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New Montgomery-Mercer identity and associated integral inequalities with applications

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Abstract. In this study, we focus on the Jensen-Mercer inequality and the Montgomery-Mercer identity to develop new error bounds for Ostrowski quadrature schemes. To wrap up this task, initially, we introduce a new equality connected to Montgomery's identity invoking the Mercer concept. We then apply the Montgomery-Mercer identity to establish some updated bounds for rectangular and mid-point schemes involving convex mappings and famously known inequalities like Hölder's and power-mean. Also, we explore some special cases that stem from our main findings, conduct numerical tests, visual analysis, and present applications related to means.

1. Introduction

In analysis, the concepts of convex sets and convex mappings were frequently explored because of their distinguished geometrical and algebraic properties. A very useful property of convex sets is that the line segment joining any two points of the set belonging to the set has very useful interpretation in many applications. One of the most significant properties of convex mapping is that local minima are also global minima. This helps us in simplifying the problem in optimization theory.
Let us continue with the notions of convex sets and mappings based on them. A set $C \subset \mathbb{R}$ is said to be convex, if

$$(1-t)a + tb \in C, \quad \forall a, b \in C, t \in [0,1].$$

Similarly, a mapping $f : C \to \mathbb{R}$ is said to be convex, if

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b), \quad \forall a, b \in C, t \in [0,1].$$

The theory of convexity is not only applicable in various fields of mathematical sciences, but its impact on the development of the theory of inequalities is particularly noteworthy. In this context, Hermite and

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Hadamard independently established a foundational result pertaining to the convexity property of the mappings. This result reads as: Let $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

Jensen explored a new result, to extend the convex mappings for n points, which is subsequently followed as:
Let f be a convex mapping on $[a, b]$, then $\forall x_i \in [a, b]$ and $0 \leq \mu_i \leq 1$, and $\sum_{i=1}^{n} \mu_i = 1$, we have

$$f\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i f(x_i).$$

In 2004 Mercer proposed a new extension of Jensen inequality, which is followed as:
If $f : [a, b] \to \mathbb{R}$ is a convex mapping, then $\forall x_i \in (a, b)$ and $0 \leq \mu_i \leq 1$, and $\sum_{i=1}^{n} \mu_i = 1$, we have

$$f\left(a+b-\sum_{i=1}^{n} \mu_i x_i\right) \leq f(a)+f(b)-\sum_{i=1}^{n} \mu_i f(x_i).$$

For more details, see [1].
In [2] authors have purported some interesting refinements of Mercer's type inequalities. In [3], authors have derived some variants of Mercer-like inequality results for convex mappings in the operator sense. For a comprehensive study related to Jensen-like inequalities from both geometric and analytical points of view, see [4].
Now we revisit the successively reported result due to Ostrowski for differentiable mapping.

Theorem 1.1. *If $f : [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b), then the following result holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,$$

$\forall x \in [a, b]$, where $\|f'\|_\infty = \sup_{s\in(a,b)} |f'(x)| < \infty$.

In 2010, Alomari et al. [5] derived some Ostrowski-type inequalities by implementing the s-convex mappings. Also, Wang et al. [6] have established some q-Fractional Ostrowski's type inequalities involving the unified convex mappings. Butt et al. [7] concluded some Ostrowski-Mercer type inequalities involving fractional operators; this is the first paper concerning this inequality. Sial et al. [8] also derived some fresh Ostrowski-Mercer-like results for differentiable convex mappings and applications. For detailed investigation, see [9–11].
Most of the researchers utilized their efforts to derive new generalizations and refinements of Mercer inequality, but Ogulmus et al. [12] examined some Hermite-Hadamard type inequalities by making use of Jensen-Mercer inequality; this was the opening venue for further research. Iscan et al. [13] have proposed various weighted Hermite-Hadamard-Mercer-like inequalities that essentially resort to the idea of convex mappings.

You et al. [14] investigated the Hermite-Hadamard-Mercer-like inequalities associated with the harmonic convexity of the mappings. In [15], authors have considered the convex mappings, generalized Jenen-Mercer inequality, and fractional concepts to conclude various fresh Hermite-Hadamard type inequalities with applications. Recently, in 2022, Faisal et al. [16] have explored several new Hermite-Hadamard-Mercer-like inequalities by considering the majorization theory and generalized Mercer inequality of the Neizgoda type. In [17], Bin-Mohsin et al. used the conception of strongly harmonic convexity in the frame of fractional concepts to obtain some Hadamard-Mercer type inequalities. In [18], Du and his colleagues investigated the Bullen's type inequalities through generalized integral operators involving convexity. Yuan et al. [19] explored the AB-fractional Simpson's-like inequalities for twice differentiable (S, P)-convex functions and applications as well. Also, Du and Peng [20] deployed the multiplicative calculus approach to determine the trapezium-type inequalities.

