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On Ulam stability of impulsive differential equations

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Abstract. In this paper, we investigate the Hyers–Ulam and Hyers–Ulam–Rassias stability of impulsive differential equations using the Banach fixed–point theorem and the Bielecki metric. To illustrate our main results, we provide three examples that demonstrate the stability properties of these equations under consideration.

1. Introduction and preliminaries

Over the past sixty years, extensive research has been conducted on the stability of various types of equations, with particular emphasis on Hyers–Ulam stability and Hyers–Ulam–Rassias stability. These stability concepts have been explored across a wide range of equations, including functional, differential, and integral equations [7, 10, 11, 23–26], to assess how small perturbations in parameters or solutions influence the structural integrity and behavior of the equation. Hyers–Ulam stability primarily investigates whether a functional equation remains stable under minor perturbations, while Hyers–Ulam–Rassias stability generalizes this notion by incorporating more complex perturbation conditions, encompassing both linear and nonlinear influences. Researchers have established rigorous criteria to ensure the existence of stable solutions, employing diverse mathematical techniques to validate these stability results. The significance of these studies extends beyond pure mathematics, finding applications in numerous scientific and engineering disciplines, where stability plays a crucial role in ensuring the accuracy and reliability of mathematical models. Practical applications of this research span diverse fields, including materials science, electronics, heat transfer, fluid dynamics, wave theory, chemical processes, and population dynamics [6, 12, 14, 27, 28, 35, 41].

The foundational results in the stability theory of functional equations [4, 29, 30] trace back to a seminal question posed by S. M. Ulam in 1940. Ulam's inquiry sought to determine conditions under which the solution of a perturbed equation remains close to that of the original equation. In essence, Ulam was investigating whether small deviations in an equation necessarily result in correspondingly small deviations in its solution. The first partial resolution of Ulam's question was provided by D. H. Hyers in 1941. Focusing on Banach spaces, Hyers addressed the stability of the additive Cauchy functional equation,

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$f(x + y) = f(x) + f(y)$. In Hyers' pioneering work [8], it was demonstrated that if a function f defined on a Banach space satisfies a certain inequality, then f must be approximately linear. Later, T. M. Rassias further generalized this work by introducing a more flexible stability concept, extending the theory to a wider class of functional equations with perturbations. Rassias' contribution broadened the applicability of the stability theory, including functional equations beyond the Cauchy equation. The combined work of Hyers and Rassias laid the groundwork for the general theory of stability in functional equations, influencing a broad spectrum of mathematical research [36, 37].

Obloza appears to be among the first researchers to investigate the Hyers–Ulam stability of linear differential equations [17, 18]. Building on this foundation, Alsina and Ger introduced a crucial result demonstrating that if a differentiable function y satisfies a specific inequality involving its derivative, then there exists another function y_0, closely approximating y, that precisely satisfies a given differential equation. Specifically, they established that if y satisfies the inequality $\left|y'(x) - y(x)\right| \leq \epsilon$, then there exists a function y_0 solving $y'(x) = y(x)$ such that the deviation between y and y_0 is bounded by at most 3ϵ for any x in the interval I. This result implies that even if a function does not exactly satisfy a differential equation, there exists a nearby function that does, thereby reinforcing the concept of stability under perturbations. The work of Alsina and Ger was further extended by Takahasi, Miura, and Miyajima [31, 32, 38–40], who established the Hyers–Ulam stability for Banach space–valued differential equations of the form $y'(x) = \lambda(x)$. Their findings highlighted that even in the presence of perturbations, an approximate solution remains in close proximity to an exact solution, a crucial property in the analysis of Banach space–valued differential equations. The stability results in this framework ensured that minor structural variations in the equation still led to solutions with similar qualitative behavior [1–3]. By extending the stability theory to Banach space–valued settings, their work significantly broadened the scope of functional analysis and differential equations.

More recently, Miura, Takahasi, and Miyajima further advanced the understanding of Hyers–Ulam stability by proving its validity for first-order linear differential equations of the form

$$y'(x) + g(x)y(x) = 0,$$

where $g(x)$ is a continuous function. This result is particularly significant as it demonstrates the applicability of Hyers–Ulam stability to a more specialized class of differential equations, extending its reach beyond the general Banach space framework. Their findings confirm that small perturbations in the equation still allow for the existence of approximate solutions that remain close to the exact solution, preserving stability even under minor structural variations. Furthermore, their research provided additional stability results for other classes of differential equations [33], particularly in cases where the equation's coefficients or structure undergo slight modifications. The general framework they developed enhances our comprehension of stability phenomena across various contexts, which is essential for both theoretical advancements and practical applications in differential equations [19–22]. Given the fundamental role of differential equations in multiple disciplines, including physics, biology and medicine, economics, engineering, and chemistry, establishing robust stability criteria is vital for ensuring accurate modeling and prediction of complex phenomena.

