, taking into account the covariant derivative of the almost complex structure F. Furthermore, in a subsequent work, Ganchev and Mihova [11] presented a characterization of the eight distinct classes of almost Norden manifolds. This characterization is based on specific conditions related to the torsion tensor of the canonical connection and its associated 1-form.

Additionally, Iscan and Salimov [17] provided a new method to enhance almost anti-Hermitian manifolds into anti-Kähler manifolds. They established that the anti-Kähler condition $\nabla^g F = 0$ is equivalent to the $\mathbb{C}$-analyticity of the anti-Hermitian metric g, meaning that the Tachibana operator ϕ_F acting on g results in zero, i.e., $\phi_F g = 0$. Building on these developments, Gezer and Cakicioglu [12] presented a new classification for anti-Kähler manifolds, considering the presence of a torsion-free connection ∇. This new classification further refines the characterization of anti-Kähler manifolds by taking into account the effects of a specific connection in the geometry.

In addition to well-known almost complex, almost tangent, and almost product structures on a differentiable manifold M, there are other polynomial structures that naturally arise as C^∞-tensor fields J of type $(1,1)$ which are roots of the algebraic equation: [13]

$$Q(J) = J^n + a_n J^{n-1} + ... + a_2 J + a_1 I = 0.$$

Here, I is the identity map on the Lie algebra of vector fields on M. Specifically, if the structure polynomial is given as:

$$Q(J) = J^2 - pJ - qI,$$

where p and q are positive integers, the solution J to this equation is referred to as a metallic structure . This name is inspired by the fact that, for different values of p and q, the positive root of the quadratic equation $x^2 - px - q = 0$ is the (p,q)-metallic number introduced by Spinadel [6], denoted as $\sigma_{p,q}$, and it can be expressed as:

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

For different values of p and q, we get different metallic numbers, and some examples include:

- For $p = q = 1$, you get the golden number $\sigma = \frac{1+\sqrt{5}}{2}$, which is the limit of the ratio of two consecutive Fibonacci numbers [24].
- For $p = 2$ and $q = 1$, we get the silver number $\sigma_{2,1} = 1 + \sqrt{2}$, which is the limit of the ratio of two consecutive Pell numbers [8].
- For $p = 3$ and $q = 1$, we get the bronze number $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$, which plays an important role in the study of dynamical systems and quasicrystals.
- For $p = 1$ and $q = 2$, we get the copper number $\sigma_{1,2} = 2$.
- For $p = 1$ and $q = 3$, we get the nickel number $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$, and so on.

Let (M, g) denote a pseudo-Riemannian manifold, and consider a g-symmetric $(1,1)$-tensor field J on M. The pair (J, g) is referred to as a metallic pseudo-Riemannian structure on M, and the triple (M, J, g) is termed a metallic pseudo-Riemannian manifold. In the event that the metallic structure J is covariantly constant with respect to the Levi-Civita connection ∇^g (meaning $\nabla^g J = 0$), the triple (M, J, g) is identified as a locally metallic pseudo-Riemannian manifold [3, 4].

In this paper, we relax the assumption of torsion-free connections and investigate integrable structures that are compatible with connections having torsion. This paper endeavors to rectify a gap in the classification of metallic pseudo-Riemannian manifolds in the context of a linear connection ∇ with torsion. The structure of this paper is as follows: In Section 2, we revisit the definition of metallic pseudo-Riemannian manifolds and introduce the concept of a conjugate connection within this framework. In Section 3, we introduce a new condition for the integrability of the metallic structure, utilizing the torsion-coupling condition (Theorem 3.2). Furthermore, specific results have been presented for cases where (∇, J) is either Codazzi-coupled or torsion-coupled. Moving on to Section 4, we delve into the Codazzi-couplings of a linear connection ∇ with a torsion tensor T^{∇} or its conjugate connections with involved tensor fields, including the pseudo-Riemannian metric g, the metallic structure J, and the twin pseudo metallic metric G. We present various results related to these couplings. Moreover, we explore the properties of a vector-valued bilinear form $\theta(X,Y) = \frac{1}{2}\left(\nabla^J_X Y - \nabla_X Y\right)$ and the $(0,3)$-tensor $C(X,Y,Z) \equiv (\nabla_Z g)(X,Y)$ under specific conditions, as well as investigating their interrelationships. In Section 5, we will initially present the conditions for the purity of the connection ∇ and its g-conjugate connection ∇^*, as well as the conditions for these to be ψ_J-operators. Here, ψ_J is a Vishnevskii operator. Furthermore, we will derive significant results by employing the torsion coupling or Codazzi coupling of the linear connection ∇ with the metallic structure J. Subsequently, for the linear connection ∇ having a torsion tensor T^{∇}, we will establish the conditions under which a metallic pseudo-Riemannian manifold becomes locally metallic pseudo-Riemannian manifold and quasi-metallic pseudo-Riemannian manifold (Theorem 5.8 and Theorem 5.10).

2. Metallic Pseudo-Riemannian Manifolds

We shall briefly recall the basic notions concerning with metallic pseudo-Riemannian manifolds given in [3]. Let M be a smooth (real) manifold and $J : TM \to TM$ be a tangent bundle isomorphism. A $(1,1)$-tensor field J on M is termed a metallic structure if $J^2 = pJ + qI$, where I is the identity operator and $p, q \in \mathbb{R}$. A manifold M with a metallic structure J and a pseudo-Riemannian metric g is termed as a metallic pseudo-Riemannian manifold (M, J, g) if the metallic structure J is a g-symmetric $(1,1)$-tensor field on M such that $g(JX, Y) = g(X, JY)$ for all vector fields X and Y on M.

Consider a metallic structure J on a smooth manifold M. Define the associated linear connections as follows:

- A linear connection ∇ on M is termed a J-connection if the metallic structure J is covariantly constant with respect to ∇, i.e., $\nabla J = 0$.
- A metallic pseudo-Riemannian manifold (M, J, g), where ∇^g is the Levi-Civita connection associated with the pseudo-Riemannian metric g, is called a locally metallic pseudo-Riemannian manifold if ∇^g is a J-connection [4].

In the classical sense, the concept of integrability for a metallic structure J is established as follows. A metallic structure J is said to be integrable if its Nijenhuis tensor field N_J vanishes, given by

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y]$$

for smooth vector fields X, Y. It has been demonstrated in [5] that for a locally metallic pseudo-Riemannian manifold (M, J, g), the metallic structure J is integrable. Additionally, Vanzura, in [26], provided necessary and sufficient conditions for the integrability of a polynomial structure J with a characteristic polynomial having only simple roots. Vanzura's findings establish that the structure J is integrable if there exists a symmetric linear J-connection ∇. Durmaz and Gezer, in [7], presented an alternative formulation for characterizing locally metallic pseudo-Riemannian manifolds: Consider a metallic pseudo-Riemannian manifold denoted by (M, J, g) with the Levi-Civita connection ∇^g. The equation $\phi_J g = 0$ is equivalent to $\nabla^g J = 0$, where $\phi_J g$ is the Tachibana operator defined as follows:

$$\left(\Phi_J g\right)(X,Y,Z) = JXg(Y,Z) - X\left(g(JY,Z)\right) + g\left((L_Y J)X, Z\right) + g\left(Y, (L_Z J)X\right)$$

for any vector fields X, Y, Z, where $(L_X J)Y = [X, JY] - J[X,Y]$ (for Tachibana operators, see [21]). Here and further, let X, Y and Z denote arbitrary differentiable vector fields on the considered manifold, or alternatively, vectors in its tangent space at any given point on the manifold.

Consider an arbitrary linear connection ∇ on a pseudo-Riemannian manifold (M, g). A symmetric $(0,2)$-tensor field ξ is termed Codazzi if it satisfies the symmetry property:

$$(\nabla_X \xi)(Y,Z) = (\nabla_Y \xi)(X,Z).$$

Alternatively, a $(1,1)$-tensor field J is Codazzi if it is self-adjoint and satisfies:

$$(\nabla_X J)Y = (\nabla_Y J)X.$$

We refer to the pairs (∇, ξ) and (∇, J) as Codazzi-coupled.

Let (M, J, g) be a metallic pseudo-Riemannian manifold. The inverse of the metallic structure J is given by:

$$J^{-1} = \frac{1}{q}(J - pI),$$

where $q \neq 0$, and I is the identity operator.

Turning our focus to another crucial element in our article, we delve into the concept of conjugate connections. The introduction of conjugate connections concerning a metric tensor field was initially proposed by Norden in the context of Weyl geometry [20]. Independently, Nagaoka and Amari [19] developed these linear connections, referring to them as dual connections, and Lauritzen utilized them in defining statistical manifolds [18].

