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# Ulam–Hyers–Rassias stability results for a coupled system of $\psi$–Hilfer nonlinear implicit fractional differential equations with multipoint boundary conditions

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**Abstract.** Over the past decade, significant progress has been made in the stability analysis of nonlinear systems. However, coupled systems of nonlinear problems remain largely unexplored in the literature. This paper investigates the stability properties of solutions for a coupled system of $\psi$–Hilfer nonlinear implicit fractional differential equations with multipoint boundary conditions over a finite interval. The stability analysis is conducted in terms of Ulam–Hyers, Ulam–Hyers–Rassias, generalized Ulam–Hyers, and generalized Ulam–Hyers–Rassias stability. The approach employs analytical techniques specifically designed for fractional differential equations, providing a rigorous evaluation of stability under different perturbations. To demonstrate the applicability of the proposed theoretical framework, several illustrative examples are provided, showcasing how the stability conditions are satisfied in practical scenarios. These examples offer valuable insights into the behavior of solutions under different parameter settings, emphasizing the robustness of the obtained stability results. More critically, this study fills a fundamental gap in the literature by extending stability analysis to coupled nonlinear implicit fractional systems. The findings contribute to a deeper theoretical understanding of fractional–order models and provide a solid foundation for future research in this evolving domain. Additionally, the results have significant implications for applications in various scientific and engineering disciplines, including control theory, mathematical biology, and signal processing, where fractional differential equations play a crucial role in modeling complex dynamical systems.

## 1. Introduction

Fractional differential equations represent a generalization of classical differential equations with integer orders, extending them to non–integer orders. This extension provides a powerful and versatile framework for modeling complex systems that exhibit memory effects and hereditary behaviors. The concept of fractional calculus originated in the late 17th century, when Gottfried Wilhelm Leibniz and Guillaume de

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L'Hôpital corresponded on the idea of derivatives of non–integer orders. Over the centuries, the theory of fractional calculus has been rigorously developed and formalized by prominent mathematicians, including Joseph Liouville, Bernhard Riemann, Hermann Weyl, and Marcel Riesz, among others. For more historical details, see [1, 20, 23, 28].

In recent years, fractional calculus has found applications across various scientific and engineering disciplines. Unlike classical differential equations, fractional differential equations are capable of capturing the dynamics of processes that exhibit anomalous diffusion, non–local behavior, and long–range temporal correlations. This makes them particularly suitable for modeling phenomena in fields such as physics, control theory, biology, finance, and engineering; see the monographs [14, 29, 35, 52, 56].

The mathematical framework of fractional differential equations involves various definitions of fractional derivatives and integrals, each tailored to specific types of problems. The most widely used definitions are the Riemann–Liouville, Caputo, and Hadamard fractional derivatives. These derivatives act as integral operators, extending the concept of differentiation to non–integer orders, thus offering a flexible approach for modeling the evolution of systems over time. Within this framework, several types of fractional derivatives have been introduced, each designed to address different characteristics of these systems. One such derivative is the $\psi$–Hilfer fractional derivative, a novel concept introduced in the early 21st century. With ongoing advancements in this area, the $\psi$–Hilfer fractional derivative is anticipated to become increasingly significant in the mathematical modeling of complex systems [12, 17, 30, 32, 33, 36, 45].

The study of existence and uniqueness of solutions to fractional differential equations is essential, as it guarantees that the models are mathematically well–posed and that their solutions are reliable for practical applications. These properties are typically established using fixed–point theorems, which serve as fundamental tools in functional analysis; see the monographs [2, 7, 9, 13, 46]. The stability analysis of fractional differential equations is critical for understanding how solutions behave under small perturbations, which is vital for the robustness of models in real–world applications. Various stability concepts have been developed, each designed for different types of perturbations and scenarios. In the context of fractional differential equations, concepts such as Lyapunov stability, asymptotic stability, Mittag–Leffler stability, Ulam–Hyers stability, and Ulam–Hyers–Rassias stability are commonly used; see [8, 11, 18, 21, 24, 26, 36–38, 40, 42–44, 49] and references cited therein.

Fractional differential equations have gained significant attention for their ability to model real–world phenomena more accurately than classical integer–order models. Their applications span various scientific and technical fields, including epidemiology, physics, finance, chemical graph theory, control theory, aerodynamics, polymer rheology, and signal processing. In epidemiology, fractional–order models better capture memory effects and hereditary properties in disease dynamics, as demonstrated in the study of an extended SEIR model using the ABC–fractional operator [54]. Similarly, in chemical graph theory, fractional boundary value problems have been explored using fixed–point techniques, emphasizing the role of fractional calculus in mathematical chemistry [55]. The analysis of nonlinear fractional boundary value problems in complex structures like the hexasilinane graph further illustrates the utility of fractional calculus in graph–theoretic problems [5].

Coupled systems of fractional differential equations have gained significant attention due to their nonlocal nature, making them highly effective in modeling complex phenomena across various fields, including bioengineering, financial economics, chaotic dynamics, and quantum evolution. These systems naturally emerge in distributed–order dynamical models, the Duffing system, the Lorenz system, the Chua circuit, anomalous diffusion, and secure communication [3, 10, 19, 22, 27, 31, 34, 41, 50, 51, 58].

Recent advancements in fractional differential equations, particularly those involving the Hilfer fractional derivative, have demonstrated substantial flexibility in generalizing classical differential operators. This has significantly contributed to the qualitative analysis of initial and boundary value problems [15, 25, 39, 47, 48, 53]. These studies underscore the growing importance of fractional differential equations in addressing real–world challenges spanning multiple disciplines.

In [32], Abdo investigated a coupled system of fractional terminal value problems incorporating the generalized Hilfer fractional derivative, formulated as:

$$\begin{cases} D_{a+}^{\varpi_1,\varrho_1;\psi}\mathfrak{F}(\xi) = g_1(\xi,\vartheta(\xi)), \quad a < \xi \leq T,\ a > 0, \\ D_{a+}^{\varpi_2,\varrho_2;\psi}\vartheta(\xi) = g_2(\xi,\mathfrak{F}(\xi)), \quad a < \xi \leq T,\ a > 0, \\ \mathfrak{F}(T) = u_1 \in \mathbb{R}, \quad \vartheta(T) = u_2 \in \mathbb{R}, \end{cases}$$

where $0 < \varpi_i < 1$, $0 \leq \varrho_i \leq 1$, $D_{a+}^{\varpi_i,\varrho_i;\psi}$ $(i = 1, 2)$ denotes the Hilfer fractional derivative of order $\varpi_i$ and type $\varrho_i$ with respect to $\psi$, and $g_1, g_2 : (a, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

In [45], Sitho et al. established existence and uniqueness results for a class of boundary value problems involving $\psi$–Hilfer–type fractional differential equations with nonlocal integro–multipoint boundary conditions described as follows:

$$\begin{cases} \left({}^{H}D_{a^+}^{\nu_1,\wp_1;\psi} + k\,{}^{H}D_{a+}^{\nu_1-1,\wp_1;\psi}\right)v(\xi) = g(\xi, v(\xi)), \quad k \in \mathbb{R}, \quad \xi \in [c, d], \\ v(c) = 0, \quad v(d) = \sum_{r=1}^{k} \lambda_r \int_a^{\varrho_r} \psi'(\tau)v(\tau)\,d\tau + \sum_{i=1}^{k} \varsigma_i v(t_i), \end{cases}$$

where ${}^{H}D_{a+}^{\nu_1,\wp_1;\psi}$ represents the $\psi$–Hilfer fractional derivative of order $\nu_1$, $1 < \nu_1 < 2$, $0 \leq \wp_1 \leq 1$, $c \geq 0$, $\lambda_r, \varsigma_i \in \mathbb{R}$, $\varrho_r, t_i \in (c, d)$, and $g : [c, d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

More recently, in [4], this boundary value problem was extended to a coupled system of $\psi$–Hilfer–type fractional differential equations with integro–multipoint boundary conditions, formulated as:

$$\begin{cases} \left({}^{H}D_{a^+}^{\nu_1,\wp_1;\psi} + k\,{}^{H}D_{a^+}^{\nu_1-1,\wp_1;\psi}\right)v(\xi) = g(\xi, v(\xi), w(\xi)), \quad \xi \in [c, d], \\ \left({}^{H}D_{a^+}^{\bar{\nu}_1,\wp_1;\psi} + k\,{}^{H}D_{a^+}^{\bar{\nu}_1-1,\wp_1;\psi}\right)w(\xi) = h(\xi, v(\xi), w(\xi)), \quad \xi \in [c, d], \\ v(c) = 0, \quad v(d) = \sum_{r=1}^{k} \lambda_r \int_a^{\varrho_r} \psi'(\tau)w(\tau)\,d\tau + \sum_{i=1}^{k} \varsigma_i w(t_i), \\ w(c) = 0, \quad w(d) = \sum_{s=1}^{q} \hbar_s \int_a^{\ell_s} \psi'(\tau)v(\tau)\,d\tau + \sum_{\tau=1}^{z} s_\tau v(\varsigma_\tau), \end{cases}$$

where ${}^{H}D_{a^+}^{\nu_1,\wp_1;\psi}$ and ${}^{H}D_{a^+}^{\bar{\nu}_1,\wp_1;\psi}$ represent the $\psi$–Hilfer fractional derivatives of orders $\nu_1$ and $\bar{\nu}_1$, with $1 < \nu_1, \bar{\nu}_1 < 2$, $0 \leq \wp_1 \leq 1$, $c \geq 0$, $\lambda_r, \mu_i, \hbar_s, s_\tau \in \mathbb{R}_+$, and $\varrho_r, t_i, \ell_s, \varsigma_\tau \in (c, d)$. The functions $g, h : [c, d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. The authors employed the fixed–point technique to establish the existence and uniqueness of the solution. The results were presented using the Banach and Krasnosel'skii fixed point theorems and the Leray–Schauder alternative.

