Filomat 39:14 (2025), 4635–4656
https://doi.org/10.2298/FIL2514635S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some sandwich-type results for analytic functions involving a multiplier transformation

H. M. Srivastava[a,*], Madan Mohan Soren[b]

[a]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*

and

Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, *Taiwan, Republic of China*

and

Center for Converging Humanities, Kyung Hee University, 26 *Kyungheedae-ro, Dongdaemun-gu, Seoul* 02447, *Republic of Korea*

and

Department of Mathematics and Informatics, Azerbaijan University, 71 *Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan*

and

Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taoyuan City 320314, *Taiwan, Republic of China*

and

Section of Mathematics, International Telematic University Uninettuno, 39 *Corso Vittorio Emanuele II, I-*00186 *Rome, Italy*

[b]*Department of Mathematics, Berhampur University, Bhanja Bihar 760007, Ganjam, Odisha, India*

Abstract. We investigate some classes of analytic functions defined on $\mathbb{U} \times \overline{\mathbb{U}}$ to obtain new results on a multiplier transformation, which involves the iterations of the Owa-Srivastava operator and its combination, associated with strong differential subordination and strong differential superordination results for analytic functions on $\mathbb{U} \times \overline{\mathbb{U}}$, $\mathbb{U}$ and $\overline{\mathbb{U}}$ being the open unit disk and its closure, respectively. Several sandwich-type results are obtained and many related recent developments of the subjects, which are addressed in this presentation, are also indicated.

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 26A33, 30C80.

Keywords. Analytic functions; Univalent functions; Dominants; Multivalent function; Owa-Srivastava operator; Sandwich-Type results; Strong differential subordination; Strong differential superordination; Subordinants.

Received: 19 November 2023; Accepted: 21 April 2025

Communicated by Dragan S. Djordjević

* Corresponding author: H. M. Srivastava

Email addresses: harimsri@math.uvic.ca, harimsri@uvic.ca (H. M. Srivastava), mms.math@buodisha.edu.in (Madan Mohan Soren)

ORCID iDs: https://orcid.org/0000-0002-9277-8092 (H. M. Srivastava), https://orcid.org/0000-0001-7344-8082 (Madan Mohan Soren)

1. Introduction, Definitions and Preliminaries

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{H}(\mathbb{U} \times \overline{\mathbb{U}})$ the class of functions which are analytic in $\mathbb{U} \times \overline{\mathbb{U}}$, where $\overline{\mathbb{U}}$ is the closure of $\mathbb{U}$ given by

$$\overline{\mathbb{U}} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| \leqq 1\} =: \mathbb{U} \cup \partial\mathbb{U}.$$

Thus, for $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$, $\xi \in \overline{\mathbb{U}}$ and $z \in \mathbb{N}$, we have

$$\mathcal{A}(\xi; n) := \{f : f(z, \xi) = z + a_{n+1}(\xi)z^{n+1} + \cdots\} \subset H(\mathbb{U} \times \overline{\mathbb{U}}),$$

where, and in what follows, the coefficients $a_k(\xi)$ are holomorphic functions in $\overline{\mathbb{U}}$ for integers $k \geqq n + 1$.

For $n \in \mathbb{N}$, $a \in \mathbb{C}$, $\xi \in \overline{\mathbb{U}}$ and $z \in \mathbb{U}$, we also set

$$\mathcal{H}_\xi[a, n] := \left\{f : f(z, \xi) \in \mathcal{H}(\mathbb{U} \times \overline{\mathbb{U}}) \quad \text{and} \quad f(z, \xi) = a + a_n(\xi)z^n + \cdots\right\}.$$

For the class $\mathcal{A}$ of functions $\mathfrak{f}$, which are analytic in $\mathbb{U}$ and normalized by

$$\mathfrak{f}(z) = z + \sum_{k=0}^{\infty} a_k z^k,$$

it is easily observed that

$$\mathcal{A} = \mathcal{A}(1; 1) \quad \text{and} \quad \mathcal{A} = \mathcal{H}_1[0, 1],$$

provided that

$$a_1(1) = 1 \quad \text{and} \quad a_k(1) = a_k \qquad (k \in \mathbb{N} \setminus \{1\}).$$

Finally, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$.

Given two functions $f, g \in \mathcal{H}$, we say that the f is subordinate to the function g, and we write $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| \leqq |z| \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

If the function g is univalent in $\mathbb{U}$, then the following equivalence holds true (see also some related recent works in [3, 12, 37]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Our motivation for the present investigation is derived from several recent works that are based upon strong differential subordination and strong differential superordination and their applications in the Geometric Function Theory of Complex Analysis. Antonino and Romaguera [7] studied the strong differential subordination which extends the concept of differential subordination from the function class $\mathcal{H}$ to the function class $\mathcal{H}_\xi(\mathbb{U} \times \overline{\mathbb{U}})$. They first studied on the Briot-Bouquet strong differential subordination. Subsequently, in the year 2006, Antonino and Romaguera [6] introduced this concept as an extension of the classical differential subordination of Miller and Mocanu [23]. The dual notions of differential subordination and differential superordination were extended and completely established by Oros (see, for example, [25] and [28]). Several examples of strong differential subordinations and strong differential superordinations of analytic functions were presented by Jeyaraman *et al.* (see, for details, [13] and [14]). In recent years, many researchers have contributed significantly in this direction by using various known operators [1, 2, 4, 5, 8, 17–20, 22, 27, 33, 40, 41]. Furthermore, in their pioneering work, Jung *et al.* [15] introduced and investigated the theory and applications of a family of multiplier transformations in their study of normalized analytic and univalent functions in the open unit disk $\mathbb{U}$. Their work has motivated and encouraged many further developments, which are based upon various families of multiplier transformations, (see, for example, [9, 10, 18, 20, 40]).