In this paper, inspired by the approach of Cădariu and Radu [15, 16], we establish the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of the impulsive differential equation given by

$$y'(x) = F(x, y(x)) + \sum_{a<x_k<x} I_k(y(x_k^-)), \quad x \in [a, b], \tag{1}$$

where a and b are fixed real numbers, $F : I \times \mathbb{C} \to \mathbb{C}$ is a continuous function, and $I_k : \mathbb{C} \to \mathbb{C}$ for $k = 1, 2, \ldots, m$. Here, $y(x_k^-)$ denotes the left–hand limit of $y(x)$ at $x = x_k$.

The formal definitions of Hyers–Ulam–Rassias stability and Hyers–Ulam stability for the impulsive differential equation under study are provided below.

Definition 1.1. *The impulsive differential equation* (1) *is said to exhibit Hyers–Ulam–Rassias stability if, for every function $y(x)$ that satisfies*

$$\left| y'(x) - F(x, y(x)) - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \le \sigma(x), \tag{2}$$

where $\sigma(x) \ge 0$ for all $x \in I$, there exists a solution $y_0(x)$ of the impulsive differential equation and a constant $C > 0$ such that

$$\left| y(x) - y_0(x) \right| \le C\sigma(x), \tag{3}$$

where C is independent of both $y(x)$ and $y_0(x)$.

Definition 1.2. *The impulsive differential equation* (1) *is considered Hyers–Ulam stable if, for every function $y(x)$ fulfilling*

$$\left| y'(x) - F(x, y(x)) - \sum_{a<x_k<x} I_k(y(x_k^-)) \right| \le \theta, \tag{4}$$

where $\theta \ge 0$ for all $x \in I$, there exists a solution $y_0(x)$ of the impulsive differential equation and a constant $C > 0$ such that

$$\left| y(x) - y_0(x) \right| \le C\theta, \tag{5}$$

where C remains independent of $y(x)$ and $y_0(x)$.

Numerous existing approaches to examining the stability of functional equations utilize a combination of fixed–point results and generalized metrics in appropriate settings. In this regard, we revisit the definition of a generalized metric on a nonempty set X.

Definition 1.3 ([34]). *Let X be a nonempty set, and let $d : X \times X \to [0, +\infty]$ be a mapping. The function d is termed a generalized metric on X if it satisfies the following properties:*

1. *$d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
3. *$d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

In the context of generalized metrics, the classical Banach fixed–point theorem remains applicable.

Theorem 1.4 ([9]). *Let (X, d) be a generalized complete metric space, and let $T : X \to X$ be a strictly contractive mapping, meaning that*

$$d(Tx, Ty) \le Ld(x, y), \quad \forall x, y \in X, \tag{6}$$

for some Lipschitz constant $0 \le L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following statements hold:

1. *The sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed–point x^* of T;*
2. *x^* is the unique fixed–point of T in the set*
$$X^* = \left\{ y \in X : d(T^k x, y) < \infty \right\}; \tag{7}$$
3. *For any $y \in X^*$,*
$$d(y, x^*) \le \frac{1}{1-L} d(Ty, y). \tag{8}$$

The structure of this manuscript is as follows: Section 2 investigates the Hyers–Ulam–Rassias stability of the impulsive differential equation (1) over a finite interval. Section 3 addresses the Hyers–Ulam stability of the same equation within a finite interval. Section 4 extends the analysis to Hyers–Ulam–Rassias stability over an infinite interval. Section 5 provides three illustrative examples, while Section 6 concludes the manuscript.

2. Hyers–Ulam–Rassias stability in the finite interval case

In this section, we explore the conditions required for the impulsive differential equation, where $x \in [a, b]$ with a and b being fixed real numbers.

We focus on the space of continuous functions $C([a, b])$ defined over $[a, b]$, equipped with a generalized Bielecki metric:

$$d(y, z) = \sup_{x \in [a,b]} \frac{|y(x) - z(x)|}{\sigma(x)}, \tag{9}$$

where $\sigma : [a, b] \to (0, \infty)$ is a non–decreasing continuous function. It is well–known that the space $C([a, b])$, when endowed with the generalized metric d, forms a complete metric space (see, e.g., [5, 13]).

Theorem 2.1. *Let $F : [a, b] \times \mathbb{C} \to \mathbb{C}$ be a continuous function such that there exists a constant $L_1 > 0$ satisfying*

$$|F(x, y) - F(x, z)| \le L_1 |y - z|, \tag{10}$$

for all $x \in [a, b]$ and $y, z \in \mathbb{C}$.