In the realm of metallic pseudo-Riemannian manifolds (M, J, g), we can define three conjugate connections, namely the g-conjugate connection ∇^* and the G-conjugate connection $\nabla^\dagger$ and $J-$conjugate connection ∇^J of a linear connection ∇:

$$Zg(X,Y) = g(\nabla_Z X, Y) + g\left(X, \nabla^*_Z Y\right) \tag{1}$$

$$ZG(X,Y) = G(\nabla_Z X, Y) + G\left(X, \nabla^\dagger_Z Y\right), \tag{2}$$

and

$$\nabla^J_X Y = J^{-1}\left(\nabla_X (JY)\right) \tag{3}$$

where $G = g(JX, Y)$ is the twin metallic pseudo-Riemannian metric on (M, J, g). One can demonstrate that ∇^* and $\nabla^\dagger$ are indeed connections, satisfying $(\nabla^*)^* = \left(\nabla^\dagger\right)^\dagger = \nabla$ [7]. When delving into conjugate connections related to tensor structures, further insights can be found in [1, 2].

Consider a vector-valued skew-symmetric bilinear form S defined as:

$$S(X,Y) = (\nabla_X J)Y - (\nabla_Y J)X.$$

It is noteworthy that the pair (∇, J) is Codazzi-coupled if and only if $S = 0$. Furthermore, utilizing the definition of torsion, S can be expressed as:

$$S(X,Y) = J\left(T^{\nabla^J}(X,Y) - T^\nabla(X,Y)\right),$$

for every vector fields X and Y. If the following equality holds, the pair (∇, J) is referred to as torsion-coupled:

$$S(X,Y) = T^\nabla(JX,Y) - JT^\nabla(X,Y)$$

or equivalently:

$$T^\nabla(JX,Y) = JT^{\nabla^J}(X,Y). \tag{4}$$

It is important to note that Codazzi coupling and torsion coupling of (∇, J) are distinct concepts that, in general, do not necessarily imply each other.

We will provide the following proposition without proof, which is easily verifiable through standard computations and supports the aforementioned statement.

Proposition 2.1. *Let ∇ be a linear connection with a torsion tensor T^∇ which satisfies $T^\nabla(JX,Y) = JT^\nabla(X,Y)$. Then, the pair (∇, J) is Codazzi-coupled if and only if the pair (∇, J) is torsion-coupled.*

In summary, torsion coupling can be seen as a more general condition compared to the Codazzi coupling of ∇ with J in the presence of torsion. Specifically, the pair (∇, J) can exhibit both Codazzi-coupling and torsion-coupling simultaneously when $T^\nabla = 0$.

3. Torsion Coupling of ∇ and Integrability of J

Let us begin with the assumption that (M, J, g) is a metallic pseudo-Riemannian manifold. In this section, we introduce a novel condition for the integrability of the metallic structure, employing the torsion coupling condition. Torsion coupling of ∇ with J is considered a generalization of the Codazzi coupling of torsion-free ∇ with J. In [7], Durmaz ve Gezer proved that Codazzi-coupling of (∇, J) with the torsion-free linear connection ∇ implies the integrability of the metallic structure J (Proposition 5). In this context, we extend this result to the case where (∇, J) is torsion-coupled when ∇ carries torsion. In other words, we demonstrate that when (∇, J) is torsion-coupled rather than Codazzi-coupled, J remains integrable. Additionally, certain results have been provided for the cases where (∇, J) is either Codazzi-coupled or torsion-coupled.

Lemma 3.1. *Let ∇ be a linear connection having a torsion tensor T^{∇} on the metallic pseudo-Riemannian manifold (M, J, g). Then the Nijenhuis tensor N_J of the metallic structure J is given by*

$$\begin{aligned} N_J(X,Y) &= \frac{p}{q} J\left(T^{\nabla^J}(JY,JX) - T^{\nabla}(JY,JX)\right) \\ &\quad - J^2 T^{\nabla^J}(X,Y) + T^{\nabla^J}(JY,JX) + JT^{\nabla}(X,JY) + JT^{\nabla}(JX,Y), \end{aligned} \tag{5}$$

where $J^2 = pJ + qI$, $(q \neq 0)$.

Proof. We start with the given equalities:

$$\begin{aligned} \nabla_X(JY) - \nabla_Y(JX) - J[X,Y] &= S(X,Y) + JT^{\nabla}(X,Y) \\ &= J\left(T^{\nabla^J}(X,Y)\right). \end{aligned} \tag{6}$$

Replacing X by JY and Y by JX in the above equation, we have

$$\nabla_{JY}\left(J^2X\right) - \nabla_{JX}\left(J^2Y\right) - J[JY,JX] = S(JY,JX) + JT^{\nabla}(JY,JX). \tag{7}$$

Also, it is possible to write the following equality:

$$(pI - J)[JY,JX] + q\left(\nabla_{JY}X - \nabla_{JX}Y\right) = (J - pI)T^{\nabla}(JY,JX) + S(JY,JX). \tag{8}$$

Now, multiplying equation (6) by $(-q)$ and adding it to equation (8), we obtain

$$\begin{aligned} &(pI - J)[JY,JX] + q\left(\nabla_{JY}X - \nabla_X(JY) - \left(\nabla_{JX}Y - \nabla_Y(JX)\right)\right) + Jq[X,Y] \\ =\ &(J - pI)T^{\nabla}(JY,JX) - JqT^{\nabla^J}(X,Y) + S(JY,JX). \end{aligned} \tag{9}$$

If the equation (9) is extended by J, we can write

$$\begin{aligned} &-q[JY,JX] + JqT^{\nabla}(JY,X) + Jq[JY,X] - JqT^{\nabla}(JX,Y) \\ &-Jq[JX,Y] + J^2q[X,Y] \\ =\ &qT^{\nabla}(JY,JX) - J^2qT^{\nabla^J}(X,Y) + (pJ + qI)T^{\nabla^J}(JY,JX) \\ &+ (-pJ - qI)T^{\nabla}(JY,JX). \end{aligned}$$

Simplifying and rearranging terms, we arrive at:

$$\begin{aligned} N_J(X,Y) &= \frac{p}{q} J\left(T^{\nabla^J}(JY,JX) - T^{\nabla}(JY,JX)\right) \\ &\quad - J^2 T^{\nabla^J}(X,Y) + T^{\nabla^J}(JY,JX) \\ &\quad + JT^{\nabla}(X,JY) + JT^{\nabla}(JX,Y). \end{aligned}$$

This is the expression for $N_J(X,Y)$ in terms of the given equalities. □

Theorem 3.2. *Let (M, J, g) be a metallic pseudo-Riemannian manifold equipped with a linear connection ∇ having a torsion tensor T^{∇}. If (∇, J) is torsion-coupled, then a metallic structure J is integrable.*

Proof. Given the torsion-coupled pair (∇, J), we have the following relations:

$$JT^{\nabla}(JX, Y) - J^2 T^{\nabla^J}(X, Y) = 0 \tag{10}$$

and

$$J^2 T^{\nabla^J}(JY, JX) = -JT^{\nabla}\left(J^2 X, JY\right). \tag{11}$$

Using the definition of the metallic structure J, we can rewrite it as

$$JT^{\nabla}(X, JY) + T^{\nabla^J}(JY, JX) = -\frac{p}{q} J\left(T^{\nabla^J}(JY, JX) + T^{\nabla}(JX, JY)\right). \tag{12}$$

Now, combining these equalities (10) and (12) with Lemma 3.1, we conclude that $N_J = 0$. □

Remark 3.3. *Let (M, J, g) be a metallic pseudo-Riemannian manifold equipped with a linear connection ∇ having a torsion tensor T^{∇}. Assume that the pair (∇, J) is torsion-coupled. Then we can write the following equality:*

$$[JX, Y] - J[X, Y] = \nabla_{JX} Y - \nabla_X JY.$$

From the definition of the Nijenhuis tensor N_J, we have

$$N_J(X, Y) = \left(\nabla_{JX} J\right) Y - (\nabla_X J) JY \tag{13}$$

such that from Theorem 3.2, it is possible to write the equality

$$\left(\nabla_{JX} J\right) Y = (\nabla_X J) JY. \tag{14}$$

Following the assertion made in Remark 3.3, we can now present the following proposition.