Now we have the famous Montgomery identity, which is utilized to formulate some classical integral inequalities. Enormous work has been done by involving this identity in recent decades.

Lemma 1.2. *If $f : [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b) and $x \in (a, b)$, then the following result holds:*

$$f(x) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \int_a^x \frac{x-a}{b-a} f'(u)\mathrm{d}u + \int_x^b \frac{x-b}{b-a} f'(u)\mathrm{d}u.$$

Equivalently, it can be transformed as: If $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b), then :

$$f(x) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = (b-a)\left[\int_0^{\frac{b-x}{b-a}} t f'(ta + (1-t)b)\mathrm{d}t + \int_{\frac{b-x}{b-a}}^1 (t-1) f'(ta + (1-t)b)\mathrm{d}t\right].$$

Authors et al. [21] utilized generalized Montgomery identity to observe a variety of new identities. By making use of these qualities, they have investigated Popoviciu and Ostrowski's like inequalities for higher order convex mappings. In [22], authors have computed some fractional bounds for Ostrowski-like inequalities concerning the Montgomery equation. In [23], Mehmood et al. deploy the Montgomery identity to acquire new extensions of Popoviciu-like inequalities aided with the new Green's mappings. In [24] Vivas Cortez et al. have obtained a new generalized Montgomery identity and established some error bounds for rectangular rule through preinvex mappings. In 2020, Kunt et al. [25] used the quantum concepts to develop the Montgomery identity and related inequalities. Ali et al. [26] examined the well-known Ostrowski inequality over rectangles in the context of q-calculus associated with Montogemery equality and coordinated convex mappings. Kalsoom et al. [27, 28] have obtained some q and (p, q) estimates considering the Montgomery identity.

The present work is organized to extend Montgomery's identity by using the concept of Mercer inequality and also to develop some upper bound of Ostrowski-Mercer inequality and mid-point type inequalities of Mercer type. We have distributed our study in three sections; our first portion is subjected to a brief introduction and basic notions, which are essential for the study. In the succeeding section, we will elaborate on our factual findings, and the final section is dedicated to numerical examples and applications.

2. Main results

In the current segment of the investigation, we discuss our main results. The following sections consist of two supplementary portions. In the starting segment, we prove our auxiliary lemma. In the proceeding portion, we examine some upper bounds of Ostrowski-Mercer-type inequalities and several interesting special cases. For our ease, we specify the space of integrable mappings by Ł$[a, b]$.

2.1. Montgomery-Mercer Identity

Here is Montgomery-Mercer identity.

Lemma 2.1. *Assume that $f : I = [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b) and $f \in$ Ł$[a, b]$, then:*

$$f(u) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz = \int_{a+b-y}^{u} \left(\frac{z-(a+b-y)}{y-x}\right) f'(z)dz + \int_u^{a+b-x} \left(\frac{z-(a+b-x)}{y-x}\right) f'(z)dz, \tag{1}$$

where $x, y, u \in [a, b]$.

Proof. Consider

$$
\begin{aligned}
I &= \int_{a+b-y}^{u} \left(\frac{z-(a+b-y)}{y-x}\right) f'(z)dz + \int_{u}^{a+b-x} \left(\frac{z-(a+b-x)}{y-x}\right) f'(z)dz \\
&= I_1 + I_2,
\end{aligned}
$$

where,

$$
\begin{aligned}
I_1 &= \int_{a+b-y}^{u} \left(\frac{z-(a+b-y)}{y-x}\right) f'(z)dz \\
&= \frac{u-(a+b-y)}{y-x} f(u) - \frac{1}{y-x}\int_{a+b-y}^{u} f(z)dz,
\end{aligned} \tag{2}
$$

and similarly, we get

$$
I_2 = -\frac{u-(a+b-x)}{y-x} f(u) - \frac{1}{y-x}\int_{u}^{a+b-x} f(z)dz. \tag{3}
$$