Additionally, consider $I_k : \mathbb{C} \to \mathbb{C}$ with a constant $L_2 > 0$ such that

$$|I_k(y) - I_k(z)| \le L_2 |y - z|, \tag{11}$$

for all $y, z \in \mathbb{C}$.

Furthermore, suppose there exists $\gamma \in \mathbb{R}$ such that

$$\int_a^x \sigma(\tau)\, d\tau \le \gamma \sigma(x), \tag{12}$$

for all $x \in [a, b]$.

If $y \in C([a, b])$ satisfies

$$\left| y'(x) - F(x, y(x)) - \sum_{a < x_k < x} I_k(y(x_k^-)) \right| \le \sigma(x), \quad x \in [a, b], \tag{13}$$

and if $\gamma(L_1 + L_2) < 1$, then there exists a unique function $y_0 \in C([a, b])$ that solves the impulsive differential equation (1)*. This solution is given by*

$$y_0(x) = y(a) + \int_a^x F(\tau, y_0(\tau))\, d\tau + \int_a^x \sum_{a < \tau_k < \tau} I_k(y_0(\tau_k^-))\, d\tau, \tag{14}$$

and satisfies the inequality

$$|y(x) - y_0(x)| \le \frac{\gamma}{1 - \gamma(L_1 + L_2)} \sigma(x), \tag{15}$$

for all $x \in [a, b]$. This result establishes the Hyers–Ulam–Rassias stability of the impulsive differential equation (1)*.*

Proof. We consider the operator $T : C([a, b]) \to C([a, b])$ defined by

$$(Ty)(x) = y(a) + \int_a^x F(\tau, y(\tau))\, d\tau + \int_a^x \sum_{a < \tau_k < \tau} I_k(y(\tau_k^-))\, d\tau, \tag{16}$$

for all $x \in [a, b]$ and $y \in C([a, b])$. It is important to note that for any continuous function y, the function Ty remains continuous. Indeed,

$$|(Ty)(x) - (Ty)(x_0)| \;=\; \Bigg| \int_a^x F(\tau, y(\tau))\, d\tau + \int_a^x \sum_{a < \tau_k < \tau} I_k(y(\tau_k^-))\, d\tau$$

$$
\begin{aligned}
&\quad - \int_a^{x_0} F(\tau, y(\tau))\, d\tau - \int_a^{x_0} \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \Big| \\
&= \Big| \int_a^x F(\tau, y(\tau))\, d\tau - \int_a^x F(\tau, y(\tau))\, d\tau \\
&\quad + \int_a^x F(\tau, y(\tau))\, d\tau - \int_a^{x_0} F(\tau, y(\tau))\, d\tau \\
&\quad + \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau - \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \\
&\quad + \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau - \int_a^{x_0} \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \Big| \\
&\leq \Big(\int_a^x \Big| F(\tau, y(\tau)) - F(\tau, y(\tau)) \Big|\, d\tau \\
&\quad + \Big| \int_{x_0}^x F(\tau, y(\tau))\, d\tau \Big| \\
&\quad + \int_a^x \sum_{a<\tau_k<\tau} \Big| I_k(y(\tau_k^-)) - I_k(y(\tau_k^-)) \Big|\, d\tau \\
&\quad + \Big| \int_{x_0}^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \Big| \Big) \to 0
\end{aligned}
$$

when $x \to x_0$.

We now proceed to demonstrate that the operator T is strictly contractive with respect to the chosen metric. Indeed, for all $y, z \in C([a, b])$ and $x \in [a, b]$, we have,

$$
\begin{aligned}
d(Ty, Tz) &= \sup_{x\in[a,b]} \frac{\Big|(Ty)(x) - (Tz)(x)\Big|}{\sigma(x)} \\
&= \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \Big| \int_a^x F(\tau, y(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \\
&\quad - \int_a^x F(\tau, z(\tau))\, d\tau - \int_a^x \sum_{a<\tau_k<\tau} I_k(z(\tau_k^-))\, d\tau \Big| \\
&\leq \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \Big| F(\tau, y(\tau)) - F(\tau, z(\tau)) \Big|\, d\tau \\
&\quad + \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \Big| I_k(y(\tau_k^-)) - I_k(z(\tau_k^-)) \Big|\, d\tau \\
&\leq \sup_{x\in[a,b]} \frac{L_1}{\sigma(x)} \int_a^x \Big| y(\tau) - z(\tau) \Big|\, d\tau \\
&\quad + \sup_{x\in[a,b]} \frac{L_2}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \Big| y(\tau_k^-) - z(\tau_k^-) \Big|\, d\tau \\
&= \sup_{x\in[a,b]} \frac{L_1}{\sigma(x)} \int_a^x \frac{\Big| y(\tau) - z(\tau) \Big|}{\sigma(\tau)} \sigma(\tau)\, d\tau
\end{aligned}
$$