Proposition 3.4. *Let (M, J, g) be a metallic pseudo-Riemannian manifold equipped with a linear connection ∇ having a torsion tensor T^{∇}. If the pair (∇, J) is torsion-coupled, then each of the pairs $\left(\nabla^J, J\right)$, $\left(\nabla^{J^{-1}}, J\right)$, $\left(\nabla, J^{-1}\right)$, $\left(\nabla^J, J^{-1}\right)$ and $\left(\nabla^{J^{-1}}, J^{-1}\right)$ is torsion-coupled, where ∇^J is a $J-$conjugate connection of the linear connection ∇.*

Proof. Utilizing the torsion coupling of a linear connection ∇ with a $(1,1)$-tensor field J, we obtain:

$$\begin{aligned} N_J(X, Y) &= \left(\nabla_{JX} J\right) Y - (\nabla_X J) JY \\ &= 0. \end{aligned}$$

Furthermore, the following expression can be computed:

$$N_J(X, Y) = J\left(T^{\nabla^J}(JX, Y) - JT^{(\nabla^J)^J}(X, Y)\right)$$

indicating that the pair $\left(\nabla^J, J\right)$ is torsion-coupled. Similar arguments can be made to establish the validity of the other expressions. □

Proposition 3.5. *Suppose we have a metallic pseudo-Riemannian manifold (M, J, g) endowed with a linear connection ∇ that possesses a torsion tensor T^{∇}. If the pair (∇, J) is torsion-coupled, then the following conditions hold:*

i) $T^{\nabla}(JX, Y) = T^{\nabla}(X, JY)$,

ii) $T^{\nabla^J}(JX, Y) = T^{\nabla^J}(X, JY)$,

iii) $T^{\nabla}\left(J^{-1}X, Y\right) = T^{\nabla}\left(X, J^{-1}Y\right)$,

iv) $T^{\nabla^J}\left(J^{-1}X, Y\right) = T^{\nabla^J}\left(X, J^{-1}Y\right)$*, where T^{∇^J} is the torsion tensor of the $J-$conjugate connection ∇^J of the linear connection ∇.*

Proof. *i*) Assuming the torsion-coupled nature of (∇, J), and referring to equality (4), we derive the following results:

$$\begin{aligned} T^{\nabla}(JX,Y) &= -J\left(T^{\nabla^J}(Y,X)\right) \\ &= -T^{\nabla}(JY,X) \\ &= T^{\nabla}(X,JY). \end{aligned}$$

ii) Utilizing the definition of torsion-coupled, we obtain:

$$\begin{aligned} T^{\nabla^J}(JX,Y) &= J^{-1}\left(T^{\nabla}\left(J^2X,Y\right)\right) \\ &= J^{-1}\left(T^{\nabla}(pJX+qX,Y)\right) \\ &= J^{-1}\left(pT^{\nabla}(JX,Y)+qT^{\nabla}(X,Y)\right) \\ &= J^{-1}\left(pT^{\nabla}(X,JY)+qT^{\nabla}(X,Y)\right) \\ &= J^{-1}\left(T^{\nabla}(X,pJY+qY)\right) \\ &= J^{-1}\left(T^{\nabla}\left(X,J^2Y\right)\right) \\ &= J^{-1}\left(T^{\nabla}(JX,JY)\right) \\ &= T^{\nabla^J}(X,JY). \end{aligned}$$

By combining (*i*) and (*ii*), the remaining equalities become readily apparent. □

Proposition 3.6. *Suppose we have a metallic pseudo-Riemannian manifold (M,J,g) endowed with a linear connection ∇ that possesses a torsion tensor T^{∇}. If the pair (∇,J) is torsion-coupled, then the following conditions hold:*
i) $T^{\nabla^{J^{-1}}}(JX,Y) = T^{\nabla^{J^{-1}}}(X,JY)$,
ii) $T^{\nabla^{J^{-1}}}\left(J^{-1}X,Y\right) = T^{\nabla^{J^{-1}}}\left(X,J^{-1}Y\right)$.

Proof. *i*) Suppose that (∇, J) are torsion-coupled. According to Proposition 3.4, we express the torsion as:

$$T^{\nabla^{J^{-1}}}(JX,Y) = JT^{\nabla}(X,Y).$$

By substituting X and Y with $J^{-1}X$ and JY, respectively, the expression becomes:

$$T^{\nabla^{J^{-1}}}(JX,Y) = T^{\nabla^{J^{-1}}}(X,JY).$$

Similar to the argument presented in (*i*), one can establish the equivalence in (*ii*). □

Consider a metallic pseudo-Riemannian manifold (M,J,g) equipped with a linear connection ∇ that possesses a torsion tensor T^{∇}. Now, let us introduce a new linear connection defined as the average of the connection ∇ and its $J-$conjugate connection ∇^J given by $\widetilde{\nabla} = \frac{1}{2}(\nabla + \nabla^J)$. In the context of (para-)Hermitian geometry, Grigorian and Zhang [15] also investigated this connection. Our setting differs, and it is crucial to emphasize that in this context, the connection $\widetilde{\nabla}$ is a (para-)complex connection, indicating that the (para-)complex structure is parallel with respect to $\widetilde{\nabla}$.

Proposition 3.7. *Suppose we have a metallic pseudo-Riemannian manifold (M,J,g) endowed with a linear connection ∇ that possesses a torsion tensor T^{∇}. If the pair (∇,J) is torsion coupled, then each of the pairs $\left(\widetilde{\nabla},J\right)$, $\left(\widetilde{\nabla},J^{-1}\right)$, $\left(\widetilde{\nabla}^J,J\right)$, $\left(\widetilde{\nabla}^J,J^{-1}\right)$, $\left(\widetilde{\nabla}^{J^{-1}},J\right)$ and $\left(\widetilde{\nabla}^{J^{-1}},J^{-1}\right)$ is torsion coupled, where $\widetilde{\nabla} = \frac{1}{2}\left(\nabla + \nabla^J\right)$.*

Proof. As the pair (∇, J) is torsion-coupled, the following equalities can be established:

$$[JX, Y] - J[X, Y] = \nabla_{JX} Y - \nabla_X JY$$

and

$$\left(\nabla_{JX} J\right) Y = (\nabla_X J) JY.$$

Consequently, we derive:

$$\begin{aligned} T^{\widetilde{\nabla}}(JX, Y) - JT^{\widetilde{\nabla}^J}(X, Y) &= \widetilde{\nabla}_{JX} Y - \widetilde{\nabla}_X JY - [JX, Y] + J[X, Y] \\ &= \frac{1}{2}\left(\nabla_{JX} Y + \nabla^J_{JX} Y\right) - \frac{1}{2}\left(\nabla_X JY + \nabla^J_X JY\right) \\ &\quad -\nabla_{JX} Y + \nabla_X JY \\ &= \frac{1}{2} J^{-1}\left(\left(\nabla_{JX} J\right) Y - (\nabla_X J) JY\right) \\ &= 0. \end{aligned}$$

This implies that the pair $\left(\widetilde{\nabla}, J\right)$ is torsion-coupled. Similar arguments can be applied to establish the validity of the other expressions. □

4. Codazzi Couplings

4.1. Codazzi coupling of ∇ with g and G

In the context of a metallic pseudo-Riemannian manifold (M, J, g) endowed with a linear connection ∇ that possesses a torsion tensor T^{∇}, consider the $(0, 3)$-tensor defined by

$$C(X, Y, Z) \equiv (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \tag{15}$$

Similarly, define another $(0, 3)$-tensor as

$$\Gamma(X, Y, Z) \equiv (\nabla_Z G)(X, Y) = ZG(X, Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y). \tag{16}$$

It is evident from the definitions of the pseudo-Riemannian metrics g and the twin metallic pseudo-Riemannian metric G that $C(X, Y, Z) = C(Y, X, Z)$ and $\Gamma(X, Y, Z) = \Gamma(Y, X, Z)$.

From the equality (15) and employing the definition of g-conjugation (resp. (16) and employing the definition of G-conjugation), it can be deduced that

$$C(X, Y, Z) = g(X, (\nabla^* - \nabla)_Z Y)$$

and

$$\Gamma(X, Y, Z) = G\left(X, \left(\nabla^{\dagger} - \nabla\right)_Z Y\right).$$

Furthermore, it is straightforward to observe that

$$C^*(X, Y, Z) \equiv \left(\nabla^*_Z g\right)(X, Y) = -C(X, Y, Z) \tag{17}$$

and

$$\Gamma^{\dagger}(X, Y, Z) \equiv \left(\nabla^{\dagger}_Z G\right)(X, Y) = -\Gamma(X, Y, Z). \tag{18}$$

The relationship between $C = \nabla g$ and $\Gamma = \nabla G$ is expressed as follows:

$$\Gamma(X, Y, Z) = C(X, JY, Z) + g(X, (\nabla_Z J) Y). \tag{19}$$

From (19), if ∇ is a J-connection, then we obtain $\Gamma(X, Y, Z) = C(X, JY, Z)$.