It follows from (2) and (3), we get (1). □

Implementing the change of variable property, we can visualize the equality (1) as:

Lemma 2.2. *Suppose that $f : I = [a,b] \to \mathbb{R}$ is differentiable mapping on (a,b), then:*

$$
\begin{aligned}
f(u) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz = (y-x)\Bigg[\int_{0}^{\frac{u-(a+b-y)}{y-x}} & t f'(t(a+b-x)+(1-t)(a+b-y))dt \\
&+ \int_{\frac{u-(a+b-y)}{y-x}}^{1} (t-1) f'(t(a+b-x)+(1-t)(a+b-y))dt\Bigg],
\end{aligned} \tag{4}
$$

for all $t \in [0,1]$ and $x, y, u \in [a,b]$.

Proof. Let

$$
\begin{aligned}
I &= \int_{0}^{\frac{u-(a+b-y)}{y-x}} t f'(t(a+b-x)+(1-t)(a+b-y))dt \\
&+ \int_{\frac{u-(a+b-y)}{y-x}}^{1} (t-1) f'(t(a+b-x)+(1-t)(a+b-y))dt \\
&= I_1 + I_2
\end{aligned}
$$

where,

$$
\begin{aligned}
I_1 &= \int_{0}^{\frac{u-(a+b-y)}{y-x}} t f'(t(a+b-x)+(1-t)(a+b-y))dt \\
&= \frac{t f'(t(a+b-x)+(1-t)(a+b-y))}{y-x}\Big|_{0}^{\frac{u-(a+b-y)}{y-x}} \\
&- \frac{1}{y-x}\int_{0}^{\frac{u-(a+b-y)}{y-x}} f(t(a+b-x)+(1-t)(a+b-y))dt \\
&= \frac{u-(a+b-y)}{(y-x)^2} f(u) - \frac{1}{(y-x)^2}\int_{a+b-y}^{u} f(z)dz,
\end{aligned} \tag{5}
$$

and similarly

$$I_2 = \int_{\frac{u-(a+b-y)}{y-x}}^{1} (t-1) f'(t(a+b-x)+(1-t)(a+b-y))dt$$
$$= -\frac{u-(a+b-x)}{(y-x)^2} f(u) - \frac{1}{(y-x)^2} \int_u^{a+b-x} f(z)dz. \tag{6}$$

Adding (5) and (6), then multiplying both sides with $(y-x)$, we get (4). □

2.2. Ostrowski-Mercer Inequality

In the following segment of the study, we utilize the result obtained in the previous section and the Jensen-Mercer inequality for the convex mappings. First, we derive Ostrowski-Mercer inequality.

Theorem 2.3. *Assume that $f : I = [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b) and $f \in Ł[a,b]$. If $|f'| \leq M$, then we have the Ostrowski-Mercer-like inequality.*

$$\left| f(u) - \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(z)dz \right| \leq \frac{M}{y-x} \left[\frac{(u-(a+b-y))^2 + ((a+b-x)-u)^2}{2} \right],$$

where $x, y, u \in [a,b]$.

Proof. Using Lemma(2.2), we can write

$$\left| f(u) - \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(z)dz \right|$$
$$\leq (y-x) \left[\int_0^{\frac{u-(a+b-y)}{y-x}} t|f'(t(a+b-x)+(1-t)(a+b-y))|dt \right.$$
$$\left. + \int_{\frac{u-(a+b-y)}{y-x}}^{1} (1-t)|f'(t(a+b-x)+(1-t)(a+b-y))|dt \right].$$

Since $|f'| \leq M$, so

$$\left| f(u) - \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(z)dz \right|$$
$$\leq M(y-x) \left[\int_0^{\frac{u-(a+b-y)}{y-x}} tdt + \int_{\frac{u-(a+b-y)}{y-x}}^{1} (1-t)dt \right]$$
$$\leq \frac{M}{y-x} \left[\frac{(u-(a+b-y))^2 + ((a+b-x)-u)^2}{2} \right],$$

which ends the proof. □

Corollary 2.4. *By plugging $u = a+b-\frac{x+y}{2}$ in Theorem 2.3, then*

$$\left| f\left(a+b-\frac{x+y}{2}\right) - \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(z)dz \right| \leq \frac{M}{4}(y-x).$$

