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On some properties of the class of charming spaces

Xing-Yu Hu[a,*], Pei Zhang[b]

[a]School of Mathematics and Statistics, Huangshan University, Huangshan Anhui, 245021 China
[b]School of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, 450000 China

Abstract. We study the properties of the class of charming spaces. It is proved that if X is the preimage of a metrizable locally Lindelöf p-space(respectively, locally s-space) under a perfect mapping, then every remainder $bX \setminus X$ of X in any compactification bX is 1-strong charming(respectively, charming). Some corollaries related to this statement are presented. It is shown that if X is a metrizable space, and X is a locally Lindelöf p-space(respectively, locally s-space), then for any compactification bX of X, the remainder $bX \setminus X$ of X is 1-strong charming (respectively, charming). It is also proved that if X is a nowhere locally compact metrizable space, then X is a locally s-space (respectively, locally Lindelöf p-space) if and only if for any (or some) compactification bX of X, the remainder $bX \setminus X$ of X is charming (respectively, 1-strong charming). Some related propositions are proved within this section. In addition, some properties of s-space are investigated.

1. Introduction

All spaces in this article are Tychonoff spaces unless stated otherwise. A *compactification* of a space X is any compact space bX containing X as a subspace, such that X is dense in bX [7]. The *remainder* of a space X is the subspace $bX \setminus X$ of a compactification bX of X. Recall that a *paratopological group* is a group with a topology such that the multiplication on the group is jointly continuous. A *topological group* G is a paratopological group such that the inverse mapping of G into itself that associates x^{-1} with $x \in G$ is continuous [6]. Recall that a space X is of *countable type* if every compact subspace B of X is contained in a compact subspace $F \subset X$, that has a countable base of open neighborhoods in X [11]. All s-spaces and metrizable spaces are of countable type. A space X is of *pointwise countable type* if, for every point $x \in X$, there exists a compact set $F(x) \subset X$ such that $x \in F(x)$ and $F(x)$ has a countable base of open neighborhoods in X [7]. It is obvious that if a space X is of countable type, then X is of pointwise countable type. M. Henriksen and J. Isbell proved that a Tychonoff space X is of countable type if and only if the remainder in any (or some) Hausdorff compactification of X is Lindelöf, as shown in [11]. Recall that the subset A of the space X is called *G_δ-set(F_σ-set)* if A is the intersection (respectively, union) of countable open (respectively, closed) set [7]. The space X has a *G_δ-diagonal (respectively, G^*_δ-diagonal)* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\bigcap_{n\in\omega} st(x, \mathcal{U}_n) = \{x\}$(respectively, $\bigcap_{n\in\omega} \overline{st(x, \mathcal{U}_n)} = \{x\}$) for each $x \in X$, where

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* Corresponding author: Xing-Yu Hu
Email addresses: 544297848@qq.com (Xing-Yu Hu), 121602580@qq.com (Pei Zhang)

$st(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$ [7]. In [3], Arhangel'skii proved that if a metrizable space X has a remainder Y with a G_δ-*diagonal*, then both X and Y are separable and metrizable.

Recall that a space X is a *p-space* if, in any (or in some) compactification bX of X, there exists a countable family $\xi = \{\gamma_n : n \in \omega\}$ of families γ_n of open subsets of bX such that $x \in \bigcap\{St(x, \gamma_n) : n \in \omega\} \subset X$ for each $x \in X$ [1]. It was shown in [1] that every p-space is of countable type, and that every metrizable space is a p-space. A mapping is said to be *perfect* if it is continuous, closed, and all fibers are compact [7]. A paracompact p-space is a preimage of a metrizable space under a perfect mapping [10]. A.V. Arhangel'skii [10] proved that a *paracompact p-space* is a preimage of a metrizable space under a perfect mapping. A *Lindelöf p-space* is a preimage of a separable and metrizable space under a perfect mapping [10]. A.V. Arhangel'skii [3] proved that if X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space. The notion of Σ-*space* was introduced in [12]. Recall that a space X is a *Lindelöf* Σ-*space* if and only if X is a continuous image of a Lindelöf p-space [13]. In [3], Arhangel'skii proved that if Y is a remainder of a paracompact p-space such that for every $y \in Y$, y is a G_δ-point in Y, then Y is a Lindelöf Σ-space. If a remainder Y of a paracompact p-space has a G_δ-diagonal, then Y is metrizable [5]. Suppose that G is a topological group and H is a closed subgroup of G. Then G/H stands for the quotient space of G that consists of left cosets xH, where $x \in G$. The spaces G/H so obtained are called *coset spaces* [6]. The space G/H need not be homeomorphic to a topological group, but it is always homogeneous and Tychonoff (a space X is called *homogeneous* if for each pair x, y of points in X, there exists a homeomorphism h of X onto itself such that $h(x) = y$)[6, P.1]) . A coset space G/H is called *compactly-fibered* if H is compact [6].

We provide some basic definitions of *source* and *s-space* from [4].