$$+\sup_{x\in[a,b]}\frac{L_2}{\sigma(x)}\int_a^x\sum_{a<\tau_k<\tau}\frac{\left|y(\tau_k^-)-z(\tau_k^-)\right|}{\sigma(\tau_k^-)}\sigma(\tau_k^-)\,d\tau$$
$$\leq \sup_{\tau\in[a,b]}\frac{\left|y(\tau)-z(\tau)\right|}{\sigma(\tau)}\sup_{x\in[a,b]}\frac{L_1}{\sigma(x)}\int_a^x\sigma(\tau)d\tau$$
$$+\sup_{\tau_k\in[a,b]}\frac{\left|y(\tau_k^-)-z(\tau_k^-)\right|}{\sigma(\tau_k^-)}\sup_{x\in[a,b]}\frac{L_2}{\sigma(x)}\int_a^x\sum_{a<\tau_k<\tau}\sigma(\tau_k^-)\,d\tau$$
$$\leq \gamma(L_1+L_2)d(y,z).$$

Since $\gamma(L_1+L_2)<1$, it follows that the operator T is strictly contractive. Consequently, we can apply the Banach fixed–point theorem mentioned earlier, which guarantees the Hyers–Ulam–Rassias stability of the impulsive differential equation (1). Furthermore, the inequality (15) directly follows from (8) and (13). □

3. Hyers–Ulam stability in the finite interval case

This section is dedicated to presenting the sufficient condition for the Hyers–Ulam stability of the impulsive differential equation (1). For a given non–decreasing continuous function $\sigma:[a,b]\to(0,\infty)$, we will continue to employ the same metric as in (9).

Theorem 3.1. *Let $F:[a,b]\times\mathbb{C}\to\mathbb{C}$ be a continuous function such that there exists a constant $L_1>0$ for which*

$$\left|F(x,y)-F(x,z)\right|\leq L_1|y-z|, \tag{17}$$

for all $x\in I$ and $y,z\in\mathbb{C}$. Moreover, let $I_k:\mathbb{C}\to\mathbb{C}$ and there exist a constant $L_2>0$ such that

$$\left|I_k(y)-I_k(z)\right|\leq L_2|y-z|, \tag{18}$$

for all $y,z\in\mathbb{C}$. Additionally, suppose there exists $\gamma\in\mathbb{R}$ such that

$$\int_a^x\sigma(\tau)d\tau\leq\gamma\sigma(x), \tag{19}$$

for all $x\in[a,b]$.

If $y\in C([a,b])$ satisfies

$$\left|y'(x)-F(x,y(x))-\sum_{a<x_k<x}I_k(y(x_k^-))d\tau\right|\leq\theta,\quad x\in[a,b], \tag{20}$$

where $\theta\geq0$ and $\gamma(L_1+L_2)<1$, then there exists a unique function $y_0\in C([a,b])$, a solution to equation (1), given by

$$y_0(x)=y(a)+\int_a^xF(\tau,y_0(\tau))d\tau+\int_a^x\sum_{a<\tau_k<\tau}I_k(y_0(\tau_k^-))\,d\tau, \tag{21}$$

and

$$\left|y(x)-y_0(x)\right|\leq\frac{(b-a)\theta\sigma(b)}{[1-\gamma(L_1+L_2)]\sigma(a)}\sigma(x), \tag{22}$$

for all $x\in[a,b]$, which implies that the impulsive differential equation (1) is Hyers–Ulam stable.

Proof. We will consider the operator $T : C([a,b]) \to C([a,b])$, defined by

$$(Ty)(x) = y(a) + \int_a^x F(\tau, y(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau, \tag{23}$$

for all $x \in [a,b]$ and $y \in C([a,b])$ (which is already well–defined).

The operator T is strictly contractive (with respect to the metric under consideration). Indeed, for all $y, z \in C([a,b])$, we have