Motivated by [30], we aim to investigate the Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability of the following coupled system of $\psi$–Hilfer nonlinear implicit fractional differential equations with multipoint boundary conditions:

$$\begin{cases} {}^{H}D_{a^+}^{\nu_1,\wp;\psi}\vartheta(\xi) = g(\xi, \mathfrak{F}(\xi), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\xi)), \quad \xi \in J = [a, b], \\ {}^{H}D_{a^+}^{\nu_2,\wp;\psi}\mathfrak{F}(\xi) = h(\xi, \vartheta(\xi), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(\xi)), \quad \xi \in J = [a, b], \\ \vartheta(a) = 0, \quad \vartheta(b) = \sum_{r=1}^{k} \omega_r\, {}^{H}D_{a^+}^{\varepsilon_r,\wp;\psi}\mathfrak{F}(\varrho_r) + \sum_{i=1}^{j} \varsigma_i \mathfrak{F}(u_i), \\ \mathfrak{F}(a) = 0, \quad \mathfrak{F}(b) = \sum_{s=1}^{q} \mu_s\, {}^{H}D_{a^+}^{\varpi_s,\wp;\psi}\vartheta(\ell_s) + \sum_{\tau=1}^{z} \lambda_\tau \vartheta(t_\tau), \end{cases} \tag{1}$$

where ${}^{H}D_{a^+}^{\nu_1,\wp;\psi}$, ${}^{H}D_{a^+}^{\nu_2,\wp;\psi}$, ${}^{H}D_{a^+}^{m,n;\psi}$, ${}^{H}D_{a^+}^{\varepsilon_r,\wp;\psi}$, and ${}^{H}D_{a^+}^{\varpi_s,\wp;\psi}$ are the $\psi$–Hilfer fractional derivatives of order $\nu_1, \nu_2, m, \varepsilon_r$, and $\varpi_s$, respectively, with $1 < \varepsilon_r, \varpi_r < m < \nu_1, \nu_2 < 2$ and type $0 \leq \wp, n \leq 1, \omega_r, \varsigma_i, \mu_s, \lambda_\tau \in \mathbb{R}_+$, $\varrho_r, u_n, \ell_s, t_\tau \in J$, and $g, h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Although substantial progress has been made in the stability analysis of nonlinear systems over the past decade, coupled systems of nonlinear implicit fractional differential equations remain largely unexplored. While stability concepts such as Ulam–Hyers and Ulam–Hyers–Rassias stability have been well studied for single equations, their application to coupled fractional systems with multipoint boundary conditions is still limited [30, 33, 45]. Existing research primarily focuses on classical and integer–order differential equations, leaving a gap in the understanding of fractional–order coupled systems, particularly in the context of the $\psi$–Hilfer derivative. This paper addresses this gap by extending stability analysis to such coupled systems and providing a comprehensive investigation of their stability under different perturbations.

The findings of this study have significant practical implications across multiple disciplines. Fractional differential equations play a crucial role in modeling complex dynamical systems in fields such as control theory, mathematical biology, and signal processing. By establishing rigorous stability conditions for coupled nonlinear implicit fractional systems, this work enhances the reliability of mathematical models used in these areas. The theoretical results obtained not only provide deeper insights into the behavior of fractional–order systems but also serve as a foundation for future research, enabling more accurate and stable modeling of real–world phenomena [6, 12, 16, 18, 36, 49, 57].

The structure of this paper is organized as follows: Section 2 introduces the fundamental concepts and definitions essential for the investigation. The definitions of Ulam–Hyers–Rassias stabilities are provided in Section 3. Section 4 discusses the stability results in the sense of Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias. Illustrative examples of the main results are presented in Section 5. Finally, the conclusion of the paper is provided in Section 6.

## 2. Basic concepts and some preliminary results

We begin by introducing some definitions related to the primary concepts of this study. These definitions, along with lemmas, play a crucial role in establishing the Ulam–Hyers–Rassias stability results of this investigation. Let $C([a,b],\mathbb{R})$ denote the space of all continuous functions from $[a,b]$ to $\mathbb{R}$, and $AC([a,b],\mathbb{R})$ represent the space of all absolutely continuous functions from $[a,b]$ to $\mathbb{R}$.

**Definition 2.1 (See [1]).** *Let $(a,b)$, where $-\infty \leq a < b \leq \infty$, denote a finite or infinite interval on the real line $\mathbb{R}$, and let $\hbar > 0$. Suppose $\psi(\xi)$ is an increasing, positive, and monotone function defined on $(a,b]$, with a continuous derivative $\psi'(\xi)$ on $(a,b)$. The $\psi$–Riemann–Liouville fractional integral $I_{a^+}^{\hbar;\psi}(\cdot)$ of a function $h \in AC^n([a,b],\mathbb{R})$ with respect to the function $\psi$ on $[a,b]$ is given by*

$$I_{a+}^{\hbar;\psi}h(\xi) = \frac{1}{\Gamma(\hbar)}\int_a^{\xi}\psi'(\tau)(\psi(\xi)-\psi(\tau))^{\hbar-1}h(\tau)\,d\tau, \quad \xi > a > 0,$$

*where $\Gamma(\cdot)$ denotes the gamma function.*

**Definition 2.2 (See [1]).** *Let $\psi'(\xi) \neq 0$, $\hbar > 0$, and $n \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order $\hbar$ for a function $h \in AC^n([a,b],\mathbb{R})$ with respect to another function $\psi$ is expressed as:*

$$\begin{aligned} D_{a^+}^{\hbar,\rho;\psi}h(\xi) &= \left(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^n I_{a^+}^{n-\hbar;\psi}h(\xi) \\ &= \frac{1}{\Gamma(n-\hbar)}\left(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^n \int_a^{\xi}\psi'(\tau)(\psi(\xi)-\psi(\tau))^{n-\hbar-1}h(\tau)\,d\tau, \end{aligned}$$

*where $n = [\hbar]+1$, and $[\hbar]$ denotes the integer part of the real number $\hbar$.*

**Definition 2.3 (See [47]).** *Let $n-1 < \hbar < n$ with $n \in \mathbb{N}$, $[a,b]$ be an interval such that $-\infty \leq a < b \leq \infty$, and suppose $h, \psi \in C^n([a,b],\mathbb{R})$ are two functions where $\psi(\xi)$ is an increasing function and $\psi'(\xi) \neq 0$ for all $\xi \in [a,b]$. The $\psi$–Hilfer fractional derivative ${}^{H}D_{a^+}^{\hbar,\rho;\psi}(\cdot)$ of a function $h$, of order $\hbar$ and type $0 \leq \rho \leq 1$, is defined as:*

$${}^{H}D_{a^+}^{\hbar,\rho;\psi}h(\xi) = I_{a^+}^{\rho(n-\hbar);\psi}\left(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^n I_{a^+}^{(1-\rho)(n-\hbar);\psi}h(\xi),$$

*where $n = [\hbar]+1$, $[\hbar]$ denotes the integer part of the real number $\hbar$, and $\gamma = \hbar + \rho(n-\hbar)$.*

**Lemma 2.4 (See [1]).** *Let $\hbar, \tau > 0$. The following semigroup property holds:*

$$I_{a^+}^{\hbar;\psi}I_{a^+}^{\tau;\psi}h(\xi) = I_{a^+}^{\hbar+\tau;\psi}h(\xi), \quad \xi > a.$$

**Lemma 2.5 (See [47]).** *If $h \in C^n([a,b],\mathbb{R})$, $n-1<\hbar<n$, $0\le\rho\le 1$, and $\gamma=\hbar+\rho(n-\hbar)$, then the following relationship holds:*

$$I_{a^+}^{\hbar;\psi}\,{}^{H}D_{a^+}^{\hbar,\rho;\psi}h(\xi)=h(\xi)-\sum_{k=1}^{n}\frac{(\psi(\xi)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}h_{\psi}^{[n-k]}I_{a^+}^{(1-\rho)(n-\hbar);\psi}h(a),$$

*for all $\xi\in[a,b]$, where*

$$h_{\psi}^{[n]}h(\xi)=\left(\frac{1}{\psi'(\xi)}\frac{d}{d\xi}\right)^{n}h(\xi).$$

**Proposition 2.6 (See [1, 47]).** *Let $\hbar\ge 0$, $\alpha>0$, and $\xi>a$. The $\psi$–fractional integral and derivative of a power function are expressed as follows:*

$$\begin{aligned}
I_{a^+}^{\hbar;\psi}(\psi(\tau)-\psi(a))^{\alpha-1}(\xi)&=\frac{\Gamma(\alpha)}{\Gamma(\alpha+\hbar)}(\psi(\xi)-\psi(a))^{\alpha+\hbar-1},\\
D_{a^+}^{\hbar,\rho;\psi}(\psi(\tau)-\psi(a))^{\alpha-1}(\xi)&=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\hbar)}(\psi(\xi)-\psi(a))^{\alpha-\hbar-1},\\
{}^{H}D_{a^+}^{\hbar,\rho;\psi}(\psi(\tau)-\psi(a))^{\alpha-1}(\xi)&=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\hbar)}(\psi(\xi)-\psi(a))^{\alpha-\hbar-1},
\end{aligned}$$

*where $\alpha>\gamma=\hbar+\rho(n-\hbar)$.*

**Lemma 2.7 (See [53]).** *Let $j-1<\hbar<j$, $k-1<m<k\le j$, $k,j\in\mathbb{N}$, $0\le n\le 1$, and $\hbar\ge m+n(k-m)$. If $h\in C^m(J,\mathbb{R})$, the following relation holds:*

$${}^{H}D_{a^+}^{m,n;\psi}I_{a^+}^{\hbar;\psi}h(\xi)=I_{a^+}^{m-\hbar;\psi}h(\xi).$$

We recall the following lemma regarding the solution to the problem below. For further details, readers may refer to the earlier work [30].