Suppose that $\mathcal{S}$ is a family of subsets of a space X. Let $\mathcal{S}_\delta$ denote the family of all sets that can be represented as the intersection of some nonempty subfamily of $\mathcal{S}$, and let $\mathcal{S}_{\delta,\sigma}$ denote the family of all sets that can be represented as the union of some subfamily of $\mathcal{S}_\delta$. $\mathcal{S}$ is called a source for a space Y in X if Y is a subspace of X such that $Y \in \mathcal{S}_{\delta,\sigma}$ [4]. Furthermore, a source $\mathcal{S}$ for Y in X is called open (respectively, closed) source if every member of $\mathcal{S}$ is an open (respectively, closed) subset of the space X [4]. A source $\mathcal{S}$ is *countable* if $\mathcal{S}$ is countable [4].

A space X is called an s-space if there exists a countable open source for X in some (respectively, every) compactification bX of X [4]. According to [4], a space X is called a Lindelöf Σ-space if there exists a countable closed source for X in some (respectively, every) compactification bX of X. It is obvious that s-spaces are open (respectively, closed) hereditary, Lindelöf Σ-spaces are closed hereditary, and all compact spaces are s-spaces. Let X and Y be subspaces of a space Z, and let γ be a family of subsets of Z such that for any distinct x, y, where $x \in X$ and $y \in Y$, there exists $P \in \gamma$ such that $x \in P$ and $y \notin P$. Then we say that γ is a T_0-*separator* in Z for the pair (X, Y) [4]. It follows from [4] that a space X is an s-space if and only if for any compactification bX of X, there exists a countable open T_0-separator γ in bX for the pair $(X, bX \setminus X)$. Clearly, a space X is a Lindelöf Σ-space if and only if for any compactification bX of X, there exists in bX a countable closed T_0-separator γ for the pair $(X, bX \setminus X)$. In [4], Arhangel'skii proved that every Čech-complete space and separable metrizable space are s-spaces. In [4], Arhangel'skii also proved that the class of s-spaces (respectively, Lindelöf Σ-spaces) is preserved by perfect mappings in both directions. It is obvious that the class of Lindelöf p-spaces is preserved by perfect mappings in both directions by Filippov's theorem in [9]. In addition, a space X is a Lindelöf p-space if and only if it is both a Lindelöf Σ-space and an s-space [4]. Clearly, all Lindelöf p-spaces are closed hereditary. In [4], Arhangel'skii also proved that if X is a nowhere locally compact space with a remainder Y, then X is a Lindelöf Σ-space if and only if Y is an s-space.

The above conclusion does not integrate the remainders with π-base and the related properties of locally Lindelöf Σ-spaces, locally Lindelöf p-spaces, and the locally s-spaces to discuss their properties as a whole. Inspired by this idea, in this article, the above result is generalized. First, we use the concept of T_0-separator to prove the conclusion in [12] again. The next step is to discuss the related properties of the coset spaces, as well as the properties of locally Lindelöf Σ-spaces, locally Lindelöf p-spaces, and locally s-spaces. We show that if X is the topological product of a family $\{X_\alpha : \alpha \in \omega\}$ of Lindelöf p-coset-spaces $X_\alpha = G_\alpha/H_\alpha$ where H_α is a compact subgroup of a topological group G_α and each X_α is a nowhere locally compact space with a remainder Y_α, for each $\alpha \in \omega$, then X is also a Lindelöf p-space. It is also proved that if X is a nowhere locally compact space with a remainder Y and X is a locally Lindelöf p-space with a compactification bX

such that the remainder $Y = bX \setminus X$ is a locally perfect, then Y and X are all locally Lindelöf p-spaces. Lastly, we combine the properties of the remainders, the π-base, and the properties of locally Lindelöf p-spaces, Lindelöf p-spaces, and metrizable spaces to discuss their properties as a whole. It is proved that if X is a nowhere locally compact locally Lindelöf p-space and metrizable, and X has a locally perfect remainder $bX \setminus X$ with the properties that every closed Lindelöf p-subspace of $bX \setminus X$ is metrizable and every compact subset of $bX \setminus X$ is a G_δ-set of $bX \setminus X$, then X and $bX \setminus X$ are all separable and metrizable spaces. Additionally, further properties of s-spaces are also explored in this context.

The set of all positive integers is denoted by $\mathbb{N}$, and w is $\mathbb{N} \cup \{0\}$. In notation and terminology we will follow [7].

2. Main results

Lemma 2.1 and Lemma 2.2 can be obtained from [12]. We use other methods to prove them again.

Lemma 2.1. ([12]) *If a space X is the union of a countable family η of its closed Lindelöf Σ-subspaces, then X is a Lindelöf Σ-space.*

Proof. Assume that $X = \bigcup_{i\in\omega} X_i$, where every X_i is a closed Lindelöf Σ-subspace of X. Let bX be a compactification of X and let bX_i be the closure of X_i in bX for each $i \in \omega$. According to the conditions given in the passage, each X_i is a closed subset of X. It follows that there exists a closed subset K_i of bX such that $K_i \cap X = X_i$. It is obvious that $\overline{K_i}^{bX} = \overline{X_i}^{bX} = bX_i$, which implies that K_i is contained in bX_i. Since each X_i is a Lindelöf Σ-space, there exists a countable closed source $\mathcal{S}_i$ for X_i in bX_i.