Proposition 4.1. *Let (M, J, g) be a metallic pseudo-Riemannian manifold equipped with a linear connection ∇ having a torsion tensor T^{∇}. (∇, J) is Codazzi-coupled if and only if $\left(\nabla^{J}, J\right)$ is Codazzi-coupled.*

Proof. From the equality (3), we have

$$\begin{aligned}\left(\nabla^{J}_{X}J\right)Y-\left(\nabla^{J}_{Y}J\right)X &= \nabla^{J}_{X}JY-J\nabla^{J}_{X}Y-\nabla^{J}_{Y}JX+J\nabla^{J}_{Y}X\\ &= pJ^{-1}\left(\left(\nabla_{X}J\right)Y-\left(\nabla_{Y}J\right)X\right)\\ &\quad -\left(\left(\nabla_{X}J\right)Y-\left(\nabla_{Y}J\right)X\right)\\ &= -qJ^{-2}\left(\left(\nabla_{X}J\right)Y-\left(\nabla_{Y}J\right)X\right).\end{aligned}$$

□

Proposition 4.2. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let G be the twin metallic pseudo-Riemannian metric, and let ∇^{*} and $\nabla^{\dagger}$ respectively denote the g-conjugate and G-conjugate of the linear connection ∇. The pair (∇^{*}, J) is Codazzi coupled if and only if the pair $\left(\nabla^{\dagger}, J\right)$ is Codazzi coupled.*

Proof. We compute

$$\begin{aligned}g\left(\left(\nabla^{*}_{X}J\right)Y-\left(\nabla^{*}_{Y}J\right)X,Z\right) &= g\left(\nabla^{*}_{X}JY-J\nabla^{*}_{X}Y-\nabla^{*}_{Y}JX+J\nabla^{*}_{Y}X,Z\right)\\ &= g\left(\nabla^{*}_{X}JY,Z\right)-g\left(\nabla^{*}_{X}Y,JZ\right)-g\left(\nabla^{*}_{Y}JX,Z\right)\\ &\quad +g\left(\nabla^{*}_{Y}X,JZ\right)\\ &= -G\left(Y,\nabla_{X}Z\right)+G\left(J^{-1}Y,\nabla_{X}JZ\right)+G\left(X,\nabla_{Y}Z\right)\\ &\quad -G\left(J^{-1}X,\nabla_{Y}JZ\right)\\ &= G\left(\nabla^{\dagger}_{X}Y,Z\right)-G\left(\nabla^{\dagger}_{X}J^{-1}Y,JZ\right)-G\left(\nabla^{\dagger}_{Y}X,Z\right)\\ &\quad +G\left(\nabla^{\dagger}_{Y}J^{-1}X,JZ\right)\\ &= \frac{1}{q}G\left(\left(\nabla^{\dagger}_{Y}J\right)X-\left(\nabla^{\dagger}_{X}J\right)Y,JZ\right),\end{aligned}$$

from which it is clear that the pair (∇^{*}, J) is Codazzi coupled if and only if the pair $\left(\nabla^{\dagger}, J\right)$ is Codazzi coupled. □

Proposition 4.3. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let ∇^{*} denote the g-conjugate of the linear connection ∇. If both (∇, J) and (∇^{*}, J) are Codazzi-coupled, then we have $C(JX, Y, Z) = C(X, JY, Z)$.*

Proof. Assume that both (∇, J) and (∇^{*}, J) are Codazzi-coupled. From equality (15), we can express

$$\begin{aligned}C\left(JX,Y,Z\right) &= \left(\nabla_{Z}g\right)\left(JX,Y\right)=Zg\left(JX,Y\right)-g\left(\nabla_{Z}JX,Y\right)-g\left(JX,\nabla_{Z}Y\right)\\ &= C\left(X,JY,Z\right)+g\left(\left(\nabla_{Z}J\right)Y,X\right)-g\left(Y,\left(\nabla_{Z}J\right)X\right)\\ &= C\left(X,JY,Z\right)+g\left(\left(\nabla_{Y}J\right)Z,X\right)-g\left(Y,\left(\nabla_{X}J\right)Z\right)\\ &= C\left(X,JY,Z\right)+g\left(Z,\left(\nabla^{*}_{Y}J\right)X-\left(\nabla^{*}_{X}J\right)Y\right)\\ &= C\left(X,JY,Z\right).\end{aligned}$$

□

Proposition 4.4. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let G be the twin metallic pseudo-Riemannian metric, and let ∇^{*} and $\nabla^{\dagger}$ respectively denote the g-conjugate and G-conjugate of the linear connection ∇.*

1) *Assuming that the pair (∇, J) is Codazzi coupled, we can state the following equivalences: $\left(\nabla^{J}, G\right)$ is Codazzi coupled $\Leftrightarrow$ (∇, g) is Codazzi coupled $\Leftrightarrow$ (∇^{*}, g) is Codazzi coupled $\Leftrightarrow$ (∇^{*}, G) is Codazzi coupled.*

2) *Assuming that the pair (∇, J) is Codazzi coupled, we can state the following equivalences: $\left(\nabla^{\dagger}, g\right)$ is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{\dagger}, G\right)$ is Codazzi coupled.*

3) *Assuming that the pair (∇^{*}, J) is Codazzi coupled, we can state the following equivalences:*

i) *(∇, g) is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled.*

ii) *$\left(\nabla^{J}, g\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{J}, G\right)$ is Codazzi coupled.*

4) *Assuming that the pairs (∇, J) and (∇^{*}, J) are Codazzi coupled, we can state the following equivalences: (∇, g) is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{J}, G\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{J}, g\right)$ is Codazzi coupled $\Leftrightarrow$ (∇^{*}, g) is Codazzi coupled $\Leftrightarrow$ (∇^{*}, G) is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{\dagger}, g\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{\dagger}, G\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{J^{-1}}, g\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{J^{-1}}, G\right)$ is Codazzi coupled.*

Proof. 1) Assuming that the pair (∇, J) is Codazzi coupled, the existence of

$$\begin{aligned}\left(\nabla^{J}_{X}G\right)(Y,Z) &= XG(Y,Z) - G\left(\nabla^{J}_{X}Y, Z\right) - G\left(Y, \nabla^{J}_{X}Z\right)\\ &= (\nabla_{X}g)(Y, JZ) - g\left((\nabla_{X}J)Y, Z\right)\end{aligned}$$

and

$$\begin{aligned}\left(\nabla^{*}_{X}G\right)(Y,Z) &= XG(Y,Z) - G\left(\nabla^{*}_{X}Y, Z\right) - G\left(Y, \nabla^{*}_{X}Z\right)\\ &= \left(\nabla^{*}_{X}g\right)(Y, JZ) - g\left((\nabla_{X}J)Y, Z\right)\end{aligned}$$

allows us to conclude that $\left(\nabla^{J}, G\right)$ is Codazzi coupled $\Leftrightarrow$ (∇, g) is Codazzi coupled and (∇^{*}, g) is Codazzi coupled $\Leftrightarrow$ (∇^{*}, G) is Codazzi coupled. Additionally, we already know that (∇, g) is Codazzi coupled $\Leftrightarrow$ (∇^{*}, g) is Codazzi coupled [9]. Thus, the proof is complete.

2) The given equality

$$\begin{aligned}\left(\nabla^{\dagger}_{X}g\right)(Y,Z) &= Xg(Y,Z) - g\left(\nabla^{\dagger}_{X}Y, Z\right) - g\left(Y, \nabla^{\dagger}_{X}Z\right)\\ &= XG\left(J^{-1}Y, Z\right) - G\left(J^{-1}\nabla^{\dagger}_{X}Y, Z\right) - G\left(J^{-1}Y, \nabla^{\dagger}_{X}Z\right)\\ &= \left(\nabla^{\dagger}_{X}G\right)\left(Y, J^{-1}Z\right) + G\left(Y, \left(\nabla^{\dagger}_{X}J^{-1}\right)Z\right)\\ &= \left(\nabla^{\dagger}_{X}G\right)\left(Y, J^{-1}Z\right) + \frac{1}{q}G\left(Z, (\nabla_{X}J)Y\right).\end{aligned}$$

implies that, based on the hypothesis, $\left(\nabla^{\dagger}, g\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^{\dagger}, G\right)$ is Codazzi coupled. Furthermore, we know that $\left(\nabla^{\dagger}, G\right)$ is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled [7]. With these considerations, it follows that $\left(\nabla^{\dagger}, g\right)$ is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled, which is also equivalent to $\left(\nabla^{\dagger}, G\right)$ being Codazzi coupled.