Henceforth we find it to be convenient to denote by $\mathcal{A}_p(\xi)$ the subclass of the functions $f(z,\xi) \in \mathcal{H}(\mathbb{U}\times\overline{\mathbb{U}})$, which are normalized by

$$f(z,\xi) = z^p + \sum_{k=1}^{\infty} a_{k+p}(\xi)\, z^{k+p} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}};\ p \in \mathbb{N}).$$

We also write $\mathcal{A}_1(\xi) := \mathcal{A}(\xi;1)$.

Each of the following definitions will be used in our present investigation.

Definition 1. (see [35]) For $f \in \mathcal{A}_p$, the general two-parameter operator $C^{t,m}$ $(t \geqq 0;\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is defined as follows:

$$C^{(0,0)} f(z) = f(z) =: C^0,$$

$$\begin{aligned} C^{(t,1)} f(z) &= (1-t) f(z) + \frac{tzf'(z)}{p} \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{p+kt}{p}\right) a_{k+p} z^{k+p} \\ &=: C^t f(z) \ \ (t \geqq 0) \end{aligned}$$

and

$$\begin{aligned} C^{(t,m)} f(z) &= C^t C^{(t,m-1)} f(z) \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{p+kt}{p}\right)^m a_{k+p} z^{k+p} \qquad (t \geqq 0;\ m \in \mathbb{N}_0). \end{aligned}$$

Definition 2. (see [32]) Let $\Omega_z^{(\lambda,p,n)}$ $(-\infty < \lambda < p;\ n \in \mathbb{N}_0)$ denote the n-times superimposition of the operator $\Omega_z^{(\lambda,p)}$. It is defined for $f \in \mathcal{A}_p$ by

$$\Omega_z^{(\lambda,p,0)} f(z) = f(z),$$

$$\Omega_z^{(\lambda,p,1)} f(z) =: \Omega_z^{(\lambda,p)} f(z)$$

and

$$\begin{aligned} \Omega_z^{(\lambda,p,n)} f(z) &= \Omega_z^{(\lambda,p)} \Omega_z^{(\lambda,p,n-1)} f(z) \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{\Gamma(p+k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(p+k+1-\lambda)}\right)^n a_{k+p} z^{k+p} \qquad (n \in \mathbb{N}). \end{aligned}$$

Definition 3. For a function $f \in \mathcal{A}_p$, the multiplier transformation $\mathcal{D}_p^{\lambda,t}(n,m) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ is defined by

$$\begin{aligned}\mathcal{D}_p^{\lambda,t}(n,m)f(z) &= \Omega_z^{(\lambda,p,n)}C^{(t,m)}f(z)\\ &= z^p + \sum_{k=1}^{\infty}\left(\frac{\Gamma(p+k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(p+k+1-\lambda)}\right)^n\left(\frac{p+kt}{p}\right)^m a_{p+k}z^{p+k}\end{aligned}$$

$\Big(m,n \in \mathbb{N}_0;\ t \geqq 0;\ -\infty < \lambda < p;\ z \in \mathbb{U}).$

Definition 4 extends the multiplier transformation $\mathcal{D}_p^{\lambda,t}(n,m)f(z)$ of Definition 3 to the functions $f \in \mathcal{A}_p(\xi)$.

Definition 4. (see [35]) For $m,n \in \mathbb{N}_0$, $t \geqq 0$, $q \geqq 1$, $-\infty < \lambda < p$ and $f \in \mathcal{A}_p(\xi)$, the general multiplier transformation $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ is defined as follows:

$$\begin{aligned}\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi) &= \Omega_z^{(\lambda,p,n)}C^{(t,m)}f(z,\xi)\\ &= z^p + \sum_{k=1}^{\infty}\left(\frac{\Gamma(p+k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(p+k+1-\lambda)}\right)^n\\ &\qquad\cdot\left(\frac{p+kt}{p}\right)^m a_{p+k}(\xi)z^{p+k} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}).\end{aligned} \tag{1}$$

Remark 1. The multiplier transformation $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ satisfies the following identity:

$$\begin{aligned}&z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)'\\ &\quad= \frac{p}{t}\Big[\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\\ &\qquad - (1-t)\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\Big].\end{aligned} \tag{2}$$

In our present investigation, we shall also make use of the following definitions and lemmas.

Definition 5. (see [25] and [27]) Assume that $f(z,\xi)$ and $F(z,\xi)$ are analytic functions in $\mathbb{U}\times\overline{\mathbb{U}}$. The function $f(z,\xi)$ is said to be strongly subordinate to $F(z,\xi)$ or the function $F(z,\xi)$ is said to be strongly superordinate to $f(z,\xi)$, which is written as follows:

$$f(z,\xi) \prec\prec F(z,\xi),$$

if there exists an analytic function w in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z,\xi) = F\Big(w(z),\xi\Big) \qquad (\forall\ \xi \in \overline{\mathbb{U}}).$$

Remark 2. *[27]* If the function $F(z,\xi)$ is univalent in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$, then

$$f(z,\xi) \prec\prec F(z,\xi) \Longleftrightarrow f(0,\xi) = F(0,\xi) \quad \text{and} \quad f\Big(\mathbb{U}\times\overline{\mathbb{U}}\Big) \subset F\Big(\mathbb{U}\times\overline{\mathbb{U}}\Big).$$

In the particular case when the function $F(z,\xi) \equiv F(z)$ and $f(z,\xi) \equiv f(z)$ are functions of z only, then the strong subordination is reduced to the usual subordination.

Definition 6. (see [19]) Denote by $\mathcal{Q}_\xi$ the set of all analytic and injective functions $s(\cdot,\xi)$ on $\mathbb{U}\times\overline{\mathbb{U}}\setminus E\big(s(z,\xi)\big)$, where

$$E\big(s(z,\xi)\big)=\left\{\zeta:\zeta\in\partial\mathbb{U}\quad\text{and}\quad\lim_{z\to\zeta}s(z,\xi)=\infty\right\},$$

and are such that

$$s'(\zeta,\xi)\neq 0\qquad\big(\zeta\in\partial\mathbb{U}\setminus E(s);\ \xi\in\overline{\mathbb{U}}\big).$$

The subclass of $\mathcal{Q}_\xi$ for which $s(0,\xi)=a$ is represented by $\mathcal{Q}_\xi(a)$.