Let $\mathcal{O}_i = \{S \cap K_i : S \in \mathcal{S}_i\}$ for each $i \in \omega$. Clearly, the $\mathcal{O}_i$ is a countable closed source for X_i in bX. Therefore $\bigcup_{i\in\omega} \mathcal{O}_i$ is a countable family of closed subsets of bX. To this end, take any two points y, z such that $y \in X$ and $z \in bX \setminus X$. There exists $P \in \bigcup_{i\in\omega} \mathcal{O}_i$ such that $y \in P$ and $z \notin P$. Hence $\bigcup_{i\in\omega} \mathcal{O}_i$ is a countable closed T_0-separator in bX for the pair $(X, bX \setminus X)$. Hence, the space X is a Lindelöf Σ-space. □

Lemma 2.2. ([12]) *If a space X is the union of a countable family γ of its Lindelöf Σ-subspaces, then X is a Lindelöf Σ-space.*

Proof. By the assumption, let $\gamma = \{X_i : i \in \omega\}$, where each X_i is a Lindelöf Σ-subspace. In addition, let Y be the sum space of γ, i.e., $Y = \bigoplus_{i\in\omega} X_i$. According to the Lemma 2.1, since each X_i is closed in X, Y is a Lindelöf Σ-space.

Take $X = \bigcup X_i$ and let $g : Y \to X$ be the natural mapping that restricts to the identity on each X_i for every $i \in \omega$. It is obvious that g is a continuous mapping. By [3], the image of a Lindelöf Σ-space under a continuous mapping is a Lindelöf Σ-space. Hence, X is a Lindelöf Σ-space. □

Since the union of countable s-spaces may not be an s-space, according to [14] and [4, Corollary 6.7], we can derive the following Proposition.

Proposition 2.3. *If a space X is the union of a countable family γ of its Lindelöf p-subspaces, then X is a Lindelöf Σ-space.*

According to [14, Corollary 3.1], we can derive the following Corollary.

Corollary 2.4. *If a space X is the union of a finite family γ of its closed (open) Lindelöf p-subspaces, then X is a Lindelöf p-space.*

Next, we discuss the related properties of coset spaces, locally Lindelöf Σ-spaces, locally Lindelöf p-spaces, and locally s-spaces.

Lemma 2.5. *The topological product of any countable family β of Lindelöf p-spaces is an s-space.*

Proof. Let $\beta = \{X_i : i \in \omega\}$ and let bX_i be a compactification of topological space X_i for each $i \in \omega$. Let $X = \prod_{i\in\omega} X_i$ and $D = \prod_{i\in\omega} bX_i$. Clearly, D is a compactification of X. Let $Y = D \setminus X$ and $M_i(i) = b_iX_i \setminus X_i$ for each $i \in \omega$. Let $M_j(i) = b_jX_j$ for each $i, j \in \omega$, $j \neq i$. Clearly, the topological product of the family $\{M_j(i), j \in \omega\}$ is a Lindelöf p-space. We name this space N_i. It is obvious that the union of the N_i is the space Y. By Proposition 2.3, the space Y is a Lindelöf Σ-space. Hence, the space X is an s-space by [4, Theorem 2.7]. □

Lemma 2.6. ([11]) *A space X is of countable type if and only if the remainder in any (or in some) compactification of X is Lindelöf.*

Proposition 2.7. *Suppose that X is the topological product of any countable family $\{X_\alpha : \alpha \in \omega\}$ of Lindelöf p-spaces X_α and X_α is an image of a topological group G_α under a perfect mapping h_α, for each $\alpha \in \omega$, then X is an s-space.*

Proof. Since the preimage of any Lindelöf p-space under a perfect mapping is also a Lindelöf p-space by Filippov's theorem in [2, Theorem 2.1], G_α is also a Lindelöf p-space. Let $G = \prod_{\alpha\in\omega} G_\alpha$, by Lemma 2.5, the space G is an s-space. By the assumption, the product of mappings h_α, $\alpha \in \omega$, is also a perfect mapping of G onto X. Hence, the space X is also an s-space by [4]. □

By [4], both the Lindelöf Σ-space and s-space are preserved by perfect mappings in both directions. We can get the following Proposition by using similar proof methods.