$$\begin{aligned}
d(Ty, Tz) &= \sup_{x\in[a,b]} \frac{\left|(Ty)(x) - (Tz)(x)\right|}{\sigma(x)} \\
&= \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \Big| \int_a^x F(\tau, y(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y(\tau_k^-))\, d\tau \\
&\quad - \int_a^x F(\tau, z(\tau))\, d\tau - \int_a^x \sum_{a<\tau_k<\tau} I_k(z(\tau_k^-))\, d\tau \Big| \\
&\leq \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \left|F(\tau, y(\tau)) - F(\tau, z(\tau))\right| d\tau \\
&\quad + \sup_{x\in[a,b]} \frac{1}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \left|I_k(y(\tau_k^-)) - I_k(z(\tau_k^-))\right| d\tau \\
&\leq \sup_{x\in[a,b]} \frac{L_1}{\sigma(x)} \int_a^x \left|y(\tau) - z(\tau)\right| d\tau \\
&\quad + \sup_{x\in[a,b]} \frac{L_2}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \left|y(\tau_k^-) - z(\tau_k^-)\right| d\tau \\
&= \sup_{x\in[a,b]} \frac{L_1}{\sigma(x)} \int_a^x \frac{\left|y(\tau) - z(\tau)\right|}{\sigma(\tau)} \sigma(\tau)\, d\tau \\
&\quad + \sup_{x\in[a,b]} \frac{L_2}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \frac{\left|y(\tau_k^-) - z(\tau_k^-)\right|}{\sigma(\tau_k^-)} \sigma(\tau_k^-)\, d\tau \\
&\leq \sup_{\tau\in[a,b]} \frac{\left|y(\tau) - z(\tau)\right|}{\sigma(\tau)} \sup_{x\in[a,b]} \frac{L_1}{\sigma(x)} \int_a^x \sigma(\tau) d\tau \\
&\quad + \sup_{\tau_k\in[a,b]} \frac{\left|y(\tau_k^-) - z(\tau_k^-)\right|}{\sigma(\tau_k^-)} \sup_{x\in[a,b]} \frac{L_2}{\sigma(x)} \int_a^x \sum_{a<\tau_k<\tau} \sigma(\tau_k^-)\, d\tau \\
&\leq \gamma(L_1 + L_2) d(y, z).
\end{aligned}$$

Due to the condition $\gamma(L_1 + L_2) < 1$, it follows that T is strictly contractive. Consequently, we can apply the Banach fixed–point theorem, which guarantees the Hyers–Ulam stability for the impulsive differential equation, with (22) being derived from (8) and (20). □

4. Hyers–Ulam–Rassias stability in the infinite interval case

Instead of considering a finite interval $[a,b]$ with $a, b \in \mathbb{R}$, we now analyze the Hyers–Ulam–Rassias stability of the impulsive differential equation (1) over the infinite interval $[a,\infty)$, for some fixed $a \in \mathbb{R}$. With the necessary adaptations, similar results can be presented for infinite intervals $(-\infty, a]$, with $a \in \mathbb{R}$, as well as for $(-\infty,\infty)$.

Let us now focus on the impulsive differential equation

$$y'(x) = F(x, y(x)) + \sum_{a<x_k<x} I_k(y(x_k^-)), \quad x \in [a, \infty), \tag{24}$$

where a is a fixed real number, $F : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ is a bounded continuous function, and $I_k : \mathbb{C} \to \mathbb{C}$ for $k = 1, 2, \dots, m$. Here, $y(x_k^-)$ denotes the left–hand limit of $y(x)$ at $x = x_k$. Our approach will be based on a recurrence procedure, leveraging the results obtained for the corresponding finite interval case.

Let us consider a fixed non–decreasing function $\sigma : [a, \infty) \to (\epsilon, \omega)$, where $\epsilon, \omega > 0$, and the space $C_b([a, \infty))$ of bounded continuous functions, endowed with the metric

$$d_b(y, z) = \sup_{x\in[a,\infty)} \frac{|y(x) - z(x)|}{\sigma(x)}. \tag{25}$$

Theorem 4.1. *Let $F : [a, \infty) \times \mathbb{C} \to \mathbb{C}$ be a bounded continuous function such that there exists a constant $L_1 > 0$ satisfying*

$$\left|F(x, y) - F(x, z)\right| \leq L_1|y - z|, \tag{26}$$

for any $x \in [a, \infty)$ and $y, z \in \mathbb{C}$.

Moreover, let $I_k : \mathbb{C} \to \mathbb{C}$ be such that there exists a constant $L_2 > 0$ satisfying

$$\left|I_k(y) - I_k(z)\right| \leq L_2|y - z|, \tag{27}$$

for all $y, z \in \mathbb{C}$.

Additionally, suppose there exists $\gamma \in \mathbb{R}$ such that

$$\int_a^x \sigma(\tau)\, d\tau \leq \gamma\sigma(x), \tag{28}$$

for all $x \in [a, \infty)$.