Theorem 2.5. *Assume that $f : I = [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b) and $f' \in Ł[a,b]$. If $|f'|$ is a convex mapping on $[a,b]$, then*

$$\left| f(u) - \frac{1}{y-x} \int_{a+b-y}^{a+b-x} f(z)dz \right| \tag{7}$$

$$\leq (y-x)\left\{\left[\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2(|f'(a)|+|f'(b)|)\right.\right.$$
$$-\left(\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3|f'(x)|+\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2-\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(y)|\right)\Bigg]$$
$$+\left[\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2(|f'(a)|+|f'(b)|)\right.$$
$$\left.\left.-\left(\frac{1}{6}-\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2+\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(x)|+\frac{1}{3}\left(\frac{(a+b-x)-u}{y-x}\right)^3|f'(y)|\right)\right]\right\}, \tag{8}$$

where $x, y, u \in [a, b]$.

Proof. Implementing the modulus property and Jensen-Mercer inequality on Lemma (2.2), we have

$$\left|f(u)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(z)dz\right|$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}}t|f'(a+b-(tx+(1-t)y))|dt\right)\right.$$
$$\left.+\left(\int_{\frac{u-(a+b-y)}{y-x}}^1(1-t)|f'(a+b-(tx+(1-t)y))|dt\right)\right]$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}}t\left(|f'(a)|+|f'(b)|-(t|f'(x)|+(1-t)|f'(y)|\right)dt\right)\right.$$
$$\left.+\left(\int_{\frac{u-(a+b-y)}{y-x}}^1(1-t)\left(|f'(a)|+|f'(b)|-(t|f'(x)|+(1-t)|f'(y)|)\right)dt\right)\right]$$
$$=(y-x)\left\{\left[\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2(|f'(a)|+|f'(b)|)\right.\right.$$
$$-\left(\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3|f'(x)|+\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2-\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(y)|\right)\Bigg]$$
$$+\left[\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2(|f'(a)|+|f'(b)|)\right.$$
$$\left.\left.-\left(\frac{1}{6}-\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2+\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(x)|+\frac{1}{3}\left(\frac{(a+b-x)-u}{y-x}\right)^3|f'(y)|\right)\right]\right\},$$

which ends the proof. □

Corollary 2.6. *By plugging $u = a + b - \frac{x+y}{2}$ in Theorem 2.5, then*

$$\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(z)dz\right|$$
$$\leq (y-x)\left[\frac{1}{4}(|f'(a)|+|f'(b)|)-\frac{1}{8}(|f'(x)|+|f'(y)|)\right].$$

Theorem 2.7. *Assume that $f : I = [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b) and $f' \in Ł[a, b]$. If $|f'|^q, q > 1$ be a convex mapping on $[a, b]$, then*

$$\left| f(u) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz \right| \tag{9}$$
$$\leq (y-x)\left\{ \left(\frac{1}{p+1}\left(\frac{u-(a+b-y)}{y-x}\right)^{p+1}\right)^{\frac{1}{p}} \left[\frac{u-(a+b-y)}{y-x}(|f'(a)|^q + |f'(b)|^q) \right.\right.$$
$$\left. -\frac{1}{2}\left(\left(\frac{u-(a+b-y)}{y-x}\right)^2 |f'(x)|^q + \left(1 - \left(\frac{(a+b-x)-u}{y-x}\right)^2\right)|f'(y)|^q \right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{p+1}\left(\frac{(a+b-x)-u}{y-x}\right)^{p+1}\right)^{\frac{1}{p}} \left[\frac{(a+b-x)-u}{y-x}(|f'(a)|^q + |f'(b)|^q) \right.$$
$$\left.\left. - \frac{1}{2}\left(\left(1 - \left(\frac{u-(a+b-y)}{y-x}\right)^2\right)|f'(x)|^q + \left(\frac{(a+b-x)-u}{y-x}\right)^2 |f'(y)|^q \right)\right]^{\frac{1}{q}} \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1 and x, y, u \in [a, b]$.