Proposition 2.8. *Suppose that X is the topological product of any countable family $\{X_\alpha : \alpha \in \omega\}$ of s-spaces X_α such that X_α is an image of a topological group G_α under a perfect mapping h_α, for each $\alpha \in \omega$, then X is also an s-space.*

Corollary 2.9. *If X is the topological product of a family $\{X_\alpha : \alpha \in \omega\}$ of Lindelöf p-coset-spaces $X_\alpha = G_\alpha/H_\alpha$ where H_α is a compact subgroup of a topological group G_α and each X_α is a nowhere locally compact space with a remainder Y_α, for each $\alpha \in \omega$, then X is also a Lindelöf p-space.*

Proof. By [6, Theorem 1.5.7], each natural mapping of G_α onto X_α is a perfect. Hence the conclusion is obvious by Proposition 2.7. □

By Proposition 2.8, We can get the following Proposition by using the similar proof methods.

Proposition 2.10. *If X is the topological product of a family $\{X_\alpha : \alpha \in \omega\}$ of s-coset-spaces $X_\alpha = G_\alpha/H_\alpha$ where H_α is a compact subgroup of a topological group G_α, for each $\alpha \in \omega$, then X is an s-space.*

Proposition 2.11. *Suppose that $X = G/H$ is a compactly-fibered coset space and X contains a dense Lindelöf p-space Z, then the Souslin number of X is countable.*

Proof. According to the condition, we can assume that f be the natural mapping of G onto G/H. Since the space H is a compact space, then the f is perfect and open, the space $Y = f^{-1}(Z)$ is a Lindelöf p-space by Filippov's theorem in [9]. In addition, the space Y is dense in G. By [6, Chapter 5, Section 7], we know that the Souslin number of Y is countable. Since the mapping f is perfect, then it is continuous. It is not difficult to verify that the Souslin number of Z is countable. Therefore $c(X) \leq \omega$, since the subspace Z is dense in X. □

Proposition 2.12. *Let $g : X \to Y$ be a perfect mapping. Then X is a locally Lindelöf Σ-space if and only if Y is a locally Lindelöf Σ-space.*

Proof. Sufficiency. Let X be a locally Lindelöf Σ-space. For each $x \in g^{-1}(y)$, one can fix a closed neighbourhood M_x of x in X such that M_x is a Lindelöf Σ-space. Since the $g : X \to Y$ is a perfect mapping, $g^{-1}(y)$ is a compact subset of X for the any point $y \in Y$. By the assumption, there exists a finite set F such that $g^{-1}(y) \subset \bigcup_{x\in F} M_x$. According to the Lemma 2.2, $\bigcup_{x\in F} M_x$ is a Lindelöf Σ-space. Since g is closed, one can fix an open neighbourhood O_y of y in Y such that $g^{-1}(O_y) \subset (\bigcup_{x\in F} M_x)^o$. Since $g^{-1}(\overline{O_y})$ is a closed subspace of $\bigcup_{x\in F} M_x$, $g^{-1}(\overline{O_y})$ is a Lindelöf Σ-space. By the assumption, the restriction of g on $g^{-1}(\overline{O_y})$ is perfect. It follows that $\overline{O_y}$ is a Lindelöf Σ-space. Hence, Y is a locally Lindelöf Σ-space.

Necessity. Assume that Y is a locally Lindelöf Σ-space, and x is an arbitrary point of X. One can fix a closed neighbourhood $N_{g(x)}$ of $g(x)$ in Y such that $N_{g(x)}$ is a Lindelöf Σ-space. By the assumption, the restriction of g on $g^{-1}(N_{g(x)})$ is perfect. By [4], $g^{-1}(N_{g(x)})$ is also a Lindelöf Σ-space. Since $g^{-1}(N_{g(x)})$ is a neighbourhood of x in X, X is a locally Lindelöf Σ-space. □

By Filippov's Theorem in [9] and Corollary 2.4, the Lindelöf p-space is preserved by perfect mappings in both directions. We can get the following Proposition by using similar proof methods.

Proposition 2.13. *Let $g : X \to Y$ be a perfect mapping. If X is a closed (open) locally Lindelöf p-space, then Y is a locally Lindelöf p-space.*

Proposition 2.14. *Let $g : X \to Y$ be a perfect mapping. If Y is a locally Lindelöf p-space, then X is also a locally Lindelöf p-space.*

Theorem 2.15. *If a space X is the union of a family γ of closed locally Lindelöf Σ-spaces and γ is locally finite in X, then X is a locally Lindelöf Σ-space.*

Proof. Take $\gamma = \{Y_i : i \in \Gamma\}$, where each Y_i is a locally Lindelöf Σ-space, and let Y be the sum space of γ, i.e., $Y = \bigoplus_{i\in\Gamma} Y_i$. It is obvious that Y is a locally Lindelöf Σ-space. Take $g : Y \to X$ be the natural mapping that restricts to the identity on each Y_i for every $i \in \omega$. By the assumption, γ is locally finite and every $Y_i \in \gamma$ is closed in X. Hence g is perfect. By Proposition 2.12, X is a locally Lindelöf Σ-space. □

By [4], the Lindelöf Σ-space and s-space are preserved by perfect mappings in both directions. We can get the following Proposition by using the similar proof methods.

Proposition 2.16. *If a space X is the union of a family γ of closed locally s-spaces and γ is locally finite in X, then X is a locally s-space.*

By Filippov's theorem in [9], the Lindelöf p-space is preserved by perfect mappings in both directions. We can get the following Proposition by using similar proof methods.