3) *i)* We compute

$$\begin{aligned}(\nabla_{X}G)(Y,Z) &= XG(Y,Z) - G(\nabla_{X}Y, Z) - G(Y, \nabla_{X}Z)\\ &= (\nabla_{X}g)(Y, JZ) + g\left(\left(\nabla^{*}_{X}J\right)Y, Z\right).\end{aligned}$$

This implies that (∇, g) is Codazzi coupled $\Leftrightarrow$ (∇, G) is Codazzi coupled.

ii) The expression

$$\begin{aligned}\left(\nabla^J_X g\right)(Y,Z) &= Xg(Y,Z) - g\left(\nabla^J_X Y, Z\right) - g\left(Y, \nabla^J_X Z\right)\\ &= \left(\nabla^J_X G\right)\left(Y, J^{-1}Z\right) + G\left(Y, \left(\nabla^J_X J^{-1}\right)Z\right)\\ &= \left(\nabla^J_X G\right)\left(Y, J^{-1}Z\right) - \frac{1}{q^2} G\left(J^2 Z, \left(\nabla^{\dagger}_X J\right)Y\right).\end{aligned}$$

coupled with Proposition 4.2, establishes that $\left(\nabla^J, g\right)$ is Codazzi coupled $\Leftrightarrow$ $\left(\nabla^J, G\right)$ is Codazzi coupled.

4) Considering (1), (2) and (3), we derive the expression (4). □

Remark 4.5. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^∇. Let ∇^* denote the g-conjugate of the linear connection ∇. Considering the $(1,2)$-tensor K given by $K(X,Y) = \nabla_X Y - \nabla^*_X Y$, if both (∇, J) and (∇^*, J) are Codazzi-coupled, then we have $K(X, JY) = JK(X,Y)$. Using Proposition 4.3, we can easily see that $K(X, JY) = JK(X,Y)$ if and only if $C(JX, Y, Z) = C(X, JY, Z)$.*

Proposition 4.6. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^∇. If the connection ∇ is a J-connection and the pair (∇, g) is Codazzi-coupled, then the following equality holds:*

$$C(JX, Y, Z) = C(X, JY, Z) = C(X, Y, JZ). \tag{20}$$

Proof. Given that the connection ∇ is a J-connection, we have $C(JX, Y, Z) = C(X, JY, Z)$. Furthermore, considering the Codazzi coupling of the pair (∇, g), we can express $C(JX, Y, Z) = C(X, Y, JZ)$, leading to the equality (20). □

Considering Proposition 4.3, we can give the following corollary.

Corollary 4.7. *In the context of a metallic pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ that possesses a torsion tensor T^∇, let the pair (∇, g) be Codazzi-coupled. If both (∇, J) and (∇^*, J) are Codazzi-coupled, then we get the equality*

$$C(JX, Y, Z) = C(X, JY, Z) = C(X, Y, JZ).$$

Proof. According to Proposition 4.3, when (∇, J) and (∇^*, J) are Codazzi-coupled, it follows that $C(JX, Y, Z) = C(X, JY, Z)$. Additionally, due to the Codazzi coupling of (∇, g), we have

$$\begin{aligned}C(JX, Y, Z) &= C(X, JY, Z)\\ &= C(Z, JY, X)\\ &= C(JZ, Y, X)\\ &= C(X, Y, JZ).\end{aligned}$$

Hence, the proof is complete. □

Considering Remark 4.5, Proposition 4.3 and Corollary 4.7, we can directly derive the following result.

Corollary 4.8. *In the context of a metallic pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ that possesses a torsion tensor T^∇, let the pair (∇, g) be Codazzi-coupled. If $K(X, JY) = JK(X,Y)$, then we have the equality*

$$C(JX, Y, Z) = C(X, JY, Z) = C(X, Y, JZ).$$

Given Corollary 4.7, one may pose the following question: If the pair (∇, J) is Codazzi-coupled, is the pair (∇^*, J) also Codazzi-coupled? The subsequent proposition is designed to address this inquiry.

Proposition 4.9. *In the context of a metallic pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ that possesses a torsion tensor T^{∇}, let the pair (∇, g) be Codazzi-coupled. If $C(JZ, Y, X) = C(Z, Y, JX)$, then the pair (∇, J) is Codazzi-coupled if and only if (∇^*, J) is Codazzi-coupled.*

Proof. We compute

$$\begin{aligned} g\left(\left(\nabla^*_X J\right) Z - \left(\nabla^*_Z J\right) X, Y\right) &= g\left(\nabla^*_X JZ - J\nabla^*_X Z - \nabla^*_Z JX + J\nabla^*_Z X, Y\right) \\ &= Xg(Y, JZ) - g(\nabla_X Y, JZ) - Xg(JY, Z) \\ &\quad + g(\nabla_X JY, Z) - Zg(JX, Y) + g(\nabla_Z Y, JX) \\ &\quad + Zg(X, JY) - g(\nabla_Z JY, X) \\ &= C(Y, JZ, X) - C(Z, JY, X) - C(JX, Y, Z) \\ &\quad + C(X, JY, Z) + g(Y, (\nabla_X J) Z - (\nabla_Z J) X). \end{aligned}$$

From the hypothesis, we have

$$g\left(\left(\nabla^*_X J\right) Z - \left(\nabla^*_Z J\right) X, Y\right) = g(Y, (\nabla_X J) Z - (\nabla_Z J) X)$$

such that (∇, J) is Codazzi-coupled if and only if (∇^*, J) is Codazzi-coupled. □

4.2. Results of ∇^J with g

In the section we will consider the $(0, 3)$-tensor defined by

$$C^J(X, Y, Z) \equiv \left(\nabla^J_Z g\right)(X, Y) = Zg(X, Y) - g\left(\nabla^J_Z X, Y\right) - g\left(X, \nabla^J_Z Y\right)$$

and $\theta(X, Y)$ as a vector-valued bilinear form given by $\theta(X, Y) = \frac{1}{2}\left(\nabla^J_X Y - \nabla_X Y\right)$ on the metallic pseudo-Riemannian manifold (M, J, g) equipped with a linear connection ∇ having a torsion tensor T^{∇}. This section is devoted to the exploration of the properties of $\theta(X, Y)$ and $C^J(X, Y, Z)$ under specific conditions, as well as the investigation of their interrelationships.

Proposition 4.10. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let ∇^J denote the J-conjugate of the linear connection ∇. Then the following expressions hold:*

i) $\theta(X, JY) + J\theta(X, Y) = p\theta(X, Y)$ or equivalently $\theta(X, JY) = -qJ^{-1}\theta(X, Y)$;

ii) $\theta(X, Y) = \theta(Y, X)$ if and only if (∇, J) is Codazzi-coupled;

iii) If the pair (∇, J) is torsion coupled, then we get

$$\theta(X, JY) = \theta(JX, Y) = -qJ^{-1}\theta(X, Y) = -\frac{1}{q}J\theta(JX, JY).$$

Proof. *i*) From hypotesis, we have

$$\begin{aligned} \theta(X, JY) &= \frac{1}{2}\left(\nabla^J_X (JY) - \nabla_X (JY)\right) \\ &= \frac{1}{2}\left(J^{-1}\left(\nabla_X J^2 Y\right) - \nabla_X JY\right) \\ &= \frac{1}{2}\left(J^{-1}\left(\left(\nabla_X J^2\right) Y + J^2 \nabla_X Y\right) - \nabla_X JY\right) \\ &= \frac{1}{2}\left(pJ^{-1}(\nabla_X J) Y - (\nabla_X J) Y\right) \\ &= p\theta(X, Y) - J\theta(X, Y). \end{aligned}$$

ii) It is clearly that

$$\theta(X,Y)-\theta(Y,X)=\frac{1}{2}J^{-1}\left(\left(\nabla_X J\right)Y-\left(\nabla_Y J\right)X\right).$$

iii) Assuming the torsion coupling of the pair (∇, J), and based on Theorem 3.2 along with equation (14), we can express the equality $\theta(X,JY)=\theta(JX,Y)$. Taking into account condition (*i*), the following equality can be formulated:

$$\theta(X,JY)=\theta(JX,Y)=-qJ^{-1}\theta(X,Y)=-\frac{1}{q}J\theta(JX,JY).$$

□

Proposition 4.11. *Let (M,J,g) be a metallic pseudo Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Then the following two equalities hold:*