**Lemma 2.8.** *Let $a\ge 0$, $1<\varepsilon_r,\varpi_s<m<\nu_1,\nu_2<2$, $0\le\wp,n\le 1$, $\theta_1=\nu_1+\wp(2-\nu_1)$, $\theta_2=\nu_2+\wp(2-\nu_2)$, and $\Upsilon\neq 0$. Then for $g,h:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, the solution of the coupled system*

$$\begin{cases}
{}^{H}D_{a^+}^{\nu_1,\wp;\psi}\vartheta(\xi)=g(\xi,\mathfrak{F}(\xi),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\xi)), & \xi\in[a,b],\\
{}^{H}D_{a^+}^{\nu_2,\wp;\psi}\mathfrak{F}(\xi)=h(\xi,\vartheta(\xi),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(\xi)), & \xi\in[a,b],\\
\vartheta(a)=0,\quad \vartheta(b)=\sum_{r=1}^{k}\omega_r\,{}^{H}D_{a^+}^{\varepsilon_r,\wp;\psi}\mathfrak{F}(\varrho_r)+\sum_{i=1}^{j}\varsigma_i\mathfrak{F}(u_i),\\
\mathfrak{F}(a)=0,\quad \mathfrak{F}(b)=\sum_{s=1}^{q}\mu_s\,{}^{H}D_{a^+}^{\varpi_s,\wp;\psi}\vartheta(\ell_s)+\sum_{\tau=1}^{z}\lambda_\tau\vartheta(t_\tau)
\end{cases}$$

*is given by*

$$\vartheta(\xi)=\begin{cases}
I_{a+}^{\nu_1;\psi}g\left(\xi,\mathfrak{F}(\xi),{}^{H}D_{a+}^{m,n;\psi}\mathfrak{F}(\xi)\right)\\
+\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\Bigg[\Psi\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi}h\left(\varrho_r,\vartheta(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(\varrho_r)\right)\\
+\sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi}h\left(u_i,\vartheta(u_i),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(u_i)\right)-I_{a+}^{\nu_1;\psi}g\left(b,\mathfrak{F}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(b)\right)\Bigg)\\
+G\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi}g\left(\ell_s,\mathfrak{F}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\ell_s)\right)\\
+\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi}g\left(t_\tau,\mathfrak{F}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(t_\tau)\right)-I_{a+}^{\nu_2;\psi}h\left(b,\vartheta(b),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(b)\right)\Bigg)\Bigg],
\end{cases}$$

$$\mathfrak{F}(\xi)=\begin{cases} I_{a+}^{\nu_2;\psi}h\left(\xi,\vartheta(\xi),\,{}^HD_{a^+}^{m,n;\psi}\vartheta(\xi)\right)\\ +\frac{(\psi(\xi)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)\cdot\Upsilon}\Bigg[\Delta\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi}h\left(\varrho_r,\vartheta(\varrho_r),\,{}^HD_{a^+}^{m,n;\psi}\vartheta(\varrho_r)\right)\\ +\sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi}h\left(u_i,\vartheta(u_i),\,{}^HD_{a^+}^{m,n;\psi}\vartheta(u_i)\right)-I_{a+}^{\nu_1;\psi}g\left(b,\mathfrak{F}(b),\,{}^HD_{a^+}^{m,n;\psi}\mathfrak{F}(b)\right)\Bigg)\\ +H\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi}g\left(\ell_s,\mathfrak{F}(\ell_s),\,{}^HD_{a^+}^{m,n;\psi}\mathfrak{F}(\ell_s)\right)\\ +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi}g\left(t_\tau,\mathfrak{F}(t_\tau),\,{}^HD_{a^+}^{m,n;\psi}\mathfrak{F}(t_\tau)\right)-I_{a+}^{\nu_2;\psi}h\left(b,\vartheta(b),\,{}^HD_{a^+}^{m,n;\psi}\vartheta(b)\right)\Bigg)\Bigg], \end{cases}$$

*where*

$$H=\frac{(\psi(b)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)},$$
$$G=\sum_{r=1}^{k}\omega_r\frac{(\psi(\varrho_r)-\psi(a))^{\theta_2-\varepsilon_r-1}}{\Gamma(\theta_2-\varepsilon_r)}+\sum_{i=1}^{j}\varsigma_i\frac{(\psi(u_i)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)},$$
$$\Delta=\sum_{s=1}^{q}\mu_s\frac{(\psi(\ell_s)-\psi(a))^{\theta_1-\varpi_s-1}}{\Gamma(\theta_1-\varpi_s)}+\sum_{\tau=1}^{z}\lambda_\tau\frac{(\psi(t_\tau)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)},$$
$$\Psi=\frac{(\psi(b)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)},$$

*and*

$$\Upsilon=H\Psi-G\Delta.$$

## 3. Definitions of Ulam–Hyers–Rassias stabilities

Let $\epsilon_1,\epsilon_2>0$, $1<\varepsilon_r,\varpi_S<m<\nu_1,\nu_2<2$, $0\le\wp,n\le 1$, and $\omega_r,\varsigma_i,\mu_s,\lambda_\tau\in\mathbb{R}_+$, $\varrho_r,u_n,\ell_s,t_\tau\in J$. Let $g,h:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous functions, and $\varphi_1,\varphi_2:J\to\mathbb{R}_+$. We consider the system of the $\psi$–Hilfer nonlinear implicit fractional problem (1) and the system of inequalities

$$\begin{cases}\left|{}^HD_{a^+}^{\nu_1,\wp;\psi}\chi(\xi)-g(\xi,\Im(\xi),\,{}^HD_{a^+}^{m,n;\psi}\Im(\xi))\right|\le\epsilon_1, & \xi\in J,\\ \left|{}^HD_{a^+}^{\nu_2,\wp;\psi}\Im(\xi)-h(\xi,\chi(\xi),\,{}^HD_{a^+}^{m,n;\psi}\chi(\xi))\right|\le\epsilon_2,\end{cases}\tag{2}$$

$$\begin{cases}\left|{}^HD_{a^+}^{\nu_1,\wp;\psi}\chi(\xi)-g(\xi,\Im(\xi),\,{}^HD_{a^+}^{m,n;\psi}\Im(\xi))\right|\le\varphi_1(\xi), & \xi\in J,\\ \left|{}^HD_{a^+}^{\nu_2,\wp;\psi}\Im(\xi)-h(\xi,\chi(\xi),\,{}^HD_{a^+}^{m,n;\psi}\chi(\xi))\right|\le\varphi_2(\xi),\end{cases}\tag{3}$$

$$\begin{cases}\left|{}^HD_{a^+}^{\nu_1,\wp;\psi}\chi(\xi)-g(\xi,\Im(\xi),\,{}^HD_{a^+}^{m,n;\psi}\Im(\xi))\right|\le\epsilon_1\varphi_1(\xi), & \xi\in J,\\ \left|{}^HD_{a^+}^{\nu_2,\wp;\psi}\Im(\xi)-h(\xi,\chi(\xi),\,{}^HD_{a^+}^{m,n;\psi}\chi(\xi))\right|\le\epsilon_2\varphi_2(\xi),\end{cases}\tag{4}$$

with multipoint boundary conditions

$$\begin{cases}\chi(a)=0, & \chi(b)=\sum_{r=1}^{k}\omega_r\,{}^HD_{a^+}^{\varepsilon_r,\wp;\psi}\Im(\varrho_r)+\sum_{i=1}^{j}\varsigma_i\Im(u_i),\\ \Im(a)=0, & \Im(b)=\sum_{s=1}^{q}\mu_s\,{}^HD_{a^+}^{\varpi_s,\wp;\psi}\chi(\ell_s)+\sum_{\tau=1}^{z}\lambda_\tau\chi(t_\tau),\end{cases}\tag{5}$$

where $1\le r\le k, 1\le i\le j, 1\le s\le q$, and $1\le\tau\le z$.

In the following Ulam–Hyers–Rassias stabilities definitions, we denote $X=C^1(J,\mathbb{R})$. For a vector $\mathfrak{F}=(\mathfrak{F}_1,\mathfrak{F}_2)>0$, this implies $\mathfrak{F}_1,\mathfrak{F}_2>0$.

**Definition 3.1.** *Problem* (1) *is said to be Ulam–Hyers stable if there exists a constant vector* $c_{g,h} = (c_g, c_h) > 0$ *such that for each* $\epsilon = (\epsilon_1, \epsilon_2) > 0$ *and for every solution* $(\chi, \mathfrak{I}) \in X \times X$ *of the inequalities* (2) *with* (5)*, there exists a solution* $(\vartheta, \mathfrak{F}) \in X \times X$ *to problem* (1) *satisfying*

$$\|(\chi, \mathfrak{I}) - (\vartheta, \mathfrak{F})\| \leq c_{g,h}\epsilon^T, \quad \xi \in J.$$

**Definition 3.2.** *Problem* (1) *is said to be generalized Ulam–Hyers stable if there exists a continuous vector function* $\sigma_{g,h} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\sigma_{g,h}(0) = 0$ *such that for every solution* $(\chi, \mathfrak{I}) \in X \times X$ *of inequalities* (2) *with* (5)*, there exists a solution* $(\vartheta, \mathfrak{F}) \in X \times X$ *to problem* (1) *satisfying*

$$\|(\chi, \mathfrak{I}) - (\vartheta, \mathfrak{F})\| \leq \sigma_{g,h}(\epsilon), \quad \xi \in J.$$