Proposition 2.17. *If a space X is the union of a family γ of closed locally Lindelöf p-spaces and γ is locally finite in X, then X is a locally Lindelöf p-space.*

Theorem 2.18. ([8, Lemma 2.6]*) If X is a Lindelöf space with the property that for every $x \in X$ there exists an open neighborhood V_x such that $\overline{V_x}$ is a Lindelöf p-space, then X is a Lindelöf p-space.*

A space X is called *charming* [3] if there exists a subspace Z of X such that Z is a Lindelöf Σ-space of X(called a Lindelöf Σ-kernel of X) and $X \setminus U$ is a Lindelöf Σ-space, for each open neighbourhood U of Z in X. In [3], Arhangel'skii proved that any image of a charming space under a continuous mapping is a charming space and any preimage of a charming space under a perfect mapping is a charming space. In [3], Arhangel'skii also proved that for every metrizable space X and every compactification bX of X, the remainder $bX \setminus X$ is a charming space.

Inspired by this concept, we will introduce some new spaces. A space X will be called *strong charming* if there exists a subspace Z of X such that Z is a compact subspace(called a compact kernel of X) and $X \setminus U$ is a Lindelöf p-space, for each open neighbourhood U of Z in X.

A space X will be called *1-strong-charming* if there exists a Lindelöf Σ-subspace Y of X(called a Lindelöf Σ-kernel of X) such that $X \setminus U$ is a Lindelöf p-space, for each open neighbourhood U of Y in X.

A space X will be called *2-charming* if there exists a subspace Z of X such that Z is a Lindelöf Σ-space(called a Lindelöf Σ-kernel of X) and $X \setminus U$ is an s-space, for each open neighbourhood U of Z in X.

We can give many such similar spaces and unify calling them as the *class of charming spaces*. Next, we investigate some properties of the class of charming spaces and discuss the relationship between the class of charming spaces and remainders.

According to [3], the next conclusions are obvious, we omit them:

Proposition 2.19. *Every strong-charming space is a 1-strong-charming space.*

Proposition 2.20. *Every strong-charming space is a charming space.*

Proposition 2.21. *Every 1-strong-charming space is a charming space and also a 2-charming space.*

Proposition 2.22. *Every 1-strong-charming space is Lindelöf.*

Proposition 2.23. *Any image of a 1-strong-charming space under a continuous mapping is a 1-strong-charming space.*

Proposition 2.24. *Any image of a strong-charming space under a continuous mapping is a strong-charming space.*

Proposition 2.25. *If a nowhere locally compact Lindelöf space X has a remainder homeomorphic to a topological group, then X is 1-strong-charming.*

Proposition 2.26. *Every 1-strong-charming topological group G has a dense subgroup that is a Lindelöf Σ-space.*

Proposition 2.27. *The Suslin number of an arbitrary 1-strong-charming topological group G is countable.*

The next we give some properties of charming space, strong charming space and 1-strong-charming space.

Lemma 2.28. *The topological product of any countable family γ of strong-charming spaces is a charming space.*

Proof. Let $\beta = \{X_i : i \in \omega\}$ and $X = \prod_{i\in\omega} X_i$. Since each X_i is a strong-charming space, one can fix a compact subspace $A_i \subset X_i$ such that the $X_i \setminus U_i$ is also a Lindelöf p-space for the each open neighbourhood U_i of A_i. Let $A = \prod_{i\in\omega} A_i$. Clearly, A is also a compact space. Hence A is a σ-compact space. Meanwhile, A is also a Lindelöf Σ-space. In addition, Let $U = \prod_{i\in L} U_i \times \prod_{i\in\omega\setminus L} X_i$, where $L \subset \omega$ is a finite set. It is obvious that U is an any canonical open neighbourhood of A. Hence, the $X \setminus U = \prod_{i\in L}(X_i \setminus U_i)$ is a Lindelöf Σ-space by [4, Corollary 6.7]. Hence X is a charming space. □

Theorem 2.29. *Let $\{X_i : i \in \omega\}$ be a sequence of topological spaces. If all spaces X_i are locally strong charming spaces, and there exists a finite set $L \subset \omega$ such that each X_i is a strong charming space for $i \in \omega \setminus L$, then the topological product space $\prod_{i\in\omega} X_i$ is a locally charming space.*

Proof. By the assumption, one can fix a neighbourhood U_i of $x \in X_i$ which is a strong charming space for each $i \in \omega$. Since each X_i is a strong charming space for $i \in \omega \setminus L$, by Lemma 2.28, the topological product $\prod_{i\in L} U_i \times \prod_{i\in\omega\setminus L} X_i$ is a charming space. Since the $\prod_{i\in L} U_i \times \prod_{i\in\omega\setminus L} X_i$ is a neighbourhood of $\{x_i\}$ in $\prod_{i\in\omega} X_i$, $\prod_{i\in\omega} X_i$ is a locally charming space. □