Lemma 1. (see [28]) *Assume that the fuction $s(z,\xi)$ is univalent in $z\in\mathbb{U}$ for all $\xi\in\overline{\mathbb{U}}$ with $s(0,\xi)=a$. Also let the functions θ and ϕ be analytic in a domain $\mathbb{D}$ containing $s(\mathbb{U}\times\overline{\mathbb{U}})$ with*

$$\phi(w)\neq 0\qquad(w\in s(\mathbb{U}\times\overline{\mathbb{U}}).$$

Set

$$Q(z,\xi)=zs'(z,\xi)\cdot\phi[s(z,\xi)]\quad\textit{and}\quad h(z,\xi)=\theta[s(z,\xi)]+Q(z,\xi)$$

and suppose that

1. *$Q(z,\xi)$ is starlike univalent in $\mathbb{U}$ for all $\xi\in\overline{\mathbb{U}}$;*
2. *$\Re\left(\frac{zh'(z,\xi)}{Q(z,\xi)}\right)>0$ for all $z\in\mathbb{U}$ and for all $\xi\in\overline{\mathbb{U}}$.*

If $r(z,\xi)\in\mathcal{H}_\xi(a,1)$ with $r(\mathcal{U}\times\overline{\mathbb{U}})\subseteq\mathbb{D}$ and

$$\theta\big(r(z,\xi)\big)+zr'(z,\xi)\phi\big(r(z,\xi)\big)\prec\prec\theta\big(s(z,\xi)\big)+zs'(z,\xi)\phi\big(s(z,\xi)\big),$$

then

$$r(z,\xi)\prec\prec s(z,\xi)\qquad(z\in\mathbb{U};\ \xi\in\overline{\mathbb{U}})$$

and $s(z,\xi)$ is the best dominant.

By setting $\theta(w)=\alpha w$ and $\phi(w)=\beta$ in Lemma 1, we get the following consequence.

Lemma 2. *Assume that the function $s(z,\xi)$ is convex univalent for all $z\in\mathbb{U}$ and for all $\xi\in\overline{\mathbb{U}}$ and $\alpha,\beta\in\mathbb{C}$ with*

$$\Re\left(1+\frac{zs''(z,\xi)}{s'(z,\xi)}\right)>\max\left\{0,-\Re\left(\frac{\alpha}{\beta}\right)\right\}.$$

If the function $r(z,\xi)$ is analytic in $z\in\mathbb{U}$ for all $\xi\in\overline{\mathbb{U}}$ and

$$\alpha r(z,\xi)+\beta zr'(z,\xi)\prec\prec\alpha s(z,\xi)+\beta zs'(z,\xi),$$

then $r(z,\xi)\prec\prec s(z,\xi)$ and $s(z,\xi)$ is the best dominant.

Lemma 3. (see [28]) *Assume that the function $s(z,\xi)$ is convex univalent in $\mathbb{U}$ for all $\xi\in\overline{\mathbb{U}}$. Also let the functions ϑ and φ be analytic in a domain $\mathbb{D}$ containing $s(\mathbb{U}\times\overline{\mathbb{U}})$. Suppose that*

1. $\Re\left(\dfrac{\vartheta'\big(s(z,\xi)\big)}{\varphi\big(s(z,\xi)\big)}\right)>0$ *for all $z\in\mathbb{U}$ and for all $\xi\in\overline{\mathbb{U}}$;*
2. *The function $Q(z,\xi)=zs'(z,\xi)\varphi\big(s(z,\xi)\big)$ is starlike univalent in $\mathbb{U}$ for all $\overline{\mathbb{U}}$.*

If $r(z,\xi) \in \mathcal{H}_\xi(a,1)$ *with* $r(\mathcal{U} \times \overline{\mathbb{U}}) \subseteq \mathbb{D}$, *and* $\vartheta\Big(r(z,\xi)\Big) + zr'(z,\xi)\varphi\Big(r(z,\xi)\Big)$ *is univalent in* $\mathbb{U}$ *for all* $\overline{\mathbb{U}}$, *and*

$$\vartheta\Big(s(z,\xi)\Big) + zs'(z,\xi)\varphi\Big(s(z,\xi)\Big) << \vartheta\Big(r(z,\xi)\Big) + zr'(z,\xi)\varphi\Big(r(z,\xi)\Big),$$

then

$$r(z,\xi) << s(z,\xi) \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $s(z,\xi)$ *is the best subordinant.*

By setting $\vartheta(w) := \alpha w$ and $\varphi(w) := \beta$ in Lemma 3, we get the following consequence.

Lemma 4. *Assume that the function* $q(z,\xi)$ *is convex univalent in* $\mathbb{U}$ *for all* $\overline{\mathbb{U}}$ *and* $\alpha, \beta \in \mathbb{C}$ *with* $\Re\left(\frac{\alpha}{\beta}\right) > 0$. *If* $r(z,\xi) \in \mathcal{H}_\xi(s(0),1) \cap Q$, $r(z) + \beta z r'(z)$ *is univalent in* $\mathbb{U}$ *for all* $\overline{\mathbb{U}}$ *and*

$$\alpha s(z,\xi) + \beta z s'(z,\xi) << \alpha r(z,\xi) + \beta z r'(z,\xi) \quad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}),$$

then $s(z,\xi) << r(z,\xi)$ *and* $s(z,\xi)$ *is the best subordinant.*

In recent years, many authors carried out researches leading to interesting results which are associated with strong differential subordination and strong differential superordination (see, for example, [11, 13, 14, 21, 28, 30, 35, 39, 43]). The main objective of the present paper is to investigate several strong differential subordination and strong differential superordination properties of analytic functions associated with some general multiplier transformations which include above-mentioned combinations with the iterations of the Owa-Srivastava operator (see [32]). Furthermore, we derive a number of sandwich-type results for these general multiplier transformations.