Proof. Employing Hölder integral inequality on Lemma 2.2, then

$$\left| f(u) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz \right|$$
$$\leq (y-x)\left[\int_0^{\frac{u-(a+b-y)}{y-x}} t|f'(t(a+b-x)+(1-t)(a+b-y))|dt \right.$$
$$\left. + \int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)|f'(t(a+b-x)+(1-t)(a+b-y))|dt \right]$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}} t^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{u-(a+b-y)}{y-x}} |f'(a+b-(tx+(1-t)y))|^q dt \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{u-(a+b-y)}{y-x}}^1 |f'(a+b-(tx+(1-t)y))|^q dt \right)^{\frac{1}{q}} \right].$$

By undertaking the Jensen-Mercer inequality, we have

$$\left| f(u) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz \right|$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}} t^p dt \right)^{\frac{1}{p}} \right.$$
$$\times \left(\int_0^{\frac{u-(a+b-y)}{y-x}} \left(|f'(a)|^q + |f'(b)|^q - (t|f'(x)|^q + (1-t)|f'(y)|^q) \right) dt \right)^{\frac{1}{q}}$$
$$+ \left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)^p dt \right)^{\frac{1}{p}}$$

$$\times\left(\int_{\frac{u-(a+b-y)}{y-x}}^{1}\left(|f'(a)|^q+|f'(b)|^q-(t|f'(x)|^q+(1-t)|f'(y)|^q)\right)dt\right)^{\frac{1}{q}}\Bigg]$$

$$=\Bigg\{\left(\frac{1}{p+1}\left(\frac{u-(a+b-y)}{y-x}\right)^{p+1}\right)^{\frac{1}{p}}\left[\frac{u-(a+b-y)}{y-x}(|f'(a)|^q+|f'(b)|^q)\right.$$

$$\left.-\frac{1}{2}\left(\left(\frac{u-(a+b-y)}{y-x}\right)^2|f'(x)|^q+\left(1-\left(\frac{(a+b-x)-u}{y-x}\right)^2\right)|f'(y)|^q\right)\right]^{\frac{1}{q}}$$

$$+\left(\frac{1}{p+1}\left(\frac{(a+b-x)-u}{y-x}\right)^{p+1}\right)^{\frac{1}{p}}\left[\frac{(a+b-x)-u}{y-x}(|f'(a)|^q+|f'(b)|^q)\right.$$

$$\left.-\frac{1}{2}\left(\left(1-\left(\frac{u-(a+b-y)}{y-x}\right)^2\right)|f'(x)|^q+\left(\frac{(a+b-x)-u}{y-x}\right)^2|f'(y)|^q\right)\right]^{\frac{1}{q}}\Bigg\},$$

which ends the proof. □

Corollary 2.8. *By plugging $u=a+b-\frac{x+y}{2}$ in Theorem 2.7, then*

$$\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(z)dz\right|$$

$$\leq(y-x)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}}\Bigg\{\left[\frac{1}{2}(|f'(a)|^q+|f'(b)|^q)-\frac{1}{2}\left(\frac{1}{4}|f'(x)|^q+\left(\frac{3}{4}\right)|f'(y)|^q\right)\right]^{\frac{1}{q}}$$

$$+\left[\frac{1}{2}(|f'(a)|^q+|f'(b)|^q)-\frac{1}{2}\left(\frac{3}{4}|f'(x)|^q+\frac{1}{4}|f'(y)|^q\right)\right]^{\frac{1}{q}}\Bigg\}.$$

Theorem 2.9. *Assume that $f: I=[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b) and $f\in Ł[a,b]$. If $|f'|^q$ is a convex mapping on $[a,b], q\geq 1$, then*

$$\left|f(u)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(z)dz\right|$$

$$\leq(y-x)\Bigg\{\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2(|f'(a)|^q+|f'(b)|^q)\right.$$

$$\left.-\left(\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3|f'(x)|^q+\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2-\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(y)|^q\right)\right]^{\frac{1}{q}}$$

$$+\left(\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2(|f'(a)|^q+|f'(b)|^q)\right.$$

$$\left.-\left(\frac{1}{6}-\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2+\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(x)|^q+\frac{1}{3}\left(\frac{(a+b-x)-u}{y-x}\right)^3|f'(y)|^q\right)\right]^{\frac{1}{q}}\Bigg\},\quad(10)$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $x,y,u\in[a,b]$.