**Definition 3.3.** *Problem* (1) *is said to be Ulam–Hyers–Rassias stable with respect to* $\varphi = (\varphi_1, \varphi_2)$ *if there exists a constant vector* $c_{g,h,\varphi} = (c_{g,\varphi_1}, c_{h,\varphi_2}) > 0$ *such that for each* $\epsilon > 0$ *and for every solution* $(\chi, \mathfrak{I}) \in X \times X$ *of inequalities* (4) *with* (5)*, there exists a solution* $(\vartheta, \mathfrak{F}) \in X \times X$ *to problem* (1) *satisfying*

$$\|(\chi, \mathfrak{I}) - (\vartheta, \mathfrak{F})\| \leq \epsilon c_{g,h,\varphi}[\varphi(\xi)]^T, \quad \xi \in J.$$

**Definition 3.4.** *Problem* (1) *is said to be generalized Ulam–Hyers–Rassias stable with respect to* $\varphi = (\varphi_1, \varphi_2)$ *if there exists a constant vector* $c_{g,h,\varphi} = (c_{g,\varphi_1}, c_{h,\varphi_2}) > 0$ *such that for every solution* $(\chi, \mathfrak{I}) \in X \times X$ *of inequalities* (3) *with* (5)*, there exists a solution* $(\vartheta, \mathfrak{F}) \in X \times X$ *to problem* (1) *satisfying*

$$\|(\chi, \mathfrak{I}) - (\vartheta, \mathfrak{F})\| \leq c_{g,h,\varphi}[\varphi(\xi)]^T, \quad \xi \in J.$$

**Remark 3.5.** *It is evident that:* (*i*) *Definition 3.1* $\Longrightarrow$ *Definition 3.2;* (*ii*) *Definition 3.3* $\Longrightarrow$ *Definition 3.4;* (*iii*) *Definition 3.3* $\Longrightarrow$ *Definition 3.1.*

**Remark 3.6.** *A vector function* $(\chi, \mathfrak{I}) \in X \times X$ *satisfies the inequalities* (2) *if and only if there exist functions* $\mathfrak{F}_1, \mathfrak{F}_2 \in C(J, \mathbb{R})$ *such that* $|\mathfrak{F}_1(\xi)| \leq \epsilon_1$ *and* $|\mathfrak{F}_2(\xi)| \leq \epsilon_2$ *for* $\xi \in J$*, and the following system holds:*

$$\begin{cases} {}^H D_{a^+}^{\nu_1,\wp;\psi}\chi(\xi) = g(\xi, \mathfrak{I}(\xi), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(\xi)) + \mathfrak{F}_1(\xi), \quad \xi \in J, \\ {}^H D_{a^+}^{\nu_2,\wp;\psi}\mathfrak{I}(\xi) = h(\xi, \chi(\xi), {}^H D_{a^+}^{m,n;\psi}\chi(\xi)) + \mathfrak{F}_2(\xi). \end{cases} \tag{6}$$

Similar observations can be made regarding inequalities (3) and (4), as discussed in Remark 3.6.

**Remark 3.7.** *If a function vector* $(\chi, \mathfrak{I}) \in X \times X$ *satisfies the inequalities* (2) *with* (5)*, then* $(\chi, \mathfrak{I})$ *also satisfies the* *following system of integral inequalities:*

$$\begin{cases} \Bigg|\chi(\xi) - I_{a+}^{\nu_1;\psi} g\left(\xi, \mathfrak{I}(\xi), {}^H D_{a+}^{m,n;\psi} \mathfrak{I}(\xi)\right) \\ -\frac{\left(\psi(\xi)-\psi(a)\right)^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\Bigg[\Psi\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r, \chi(\varrho_r), {}^H D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) \\ +\sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i, \chi(u_i), {}^H D_{a^+}^{m,n;\psi}\chi(u_i)\right) - I_{a+}^{\nu_1;\psi} g\left(b, \mathfrak{I}(b), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Bigg) \\ +G\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{I}(\ell_s), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) \\ +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{I}(t_\tau), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b, \chi(b), {}^H D_{a^+}^{m,n;\psi}\chi(b)\right)\Bigg)\Bigg]\Bigg| \\ \leq \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)}, \\ \Bigg|\mathfrak{I}(\xi) - I_{a+}^{\nu_2;\psi} h\left(\xi, \chi(\xi), {}^H D_{a^+}^{m,n;\psi} \chi(\xi)\right) \\ -\frac{\left(\psi(\xi)-\psi(a)\right)^{\theta_2-1}}{\Gamma(\theta_2)\cdot\Upsilon}\Bigg[\Delta\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r, \chi(\varrho_r), {}^H D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) \\ +\sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i, \chi(u_i), {}^H D_{a^+}^{m,n;\psi}\chi(u_i)\right) - I_{a+}^{\nu_1;\psi} g\left(b, \mathfrak{I}(b), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Bigg) \\ +H\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{I}(\ell_s), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) \\ +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{I}(t_\tau), {}^H D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b, \chi(b), {}^H D_{a^+}^{m,n;\psi}\chi(b)\right)\Bigg)\Bigg]\Bigg| \\ \leq \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}. \end{cases}$$

We can make similar observations for the solutions of the inequalities (3) and (4) with (5).

## 4. Main results

We assume the following conditions to prove the Ulam–Hyers–Rassias stabilities of the problem (1):

(H1) The functions $g, h : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, and there exist positive constants $\mathfrak{L}_1, \mathfrak{L}_2 > 0$ such that for all $\xi \in J$ and $\vartheta_i, \mathfrak{y}_i \in \mathbb{R}$ $(i = 1, 2)$, we have

$$\|g(\xi, \mathfrak{y}_1, \bar{\mathfrak{y}}_1) - g(\xi, \mathfrak{y}_2, \bar{\mathfrak{y}}_2)\| \leq \mathfrak{L}_1\Big(\|\mathfrak{y}_1 - \mathfrak{y}_2\| + \|\bar{\mathfrak{y}}_1 - \bar{\mathfrak{y}}_2\|\Big),$$

$$\|h(\xi, \vartheta_1, \bar{\vartheta}_1) - h(\xi, \vartheta_2, \bar{\vartheta}_2)\| \leq \mathfrak{L}_2\Big(\|\vartheta_1 - \vartheta_2\| + \|\bar{\vartheta}_1 - \bar{\vartheta}_2\|\Big).$$

(H2) The functions $\varphi_1, \varphi_2 : J \to \mathbb{R}_+$ are increasing and continuous, and there exist constants $c_{\varphi_1}, c_{\varphi_2} > 0$ such that

$$I_{a^+}^{\nu_1;\psi}\varphi_1(\xi) \leq c_{\varphi_1}\varphi_1(\xi), \quad I_{a^+}^{\nu_2;\psi}\varphi_2(\xi) \leq c_{\varphi_2}\varphi_2(\xi), \quad \xi \in J.$$

For convenience, we introduce the following notations:

$$\mathfrak{R}(\bar{\ell}, \tau) = \frac{(\psi(\bar{\ell}) - \psi(a))^{\tau}}{\Gamma(\tau + 1)}, \quad \mathfrak{D} = \frac{\mathfrak{R}(b, \theta_2 - 1)}{|\Upsilon|}, \quad \text{where } \Upsilon \neq 0,$$

$$L = \mathfrak{R}(b, \nu_1), \quad M = \mathfrak{R}(b, \nu_2), \quad N = \frac{\mathfrak{R}(b, \theta_1 - 1)}{|\Upsilon|}, \quad \text{where } \Upsilon \neq 0,$$

$$O = \sum_{r=1}^{k} \omega_r \mathfrak{R}(\varrho_r, \nu_2 - \varepsilon_r) + \sum_{i=1}^{j} \varsigma_i \mathfrak{R}(u_i, \nu_2),$$

$$P = \sum_{s=1}^{q} \mu_s \mathfrak{R}(\ell_s, \nu_1 - \varpi_s) + \sum_{\tau=1}^{z} \lambda_\tau \mathfrak{R}(t_\tau, \nu_1),$$

$$Q = |\Psi|\mathfrak{L}_2 O + |G|\mathfrak{L}_2 M, \quad R = |\Psi|\mathfrak{L}_1 L + |G|\mathfrak{L}_1 P,$$

$$S = \mathfrak{L}_1 L + NR, \quad U = NQ, \quad V = |\Delta|\mathfrak{L}_2 O + |H|\mathfrak{L}_2 M,$$

$$W = |\Delta|\mathfrak{L}_1 L + |H|\mathfrak{L}_1 P, \quad Y = \mathfrak{L}_2 M + \mathfrak{D} V, \quad Z = \mathfrak{D} W.$$

### 4.1. Ulam–Hyers and generalized Ulam–Hyers stabilities in the finite interval case

In this subsection, we will demonstrate the Ulam–Hyers and generalized Ulam–Hyers stabilities of problem (1) over a finite interval.