The organization of this paper is given as follows. In Section 1, we present the introduction, definitions and preliminaries that provide the foundation of our paper as well as that are needed to prove our main results and their consequences. In Section 2, we prove our main results associated with strong differential subordination by using the general multiplier transformations and deduce some corollaries and consequences of our main results. In Section 3, we present strong superordination results along with and, by suitably combining them with the results of Section 2, we obtain a number of sandwich-type results in Section 4. Finally, in the concluding section (Section 5), we present our remarks and observations which are based upon the subject-matter of this paper.

2. Results Associated with Strong Subordination

Unless mentioned otherwise, we assume hereafter that $m,\ n \in \mathbb{N}_0$, $t \geqq 0$, $q \geqq 1$ and $\infty < \lambda < p$. Our first strong subordination result is asserted by Theorem 1 below.

Theorem 1. *Let the function* $s(z,\xi) \in \mathcal{A}_p(\xi)$ *be univalent in* $\mathbb{U} \times \overline{\mathbb{U}}$ *with* $s(0,\xi) = 1$ *and*

$$\Re\left(1 + \frac{zs''(z,\xi)}{s'(z,\xi)}\right) > \max\left\{0, -\frac{p}{t}\Re\left(\frac{1}{\delta}\right)\right\} \quad (\delta > 0). \tag{3}$$

If the function $f \in \mathcal{A}_p(\xi)$ *satisfies the following strong subordination condition*:

$$(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}$$
$$<< s(z,\xi) + \frac{\delta t}{p}zs'(z,\xi), \tag{4}$$

where $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ *is defined in* (1), *then*

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \prec\prec s(z,\xi) \qquad (z\in\mathbb{U};\ \xi\in\overline{\mathbb{U}}) \tag{5}$$

and $s(z,\xi)$ *is the best dominant.*

Proof. Let

$$r(z,\xi)=\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \qquad (z\in\mathbb{U};\ \xi\in\overline{\mathbb{U}}). \tag{6}$$

Then, upon differentiating (6) with respect to z, we find that

$$pz^p r(z,\xi)+z^{p+1}r'(z,\xi)=z\Big(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\Big)',$$

which, by applying the identity (2) and after some simplification, yields

$$r(z,\xi)+\frac{t}{p}zr'(z,\xi)=\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}. \tag{7}$$

Therefore, by the hypothesis (4), we have

$$r(z,\xi)+\frac{\delta t}{p}zr'(z,\xi)\prec\prec s(z,\xi)+\frac{\delta t}{p}zs'(z,\xi).$$

Finally, we apply Lemma 2 with $\alpha=1$ and $\beta=\frac{\delta t}{p}$. We are thus led to the required result (5) of Theorem 1. The proof of Theorem 1 is completed. □

If, in Theorem 1, we set

$$s(z,\xi)=\frac{1+A\xi z}{1+B\xi z},\ -1\leqq B<A\leqq 1 \quad \text{and} \quad s(z,\xi)=\frac{1+\xi z}{1-\xi z},$$

then we have the following corollaries.

Corollary 1. *Let* $\delta\in\mathbb{C}\setminus\{0\}$ $(1\leqq B<A\leqq 1)$ *and*

$$\frac{|B|-1}{|B|+1}<\frac{p}{t}\Re\left(\frac{1}{\delta}\right).$$

If $f\in\mathcal{A}_p(\xi)$ *satisfies the following strong subordination:*

$$(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p}+\delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}$$
$$\prec\prec \frac{1+A\xi z}{1+B\xi z}+\frac{\delta t}{p}\frac{(A-B)\xi z}{(1+B\xi z)^2},$$

where $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ *is defined in* (1), *then*

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p}\prec\prec\frac{1+A\xi z}{1+B\xi z} \qquad (z\in\mathbb{U};\ \xi\in\overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ *is the best dominant.*

Corollary 2. *Let $\delta \in \mathbb{C} \setminus \{0\}$ with $\Re\left(\frac{1}{\delta}\right) > 0$. If $f \in \mathcal{A}_p(\xi)$ satisfies the following strong subordination:*

$$(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p}+\delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \\ \prec\prec \frac{1+\xi z}{1-\xi z}+\frac{\delta t}{p}\frac{2\xi z}{(1-\xi z)^2},$$

then

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \prec\prec \frac{1+\xi z}{1-\xi z} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+\xi z}{1-\xi z}$ is the best dominant.

Theorem 2. *Let the function $s(z,\xi)$ be univalent in $\mathbb{U}$ for all $\overline{\mathbb{U}}$ with $s(0,\xi) = 1$ and*

$$\Re\left(1+\frac{zs''(z,\xi)}{s'(z,\xi)}-\frac{zs'(z,\xi)}{s(z,\xi)}\right) > 0. \tag{8}$$

Also let $\eta \in \mathbb{C} \setminus \{0\}$ and $\nu, \mu \in \mathbb{C}$ with $\nu + \mu \neq 0$. If $f \in \mathcal{A}_p(\xi)$ satisfies the following strong subordination:

$$\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and

$$\eta\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)'+\mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}-p\right) \\ \prec\prec \frac{zs'(z,\xi)}{s(z,\xi)}, \tag{9}$$

then

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^{\eta} \prec\prec s(z,\xi)$$

and $s(z,\xi)$ is the best dominant.