If $y \in C_b([a, \infty))$ is such that

$$\left|y'(x) - F(x, y(x)) - \sum_{a<x_k<x} I_k(y(x_k^-))\, d\tau\right| \leq \sigma(x), \quad x \in [a, \infty), \tag{29}$$

and $\gamma(L_1 + L_2) < 1$, then there exists a unique function $y_0 \in C_b([a, \infty))$, the solution to equation (24)*, given by*

$$y_0(x) = y(a) + \int_a^x F(\tau, y_0(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y_0(\tau_k^-))\, d\tau, \tag{30}$$

and

$$\left|y(x) - y_0(x)\right| \leq \frac{\gamma}{1 - \gamma(L_1 + L_2)}\sigma(x) \tag{31}$$

for all $x \in [a, \infty)$, which means that the impulsive differential equation (24) *is Hyers–Ulam–Rassias stable.*

Proof. For any $n \in \mathbb{N}$, we define $I_n = [a, a + n]$. By Theorem 2.1, there exists a unique bounded continuous function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = y(a) + \int_a^x F(\tau, y_{0,n}(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y_{0,n}(\tau_k^-))\, d\tau \tag{32}$$

and

$$|y(x) - y_{0,n}(x)| \leq \frac{\gamma}{1 - \gamma(L_1 + L_2)}\sigma(x) \tag{33}$$

for all $x \in I_n$. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$, then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \cdots . \tag{34}$$

For any $x \in [a, \infty)$, we define $n(x) \in \mathbb{N}$ as $n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}$. We also define a function $y_0 : [a, \infty) \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x). \tag{35}$$

For any $x_1 \in [a, \infty)$, let $n_1 = n(x_1)$. Then $x_1 \in \operatorname{Int} I_{n_1+1}$ and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem 2.1, y_{0,n_1+1} is continuous at x_1, and so is y_0.

Now, we will show that y_0 satisfies

$$y_0(x) = y(a) + \int_a^x F(\tau, y_0(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y_0(\tau_k^-))\, d\tau \tag{36}$$

and

$$|y(x) - y_0(x)| \leq \frac{\gamma}{1 - \gamma(L_1 + L_2)} \sigma(x) \tag{37}$$

for all $x \in [a, \infty)$. For an arbitrary $x \in [a, \infty)$, we choose $n(x)$ such that $x \in I_{n(x)}$. By (32) and (35), we have

$$\begin{aligned} y_0(x) = y_{0,n(x)}(x) &= y(a) + \int_a^x F(\tau, y_{0,n(x)}(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y_{0,n(x)}(\tau_k^-))\, d\tau \\ &= y(a) + \int_a^x F(\tau, y_0(\tau))\, d\tau + \int_a^x \sum_{a<\tau_k<\tau} I_k(y_0(\tau_k^-))\, d\tau. \end{aligned} \tag{38}$$

Note that $n(\tau) \leq n(x)$ for any $\tau \in I_{n(x)}$, and it follows from (34) that

$$y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau),$$

so the last equality in (38) holds.

To prove (37), by (35) and (33), we have that for all $x \in [a, \infty)$,

$$|y(x) - y_0(x)| = |y(x) - y_{0,n(x)}(x)| \leq \frac{\gamma}{1 - \gamma(L_1 + L_2)} \sigma(x). \tag{39}$$

Finally, we prove the uniqueness of y_0. Let us consider another bounded continuous function y_1 which satisfies (30) and (31) for all $x \in [a, \infty)$. By the uniqueness of the solution on $I_{n(x)}$ for any $n(x) \in \mathbb{N}$, we have that $y_{0|_{I_{n(x)}}} = y_{0,n(x)}$ and $y_{1|I_{n(x)}}$ satisfies (30) and (31) for all $x \in I_{n(x)}$, so

$$y_0(x) = y_{0|_{I_{n(x)}}}(x) = y_{1|I_{n(x)}}(x) = y_1(x).$$

□

5. Examples

To illustrate that the conditions of the above results are possible to attain, we will present some examples.

Example 5.1. *Consider the impulsive differential equation*

$$y'(x) = \frac{1}{30} + \frac{1}{300}(x + y - 2) + \sum_{0<\frac{1}{11}<2} \frac{\left|y\left(\frac{1}{11}^-\right)\right|}{2 + \left|y\left(\frac{1}{11}^-\right)\right|}, \tag{40}$$

for $x \in [0,2]$. It is clear that all the conditions of Theorem 2.1 are satisfied. Specifically, let $F : [0,2] \times \mathbb{C} \to \mathbb{C}$ be defined by

$$\begin{aligned} \left|F(x,y_1) - F(x,y_2)\right| &= \left|\frac{1}{300}(x+y_1-2) - \frac{1}{300}(x+y_2-2)\right| \\ &\leq \frac{1}{300}|y_1 - y_2| \\ &= L_1|y_1 - y_2|. \end{aligned}$$