$$C^J(JX,Y,Z)=C(X,JY,Z)-2pg(\theta(Z,X),Y),$$

where $C^J(X,Y,Z)=\left(\nabla^J_Z g\right)(X,Y)$, and

$$C(X,Y,Z)-C^J(X,Y,Z)=2\left(g(\theta(Z,X),Y)+g(X,\theta(Z,Y))\right).$$

Proof. Using the definition of $C^J=\nabla^J g$, we obtain

$$\begin{aligned}
C^J(JX,Y,Z) &= Zg(JX,Y)-g\left(\nabla^J_Z JX,Y\right)-g\left(JX,\nabla^J_Z Y\right)\\
&= Zg(X,JY)-g\left(J^{-1}\left(\nabla_Z J^2X\right),Y\right)-g(X,\nabla_Z JY)\\
&= Zg(X,JY)-g(\nabla_Z X,JY)-g(X,\nabla_Z JY)-pg\left(J^{-1}(\nabla_Z J)X,Y\right)\\
&= C(X,JY,Z)-2pg(\theta(Z,X),Y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
C^J(X,Y,Z) &= Zg(X,Y)-g\left(\nabla^J_Z X,Y\right)-g\left(X,\nabla^J_Z Y\right)\\
&= Zg(X,Y)-g(\nabla_Z X+2\theta(Z,X),Y)-g(X,\nabla_Z Y+2\theta(Z,Y))\\
&= C(X,Y,Z)-2\left(g(\theta(Z,X),Y)+g(X,\theta(Z,Y))\right).
\end{aligned}$$

This completes the proof. □

Proposition 4.12. *In the context of a metallic pseudo Riemannian manifold (M,J,g) with a linear connection ∇ that possesses a torsion tensor T^{∇}, the following statements are equivalent:*

a) $C(X,JY,Z)+C(JX,Y,Z)=2pg(\theta(Z,X),Y)$.

b) $C^J(X,Y,Z)=-C(X,Y,Z)$.

c) *C and θ satisfy*

$$C(X,Y,Z)=g(\theta(Z,X),Y)+g(X,\theta(Z,Y)).$$

d) *For $\widetilde{\nabla}=\frac{1}{2}\left(\nabla+\nabla^J\right)$, we have $\widetilde{\nabla}g=0$.*

Proof. The proof is straightforward and easily derived from Proposition 4.11. □

Proposition 4.13. *Suppose (M,J,g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let ∇^J denote the J-conjugate of the linear connection ∇. Then*

i) *If both $\left(\nabla^J,g\right)$ and (∇,J) are Codazzi-coupled, we get*

$$C(X,JY,JZ)=C(Z,JY,JX).$$

ii) *If both* $\left(\nabla^J, g\right)$ *and* (∇^*, J) *are Codazzi-coupled, we get*

$$C(X, JY, Z) = C(X, JZ, Y).$$

iii) *If both* (∇, g) *and* (∇, J) *are Codazzi-coupled, we get*

$$C^J(JX, Y, Z) = C^J(JZ, Y, X).$$

iv) *If both* (∇, g) *and* (∇^*, J) *are Codazzi-coupled, we get*

$$C^J(JX, Y, JZ) = C^J(JX, Z, JY),$$

where $C^J = \nabla^J g$.

Proof. *i*) By definition of $C^J = \nabla^J g$, we derive

$$C^J(X, Y, Z) = C\left(J^{-1}X, JY, Z\right) + \frac{p}{q} g((\nabla_Z J)X, Y). \tag{21}$$

If $\left(\nabla^J, g\right)$ is Codazzi-coupled, we can express

$$C\left(J^{-1}X, JY, Z\right) - C\left(J^{-1}Z, JY, X\right) + \frac{p}{q}\left(g((\nabla_Z J)X, Y) - g((\nabla_X J)Z, Y)\right) = 0.$$

Substitute X and Z byJX and JZ, respectively, from the hypothesis, we obtain

$$C(X, JY, JZ) = C(Z, JY, JX).$$

ii) Given that $\left(\nabla^J, g\right)$ is Codazzi-coupled, from equation (21), we can express the following equality:

$$\begin{aligned} 0 &= C\left(J^{-1}X, JY, Z\right) - C\left(J^{-1}X, JZ, Y\right) + \frac{p}{q}\left((g(\nabla_Z J)X, Y) - (\nabla_Y J)X, Z\right) \\ &= C\left(J^{-1}X, JY, Z\right) - C\left(J^{-1}X, JZ, Y\right) \\ &\quad + \frac{p}{q}\left(g\left(X, \left(\nabla^*_Z J\right)Y\right) - g\left(X, \left(\nabla^*_Y J\right)Z\right)\right). \end{aligned} \tag{22}$$

By substituting X with JX in equation (22), we obtain

$$C(X, JY, Z) = C(X, JZ, Y),$$

where (∇^*, J) is Codazzi-coupled.

iii) From Proposition 4.11, it is established that

$$C^J(JX, Y, Z) = C(X, JY, Z) - pg\left((\nabla_Z J)X, J^{-1}Y\right)$$

and

$$C^J(JZ, Y, X) = C(Z, JY, X) - pg\left((\nabla_X J)Z, J^{-1}Y\right).$$

Given that both (∇, g) and (∇, J) are Codazzi-coupled, we get

$$\begin{aligned} C^J(JX, Y, Z) - C^J(JZ, Y, X) &= C(X, JY, Z) - C(Z, JY, X) \\ &\quad + pg\left((\nabla_X J)Z - (\nabla_Z J)X, J^{-1}Y\right) \\ &= 0, \end{aligned}$$

which implies

$$C^J(JX,Y,Z) = C^J(JZ,Y,X).$$

iv) From Proposition 4.11, we know that

$$C(X,Y,Z) = C^J\left(JX, J^{-1}Y, Z\right) + pg\left((\nabla_Z J)X, J^{-2}Y\right)$$

and

$$C(X,Z,Y) = C^J\left(JX, J^{-1}Z, Y\right) + pg\left((\nabla_Y J)X, J^{-2}Z\right).$$

Subtracting the second equation from the first, we get

$$\begin{aligned} C(X,Y,Z) - C(X,Z,Y) &= C^J\left(JX, J^{-1}Y, Z\right) - C^J\left(JX, J^{-1}Z, Y\right) \\ &\quad + p\left(g\left((\nabla_Z J)X, J^{-2}Y\right) - g\left((\nabla_Y J)X, J^{-2}Z\right)\right) \\ &= C^J\left(JX, J^{-1}Y, Z\right) - C^J\left(JX, J^{-1}Z, Y\right) \\ &\quad + p\left(g\left(\left(\nabla^*_Z J\right)J^{-2}Y, X\right) - g\left(\left(\nabla^*_Y J\right)J^{-2}Z, X\right)\right) \\ &= C^J\left(JX, J^{-1}Y, Z\right) - C^J\left(JX, J^{-1}Z, Y\right) \\ &\quad + \frac{p}{q^2} g\left(\left(\nabla^*_Z J\right)Y - \left(\nabla^*_Y J\right)Z, J^2 X\right). \end{aligned}$$

Given the hypothesis, we have

$$C^J\left(JX, J^{-1}Y, Z\right) = C^J\left(JX, J^{-1}Z, Y\right).$$

Substituting Y and Z with JY and JZ, respectively, we have

$$C^J(JX,Y,JZ) = C^J(JX,Z,JY).$$

□

5. Torsion coupling and purity of ∇^*

In the setting of a metallic pseudo Riemannian manifold (M,J,g) with a linear connection ∇ that possesses a torsion tensor T^∇, this section will initially present the conditions for the purity of the connection ∇ and its g-conjugate connection ∇^*, as well as the conditions for these to be ψ_J−operators. Here ψ_J is a Vishnevskii operator applied to pure connections. The Vishnevskii operator was first introduced by Vishnevskii [27] in the context of integrable structures. For detailed information on Vishnevskii operators and their applications, we refer to [21]. Furthermore, in this section, we will derive significant results by employing the torsion coupling or Codazzi coupling of the linear connection ∇ with the metallic structure J. Subsequently, for the linear connection ∇ having a torsion tensor T^∇, we will establish the conditions under which the Tachibana operator $\Phi_J g$ and the cyclic sum of $\Phi_J g$ become zero. These will respectively lead to the identification of two novel classifications for locally metallic pseudo-Riemannian manifolds and quasi-metallic pseudo-Riemannian manifolds. The Tachibana operators were initially introduced and examined by Tachibana in [25]. Subsequently, these operators found applications in pure geometry and in the theory of lifts ([7],[17],[21],[23]).