Proof. Let the function $r(z,\xi)$ be defined by

$$r(z,\xi) := \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^{\eta}. \tag{10}$$

It is clear that $r(z,\xi)$ is analytic in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$. Now, by logarithmic differentiation in (10), we obtain

$$\frac{zr'(z,\xi)}{r(z,\xi)} = \eta\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)'+\mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}-p\right). \tag{11}$$

We now set

$$\theta(w) = 1 \quad \text{and} \quad \phi(w) = \frac{1}{w},$$

$$\Phi(z,\xi) = zs'(z,\xi)\phi\Big(s(z,\xi)\Big) = \frac{zs'(z,\xi)}{s(z,\xi)}$$

and

$$h(z,\xi) = \theta\Big(s(z,\xi)\Big) + \Phi(z,\xi) = 1 + \frac{zs'(z,\xi)}{s(z,\xi)}.$$

Thus, from the hypothesis (8), we see that the function $\Phi(z,\xi)$ is starlike in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$ and

$$\Re\left(\frac{zh'(z,\xi)}{\Phi(z,\xi)}\right) = \Re\left(1 + \frac{zs''(z,\xi)}{s'(z,\xi)} - \frac{zs'(z,\xi)}{s(z,\xi)}\right) > 0.$$

Therefore, the relation (11) can be written as follows:

$$\begin{aligned}\theta\Big(r(z,\xi)\Big) &+ zr'(z,\xi)\phi\Big(r(z,\xi)\Big)\\ &= 1 + \eta\left(\frac{\nu z\Big(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\Big)' + \mu z\Big(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\Big)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right),\end{aligned}$$

which, in view of (9), yields

$$\theta\Big(r(z,\xi)\Big) + zr'(z,\xi)\phi\Big(r(z,\xi)\Big) \prec\prec 1 + \frac{zs'(z,\xi)}{s(z,\xi)} = \theta\Big(s(z,\xi)\Big) + zs'(z,\xi)\phi\Big(s(z,\xi)\Big).$$

Finally, by applying Lemma 1, we obtain

$$r(z,\xi) \prec\prec s(z,\xi),$$

and $s(z,\xi)$ is the best dominant. This completes the proof of Theorem 2. □

Setting

$$\nu = 0, \quad \mu = 1 \quad \text{and} \quad s(z,\xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 2 and assuming that (8) holds true, we can deduce the following result.

Corollary 3. *Let $\eta \in \mathbb{C} \setminus \{0\}$ and $-1 \leqq B < A \leqq 1$. If $f \in \mathcal{A}_p(\xi)$ satisfies the following strong subordination:*

$$\eta\left(\frac{z\Big(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\Big)'}{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right) \prec\prec \frac{(A-B)\xi z}{(1+A\xi z)(1+B\xi z)},$$

then

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^{\eta} \prec\prec \frac{1+A\xi z}{1+B\xi z} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ is the best dominant.

Theorem 3. *Let the function $s(z,\xi)$ be univalent in $\mathbb{U}$ for all $\overline{\mathbb{U}}$ with $s(0,\xi) = 1$ and*

$$\Re\left(1 + \frac{zs''(z,\xi)}{s'(z,\xi)} - \frac{zs'(z,\xi)}{s(z,\xi)}\right) > 0. \tag{12}$$

Also let $\eta \in \mathbb{C} \setminus \{0\}$ *and* $\nu, \mu \in \mathbb{C}$ *with* $\nu + \mu \neq 0$. *Suppose that* $f \in \mathcal{A}_p(\xi)$ *satisfies the following condition*:

$$\frac{\nu \mathcal{D}_p^{\lambda,t}(n+q-1, m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)}{(\nu+\mu) z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and set

$$\Delta(z,\xi) = \left(\frac{\nu \mathcal{D}_p^{\lambda,t}(n+q-1, m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)}{(\nu+\mu) z^p} \right)^{\eta} + \eta \left(\frac{\nu z \Big(\mathcal{D}_p^{\lambda,t}(n+q-1, m-1)\Omega_z^{(\lambda,p)} f(z,\xi)\Big)' + \mu z \Big(\mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)\Big)'}{\nu \mathcal{D}_p^{\lambda,t}(n+q-1, m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)} - p \right). \tag{13}$$

If

$$\Delta(z,\xi) \prec\prec s(z,\xi) + \frac{z s'(z,\xi)}{s(z,\xi)} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}), \tag{14}$$

then

$$\left(\frac{\nu \mathcal{D}_p^{\lambda,t}(n+q-1, m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)}{(\nu+\mu) z^p} \right)^{\eta} \prec\prec s(z,\xi)$$

and $s(z,\xi)$ *is the best dominant.*

Proof. Following the lines of the proof of Theorem 2, let $r(z,\xi)$ be defined in (10) and

$$\theta(w) = w \quad \text{and} \quad \phi(w) = \frac{1}{w},$$

$$\Phi(z,\xi) = z s'(z,\xi) \phi\big(s(z,\xi)\big) = \frac{z s'(z,\xi)}{s(z,\xi)}$$

and

$$h(z,\xi) = \theta\big(s(z,\xi)\big) + \Phi(z,\xi) = s(z,\xi) + \frac{z s'(z,\xi)}{s(z,\xi)}.$$

Hence, from (10) and (11), the condition (14) can be written as follows:

$$\theta\big(r(z,\xi)\big) + z r'(z,\xi) \phi\big(r(z,\xi)\big) \prec\prec \theta\big(s(z,\xi)\big) + z s'(z,\xi) \phi\big(s(z,\xi)\big).$$

Therefore, by applying Lemma 1, we get the required result. □

Setting

$$\nu = 0, \quad \mu = 1 \quad \text{and} \quad s(z,\xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 3 and assuming that (12) holds true, we have the following result.

Corollary 4. *Let* $f \in \mathcal{A}_p(\xi)$ *satisfy the following condition*:

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q, m) f(z,\xi)}{z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\eta \in \mathbb{C} \setminus \{0\}$. *If*

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^\eta + \eta\left(\frac{z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right)$$
$$\prec\prec \frac{1+A\xi z}{1+B\xi z} + \frac{(A-B)\xi z}{(1+A\xi z)(1+B\xi z)},$$

then

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^\eta \prec\prec \frac{1+A\xi z}{1+B\xi z} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ *is the best dominant.*

Theorem 4. *Let* $\alpha, \eta \in \mathbb{C} \setminus \{0\}$ *and* $\nu, \mu \in \mathbb{C}$ *with* $\nu + \mu \neq 0$. *Also let the function* $s(z,\xi)$ *be univalent in* $\mathbb{U}$ *for all* $\overline{\mathbb{U}}$ *with* $s(0,\xi) = 1$ *and*

$$\Re\left(1 + \frac{zs''(z,\xi)}{s'(z,\xi)}\right) > \max\{0, -\Re(\alpha)\}. \tag{15}$$