**Theorem 4.1.** *If assumption* (*H*1) *holds, then the problem* (1) *is Ulam–Hyers stable, and as a result, it is also generalized Ulam–Hyers stable.*

*Proof.* Let $(\chi, \mathfrak{I}) \in X \times X$ be a solution of inequalities (2) with (5). We define $(\vartheta, \mathfrak{F}) \in X \times X$ as the unique solution of the following problem:

$$\begin{cases} {}^H D_{a^+}^{\nu_1,\wp;\psi} \vartheta(\xi) = g(\xi, \mathfrak{F}(\xi), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(\xi)), \quad \xi \in [a,b], \\ {}^H D_{a^+}^{\nu_2,\wp;\psi} \mathfrak{F}(\xi) = h(\xi, \vartheta(\xi), {}^H D_{a^+}^{m,n;\psi} \vartheta(\xi)), \quad \xi \in [a,b], \\ \vartheta(a) = 0, \quad \vartheta(b) = \sum_{r=1}^{k} \omega_r \, {}^H D_{a^+}^{\varepsilon_r,\wp;\psi} \mathfrak{F}(\varrho_r) + \sum_{i=1}^{j} \varsigma_i \mathfrak{F}(u_i), \\ \mathfrak{F}(a) = 0, \quad \mathfrak{F}(b) = \sum_{s=1}^{q} \mu_s \, {}^H D_{a^+}^{\varpi_s,\wp;\psi} \vartheta(\ell_s) + \sum_{\tau=1}^{z} \lambda_\tau \vartheta(t_\tau). \end{cases}$$

Thus, we obtain

$$\vartheta(\xi) = \begin{cases} I_{a+}^{\nu_1;\psi} g\left(\xi, \mathfrak{F}(\xi), {}^H D_{a+}^{m,n;\psi} \mathfrak{F}(\xi)\right) \\ + \frac{(\psi(\xi)-\psi(a))^{\theta_1 - 1}}{\Gamma(\theta_1)\cdot\Upsilon} \Bigg[ \Psi\Bigg( \sum_{r=1}^{k} \omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r, \vartheta(\varrho_r), {}^H D_{a^+}^{m,n;\psi} \vartheta(\varrho_r)\right) \\ + \sum_{i=1}^{j} \varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i, \vartheta(u_i), {}^H D_{a^+}^{m,n;\psi} \vartheta(u_i)\right) - I_{a+}^{\nu_1;\psi} g\left(b, \mathfrak{F}(b), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(b)\right) \Bigg) \\ + G\Bigg( \sum_{s=1}^{q} \mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{F}(\ell_s), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(\ell_s)\right) \\ + \sum_{\tau=1}^{z} \lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{F}(t_\tau), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b, \vartheta(b), {}^H D_{a^+}^{m,n;\psi} \vartheta(b)\right) \Bigg) \Bigg], \end{cases}$$

$$\mathfrak{F}(\xi) = \begin{cases} I_{a+}^{\nu_2;\psi} h\left(\xi, \vartheta(\xi), {}^H D_{a^+}^{m,n;\psi} \vartheta(\xi)\right) \\ + \frac{(\psi(\xi)-\psi(a))^{\theta_2 - 1}}{\Gamma(\theta_2)\cdot\Upsilon} \Bigg[ \Delta\Bigg( \sum_{r=1}^{k} \omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r, \vartheta(\varrho_r), {}^H D_{a^+}^{m,n;\psi} \vartheta(\varrho_r)\right) \\ + \sum_{i=1}^{j} \varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i, \vartheta(u_i), {}^H D_{a^+}^{m,n;\psi} \vartheta(u_i)\right) - I_{a+}^{\nu_1;\psi} g\left(b, \mathfrak{F}(b), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(b)\right) \Bigg) \\ + H\Bigg( \sum_{s=1}^{q} \mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{F}(\ell_s), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(\ell_s)\right) \\ + \sum_{\tau=1}^{z} \lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{F}(t_\tau), {}^H D_{a^+}^{m,n;\psi} \mathfrak{F}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b, \vartheta(b), {}^H D_{a^+}^{m,n;\psi} \vartheta(b)\right) \Bigg) \Bigg]. \end{cases}$$

Based on the relations above and Remark 3.6, we conclude that

$$
\begin{aligned}
|\chi(\xi)-\vartheta(\xi)| \leq\ & \Bigg|\chi(\xi)-I_{a+}^{\nu_1;\psi} g\left(\xi, \mathfrak{I}(\xi), {}^{H}D_{a+}^{m,n;\psi}\mathfrak{I}(\xi)\right) \\
& -\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\Bigg[\Psi\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r, \chi(\varrho_r), {}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) \\
& +\sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i, \chi(u_i), {}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right)-I_{a+}^{\nu_1;\psi} g\left(b, \mathfrak{I}(b), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Bigg) \\
& +G\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{I}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) \\
& +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{I}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right)-I_{a+}^{\nu_2;\psi} h\left(b, \chi(b), {}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)\Bigg)\Bigg]\Bigg| \\
& +I_{a+}^{\nu_1;\psi}\left|g\left(\xi, \mathfrak{I}(\xi), {}^{H}D_{a+}^{m,n;\psi}\mathfrak{I}(\xi)\right)-g\left(\xi, \mathfrak{F}(\xi), {}^{H}D_{a+}^{m,n;\psi}\mathfrak{F}(\xi)\right)\right| \\
& +\left|\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\right|\Bigg[|\Psi|\Bigg(\sum_{r=1}^{k}|\omega_r|\, I_{a+}^{\nu_2-\varepsilon_r;\psi} \\
& \times\left|h\left(\varrho_r, \chi(\varrho_r), {}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right)-h\left(\varrho_r, \vartheta(\varrho_r), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(\varrho_r)\right)\right| \\
& +\sum_{i=1}^{j}|\varsigma_i|\, I_{a+}^{\nu_2;\psi}\left|h\left(u_i, \chi(u_i), {}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right)-h\left(u_i, \vartheta(u_i), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(u_i)\right)\right| \\
& +I_{a+}^{\nu_1;\psi}\left|g\left(b, \mathfrak{I}(b), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)-g\left(b, \mathfrak{F}(b), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(b)\right)\right|\Bigg) \\
& +|G|\Bigg(\sum_{s=1}^{q}|\mu_s|\, I_{a+}^{\nu_1-\varpi_s;\psi}\left|g\left(\ell_s, \mathfrak{I}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right)-g\left(\ell_s, \mathfrak{F}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\ell_s)\right)\right| \\
& +\sum_{\tau=1}^{z}|\lambda_\tau|\, I_{a+}^{\nu_1;\psi}\left|g\left(t_\tau, \mathfrak{I}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right)-g\left(t_\tau, \mathfrak{F}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(t_\tau)\right)\right| \\
& +I_{a+}^{\nu_2;\psi}\left|h\left(b, \chi(b), {}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)-h\left(b, \vartheta(b), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(b)\right)\right|\Bigg)\Bigg] \\
\leq\ & \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)}+\mathfrak{R}(b,\nu_1)\mathfrak{L}_1|\mathfrak{I}(\xi)-\mathfrak{F}(\xi)| \\
& +\frac{\mathfrak{R}(b,\theta_1-1)}{|\Upsilon|}\Bigg[|\Psi|\Bigg(\sum_{r=1}^{k}\omega_r\mathfrak{R}(\varrho_r,\nu_2-\varepsilon_r)\mathfrak{L}_2|\chi(\varrho_r)-\vartheta(\varrho_r)| \\
& +\sum_{i=1}^{j}\varsigma_i\mathfrak{R}(u_i,\nu_2)\mathfrak{L}_2|\chi(u_i)-\vartheta(u_i)|+\mathfrak{R}(b,\nu_1)\mathfrak{L}_1|\mathfrak{I}(b)-\mathfrak{F}(b)|\Bigg) \\
& +|G|\Bigg(\sum_{s=1}^{q}\mu_s\mathfrak{R}(\ell_s,\nu_1-\varpi_s)\mathfrak{L}_1|\mathfrak{I}(\ell_s)-\mathfrak{F}(\ell_s)| \\
& +\sum_{\tau=1}^{z}\lambda_\tau\mathfrak{R}(t_\tau,\nu_1)\mathfrak{L}_1|\mathfrak{I}(t_\tau)-\mathfrak{F}(t_\tau)|+\mathfrak{R}(b,\nu_2)\mathfrak{L}_2|\chi(b)-\vartheta(b)|\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)} + \mathfrak{L}_1 \mathfrak{R}(b,\nu_1)\|\mathfrak{I}-\mathfrak{F}\| \\
&\quad + \frac{\mathfrak{R}(b,\theta_1-1)}{|\Upsilon|}\Bigg[|\Psi|\Bigg(\mathfrak{L}_2\Bigg\{\sum_{r=1}^{k}\omega_r \mathfrak{R}(\varrho_r,\nu_2-\varepsilon_r) + \sum_{i=1}^{j}\varsigma_i \mathfrak{R}(u_i,\nu_2)\Bigg\}\|\chi-\vartheta\| + \mathfrak{L}_1\mathfrak{R}(b,\nu_1)\|\mathfrak{I}-\mathfrak{F}\|\Bigg) \\
&\quad + |G|\Bigg(\mathfrak{L}_1\Bigg\{\sum_{s=1}^{q}\mu_s \mathfrak{R}(\ell_s,\nu_1-\varpi_s) + \sum_{\tau=1}^{z}\lambda_\tau \mathfrak{R}(t_\tau,\nu_1)\Bigg\}\|\mathfrak{I}-\mathfrak{F}\| + \mathfrak{L}_2\mathfrak{R}(b,\nu_2)\|\chi-\vartheta\|\Bigg)\Bigg] \\
&= \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)} + \mathfrak{L}_1 L\|\mathfrak{I}-\mathfrak{F}\| + N\Big[\Big(|\Psi|\mathfrak{L}_2 O + |G|\mathfrak{L}_2 M\Big)\|\chi-\vartheta\| + \Big(|\Psi|\mathfrak{L}_1 L + |G|\mathfrak{L}_1 P\Big)\|\mathfrak{I}-\mathfrak{F}\|\Big] \\
&= \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)} + (\mathfrak{L}_1 L + NR)\|\mathfrak{I}-\mathfrak{F}\| + NQ\|\chi-\vartheta\|.
\end{aligned}
$$

Hence

$$
\|\chi-\vartheta\| \leq \frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)} + S\|\mathfrak{I}-\mathfrak{F}\| + U\|\chi-\vartheta\|. \tag{7}
$$