Proof. Employing the power-mean integral inequality and the convexity of $|f'|^q$ on equality (2.2), we obtain

$$\left|f(u)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x}f(z)dz\right|$$

$$\leq (y-x)\left[\int_0^{\frac{u-(a+b-y)}{y-x}} t|f'(t(a+b-x)+(1-t)(a+b-y))|dt\right.$$
$$\left.+\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)|f'(t(a+b-x)+(1-t)(a+b-y))|dt\right]$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{u-(a+b-y)}{y-x}} t|f'(a+b-(tx+(1-t)y))|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)|f'(a+b-(tx+(1-t)y))|^q dt\right)^{\frac{1}{q}}\right].$$

By undertaking the Jensen-Mercer inequality, we have

$$\left|f(u)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz\right|$$
$$\leq (y-x)\left[\left(\int_0^{\frac{u-(a+b-y)}{y-x}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{u-(a+b-y)}{y-x}} t\left(|f'(a)|^q+|f'(b)|^q-(t|f'(x)|^q+(1-t)|f'(y)|^q)\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{u-(a+b-y)}{y-x}}^1 (1-t)\left(|f'(a)|^q+|f'(b)|^q-(t|f'(x)|^q+(1-t)|f'(y)|^q)\right)dt\right)^{\frac{1}{q}}\right]$$
$$=(y-x)\left\{\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2(|f'(a)|^q+|f'(b)|^q)\right.\right.$$
$$\left.-\left(\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3|f'(x)|^q+\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2-\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(y)|^q\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(\frac{(a+b-x)-u}{y-x}\right)^2(|f'(a)|^q+|f'(b)|^q)\right.$$
$$\left.\left.-\left(\frac{1}{6}-\left(\frac{1}{2}\left(\frac{u-(a+b-y)}{y-x}\right)^2+\frac{1}{3}\left(\frac{u-(a+b-y)}{y-x}\right)^3\right)|f'(x)|^q+\frac{1}{3}\left(\frac{(a+b-x)-u}{y-x}\right)^3|f'(y)|^q\right)\right]^{\frac{1}{q}}\right\},$$

which ends the proof. □

Corollary 2.10. *By plugging $u=a+b-\frac{x+y}{2}$ in Theorem 2.9, then*

$$\left|f\left(a+b-\frac{x+y}{2}\right)-\frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz\right|$$
$$\leq (y-x)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{\left[\frac{1}{8}(|f'(a)|^q+|f'(b)|^q)-\left(\frac{1}{24}|f'(x)|^q+\left(\frac{1}{12}\right)|f'(y)|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{1}{8}(|f'(a)|^q+|f'(b)|^q)-\left(\frac{1}{12}|f'(x)|^q+\frac{1}{24}|f'(y)|^q\right)\right]^{\frac{1}{q}}\right\}.$$

3. Applications

Here, we present some new relations between special means and numerical examples to support the major findings. First of all, we recall some binary relations between two numbers.

$$A(a,b) = \frac{a+b}{2}.$$

$$A_w = (w_1, w_2; a, b) = \frac{w_1 a + w_2 b}{w_1 + w_2}.$$

$$Ł_n(a,b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}}.$$

Proposition 3.1. *Owing to the assumptions of Theorem 2.5, then relation holds :*

$$\left|(2A(a,b) - A(x,y))^n - Ł_n^n(a+b-x,\ a+b-y)\right| \leq \left[\frac{nA(|a|^{n-1}, |b|^{n-1})}{2} - \frac{nA(|x|^{n-1}, |y|^{n-1})}{4}\right].$$

Proof. The proof is straight away by taking $f(x) = x^n$. □

Proposition 3.2. *Owing to the assumptions of Theorem 2.7, then relation holds :*

$$\begin{aligned}
&\left|(2A(a,b) - A(x,y))^n - Ł_n^n(a+b-x,\ a+b-y)\right| \\
&\leq n(y-x)\left(\frac{1}{(1+p)2^{p+1}}\right)^{\frac{1}{p}}\left[\left(A(|a|^{(n-1)q}, |b|^{(n-1)q}) - \frac{A_w(\frac{1}{4}, \frac{3}{4}; |x|^{(n-1)q}, |y|^{(n-1)q})}{2}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(A(|a|^{(n-1)q}, |b|^{(n-1)q}) - \frac{A_w(\frac{3}{4}, \frac{1}{4}; |x|^{(n-1)q}, |y|^{(n-1)q})}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. The proof is straight away by taking $f(x) = x^n$. □