Additionally, we have

$$I_k(y(x_k^-)) = \Delta y \mid_{x=x_k} .$$

Thus,

$$\Delta y\Big|_{x=\frac{1}{11}} = I_k\left(y\left(\frac{1}{11}^-\right)\right) = \frac{\left|y\left(\frac{1}{11}^-\right)\right|}{2+\left|y\left(\frac{1}{11}^-\right)\right|}.$$

Clearly,

$$\begin{aligned} \left|I_k(y_1) - I_k(y_2)\right| &= \left|\frac{y_1}{2+y_1} - \frac{y_2}{2+y_2}\right| \\ &= \left|\frac{y_1(2+y_2) - y_2(2+y_1)}{(2+y_1)(2+y_2)}\right| \\ &= \left|\frac{2y_1 - 2y_2}{(2+y_1)(2+y_2)}\right| \\ &= \left|\frac{2(y_1 - y_2)}{(2+y_1)(2+y_2)}\right| \\ &\leq \frac{1}{2}|y_1 - y_2| \\ &= L_2|y_1 - y_2|. \end{aligned}$$

Let $y \in C([0,2])$ be such that

$$\left|y'(x) - \frac{1}{30} - \frac{1}{300}(x+y-2) - \sum_{0<\frac{1}{11}<2} \frac{\left|y\left(\frac{1}{11}^-\right)\right|}{2+\left|y\left(\frac{1}{11}^-\right)\right|}\right| \leq e^{3x} = \sigma(x),$$

for all $x \in [0,2]$.

It follows that

$$\left|\int_0^x \sigma(\tau)\,d\tau\right| = \left|\int_0^x e^{3\tau}\,d\tau\right| = \frac{e^{3x}}{3} - \frac{1}{3} \leq \frac{1}{2}e^{3x} = \gamma\sigma(x),$$

for all $x \in [0,2]$.

Therefore, this exhibits the Hyers–Ulam–Rassias stability of the impulsive differential equation (40)*. Thus, by Theorem 2.1, there exists a unique continuous function $y_0 \in C([0,2])$ such that*

$$|y(x) - y_0(x)| \leq \frac{\frac{1}{2}}{1 - \frac{1}{2}\left[\frac{1}{300} + \frac{1}{2}\right]}e^{3x} = \frac{\gamma}{1 - \gamma(L_1 + L_2)}\sigma(x).$$

Example 5.2. *Consider the impulsive differential equation,*

$$y'(x) = \frac{1}{10} + \frac{1}{100}\cos(y(x)) + \sum_{0<\frac{1}{13}<2} \frac{\left|y\left(\frac{1}{13}^{-}\right)\right|}{4+\left|y\left(\frac{1}{13}^{-}\right)\right|}, \tag{41}$$

for $x \in [0,2]$. *It is clear that all the conditions of Theorem 2.1 are satisfied. Specifically, define* $F : [0,2]\times\mathbb{C} \to \mathbb{C}$ *such that*

$$\begin{aligned}
\left|F(x,y_1) - F(x,y_2)\right| &= \left|\frac{1}{100}\cos(y_1) - \frac{1}{100}\cos(y_2)\right| \\
&\leq \frac{1}{100}|y_1 - y_2| \\
&= L_1|y_1 - y_2|.
\end{aligned}$$

Additionally,

$$I_k(y(x_k^-)) = \Delta y\,|_{x=x_k}.$$

Thus,

$$\Delta y\Big|_{x=\frac{1}{13}} = I_k\left(y\left(\frac{1}{13}^{-}\right)\right) = \frac{\left|y\left(\frac{1}{13}^{-}\right)\right|}{4+\left|y\left(\frac{1}{13}^{-}\right)\right|}.$$

Clearly,

$$\begin{aligned}
\left|I_k(y_1) - I_k(y_2)\right| &= \left|\frac{y_1}{4+y_1} - \frac{y_2}{4+y_2}\right| \\
&= \left|\frac{y_1(4+y_2) - y_2(4+y_1)}{(4+y_1)(4+y_2)}\right| \\
&= \left|\frac{4y_1 - 4y_2}{(4+y_1)(4+y_2)}\right| \\
&= \left|\frac{4(y_1 - y_2)}{(4+y_1)(4+y_2)}\right| \\
&\leq \frac{1}{4}|y_1 - y_2| \\
&= L_2|y_1 - y_2|.
\end{aligned}$$

Let $y \in C([0,2])$ *be such that*

$$\left|y'(x) - \frac{1}{10} - \frac{1}{100}\cos(y(x)) - \sum_{0<\frac{1}{13}<2} \frac{\left|y\left(\frac{1}{13}^{-}\right)\right|}{4+\left|y\left(\frac{1}{13}^{-}\right)\right|}\right| \leq e^{13x} = \sigma(x),$$

for all $x \in [0,2]$. *Clearly,*

$$\left|\int_0^x \sigma(\tau)\,d\tau\right| = \left|\int_0^x e^{13\tau}\,d\tau\right| = \frac{e^{13x}}{13} - \frac{1}{13} \leq \frac{1}{13}e^{13x} = \gamma\sigma(x),$$

for all $x \in [0,2]$.