Proposition 2.30. *If a space X is the union of a charming space and countable family η of its Lindelöf p-spaces, then X is also a charming space.*

Proof. Let $\eta = \{Y_i : i \in \omega\}$, suppose that $X = X_1 \cup \bigcup_{i\in\omega} Y_i$, where X_1 is a charming subspace of X and each Y_i is a Lindelöf p-space. Let Z be a Lindelöf Σ-subspace of X_1 for each $i \in \omega$, by the assumption, the $X_1 \setminus U$ is a Lindelöf Σ-subspace of X_1 for each open neighbourhood U of Z in X_1. According to the Proposition 2.3, $\bigcup_{i\in\omega} Y_i$ is a Lindelöf Σ-subspace of X, hence, $X \setminus U = (X_1 \setminus U) \cup \bigcup_{i\in\omega} Y_i$ is also a Lindelöf Σ-subspace of X. Hence X is also a charming space. □

Corollary 2.31. *If a space X is the union of a charming space and countable family η of its Lindelöf Σ-spaces, then X is also a charming space.*

Theorem 2.32. *Let $f : X \to Y$ be a perfect mapping. Then X is 2-charming if and only if Y is also 2-charming.*

Proof. Sufficiency. Suppose that Y is a 2-charming space and fix a Lindelöf Σ-space D of Y. Since the mapping g is perfect, the set $g^{-1}(D)$ is a Lindelöf Σ-subspace of X by the [4]. Take any open subset O of X such that $g^{-1}(D) \subset O$. Then $X \setminus O$ is closed in X and $X \setminus O \subset g^{-1}(g(X \setminus O))$. Since $X \setminus O$ is closed and $(X \setminus O) \cap g^{-1}(D) = \emptyset$, the set $g(X \setminus O)$ is also closed and $g(X \setminus O) \cap D = \emptyset$. Then $g(X \setminus O)$ is a s-subspace of Y by the definition of 2-charming space. It follows that $g^{-1}(g(X \setminus O))$ is a s-subspace of X by [4]. Since s-space is open(respectively, closed) hereditary. Hence $X \setminus O$ is also an s-space. Clearly, X is a 2-charming space.

Necessity. Suppose that X is a 2-charming space and fix a Lindelöf Σ-space M of X. Since the mapping g is continuous, the set $g(M)$ is a Lindelöf Σ-space of Y.

Take any open subset V of Y such that $g(M) \subset V$, by the assumption, $X \setminus f^{-1}(V)$ is an s-space. Since the mapping $h = g|(X \setminus g^{-1}(V)) : X \setminus g^{-1}(V) \to Y \setminus V$ is a perfect mapping, by [4], $Y \setminus V$ is also an s-space. Thus Y is 2-charming.

□

By [4], Lindelöf Σ-space and s-space are preserved by perfect mappings in both directions. We can get the following Proposition by using the similar proof methods.

Proposition 2.33. *Let $f : X \to Y$ be a perfect mapping. Then X is charming if and only if Y is charming.*

By Filippov's theorem in [9], the Lindelöf p-space is preserved by perfect mappings in both directions. We can get the following Proposition by using similar proof methods.

Proposition 2.34. *Let $f : X \to Y$ be a perfect mapping. Then X is 1-strong-charming if and only if Y is 1-strong-charming.*

Corollary 2.35. *Let X be a space. Then there exists a compactification bX of X such that the remainder $bX \setminus X$ of X is 2-charming if and only if for any compactification $c(X)$ of X, the remainder $c(X) \setminus X$ of X is 2-charming.*

Proof. The sufficiency is clear.

Necessity. By the assumption, the remainder $bX \setminus X$ of X is 2-charming. The remainder $\beta X \setminus X$ of X in the Čech-Stone compactification βX is a perfect preimage of $bX \setminus X$. According to the Theorem 2.32, $\beta X \setminus X$ is 2-charming and a remainder Y of X in arbitrary compactification $c(X)$ of X is an image of $\beta X \setminus X$ under a perfect mapping. Clearly, Y is 2-charming by Theorem 2.32. □

By Corollary 2.35, Proposition 2.33 and Proposition 2.34, we can get the following Proposition by using similar proof methods.

Proposition 2.36. *Let X be a space. Then there exists a compactification bX of X such that the remainder $bX \setminus X$ of X is charming if and only if for any compactification $c(X)$ of X, the remainder $c(X) \setminus X$ of X is charming.*

Proposition 2.37. *Let X be a space. Then there exists a compactification bX of X such that the remainder $bX \setminus X$ of X is 1-strong-charming if and only if for any compactification $c(X)$ of X, the remainder $c(X) \setminus X$ of X is 1-strong-charming.*

By an argument similar to the proofs of Corollary 2.35, Proposition 2.36, and Proposition 2.37, we have the following result.