Proposition 3.3. *If the assumptions of Theorem 2.9 are satisfied, then following relation holds:*

$$\begin{aligned}
&\left|(2A(a,b) - A(x,y))^n - Ł_n^n(a+b-x,\ a+b-y)\right| \\
&\leq n(y-x)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left\{\left[\frac{1}{4}A(|a|^{(n-1)q}, |b|^{(n-1)q}) - \left(\frac{1}{24}|x|^{(n-1)q} + \left(\frac{1}{12}\right)|y|^{(n-1)q}\right)\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{4}A(|a|^{(n-1)q}, |b|^{(n-1)q}) - \left(\frac{1}{12}|x|^{(n-1)q} + \frac{1}{24}|y|^{(n-1)q}\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. The proof is straight away by taking $f(x) = x^n$. □

Example 3.4. *If we choose $u = a + b - \frac{x+y}{2}$ & $f(l) = l^2$ with $a = 1, x = 2, y = 3$ and $b = 4$ in Theorem 2.5 then we have*

$$\left|f\left(a+b-\frac{x+y}{2}\right) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz\right| = 0.0833,$$

and

$$(y-x)\left[\frac{1}{4}(|f'(a)| + |f'(b)|) - \frac{1}{8}\left(|f'(x)| + |f'(y)|\right)\right] = 1.25.$$

This implies that $0.0833 < 1.25$

Example 3.5. *If we choose $u = a + b - \frac{x+y}{2}$ & $f(l) = l^2$ with $a = 1, x = 2, y = 3, p = 2 = q$ and $b = 4$ in Theorem 2.7 then we have*

$$\left| f\left(a + b - \frac{x+y}{2}\right) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz \right| = 0.0833$$

And

$$(y-x)\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}} \left\{ \left[\frac{1}{2}(|f'(a)|^q + |f'(b)|^q) - \frac{1}{2}\left(\frac{1}{4}|f'(x)|^q + \left(\frac{3}{4}\right)|f'(y)|^q\right)\right]^{\frac{1}{q}} + \left[\frac{1}{2}(|f'(a)|^q + |f'(b)|^q) - \frac{1}{2}\left(\frac{3}{4}|f'(x)|^q + \frac{1}{4}|f'(y)|^q\right)\right]^{\frac{1}{q}} \right\} = 1.867.$$

This implies that $0.0833 < 1.867$

Example 3.6. *If we choose $u = a + b - \frac{x+y}{2}$ & $f(l) = l^2$ with $a = 1, x = 2, y = 3, q = 2$ and $b = 4$ in Theorem 2.9 then we have*

$$\left| f\left(a + b - \frac{x+y}{2}\right) - \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(z)dz \right| = 0.0833$$

And

$$(y-x)\left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left\{ \left[\frac{1}{8}(|f'(a)|^q + |f'(b)|^q) - \left(\frac{1}{24}|f'(x)|^q + \left(\frac{1}{12}\right)|f'(y)|^q\right)\right]^{\frac{1}{q}} + \left[\frac{1}{8}(|f'(a)|^q + |f'(b)|^q) - \left(\frac{1}{12}|f'(x)|^q + \frac{1}{24}|f'(y)|^q\right)\right]^{\frac{1}{q}} \right\} = 1.617.$$

This implies that $0.0833 < 1.617$

4. Numerical Examples and Simulations

In this section, we validate our results with numerical examples and simulations. First, we give the graphical demonstration of Theorem 2.5.

- We choose $f(x) = x^n$ with $a = 1, x = 2, y = 3$ and $b = 4$ in Theorem 2.5, then

$$\left| \left(\frac{5}{2}\right)^n - \left[\frac{3^{n+1}}{n+1} - \frac{2^{n+1}}{n+1}\right] \right| \le \frac{n}{4}\left(1 + |4^{n-1}|\right) - \frac{n}{8}\left(|2^{n-1}| + |3^{n-1}|\right).$$

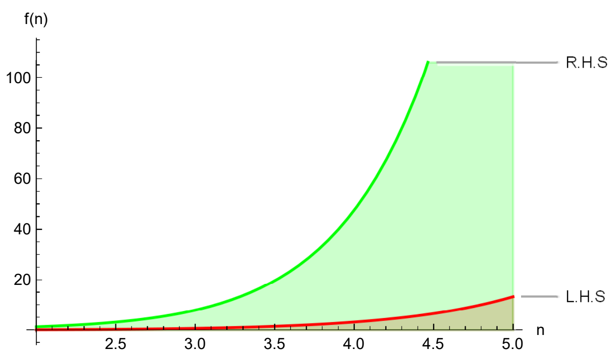

Figure 1: Figure clearly provides the validation of Theorem 2.5.