Therefore, this exhibits the Hyers–Ulam–Rassias stability of the impulsive differential equation (41). *Thus, Theorem 2.1 guarantees the existence of a unique continuous function* $y_0 \in C([0,2])$ *such that*

$$|y(x) - y_0(x)| \leq \frac{\frac{1}{13}}{1 - \frac{1}{13}\left[\frac{1}{100} + \frac{1}{4}\right]}e^{13x} = \frac{\gamma}{1-\gamma(L_1+L_2)}\sigma(x).$$

Example 5.3. *Consider the impulsive differential equation,*

$$y'(x) = \frac{1}{20} + \frac{1}{200}(x^2 - 4x + y) + \sum_{0<\frac{1}{12}<2} \frac{\left|y\left(\frac{1}{12}^-\right)\right|}{3 + \left|y\left(\frac{1}{12}^-\right)\right|}, \tag{42}$$

for any $x \in [0,2]$. It is clear that all the conditions of Theorem 2.1 are satisfied. Specifically, let $F : [0,2] \times \mathbb{C} \to \mathbb{C}$ be such that

$$\begin{aligned} \left|F(x,y_1) - F(x,y_2)\right| &= \left|\frac{1}{200}(x^2 - 4x + y_1) - \frac{1}{200}(x^2 - 4x + y_2)\right| \\ &\leq \frac{1}{200}|y_1 - y_2| \\ &= L_1|y_1 - y_2|. \end{aligned}$$

Moreover,

$$I_k(y(x_k^-)) = \Delta y\mid_{x=x_k}.$$

Thus,

$$\Delta y\Big|_{x=\frac{1}{12}} = I_k\left(y\left(\frac{1}{12}^-\right)\right) = \frac{\left|y\left(\frac{1}{12}^-\right)\right|}{3 + \left|y\left(\frac{1}{12}^-\right)\right|}.$$

Clearly,

$$\begin{aligned} \left|I_k(y_1) - I_k(y_2)\right| &= \left|\frac{y_1}{3+y_1} - \frac{y_2}{3+y_2}\right| \\ &= \left|\frac{y_1(3+y_2) - y_2(3+y_1)}{(3+y_1)(3+y_2)}\right| \\ &= \left|\frac{3y_1 - 3y_2}{(3+y_1)(3+y_2)}\right| \\ &= \left|\frac{3(y_1 - y_2)}{(3+y_1)(3+y_2)}\right| \\ &\leq \frac{1}{3}|y_1 - y_2| \\ &= L_2|y_1 - y_2|. \end{aligned}$$

Let $y \in C([0,2])$ be such that

$$\left|y'(x) - \frac{1}{20} - \frac{1}{200}(x^2 - 4x + y) - \sum_{0<\frac{1}{12}<2} \frac{\left|y\left(\frac{1}{12}^-\right)\right|}{3 + \left|y\left(\frac{1}{12}^-\right)\right|}\right| \leq e^{3x} = \sigma(x),$$

for all $x \in [0,2]$.
Clearly,

$$\left|\int_0^x \sigma(\tau)\, d\tau\right| = \left|\int_0^x e^{3\tau}\, d\tau\right| = \frac{e^{3x}}{3} - \frac{1}{3} \leq \frac{1}{3}e^{3x} = \gamma\sigma(x),$$

for all $x \in [0,2]$.

Therefore, this exhibits the Hyers–Ulam–Rassias stability of the impulsive differential equation (42). *Thus, Theorem 2.1 guarantees the existence of a unique continuous function $y_0 \in C([0,2])$ such that*

$$|y(x) - y_0(x)| \leq \frac{\frac{1}{3}}{1 - \frac{1}{3}\left[\frac{1}{200} + \frac{1}{3}\right]}e^{3x} = \frac{\gamma}{1 - \gamma(L_1 + L_2)}\sigma(x).$$

6. Conclusion

In this paper, we have established the Hyers–Ulam and Hyers–Ulam–Rassias stability of impulsive differential equations by employing the Banach fixed–point theorem and the Bielecki metric. Our theoretical findings are supported by three illustrative examples that highlight the stability behavior of the considered equations. These results not only validate the effectiveness of the applied mathematical tools but also contribute to a deeper understanding of the stability properties of impulsive differential equations, paving the way for further research in this area.

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