Proposition 5.1. *Let (M,J,g) be a metallic pseudo Riemannian manifold and ∇ be a linear connection that possesses a torsion tensor T^∇. If $C(Y,Z,JX) = C(JY,Z,X)$, then the pair (∇, J) is torsion-coupled if and only if the pair (∇^*, J) is torsion-coupled, where $C = \nabla g$.*

Proof. Assume that $C(Y,Z,JX) = C(JY,Z,X)$. Then, we can write

$$\begin{aligned}
& g\left(T^{\nabla^*}(JX,Y) - JT^{(\nabla^*)^J}(X,Y), Z\right) \\
= & g\left(\nabla^*_{JX}Y - \nabla^*_X JY - [JX,Y] + J[X,Y], Z\right) \\
= & JXg(Y,Z) - g\left(\nabla_{JX}Z, Y\right) - g\left(\nabla_{JX}Y, Z\right) \\
& -Xg(JY,Z) + g(\nabla_X Z, JY) + g(\nabla_X JY, Z) \\
& +g((\nabla_Y J)X - (\nabla_X J)Y, Z) \\
& +g\left(T^{\nabla}(JX,Y) - JT^{\nabla}(X,Y), Z\right) \\
= & C(Y,Z,JX) - C(JY,Z,X) + g((\nabla_Y J)X - (\nabla_X J)Y, Z) \\
& +g\left(T^{\nabla}(JX,Y) - JT^{\nabla}(X,Y), Z\right) \\
= & g\left(T^{\nabla}(JX,Y) - JT^{\nabla^J}(X,Y), Z\right)
\end{aligned}$$

which completes the proof. □

Corollary 5.2. *Let (M,J,g) be a metallic pseudo Riemannian manifold and ∇ be a linear connection that possesses a torsion tensor T^{∇}. Then $C(Y,Z,JX) = C(JY,Z,X)$ if and only if $K(JX,Y) = K(X,JY)$, where $C = \nabla g$.*

Proposition 5.3. *In the context of a metallic pseudo Riemannian manifold (M,J,g) with a linear connection ∇ that possesses a torsion tensor T^{∇}, let the pair (∇,g) be Codazzi-coupled. If the pairs (∇,J) and (∇^*,J) are both Codazzi-coupled, then the equality*

$$K(JX,Y) = JK(X,Y) = K(X,JY)$$

holds, where $K(X,Y) = \nabla_X Y - \nabla^*_X Y$.

Proof. Assuming that (∇,g) is Codazzi-coupled, we derive the expression:

$$\begin{aligned}
g(K(JX,Y) - JK(X,Y), Z) &= g\left(\nabla_{JX}Y - \nabla^*_{JX}Y, Z\right) - g\left(J\nabla_X Y - J\nabla^*_X Y, Z\right) \\
&= -JXg(Y,Z) + g\left(\nabla_{JX}Y, Z\right) + g\left(\nabla_{JX}Z, Y\right) \\
&\quad +Xg(Y,JZ) - g(\nabla_X Y, JZ) - g(Y, \nabla_X JZ) \\
&= -C(Y,Z,JX) + C(Y,JZ,X) \\
&= -C(JX,Z,Y) + C(X,JZ,Y).
\end{aligned}$$

From Proposition 4.3, we deduce $K(JX,Y) = JK(X,Y)$. Moreover, considering Remark 4.5, we have $K(X,JY) = JK(X,Y)$, allowing us to express the following equality:

$$K(JX,Y) = JK(X,Y) = K(X,JY).$$

□

In the context of a metallic pseudo-Riemannian manifold (M,J,g) equipped with a linear connection ∇ featuring a torsion tensor T^{∇}, the linear connection ∇ is termed a pure connection with respect to J under the condition:

$$\nabla_{JX}Y = \nabla_X JY = J\nabla_X Y.$$

The following theorems we are about to present are analogous of the Theorems provided by Salimov for manifolds with torsion-free connections [21]. Here, we have formulated these results for manifolds with torsion, and during the proof, we have applied Salimov's proof methodology directly, which is given in [21].

Theorem 5.4. *Let (M, J, g) be a metallic pseudo Riemannian manifold and ∇ be a linear connection having a torsion tensor T^{∇}. If the torsion tensor T^{∇} of $J-$connection ∇ is pure, then we have $\psi_{JX}Y = \phi_{JX}Y$. Here $\psi_{JX}Y = \nabla_{JX}Y - J(\nabla_X Y)$ and $\phi_{JX}Y = [JX, Y] - J[X, Y]$ are respectively called Vishnevskii and Tachibana operators.*

Proof. We have

$$\begin{aligned}
\phi_{JX}Y &= [JX, Y] - J[X, Y] \\
&= \nabla_{JX}Y - \nabla_Y JX - T^{\nabla}(JX, Y) - J\nabla_X Y + J\nabla_Y X + JT^{\nabla}(X, Y) \\
&= \nabla_{JX}Y - J\nabla_X Y - T^{\nabla}(JX, Y) + JT^{\nabla}(X, Y) - (\nabla_Y J) X \\
&= \psi_{JX}Y - T^{\nabla}(JX, Y) + JT^{\nabla}(X, Y) - (\nabla_Y J) X.
\end{aligned}$$

Since the torsion tensor T^{∇} of the $J-$connection ∇ is pure, we get $\psi_{JX}Y = \phi_{JX}Y$. □

Theorem 5.5. *Let (M, J, g) be a metallic pseudo Riemannian manifold, ∇ be a linear connection having a torsion tensor T^{∇} and ∇ be a pure connection with respect to J. The curvature tensor R^{∇} of the linear connection ∇ is a pure tensor field with respect to J if and only if*

$$\begin{aligned}
\psi_{JX}(\nabla_Y Z) &= \nabla_{JX}\nabla_Y Z - J\nabla_X\nabla_Y Z \\
&= 0,
\end{aligned}$$

for any vector fields $X, Y, Z \in Ker\psi_J$, where the torsion tensor T^{∇} of the linear connection ∇ is pure and ψ_J is a Vishnevskii operator applied to pure connections.

Proof. Let (M, J, g) be a metallic pseudo Riemannian manifold, ∇ be a linear connection having a torsion tensor T^{∇} . Given that ∇ is a pure connection with respect to J, we have

$$JR(X, Y)Z = R(X, Y)JZ.$$

From Theorem 5.4, we get

$$0 = \psi_{JX}Y = \phi_{JX}Y = [JX, Y] - J[X, Y]$$

for any vector fields $X, Y, Z \in Ker\psi_J$. Moreover, from the definition of the curvature tensor field R, we conclude that

$$R(X, JY)Z = JR(X, Y)Z - \psi_{JY}(\nabla_X Z) \tag{23}$$

and

$$R(JX, Y)Z = \psi_{JX}(\nabla_Y Z) + JR(X, Y)Z.$$

From the above expressions, the curvature tensor field R of ∇ is a pure tensor field with respect to J if and only if $\psi_{JX}(\nabla_Y Z) = \nabla_{JX}\nabla_Y Z - J\nabla_X\nabla_Y Z = 0$ for any vector fields $X, Y, Z \in Ker\psi_J$. □

From equation (23), it is evident that $\psi_{JX}(\nabla_Y Z)$ is a $(1, 3)$-tensor field on (M, J, g). Therefore, going forward, we denote $(\psi_J\nabla)(X, Y, Z)$ for $\psi_{JX}(\nabla_Y Z)$:

$$(\psi_J\nabla)(X, Y, Z) = \nabla_{JX}\nabla_Y Z - J\nabla_X\nabla_Y Z.$$

It is worth noting that, in the case where a pure connection ∇ serves as a connection for Kähler–Norden manifolds, it automatically implies that $\psi_J\nabla = 0$ [21, 22]. Therefore, investigating the validity of the condition $\psi_J\nabla = 0$ in our setting becomes crucial.