Proposition 2.38. *Let $f : X \to Y$ be a perfect mapping. Let bX and bY be compactifications of X and Y, respectively. Then the remainder $bX \setminus X$ of X is charming if and only if the remainder $bY \setminus Y$ of Y is charming.*

Proposition 2.39. *Let $f : X \to Y$ be a perfect mapping. Let bX and bY be compactifications of X and Y, respectively. Then the remainder $bX \setminus X$ of X is 2-charming if and only if the remainder $bY \setminus Y$ of Y is 2-charming.*

Proposition 2.40. *Let $f : X \to Y$ be a perfect mapping. Let bX and bY be compactifications of X and Y, respectively. Then the remainder $bX \setminus X$ of X is 1-strong-charming if and only if the remainder $bY \setminus Y$ of Y is 1-strong-charming.*

Theorem 2.41. *Let X be a metrizable space. If X is a locally Lindelöf p-space, then for any compactification bX of X the remainder $bX \setminus X$ of X is 1-strong charming.*

Proof. Since the space X is a metrizable space, hence one can fix a completion M of X such that M is a Čech-complete space. In addition, the space X is a dense subspace of M. Let B be any compactification of M, it is obvious that the space B is a compactification of M and the space $B \setminus M$ is a remainder of M in B. Hence the space $B \setminus M$ is σ-compact space, since the space M is a Čech-complete space. Since the space $B \setminus X$ is also a remainder of X in B and $B \setminus M \subset B \setminus X$. Therefore, by [3], the space $B \setminus M$ is a Lindelöf Σ-subspace of $B \setminus X$.

Let O be any open neighbourhood of $B \setminus M$, then $(B \setminus X) \setminus O$ is closed in $B \setminus X$. Hence X is of countable type, since X is metrizable. By Lemma 2.6, the remainder $B \setminus X$ is Lindelöf. Hence the space $(B \setminus X) \setminus O$ is Lindelöf.

By assumption, one can fix an open neighbourhood O_y of $y \in ((B \setminus X) \setminus U)$ such that $\overline{O_y} \bigcap (B \setminus M) = \emptyset$. Since X is metrizable and locally Lindelöf p-space, hence by Proposition 2.17 $X = \bigcup_{\alpha \in \Lambda} X_\alpha$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a locally finite family in X and each X_α is closed locally Lindelöf p-space. Clearly, one can fix a finite set $C \subset \Lambda$ such that $O_y \cap X = \bigcup\{O_y \cap X_i : i \in C\}$. Hence $O_y \cap X$ is Lindelöf p-space and metrizable space and $\overline{O_y \cap X}^{(X)}$ is also Lindelöf p-space and metrizable.

Since the space $\overline{\overline{O_y \cap X}^{(X)}} \setminus \overline{O_y \cap X}^{(X)}$ is a remainder of $\overline{O_y \cap X}^{(X)}$. Hence the remainder $\overline{\overline{O_y \cap X}^{(X)}} \setminus \overline{O_y \cap X}^{(X)}$ is a Lindelöf p-subspace of $bX \setminus X$ by the [2, Theorem 2.1]. Therefore, the space $\overline{O}_y \cap ((bX \setminus X) \setminus O) = (\overline{O}_y \setminus X) \cap ((bX \setminus X) \setminus O)$ is also a Lindelöf p-space. It is obvious that $(bX \setminus X) \setminus O$ is a locally Lindelöf p-space and $(bX \setminus X) \setminus O$ is a Lindelöf subspace of $bX \setminus X$. By Theorem 2.18, the space $(bX \setminus X) \setminus O$ is also a Lindelöf p-subspace of $bX \setminus X$. Clearly, the remainder $bX \setminus X$ is a 1-strong-charming space. □

By [4, Theorem 2.7], we can get the following Corollary by using similar proof methods.

Corollary 2.42. *Let X be a metrizable space. If X is a locally s-space, then for any compactification bX of X the remainder $bX \setminus X$ of X is charming.*

By Propositions 2.36, 2.37, 2.38, 2.40, Theorem 2.41, and Corollary 2.42, we have the following result.

Corollary 2.43. *If X is preimage of a metrizable locally Lindelöf p-space under a perfect mapping, then every remainder $bX \setminus X$ of X in any compactification bX is 1-strong charming.*

Corollary 2.44. *If X is preimage of a metrizable locally s-space under a perfect mapping, then every remainder $bX \setminus X$ of X in any compactification bX is charming.*

Theorem 2.45. *If X is a nowhere locally compact metrizable space and there exists a compactification bX of X such that the remainder $bX \setminus X$ is charming, then X is a locally s-space.*

Proof. By the assumption, one can fix a Lindelöf Σ-subspace B of $bX \setminus X$ such that for any open neighbourhood U of B in $bX \setminus X$, the set $(bX \setminus X) \setminus U$ is a Lindelöf Σ-subspace of $bX \setminus X$.