Similarly

$$
\begin{aligned}
|\mathfrak{I}(\xi)-\mathfrak{F}(\xi)| \leq\ & \Bigg|\mathfrak{I}(\xi) - I_{a+}^{\nu_2;\psi} h\left(\xi,\chi(\xi),{}^{H}D_{a^+}^{m,n;\psi}\chi(\xi)\right) \\
& - \frac{(\psi(\xi)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)\cdot\Upsilon}\Bigg[\Delta\Bigg(\sum_{r=1}^{k}\omega_r I_{a+}^{\nu_2-\varepsilon_r;\psi} h\left(\varrho_r,\chi(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) \\
& + \sum_{i=1}^{j}\varsigma_i I_{a+}^{\nu_2;\psi} h\left(u_i,\chi(u_i),{}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right) - I_{a+}^{\nu_1;\psi} g\left(b,\mathfrak{I}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Bigg) \\
& + H\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s,\mathfrak{I}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) \\
& + \sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau,\mathfrak{I}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b,\chi(b),{}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)\Bigg)\Bigg]\Bigg| \\
& + I_{a+}^{\nu_2;\psi}\left|h\left(\xi,\chi(\xi),{}^{H}D_{a+}^{m,n;\psi}\chi(\xi)\right) - h\left(\xi,\vartheta(\xi),{}^{H}D_{a+}^{m,n;\psi}\vartheta(\xi)\right)\right| \\
& + \left|\frac{(\psi(\xi)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)\cdot\Upsilon}\right|\Bigg[|\Delta|\Bigg(\sum_{r=1}^{k}|\omega_r| I_{a+}^{\nu_2-\varepsilon_r;\psi} \\
& \times\left|h\left(\varrho_r,\chi(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) - h\left(\varrho_r,\vartheta(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(\varrho_r)\right)\right| \\
& + \sum_{i=1}^{j}|\varsigma_i| I_{a+}^{\nu_2;\psi}\left|h\left(u_i,\chi(u_i),{}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right) - h\left(u_i,\vartheta(u_i),{}^{H}D_{a^+}^{m,n;\psi}\vartheta(u_i)\right)\right| \\
& + I_{a+}^{\nu_1;\psi}\left|g\left(b,\mathfrak{I}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right) - g\left(b,\mathfrak{F}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(b)\right)\right|\Bigg) \\
& + |H|\Bigg(\sum_{s=1}^{q}|\mu_s| I_{a+}^{\nu_1-\varpi_s;\psi}\left|g\left(\ell_s,\mathfrak{I}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) - g\left(\ell_s,\mathfrak{F}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\ell_s)\right)\right| \\
& + \sum_{\tau=1}^{z}|\lambda_\tau| I_{a+}^{\nu_1;\psi}\left|g\left(t_\tau,\mathfrak{I}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - g\left(t_\tau,\mathfrak{F}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(t_\tau)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+I_{a+}^{\nu_2;\psi}\left|h\left(b,\chi(b),\,{}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)-h\left(b,\vartheta(b),\,{}^{H}D_{a^+}^{m,n;\psi}\vartheta(b)\right)\right|\Big)\Big]\\
\leq\;& \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}+\mathfrak{R}(b,\nu_2)\mathfrak{L}_2|\chi(\xi)-\vartheta(\xi)|\\
&+\frac{\mathfrak{R}(b,\theta_2-1)}{|\Upsilon|}\Bigg[|\Delta|\Bigg(\sum_{r=1}^{k}\omega_r\mathfrak{R}(\varrho_r,\nu_2-\varepsilon_r)\mathfrak{L}_2|\chi(\varrho_r)-\vartheta(\varrho_r)|\\
&+\sum_{i=1}^{j}\varsigma_i\mathfrak{R}(u_i,\nu_2)\mathfrak{L}_2|\chi(u_i)-\vartheta(u_i)|+\mathfrak{R}(b,\nu_1)\mathfrak{L}_1|\mathfrak{I}(b)-\mathfrak{F}(b)|\Bigg)\\
&+|H|\Bigg(\sum_{s=1}^{q}\mu_s\mathfrak{R}(\ell_s,\nu_1-\varpi_s)\mathfrak{L}_1|\mathfrak{I}(\ell_s)-\mathfrak{F}(\ell_s)|\\
&+\sum_{\tau=1}^{z}\lambda_\tau\mathfrak{R}(t_\tau,\nu_1)\mathfrak{L}_1|\mathfrak{I}(t_\tau)-\mathfrak{F}(t_\tau)|+\mathfrak{R}(b,\nu_2)\mathfrak{L}_2|\chi(b)-\vartheta(b)|\Bigg)\Bigg]\\
\leq\;& \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}+\mathfrak{L}_2\mathfrak{R}(b,\nu_2)\|\chi-\vartheta\|\\
&+\frac{\mathfrak{R}(b,\theta_2-1)}{|\Upsilon|}\Bigg[|\Delta|\Bigg(\mathfrak{L}_2\Bigg\{\sum_{r=1}^{k}\omega_r\mathfrak{R}(\varrho_r,\nu_2-\varepsilon_r)+\sum_{i=1}^{j}\varsigma_i\mathfrak{R}(u_i,\nu_2)\Bigg\}\|\chi-\vartheta\|+\mathfrak{L}_1\mathfrak{R}(b,\nu_1)\|\mathfrak{I}-\mathfrak{F}\|\Bigg)\\
&+|H|\Bigg(\mathfrak{L}_1\Bigg\{\sum_{s=1}^{q}\mu_s\mathfrak{R}(\ell_s,\nu_1-\varpi_s)+\sum_{\tau=1}^{z}\lambda_\tau\mathfrak{R}(t_\tau,\nu_1)\Bigg\}\|\mathfrak{I}-\mathfrak{F}\|+\mathfrak{L}_2\mathfrak{R}(b,\nu_2)\|\chi-\vartheta\|\Bigg)\Bigg]\\
=\;& \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}+\mathfrak{L}_2M\|\chi-\vartheta\|+\mathfrak{D}_1\Big[\Big(|\Delta|\mathfrak{L}_2O+|H|\mathfrak{L}_2M\Big)\|\chi-\vartheta\|+\Big(|\Delta|\mathfrak{L}_1L+|H|\mathfrak{L}_1P\Big)\|\mathfrak{I}-\mathfrak{F}\|\Big]\\
=\;& \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}+(\mathfrak{L}_2M+\mathfrak{D}_1V)\|\chi-\vartheta\|+\mathfrak{D}_1W\|\mathfrak{I}-\mathfrak{F}\|.
\end{aligned}
$$

Hence

$$
\|\mathfrak{I}-\mathfrak{F}\| \;\leq\; \frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}+Y\|\chi-\vartheta\|+Z\|\mathfrak{I}-\mathfrak{F}\|. \tag{8}
$$

From (7) and (8), by letting $c_{g,h}=(c_g,c_h)$, for each $\epsilon=(\epsilon_1,\epsilon_2)>0$, it follows that

$$
\begin{aligned}
\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\| \;\leq\;& \left(\frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)}+\frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}\right)+(S+Y)\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\|+(U+Z)\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\|\\
\leq\;& \frac{\left(\frac{\epsilon_1 b^{\nu_1}}{\Gamma(\nu_1+1)}+\frac{\epsilon_2 b^{\nu_2}}{\Gamma(\nu_2+1)}\right)}{\Big[1-(S+Y+U+Z)\Big]}\\
\leq\;& c_g\epsilon_1+c_h\epsilon_2=c_{g,h}\epsilon^T,
\end{aligned}
$$

with

$$
c_g=\frac{\frac{b^{\nu_1}}{\Gamma(\nu_1+1)}}{\Big[1-(S+Y+U+Z)\Big]},
$$

$$
c_h=\frac{\frac{b^{\nu_2}}{\Gamma(\nu_2+1)}}{\Big[1-(S+Y+U+Z)\Big]}.
$$

Hence, problem (1) is Ulam–Hyers stable. Furthermore, it is generalized Ulam–Hyers stable, since

$$\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\|\leq\sigma_{g,h}(\epsilon),$$

with $\sigma_{g,h}(\epsilon)=c_{g,h}\epsilon^{T}$ and $\sigma_{g,h}(0)=0$. This completes the proof. □

*4.2. Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stabilities in the finite interval case*

In this subsection, we will demonstrate the Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stabilities of problem (1) over a finite interval.

**Theorem 4.2.** *If assumption* (H1) *holds, and there exists a function* $\varphi=(\varphi_1,\varphi_2)$, *where* $\varphi_i\in C([a,b],\mathbb{R}_+)$ *for* $i=1,2$, *satisfying* (H2), *then the problem* (1) *is Ulam–Hyers–Rassias stable, and hence generalized Ulam–Hyers–Rassias stable with respect to* $\varphi$.

*Proof.* Let $(\chi,\mathfrak{I})\in X\times X$ be a solution of the inequalities (4) with (5). We denote by $(\vartheta,\mathfrak{F})\in X\times X$ the unique solution of the problem (1). By Remark 3.6, it follows that

$$\begin{cases}\Big|\chi(\xi)-I_{a+}^{\nu_1;\psi}g\left(\xi,\mathfrak{I}(\xi),{}^{H}D_{a+}^{m,n;\psi}\mathfrak{I}(\xi)\right)\\ -\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\Big[\Psi\Big(\sum_{r=1}^{k}\omega_rI_{a+}^{\nu_2-\varepsilon_r;\psi}h\left(\varrho_r,\chi(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right)\\ +\sum_{i=1}^{j}\varsigma_iI_{a+}^{\nu_2;\psi}h\left(u_i,\chi(u_i),{}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right)-I_{a+}^{\nu_1;\psi}g\left(b,\mathfrak{I}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Big)\\ +G\Big(\sum_{s=1}^{q}\mu_sI_{a+}^{\nu_1-\varpi_s;\psi}g\left(\ell_s,\mathfrak{I}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right)\\ +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi}g\left(t_\tau,\mathfrak{I}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right)-I_{a+}^{\nu_2;\psi}h\left(b,\chi(b),{}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)\Big)\Big]\Big|\\ \leq\epsilon_1I_{a^+}^{\nu_1;\psi}\varphi_1(\xi),\\ \Big|\mathfrak{I}(\xi)-I_{a+}^{\nu_2;\psi}h\left(\xi,\chi(\xi),{}^{H}D_{a^+}^{m,n;\psi}\chi(\xi)\right)\\ -\frac{(\psi(\xi)-\psi(a))^{\theta_2-1}}{\Gamma(\theta_2)\cdot\Upsilon}\Big[\Delta\Big(\sum_{r=1}^{k}\omega_rI_{a+}^{\nu_2-\varepsilon_r;\psi}h\left(\varrho_r,\chi(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right)\\ +\sum_{i=1}^{j}\varsigma_iI_{a+}^{\nu_2;\psi}h\left(u_i,\chi(u_i),{}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right)-I_{a+}^{\nu_1;\psi}g\left(b,\mathfrak{I}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Big)\\ +H\Big(\sum_{s=1}^{q}\mu_sI_{a+}^{\nu_1-\varpi_s;\psi}g\left(\ell_s,\mathfrak{I}(\ell_s),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right)\\ +\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi}g\left(t_\tau,\mathfrak{I}(t_\tau),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right)-I_{a+}^{\nu_2;\psi}h\left(b,\chi(b),{}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)\Big)\Big]\Big|\\ \leq\epsilon_2I_{a^+}^{\nu_2;\psi}\varphi_2(\xi)\end{cases}$$