Here we show a graphical description of Theorem 2.7.

- We choose $f(x) = x^n$ with $a = 1, x = 2, y = 3$ and $b = 4$ in Theorem 2.7, then

$$\left|\left(\frac{5}{2}\right)^n - \left[\frac{3^{n+1}}{n+1} - \frac{2^{n+1}}{n+1}\right]\right|$$
$$\leq n\left(\frac{1}{24}\right)^{\frac{1}{2}}\left\{\left[\frac{1}{2}(1+|4^{n-1}|^2) - \frac{1}{8}\left(|2^{n-1}|^2 + 3|3^{n-1}|^2\right)\right]^{\frac{1}{2}}\right.$$
$$\left.+\left[\frac{1}{2}(1+|4^{n-1}|^2) - \frac{1}{8}\left(3|2^{n-1}|^2 + |3^{n-1}|^2\right)\right]^{\frac{1}{2}}\right\}.$$

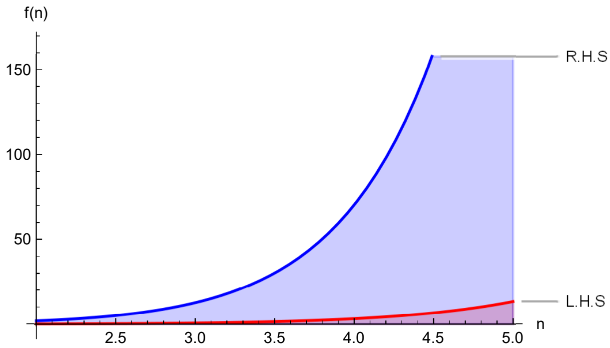

Figure 2: Figure clearly provides the validation of Theorem 2.7.

Here we show a graphical description of Theorem 2.9.

- We choose $f(x) = x^n$ with $a = 1, x = 2, y = 3$ and $b = 4$ in Theorem 2.9, then

$$\left|\left(\frac{5}{2}\right)^n - \left[\frac{3^{n+1}}{n+1} - \frac{2^{n+1}}{n+1}\right]\right| \leq n\left(\frac{1}{8}\right)$$
$$\leq n\left(\frac{1}{8}\right)^{\frac{1}{2}}\left\{\left[\frac{1}{8}(1+|4^{n-1}|^2) - \left(\frac{1}{24}|2^{n-1}|^2 + \frac{1}{12}|3^{n-1}|^2\right)\right]^{\frac{1}{2}}\right.$$
$$\left.+\left[\frac{1}{8}(1+|4^{n-1}|^2) - \left(\frac{1}{12}|2^{n-1}|^2 + \frac{1}{24}|3^{n-1}|^2\right)\right]^{\frac{1}{2}}\right\}.$$

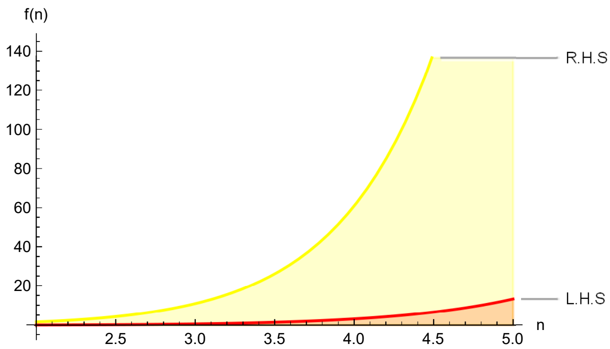

Figure 3: Figure clearly provides the validation of Theorem 2.9.

5. Conclusion

Mathematical inequalities are a very active area of analysis due to their rigorous application in both applied and pure mathematics. Error estimation via integral inequalities in association with convexity and its generalizations of numerical quadrature rules is a frequently studied problem nowadays. Many researchers have devoted their efforts to finding some precise bounds involving Montgomery identity, Hermite-Hadamard inequality, Fink identity, green mappings, Taylor series, etc. Inspired by ongoing research, we have developed a new Montgomery-Mercer identity and some upper bounds for the Ostrowski-Mercer inequality for differentiable convex mappings. In future, this identity will be helpful in deriving many mathematical inequalities involving different classes of convexity, bounding property of the mapping, Lipschitz mappings, and bounded variation. The current study is the first attempt regarding Montgomery-Mercer's identity, and hopefully, this identity will act as a base point for future research.

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