In addition, one can fix an open subset U_x of bX such that $x \in U_x$ and $\overline{U_x} \cap B = \emptyset$ for any point $x \in X$. Since X is a nowhere locally compact, the space $U_x \cap (bX \setminus X) \neq \emptyset$ and the space $\overline{U}_x = \overline{U_x \cap (bX \setminus X)}$ is a compactification of the $\overline{U_x \cap (bX \setminus X)}^{(bX \setminus X)}$ of $bX \setminus X$. Clearly, the space $\overline{U_x \cap (bX \setminus X)}^{(bX \setminus X)} \cap B = \emptyset$, since $\overline{U_x} \cap B = \emptyset$. It implies that $\overline{U_x \cap (bX \setminus X)}^{(bX \setminus X)}$ is a Lindelöf Σ-subspace of $bX \setminus X$, since $\overline{U_x \cap (bX \setminus X)}^{(bX \setminus X)}$ is a closed subspace of $bX \setminus X$.

By [4, Corollary 2.9], the space $\overline{U_x} \setminus \overline{U_x \cap (bX \setminus X)}^{(bX \setminus X)} = \overline{U_x} \setminus (\overline{U_x} \cap (bX \setminus X)) \subset X$ is an s-space. Hence, $\overline{U_x} \setminus (\overline{U_x} \cap (bX \setminus X)) = \overline{U_x \cap X}^{(X)}$ is an s-space. Clearly, $U_x \cap X$ is an s-space. Therefore X is a locally s-space. □

By [2, Theorem 2.1], we can get the following Proposition by using similar proof methods.

Proposition 2.46. *If X is a nowhere locally compact metrizable space and there exists a compactification bX of X such that the remainder $bX \setminus X$ is 1-strong-charming, then X is a locally Lindelöf p-space.*

By Theorem 2.41, Corollary 2.42, Theorem 2.45 and Proposition 2.46. We have the following result.

Corollary 2.47. *Let X be a nowhere locally compact metrizable space. Then X is a locally s-space if and only if for any (or some) compactification bX of X, the remainder $bX \setminus X$ of X is charming.*

Corollary 2.48. *Let X be a nowhere locally compact metrizable space. Then X is a locally Lindelöf p-space if and only if for any (or some) compactification bX of X, the remainder $bX \setminus X$ of X is 1-strong charming.*

By [3, Theorem 5.7], we have the following result.

Proposition 2.49. *Every Lindelöf remainder of a compactly-fibered coset space is a charming space.*

Recall that a family $\mathcal{U}$ of non-empty open subsets of a space X is called a *π-base* of a point $x \in X$, if for any non-empty open neighborhood V of x there is $U \in \mathcal{U}$ such that $U \subset V$. The *π-character* of x in X is defined by $\pi_X(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a π-base of the point $x\}$. If $\sup\{\pi_X(x, X)\colon x \in X\}$ is countable, then X is called to have *countable π-character*.

Lastly, we combine the remainders with the π-base and the locally Lindelöf p-space, Lindelöf p-space, metrizable to discuss their properties as a whole.

Theorem 2.50. *(*[14, Theorem 4.6]*) Suppose that X is a locally s-space with a compactification bX such that the remainder $Y = bX \setminus X$ is locally perfect. Then Y is a Lindelöf Σ-space, and X is an s-space.*

Lemma 2.51. *Let X be a nowhere locally compact paracompact p-space such that the remainder $bX \setminus X$ of X in a compactification bX is charming. If every closed s-subspace of $bX \setminus X$ is metrizable and every compact subset of $bX \setminus X$ is a G_δ-set of $bX \setminus X$, then $bX \setminus X$ is a Lindelöf Σ-space and X has a countable π-base.*

Proof. By the assumption, $bX \setminus X = \bigcup\{F_n : n \in \omega\}$ such that F_n is a closed s-subspace of $bX \setminus X$ for every $n \in \omega$. Clearly, $bX \setminus X$ has a countable network since every closed s-subspace of $bX \setminus X$ is metrizable. The Souslin number $c(bX)$ of bX is countable and the Souslin number $c(X)$ of X is countable, since X is nowhere locally compact. Then X is an s-space. Clearly the space $bX \setminus X$ is a Lindelöf Σ-space by [4, Theorem 2.7].

It is obvious that X and $bX \setminus X$ are separable and metrizable spaces, since every s-space with a countable network is metrizable [4]. Since the $bX \setminus X$ is a Lindelöf Σ-space and has a countable network, hence $bX \setminus X$ has a countable base by [4, Corollary 6.6, Corollary 6.7]. Then X has a countable π-base. □

Theorem 2.52. *Let X be a nowhere locally compact locally s-space and metrizable space. If X has a locally perfect remainder $bX \setminus X$ with the properties that every closed s-subspace of $bX \setminus X$ is metrizable and every compact subset of $bX \setminus X$ is a G_δ-set of $bX \setminus X$, then X and $bX \setminus X$ are all separable and metrizable spaces.*

Proof. Since X is a nowhere locally compact space, locally s-space, and metrizable space, Theorem 2.42 implies that its remainder $bX \setminus X$ is a charming space. By Theorem 2.50, X is an s-space, and Lemma 2.51 further shows that both X and $bX \setminus X$ are separable and metrizable spaces. □

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