for all $\xi\in[a,b]$.
From the above relations, it follows that

$$\begin{aligned}|\chi(\xi)-\vartheta(\xi)|\quad\leq\quad&\Big|\chi(\xi)-I_{a+}^{\nu_1;\psi}g\left(\xi,\mathfrak{I}(\xi),{}^{H}D_{a+}^{m,n;\psi}\mathfrak{I}(\xi)\right)\\ &-\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\Big[\Psi\Big(\sum_{r=1}^{k}\omega_rI_{a+}^{\nu_2-\varepsilon_r;\psi}h\left(\varrho_r,\chi(\varrho_r),{}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right)\\ &+\sum_{i=1}^{j}\varsigma_iI_{a+}^{\nu_2;\psi}h\left(u_i,\chi(u_i),{}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right)-I_{a+}^{\nu_1;\psi}g\left(b,\mathfrak{I}(b),{}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right)\Big)\end{aligned}$$

$$
\begin{aligned}
&+G\Bigg(\sum_{s=1}^{q}\mu_s I_{a+}^{\nu_1-\varpi_s;\psi} g\left(\ell_s, \mathfrak{I}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right)\\
&+\sum_{\tau=1}^{z}\lambda_\tau I_{a+}^{\nu_1;\psi} g\left(t_\tau, \mathfrak{I}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - I_{a+}^{\nu_2;\psi} h\left(b, \chi(b), {}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right)\Bigg)\Bigg]\Bigg\|\\
&+I_{a+}^{\nu_1;\psi}\left|g\left(\xi, \mathfrak{I}(\xi), {}^{H}D_{a+}^{m,n;\psi}\mathfrak{I}(\xi)\right) - g\left(\xi, \mathfrak{F}(\xi), {}^{H}D_{a+}^{m,n;\psi}\mathfrak{F}(\xi)\right)\right|\\
&+\left|\frac{(\psi(\xi)-\psi(a))^{\theta_1-1}}{\Gamma(\theta_1)\cdot\Upsilon}\right|\Bigg[|\Psi|\Bigg(\sum_{r=1}^{k}|\omega_r|\, I_{a+}^{\nu_2-\varepsilon_r;\psi}\\
&\times\left|h\left(\varrho_r, \chi(\varrho_r), {}^{H}D_{a^+}^{m,n;\psi}\chi(\varrho_r)\right) - h\left(\varrho_r, \vartheta(\varrho_r), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(\varrho_r)\right)\right|\\
&+\sum_{i=1}^{j}|\varsigma_i|\, I_{a+}^{\nu_2;\psi}\left|h\left(u_i, \chi(u_i), {}^{H}D_{a^+}^{m,n;\psi}\chi(u_i)\right) - h\left(u_i, \vartheta(u_i), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(u_i)\right)\right|\\
&+I_{a+}^{\nu_1;\psi}\left|g\left(b, \mathfrak{I}(b), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(b)\right) - g\left(b, \mathfrak{F}(b), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(b)\right)\right|\Bigg)\\
&+|G|\Bigg(\sum_{s=1}^{q}\left|\mu_s\right| I_{a+}^{\nu_1-\varpi_s;\psi}\left|g\left(\ell_s, \mathfrak{I}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(\ell_s)\right) - g\left(\ell_s, \mathfrak{F}(\ell_s), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(\ell_s)\right)\right|\\
&+\sum_{\tau=1}^{z}|\lambda_\tau|\, I_{a+}^{\nu_1;\psi}\left|g\left(t_\tau, \mathfrak{I}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{I}(t_\tau)\right) - g\left(t_\tau, \mathfrak{F}(t_\tau), {}^{H}D_{a^+}^{m,n;\psi}\mathfrak{F}(t_\tau)\right)\right|\\
&+I_{a+}^{\nu_2;\psi}\left|h\left(b, \chi(b), {}^{H}D_{a^+}^{m,n;\psi}\chi(b)\right) - h\left(b, \vartheta(b), {}^{H}D_{a^+}^{m,n;\psi}\vartheta(b)\right)\right|\Bigg)\Bigg]\\
\leq\;& \epsilon_1 c_{\varphi_1}\varphi_1(\xi) + \mathfrak{L}_1\mathfrak{R}(b,\nu_1)\|\mathfrak{I}-\mathfrak{F}\|\\
&+\frac{\mathfrak{R}(b,\theta_1-1)}{|\Upsilon|}\Bigg[|\Psi|\Bigg(\mathfrak{L}_2\Bigg\{\sum_{r=1}^{k}\omega_r\mathfrak{R}(\varrho_r,\nu_2-\varepsilon_r)+\sum_{i=1}^{j}\varsigma_i\mathfrak{R}(u_i,\nu_2)\Bigg\}\|\chi-\vartheta\| + \mathfrak{L}_1\mathfrak{R}(b,\nu_1)\|\mathfrak{I}-\mathfrak{F}\|\Bigg)\\
&+|G|\Bigg(\mathfrak{L}_1\Bigg\{\sum_{s=1}^{q}\mu_s\mathfrak{R}(\ell_s,\nu_1-\varpi_s)+\sum_{\tau=1}^{z}\lambda_\tau\mathfrak{R}(t_\tau,\nu_1)\Bigg\}\|\mathfrak{I}-\mathfrak{F}\| + \mathfrak{L}_2\mathfrak{R}(b,\nu_2)\|\chi-\vartheta\|\Bigg)\Bigg].
\end{aligned}
$$

Hence

$$\|\chi-\vartheta\| \leq \epsilon_1 c_{\varphi_1}\varphi_1(\xi) + S\|\mathfrak{I}-\mathfrak{F}\| + U\|\chi-\vartheta\|. \tag{9}$$

Similarly,

$$\|\mathfrak{I}-\mathfrak{F}\| \leq \epsilon_2 c_{\varphi_2}\varphi_2(\xi) + Y\|\chi-\vartheta\| + Z\|\mathfrak{I}-\mathfrak{F}\|. \tag{10}$$

From (9) and (10), and defining $c_{g,h,\varphi} = (c_{g,\varphi_1}, c_{h,\varphi_2})$, for each $\epsilon = (\epsilon_1, \epsilon_2) > 0$, it follows that

$$
\begin{aligned}
\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\| &\leq \left(\epsilon_1 c_{\varphi_1}\varphi_1(\xi)+\epsilon_2 c_{\varphi_2}\varphi_2(\xi)\right) + (S+Y)\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\| + (U+Z)\|(\chi,\mathfrak{I})-(\vartheta,\mathfrak{F})\|\\
&\leq \frac{\left(\epsilon_1 c_{\varphi_1}\varphi_1(\xi)+\epsilon_2 c_{\varphi_2}\varphi_2(\xi)\right)}{\left[1-(S+Y+U+Z)\right]}\\
&\leq c_{g,\varphi_1}\epsilon_1\varphi_1(\xi) + c_{h,\varphi_2}\epsilon_2\varphi_2(\xi) = \epsilon c_{g,h,\varphi}[\varphi(\xi)]^T,
\end{aligned}
$$

with

$$c_{g,\varphi_1} = \frac{c_{\varphi_1}}{\left[1-(S+Y+U+Z)\right]},$$

$$c_{h,\varphi_2} = \frac{c_{\varphi_2}}{\Big[1-(S+Y+U+Z)\Big]}.$$

Hence, problem (1) is Ulam–Hyers–Rassias stable with respect to $\varphi$. Moreover, it is generalized Ulam–Hyers–Rassias stable with respect to $\varphi$. If we take $\epsilon = 1$, then

$$\|(\chi, \mathfrak{I}) - (\vartheta, \mathfrak{F})\| \le c_{g,h,\varphi}[\varphi(\xi)]^T.$$

This completes the proof. □

## 5. Illustrative examples

In this section, we provide two examples to demonstrate the results derived earlier.