Suppose that $f \in \mathcal{A}_p(\xi)$ *satisfies the following condition:*

$$\frac{\nu D_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu D_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and set

$$\nabla(z,\xi) = \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta$$
$$\cdot\left[\alpha + \eta\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)' + \mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right)\right]. \tag{16}$$

If

$$\nabla(z,\xi) \prec\prec \alpha s(z,\xi) + zs'(z,\xi) \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}), \tag{17}$$

then

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta \prec\prec s(z,\xi)$$

and $s(z,\xi)$ *is the best dominant.*

Proof. The proof is similar to that of Theorem 2. Let $r(z,\xi)$ be defined in (10). From (11), we have

$$zr'(z,\xi) = \eta r(z,\xi)$$
$$\cdot\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)' + \mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right). \tag{18}$$

Thus, if we set

$$\theta(w) = \alpha w \quad \text{and} \quad \phi(w) = 1,$$

$$\Phi(z,\xi) = zs'(z,\xi)\phi\Big(s(z,\xi)\Big) = zs'(z,\xi)$$

and

$$h(z,\xi) = \theta\Big(s(z,\xi)\Big) + \Phi(z,\xi) = \alpha s(z,\xi) + zs'(z,\xi),$$

it is clear from (15) that the function $\Phi(z,\xi)$ is starlike in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$ and

$$\Re\left(\frac{zh'(z,\xi)}{\Phi(z,\xi)}\right) = \Re\left(\alpha + 1 + \frac{zs''(z,\xi)}{s'(z,\xi)}\right) > 0.$$

Now, from (10) and (18), we get

$$\theta\Big(r(z,\xi)\Big) + zr'(z,\xi)\phi\Big(r(z,\xi)\Big) = \alpha r(z,\xi) + zr'(z,\xi) = \nabla(z,\xi),$$

where $\nabla(z,\xi)$ is defined in (16). The equation (17) can be written as follows:

$$\theta\Big(r(z,\xi)\Big) + zr'(z,\xi)\phi\Big(r(z,\xi)\Big) \prec\prec \theta\Big(s(z,\xi)\Big) + zs'(z,\xi)\phi\Big(s(z,\xi)\Big).$$

Therefore, by applying Lemma 1, we get

$$r(z,\xi) \prec\prec s(z,\xi),$$

and $s(z,\xi)$ is the best dominant. The proof of Theorem 4 is completed. □

Setting

$$\nu = 0, \quad \mu = 1, \quad \alpha, \eta \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad s(z,\xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 4 and assuming that (15) holds true, we can deduce the following result.

Corollary 5. *Let $f \in \mathcal{A}_p(\xi)$ satisfy the following condition*:

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \neq 0 \quad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}).$$

If

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^{\eta} \cdot \left[\alpha + \eta\left(\frac{z\Big(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\Big)'}{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right)\right] \prec\prec \frac{1 + A\xi z}{1 + B\xi z} + \frac{(A-B)\xi z}{(1 + A\xi z)(1 + B\xi z)},$$

then

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^{\eta} \prec\prec \frac{1 + A\xi z}{1 + B\xi z} \quad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ is the best dominant.

3. Results Associated with Strong Superordination

Here, in this section, we present the following strong superordination results.

Theorem 5. *Let the function $s(z,\xi) \in \mathcal{A}_p(\xi)$ be convex univalent in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$ with $s(0,\xi) = 1$. Also let $f \in \mathcal{A}_p(\xi)$,*

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \in \mathcal{H}_\xi(1,1) \cap \boldsymbol{Q}_\xi$$

and

$$(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}$$

be univalent in $\mathbb{U} \times \overline{\mathbb{U}}$, where $D_p^{\lambda,t}(n,m)f(z,\xi)$ is defined in (1)*. If*

$$\begin{aligned} &s(z,\xi) + \frac{\delta t}{p} z s'(z,\xi) \\ &\qquad \prec\prec (1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}, \end{aligned} \tag{19}$$

then

$$s(z,\xi) \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $s(z,\xi)$ is the best subordinant.

Proof. Let the function $r(z,\xi)$ be defined in (6). Then, by using (7), we find from Theorem 1 that

$$\begin{aligned} &(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \\ &\qquad = r(z,\xi) + \frac{\delta t}{p} z r'(z,\xi). \end{aligned}$$

Since the hypothesis (19) reduces to the following form:

$$s(z,\xi) + \frac{\delta t}{p} z s'(z,\xi) \prec\prec r(z,\xi) + \frac{\delta t}{p} z r'(z,\xi).$$

Thus, by applying Lemma 4, we get

$$s(z,\xi) \prec\prec r(z,\xi)$$

or, equivalently,

$$s(z,\xi) \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $s(z,\xi)$ is the best subordinant. The proof of Theorem 5 is evidently completed. □

Setting

$$s(z,\xi) = \frac{1+A\xi z}{1+B\xi z} \qquad (-1 \leqq B < A \leqq 1)$$

and

$$s(z,\xi) = \frac{1+\xi z}{1-\xi z}$$

in Theorem 5, we have the following corollaries.