Theorem 5.6. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇}. Let the pair (∇, g) be Codazzi-coupled. The connection ∇ is a pure connection with respect to J if and only if the connection ∇^* is a pure connection with respect to J, where ∇^* denotes the g-conjugate of the linear connection ∇.*

Proof. Assume that ∇ is a pure connection. In this case, we have $\nabla^*_X JY = J\nabla^*_X Y$ as shown by Durmaz and Gezer [7]. Using the given hypothesis, we can express

$$\begin{aligned} g\left(\nabla^*_{JX} Y - \nabla^*_X JY, Z\right) &= g\left(\nabla^*_{JX} Y, Z\right) - g\left(\nabla^*_X JY, Z\right) \\ &= JX\left(g\left(Z, Y\right)\right) - g\left(\nabla_{JX} Z, Y\right) - g\left(\nabla^*_X JY, Z\right) \\ &= JX\left(g\left(Z, Y\right)\right) - g\left(\nabla_{JX} Z, Y\right) - X\left(g\left(Z, JY\right)\right) \\ &\quad + g\left(\nabla_X Z, JY\right) \\ &= JX\left(g\left(Z, Y\right)\right) - X\left(g\left(JY, Z\right)\right). \end{aligned}$$

Using Proposition 4.6, where $C\left(JZ, Y, X\right) = C\left(Z, Y, JX\right)$, we have

$$C\left(Z, Y, JX\right) - C\left(JZ, Y, X\right) = JX\left(g\left(Y, Z\right)\right) - X\left(g\left(JY, Z\right)\right).$$

Thus, we can rewrite this as

$$\nabla^*_{JX} Y = \nabla^*_X JY = J\nabla^*_X Y,$$

demonstrating that the connection ∇^* is also a pure connection. The reverse proof is straightforward. □

Theorem 5.7. *Suppose (M, J, g) is a metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^∇. Let the pair (∇, g) be Codazzi-coupled. If $JX\left(Y\left(g\left(Z, W\right)\right)\right) = X\left(Y\left(g\left(Z, JW\right)\right)\right)$, then $\psi_J \nabla = 0$ if and only if $\psi_J \nabla^* = 0$, where ∇^* is the g-conjugate connection of ∇ and the connection ∇ is pure connection with respect to J.*

Proof. Given that (∇, g) is Codazzi-coupled and the connection ∇ is a pure connection, we obtain:

$$\begin{aligned} g((\psi_J \nabla)(X, Y, Z), W) &= g\left(\nabla_{JX} \nabla_Y Z - J\nabla_X \nabla_Y Z, W\right) \\ &= g\left(\nabla_{JX} \nabla_Y Z, W\right) - g\left(J\nabla_X \nabla_Y Z, W\right) \\ &= JX\left(Y\left(g\left(Z, W\right)\right)\right) - X\left(Y\left(g\left(Z, JW\right)\right)\right) \\ &\quad - \left(JX\left(g\left(Z, \nabla^*_Y W\right)\right) - Xg\left(Z, \nabla^*_Y JW\right)\right) \\ &= JX\left(Y\left(g\left(Z, W\right)\right)\right) - X\left(Y\left(g\left(Z, JW\right)\right)\right) \\ &\quad - g\left(\nabla^*_{JX} \nabla^*_Y W - J\nabla^*_X \nabla^*_Y W, Z\right) \\ &= JX\left(Y\left(g\left(Z, W\right)\right)\right) - X\left(Y\left(g\left(Z, JW\right)\right)\right) \\ &\quad - g\left((\psi_J \nabla^*)(X, Y, W), Z\right) \end{aligned}$$

Therefore, this concludes the proof. □

Consider a metallic pseudo-Riemannian manifold (M, J, g) with ∇^g as the Levi-Civita connection of g. In [7], the authors established the equivalence between $\Phi_J g = 0$ and $\nabla^g J = 0$, where Φ_J is the Tachibana operator defined by

$$\left(\Phi_J g\right)(X, Y, Z) = JXg\left(Y, Z\right) - X\left(g\left(JY, Z\right)\right) + g\left(\left(L_Y J\right) X, Z\right) + g\left(Y, \left(L_Z J\right) X\right).$$

Here $\left(L_X J\right) Y = [X, JY] - J[X, Y]$. Thus, they provided an alternative characterization for locally metallic pseudo-Riemannian manifolds.

Theorem 5.8. *In the context of a metallic pseudo Riemannian manifold (M, J, g) with a linear connection ∇ that possesses a torsion tensor T^∇, let G be the twin metallic pseudo-Riemannian metric. Suppose that the pair (∇, J) is torsion-coupled. Then $\Phi_J g = 0$ if and only if $C\left(Y, Z, JX\right) = -\Gamma^*\left(Y, Z, X\right)$, where $\Gamma^* = \nabla^* G$.*

Proof. Given that (∇, J) is torsion-coupled, the relationship is expressed as:

$$[Y, JX] - J[Y, X] = \nabla_X JY - \nabla_{JX} Y.$$

Utilizing the definition of the operator $\Phi_J g$, we obtain:

$$\begin{aligned}
\left(\Phi_J g\right)(X, Y, Z) &= JXg(Y, Z) - Xg(JY, Z) + g((L_Y J)X, Z) + g(Y, (L_Z J)X) \\
&= JXg(Y, Z) - Xg(JY, Z) + g([Y, JX] - J[Y, X], Z) \\
&\quad + g(Y, [Z, JX] - J[Z, X]) \\
&= C(Y, Z, JX) - C(JY, Z, X) + g(Y, (\nabla_X J)Z) \\
&= C(Y, Z, JX) + \Gamma^*(Y, Z, X).
\end{aligned}$$

Therefore, it is evident that $\Phi_J g = 0$ if and only if $C(Y, Z, JX) = -\Gamma^*(Y, Z, X)$. □

The Theorem 5.8 directly implies the following result, which provides a novel characterization of locally metallic pseudo-Riemannian manifolds through the utilization of the Tachibana operator and torsion coupling. In [7], the authors showed that a metallic pseudo-Riemannian manifold (M, J, g) with a torsion-free linear connection ∇ constitutes a locally metallic pseudo-Riemannian manifold if the torsion-free linear connection ∇ is a J-connection and (∇, g) is Codazzi-coupled. In here, we extend this result under the condition of torsion coupling.

Corollary 5.9. *In the context of a metallic pseudo Riemannian manifold (M, J, g) with a linear connection ∇ that possesses a torsion tensor T^{∇}, let G be the twin metallic pseudo-Riemannian metric. Suppose that the pair (∇, J) is torsion-coupled. Then the metallic pseudo Riemannian manifold (M, J, g) constitutes a locally metallic pseudo-Riemannian manifold if and only if $C(Y, Z, JX) = -\Gamma^*(Y, Z, X)$, where $\Gamma^* = \nabla^* G$.*

If a non-integrable metallic pseudo-Riemannian manifold (M, J, g) satisfies

$$\left(\Phi_J g\right)(X, Y, Z) + \left(\Phi_J g\right)(Z, X, Y) + \left(\Phi_J g\right)(Y, Z, X) = 0,$$

then we refer to the triple (M, J, g) as a quasi metallic pseudo-Riemannian manifold [7]. Thus, the following theorem concludes the article.

Theorem 5.10. *Suppose (M, J, g) is a non-integrable metallic pseudo-Riemannian manifold with a linear connection ∇ that possesses a torsion tensor T^{∇} and the pair (∇, J) is torsion-coupled. Assuming that the pair $(\triangledown, g)$ is Codazzi-coupled, the triple (M, J, g) is a quasi metallic pseudo-Riemannian manifold if and only if*

$$g\left(\left(\nabla^*_X J\right)Y, Z\right) + g\left(\left(\nabla^*_Z J\right)X, Y\right) + g\left(\left(\nabla^*_Y J\right)Z, X\right) = 0,$$

where ∇^* denotes the g-conjugate of the linear connection ∇.

Proof. Suppose that (∇, J) is torsion-coupled. According to Theorem 5.8, we obtain the expressions:

$$\begin{aligned}
\left(\Phi_J g\right)(X, Y, Z) &= C(Y, Z, JX) - C(JY, Z, X) + g(Y, (\nabla_X J)Z), \\
\left(\Phi_J g\right)(Z, X, Y) &= C(X, Y, JZ) - C(JX, Y, Z) + g(X, (\nabla_Z J)Y), \\
\left(\Phi_J g\right)(Y, Z, X) &= C(Z, X, JY) - C(JZ, X, Y) + g(Z, (\nabla_Y J)X).
\end{aligned}$$

If $(\triangledown, g)$ is Codazzi-coupled, then the equation

$$\begin{aligned}
&\left(\Phi_J g\right)(X, Y, Z) + \left(\Phi_J g\right)(Z, X, Y) + \left(\Phi_J g\right)(Y, Z, X) \\
= \; & g\left(\left(\nabla^*_X J\right)Y, Z\right) + g\left(\left(\nabla^*_Z J\right)X, Y\right) + g\left(\left(\nabla^*_Y J\right)Z, X\right)
\end{aligned}$$

holds, providing the result. □

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