**Example 5.1.** *Consider the coupled system of the $\psi$–Hilfer nonlinear implicit fractional multipoint boundary value problem*

$$\begin{cases} {}^HD^{\frac{9}{5},\frac{1}{10};e^{\frac{2\xi}{5}}}\vartheta(\xi) = g\Big(\xi, \mathfrak{F}(\xi), {}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\mathfrak{F}(\xi)\Big), \quad \xi \in [0,2],\\ {}^HD^{\frac{17}{10},\frac{1}{10};e^{\frac{2\xi}{5}}}\mathfrak{F}(\xi) = h\Big(\xi, \vartheta(\xi), {}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\vartheta(\xi)\Big), \quad \xi \in [0,2],\\ \vartheta(0) = 0, \quad \mathfrak{F}(0) = 0,\\ \vartheta(2) = \sum_{r=1}^{3}\left(\frac{3r}{5r+1}\right){}^HD^{\frac{6r}{5},\frac{1}{10};e^{\frac{2\xi}{5}}}\mathfrak{F}\left(\frac{r}{2}\right) + \sum_{i=1}^{4}\left(\frac{i}{3}\right)\mathfrak{F}\left(\frac{2i}{41}\right),\\ \mathfrak{F}(2) = \sum_{s=1}^{4}\left(\frac{s}{2}\right){}^HD^{\frac{11}{10s},\frac{1}{10};e^{\frac{2\xi}{5}}}\vartheta\left(\frac{2}{s}\right) + \sum_{\tau=1}^{5}\left(\frac{\tau}{9}\right)\vartheta\left(\frac{\tau}{3}\right). \end{cases} \tag{11}$$

*Here* $\nu_1 = \frac{9}{5}$, $\nu_2 = \frac{17}{10}$, $\wp = \frac{1}{10}$, $m = \frac{3}{2}$, $n = \frac{1}{9}$, $a = 0$, $b = 2$, $k = 3$, $j = 4$, $q = 4$, $z = 5$, $\omega_r = \frac{3r}{5r+1}$, $\varrho_r = \frac{r}{2}$, $\varepsilon_r = \frac{6r}{5}$, $\varsigma_i = \frac{i}{3}$, $u_i = \frac{2i}{41}$, $\mu_s = \frac{s}{2}$, $\ell_s = \frac{2}{s}$, $\varpi_s = \frac{11}{10s}$, $\lambda_\tau = \frac{\tau}{9}$, $t_\tau = \frac{\tau}{3}$, $\psi(\xi) = e^{\frac{2\xi}{5}}$.

*From the data, we compute that* $\theta_1 = 6.572$, $\theta_2 = 1.73$, $H \approx 0.0112$, $G \approx -64.4469$, $\Psi \approx 1.2253$, $\Delta \approx 0.0238$, *and* $\Upsilon = H\Psi - G\Delta \approx 1.5476$.

*Consider the functions*

$$g\Big(\xi, \mathfrak{F}, {}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\mathfrak{F}\Big) = \frac{\xi}{\xi^3+\xi^2+3\xi+2} + \frac{\sin|\mathfrak{F}|}{1000} + \frac{\cos|{}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\mathfrak{F}|}{200},$$

$$h\Big(\xi, \vartheta, {}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\vartheta\Big) = \frac{15}{31\log|\xi|}\cot^{-1}|\vartheta| + \frac{{}^HD^{\frac{3}{2},\frac{1}{9};e^{\frac{2\xi}{5}}}\vartheta}{500}.$$

*For* $\vartheta_1, \bar{\vartheta}_1, \mathfrak{y}_1, \bar{\mathfrak{y}}_1, \vartheta_2, \bar{\vartheta}_2, \mathfrak{y}_2, \bar{\mathfrak{y}}_2 \in \mathbb{R}$ *and* $\xi \in [0,2]$, *we have*

$$\|g(\xi, \mathfrak{y}_1, \bar{\mathfrak{y}}_1) - g(\xi, \mathfrak{y}_2, \bar{\mathfrak{y}}_2)\| \le \frac{1}{100}\Big(\|\mathfrak{y}_1 - \mathfrak{y}_2\| + \|\bar{\mathfrak{y}}_1 - \bar{\mathfrak{y}}_2\|\Big),$$

$$\|h(\xi, \vartheta_1, \bar{\vartheta}_1) - h(\xi, \vartheta_2, \bar{\vartheta}_2)\| \le \frac{3}{5}\Big(\|\vartheta_1 - \vartheta_2\| + \|\bar{\vartheta}_1 - \bar{\vartheta}_2\|\Big).$$

*From* $(H1)$ *we have* $\mathfrak{L}_1 = \frac{1}{100}$, $\mathfrak{L}_2 = \frac{3}{5}$. *Hence, all the conditions of Theorem 4.1 are satisfied. Therefore, the problem* (11) *is both Ulam–Hyers stable and generalized Ulam–Hyers stable.*

**Example 5.2.** *Consider the coupled system of the $\psi$-Hilfer nonlinear implicit fractional multipoint boundary value problem*

$$\begin{cases} {}^HD^{\frac{19}{10},\frac{1}{5};\frac{\sin(\xi)}{3}}\vartheta(\xi) = g\Big(\xi, \mathfrak{F}(\xi), {}^HD^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\mathfrak{F}(\xi)\Big), \quad \xi \in [0,1],\\ {}^HD^{\frac{7}{5},\frac{1}{5};\frac{\sin(\xi)}{3}}\mathfrak{F}(\xi) = h\Big(\xi, \vartheta(\xi), {}^HD^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\vartheta(\xi)\Big), \quad \xi \in [0,1],\\ \vartheta(0) = 0, \quad \mathfrak{F}(0) = 0,\\ \vartheta(1) = \sum_{r=1}^{2}\left(\frac{13r}{r+1}\right){}^HD^{\frac{11}{10r},\frac{3}{10};\frac{\sin(\xi)}{3}}\mathfrak{F}\left(\frac{r}{4}\right) + \sum_{i=1}^{3}\left(\frac{i}{23}\right)\mathfrak{F}\left(\frac{i}{5}\right),\\ \mathfrak{F}(1) = \sum_{s=1}^{3}\left(\frac{2}{3s}\right){}^HD^{\frac{6s}{5},\frac{3}{10};\frac{\sin(\xi)}{3}}\vartheta\left(\frac{3s}{2}\right) + \sum_{\tau=1}^{4}\left(\frac{\tau}{5}\right)\vartheta\left(\frac{\tau}{100}\right). \end{cases} \tag{12}$$

*where* $\nu_1 = \frac{19}{10}$, $\nu_2 = \frac{7}{5}$, $\wp = \frac{1}{5}$, $m = \frac{6}{5}$, $n = \frac{4}{5}$, $a = 0$, $b = 1$, $k = 2$, $j = 3$, $q = 3$, $z = 4$, $\omega_r = \frac{13r}{r+1}$, $\varrho_r = \frac{r}{4}$, $\varepsilon_r = \frac{11}{10r}$, $\varsigma_i = \frac{i}{23}$, $u_i = \frac{i}{5}$, $\mu_s = \frac{2}{3s}$, $\ell_s = \frac{3s}{2}$, $\varpi_s = \frac{6s}{5}$, $\lambda_\tau = \frac{\tau}{5}$, $t_\tau = \frac{\tau}{100}$, *and* $\psi(\xi) = \frac{\sin(\xi)}{3}$.

*From the provided data, we compute that* $\theta_1 = 1.92, \theta_2 = 1.52, H \approx 0.0091, G \approx 128.3282, \Psi \approx 0.0791, \Delta \approx -37360.0091$, *and* $\Upsilon = H\Psi - G\Delta \approx 4794342.721$.

*Consider the functions*

$$g\left(\xi, \mathfrak{F}, {}^{H}D^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\mathfrak{F}\right) = \frac{1}{\sqrt{500+\xi^3}} + \frac{\mathfrak{F}}{\xi^3+1000} + \frac{{}^{H}D^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\mathfrak{F}}{20},$$

$$h\left(\xi, \vartheta, {}^{H}D^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\vartheta\right) = \frac{|\vartheta|}{\xi(500+|\vartheta|)} + \frac{\cos\left({}^{H}D^{\frac{6}{5},\frac{4}{5};\frac{\sin(\xi)}{3}}\vartheta\right)}{10}.$$

*For* $\vartheta_1, \bar{\vartheta}_1, \mathfrak{y}_1, \bar{\mathfrak{y}}_1, \vartheta_2, \bar{\vartheta}_2, \mathfrak{y}_2, \bar{\mathfrak{y}}_2 \in \mathbb{R}$ *and* $\xi \in [0,1]$, *we have the following inequalities:*

$$\|g(\xi, \mathfrak{y}_1, \bar{\mathfrak{y}}_1) - g(\xi, \mathfrak{y}_2, \bar{\mathfrak{y}}_2)\| \leq \frac{1}{2}\left(\|\mathfrak{y}_1 - \mathfrak{y}_2\| + \|\bar{\mathfrak{y}}_1 - \bar{\mathfrak{y}}_2\|\right),$$

$$\|h(\xi, \vartheta_1, \bar{\vartheta}_1) - h(\xi, \vartheta_2, \bar{\vartheta}_2)\| \leq \frac{1}{3}\left(\|\vartheta_1 - \vartheta_2\| + \|\bar{\vartheta}_1 - \bar{\vartheta}_2\|\right).$$

*From assumption* (*H*1), *we obtain* $\mathfrak{L}_1 = \frac{1}{2}$ *and* $\mathfrak{L}_2 = \frac{1}{3}$. *Therefore, all the conditions of Theorem 4.2 are satisfied. Moreover, if there exists a function* $\varphi = (\varphi_1, \varphi_2)$, *where* $\varphi_i \in C([0,1], \mathbb{R}_+)$ *for* $i = 1, 2$, *satisfying assumption* (*H*2), *then the problem* (12) *is both Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable on the interval* $[0,1]$ *with respect to* $\varphi$.

## 6. Conclusion

This study delves into the stability analysis of coupled systems of $\psi$–Hilfer nonlinear implicit fractional differential equations with multipoint boundary conditions. By establishing results on Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability , the paper addresses significant gaps in the existing literature, offering a comprehensive framework for understanding these complex systems. The theoretical findings are substantiated with illustrative examples, reinforcing their applicability and relevance.

The results not only enhance the theoretical foundation of fractional differential systems but also pave the way for future research, particularly in addressing the stability of nonlinear coupled systems. This work marks a significant step forward in advancing the understanding of fractional systems, providing a valuable resource for both researchers and practitioners in this evolving field.

### Ethics declarations

### Conflict of interest

The authors declare that they have no conflicts of interest.

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