Corollary 6. *Let* $\delta \in \mathbb{C} \setminus \{0\}$ $(-1 \leqq B < A \leqq 1)$ *and*

$$\frac{|B|-1}{|B|+1} < \frac{p}{t}\Re\left(\frac{1}{\delta}\right).$$

If $f \in \mathcal{A}_p(\xi)$ *satisfies the following condition*:

$$\frac{1+A\xi z}{1+B\xi z} + \frac{\delta t}{p}\frac{(A-B)\xi z}{(1+B\xi z)^2} \prec\prec (1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p},$$

where $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ *is defined in* (1), *then*

$$\frac{1+A\xi z}{1+B\xi z} \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ *is the best subordinant.*

Corollary 7. *Let* $\delta \in \mathbb{C} \setminus \{0\}$ *and* $\frac{p}{t}\Re\left(\frac{1}{\delta}\right) > 0$. *If* $f \in \mathcal{A}_p(\xi)$ *satisfies the following condition*:

$$\frac{1+\xi z}{1-\xi z} + \frac{\delta t}{p}\frac{2\xi z}{(1-\xi z)^2} \prec\prec (1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}$$

where $\mathcal{D}_p^{\lambda,t}(n,m)f(z,\xi)$ *is defined in* (1), *then*

$$\frac{1+\xi z}{1-\xi z} \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+\xi z}{1-\xi z}$ *is the best subordinant.*

Theorem 6. *Let the function* $s(z,\xi) \in \mathcal{A}_p(\xi)$ *be convex univalent in* $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$ *with* $s(0,\xi) = 1$ *and* (8) *holds true. Also let* $\eta \in \mathbb{C} \setminus \{0\}$ *and* $\nu, \mu \in \mathbb{C}$ *with* $\nu + \mu \neq 0$. *Assume that* $f \in \mathcal{A}_p(\xi)$ *such that*

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^{\eta} \in \mathcal{H}_\xi(1,1) \cap \mathcal{Q}_\xi$$

and

$$\eta\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)' + \mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right)$$

is univalent in $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$. *If*

$$\frac{zs'(z,\xi)}{s(z,\xi)} \prec\prec \eta\left(\frac{\nu z\left(\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)\right)' + \mu z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right),$$

then

$$s(z,\xi) \prec\prec \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^{\eta}$$

and $s(z,\xi)$ *is the best subordinant.*

Proof. Set $\vartheta(w) = 1$ and $\varphi(w) = \frac{1}{w}$ and note that $\vartheta(w)$ and $\varphi(w)$ are analytic in $\mathbb{C} \setminus \{0\}$. Hence, clearly, Theorem 6 immediately follows as an application of Lemma 3. □

Setting

$$\nu = 0, \quad \mu = 1 \quad \text{and} \quad s(z,\xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 6 and assuming that (8) holds true, we get the following result.

Corollary 8. *Let $\eta \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}_p(\xi)$ satisfies the following strong superordination:*

$$\frac{(A-B)\xi z}{(1 + A\xi z)(1 + B\xi z)} \prec\prec \eta \left(\frac{z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p \right),$$

then

$$\frac{1 + A\xi z}{1 + B\xi z} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \right)^{\eta} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ is the best subordinant.

Theorem 7. *Let $s(z,\xi) \in \mathcal{A}_p(\xi)$ be a convex univalent function in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$ with $s(0,\xi) = 1$. Suppose that $s(z,\xi)$ satisfies* (12) *and $\Re\left(s(z,\xi)\right) > 0$. Also let $\eta \in \mathbb{C} \setminus \{0\}$ and $\nu, \mu \in \mathbb{C}$ with $\nu + \mu \neq 0$. Assume that $f \in \mathcal{A}_p(\xi)$ such that*

$$\left(\frac{\nu \mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p} \right)^{\eta} \in \mathcal{H}_\xi(1,1) \cap \mathcal{Q}_\xi$$

and $\Delta(z,\xi)$ defined in (13) *is univalent in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$. If*

$$s(z,\xi) + \frac{zs'(z,\xi)}{s(z,\xi)} \prec\prec \Delta(z,\xi) \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}),$$

then

$$s(z,\xi) \prec\prec \left(\frac{\nu \mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu \mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p} \right)^{\eta},$$

where $s(z,\xi)$ is the best subordinant.

Proof. Our demonstration of Theorem 7 would run analogous to the proof of Theorem 6, so we omit the details involved. □

Setting

$$\nu = 0, \quad \mu = 1 \quad \text{and} \quad s(z,\xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 7 and assuming that (12) holds true, we can deduce the following result.

Corollary 9. *Let $f \in \mathcal{A}_p(\xi)$ satisfy the following condition:*

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and let $\eta \in \mathbb{C} \setminus \{0\}$. *If*

$$\frac{1 + A\xi z}{1 + B\xi z} + \frac{(A - B)\xi z}{(1 + A\xi z)(1 + B\xi z)} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)}{z^p}\right)^{\eta} + \eta\left(\frac{z\left(\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)\right)'}{\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)} - p\right),$$

then

$$\frac{1 + A\xi z}{1 + B\xi z} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)}{z^p}\right)^{\eta} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ *is the best subordinant.*

Theorem 8. *Let* $\alpha, \eta \in \mathbb{C} \setminus \{0\}$ *and* $\nu, \mu \in \mathbb{C}$ *with* $\nu + \mu \neq 0$. *Also let the function* $s(z, \xi)$ *be univalent in* $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$ *with* $s(0, \xi) = 1$ *and*

$$\Re\left(\frac{\alpha}{\eta}\right) > 0.$$

Assume that $f \in \mathcal{A}_p(\xi)$ *such that*

$$\frac{\nu\mathcal{D}_p^{\lambda,t}(n + q - 1, m - 1)f(z, \xi) + \mu\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)}{(\nu + \mu)z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n + q - 1, m - 1)\Omega_z^{(\lambda,p)}f(z, \xi) + \mu\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)}{(\nu + \mu)z^p}\right)^{\eta} \in \mathcal{H}_\xi(1, 1) \cap \mathcal{Q}_\xi,$$

and that $\nabla(z, \xi)$ *defined in* (16) *is univalent in* $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$. *If*

$$s(z, \xi) + \frac{zs'(z, \xi)}{s(z, \xi)} \prec\prec \nabla(z, \xi) \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}),$$

then

$$s(z, \xi) \prec\prec \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n + q - 1, m - 1)\Omega_z^{(\lambda,p)}f(z, \xi) + \mu\mathcal{D}_p^{\lambda,t}(n + q, m)f(z, \xi)}{(\nu + \mu)z^p}\right)^{\eta}$$

and $s(z, \xi)$ *is the best subordinant.*

Proof. Just as in our proofs of Theorem 6 and Theorem 7, Theorem 8 follows from Lemma 3. The details are being omitted here. □

Setting

$$\nu = 0, \quad \mu = 1, \quad \alpha, \eta \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad s(z, \xi) = \frac{1 + A\xi z}{1 + B\xi z} \quad (-1 \leqq B < A \leqq 1)$$

in Theorem 8 and assuming that (15) holds true, we get the following result.

Corollary 10. *Let $f \in \mathcal{A}_p(\xi)$ satisfy the following condition*:

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \neq 0 \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}).$$

If

$$\frac{1+A\xi z}{1+B\xi z} + \frac{(A-B)\xi z}{(1+A\xi z)(1+B\xi z)} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^{\eta} \cdot \left[\alpha + \eta\left(\frac{z\left(\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)\right)'}{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)} - p\right)\right],$$

then

$$\frac{1+A\xi z}{1+B\xi z} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^{\eta} \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}})$$

and $\frac{1+A\xi z}{1+B\xi z}$ *is the best subordinant.*

4. Sandwich-Type Results

By combining Theorem 1 with Theorem 5, we have the following sandwich-type results.

Theorem 9. *Let the functions $s_1(z,\xi)$ and $s_2(z,\xi)$ be convex univalent in $\mathbb{U} \times \overline{\mathbb{U}}$ with*

$$s_1(0,\xi) = s_2(0,\xi) = 1 \quad \text{and} \quad \frac{t}{p}\ \Re(\delta) > 0.$$

Also let $f \in \mathcal{A}_p(\xi)$ and

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \in \mathcal{H}_\xi(1,1) \cap \mathcal{Q}_\xi.$$

If the function

$$(1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}$$

is univalent in $\mathbb{U} \times \overline{\mathbb{U}}$, and

$$s_1(z,\xi) + \frac{\delta t}{p} z s_1'(z,\xi) \prec\prec (1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \prec\prec s_2(z,\xi) + \frac{\delta t}{p} z s_2'(z,\xi),$$

then

$$s_1(z,\xi) \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)}{z^p} \prec\prec s_2(z,\xi) \qquad (z \in \mathbb{U};\ \xi \in \overline{\mathbb{U}}),$$

where $s_1(z,\xi)$ and $s_2(z,\xi)$ are the best subordinant and the best dominant, respectively.

If we set

$$s_1(z,\xi) = \frac{1+A_1\xi z}{1+B_1\xi z} \quad \text{and} \quad s_1(z,\xi) = \frac{1+A_2\xi z}{1+B_2\xi z} \quad (-1 \leqq B_2 < B_1 < A_1 \leqq A_2 \leqq 1)$$

in Theorem 9, then we have the following corollary.

Corollary 11. *If $f \in \mathcal{A}_p(\xi)$,*

$$\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi)}{z^p} \in \mathcal{H}_\xi(1,1) \cap \mathcal{Q}_\xi$$

and

$$\begin{aligned}
&\frac{1+A_1\xi z}{1+B_1\xi z} + \frac{\delta t}{p}\frac{(A_1-B_1)\xi z}{(1+B_1\xi z)^2} \\
&\qquad \prec\prec (1-\delta)\frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi)}{z^p} + \delta\frac{\mathcal{D}_p^{\lambda,t}(n+q,m) f(z,\xi)}{z^p} \\
&\qquad \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z} + \frac{\delta t}{p}\frac{(A_2-B_2)\xi z}{(1+B_2\xi z)^2},
\end{aligned}$$

then

$$\frac{1+A_1\xi z}{1+B_1\xi z} \prec\prec \frac{\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi)}{z^p} \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z},$$

where

$$\frac{1+A_1\xi z}{1+B_1\xi z} \quad \textit{and} \quad \frac{1+A_2\xi z}{1+B_2\xi z}$$

are the best subordinant and the best dominant, respectively.

Next, by combining Theorem 3 with Theorem 7, we have the following sandwich-type results.

Theorem 10. *Let the functions $s_1(z,\xi)$ and $s_2(z,\xi)$ be convex univalent in $\mathbb{U} \times \overline{\mathbb{U}}$ with*

$$s_1(0,\xi) = s_2(0,\xi) = 1, \quad \Re\Big(s_1(z,\xi)\Big) > 0 \quad \textit{and} \quad s_2(z,\xi) \neq 0$$

and

$$\Re\left(1 + \frac{zs_j''(z,\xi)}{s_j'(z,\xi)} - \frac{zs_j'(z,\xi)}{s_j(z,\xi)}\right) > 0 \qquad (j := 1,2).$$

Also let $\eta \in \mathbb{C} \setminus \{0\}$ and $\nu, \mu \in \mathbb{C}$ with $\nu + \mu \neq 0$. Suppose that the function $f \in \mathcal{A}_p(\xi)$ such that

$$\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m) f(z,\xi)}{(\nu+\mu)z^p} \neq 0$$

and

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)} f(z,\xi) + \mu\mathcal{D}_p^{\lambda,t}(n+q,m) f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta \in \mathcal{H}_\xi(1,1) \cap \mathcal{Q}_\xi.$$

Suppose also that $\Delta(z,\xi)$ defined in (13) is univalent in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$. If

$$s_1(z,\xi) + \frac{zs_1'(z,\xi)}{s_1(z,\xi)} \prec\prec \Delta(z,\xi) \prec\prec s_2(z,\xi) + \frac{zs_2'(z,\xi)}{s_2(z,\xi)},$$

then

$$s_1(z,\xi) \prec\prec \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta \prec\prec s_2(z,\xi),$$

where $s_1(z,\xi)$ and $s_2(z,\xi)$ are the best subordinant and the best dominant, respectively.

Setting

$$\nu=0,\quad \mu=1,\quad s_1(z,\xi)=\frac{1+A_1\xi z}{1+B_1\xi z} \quad \text{and} \quad s_1(z,\xi)=\frac{1+A_2\xi z}{1+B_2\xi z}$$
$$(-1 \leqq B_2 < B_1 < A_1 \leqq A_2 \leqq 1)$$

in Theorem 10, we are led to the following corollary.

Corollary 12. *Let $\eta \in \mathbb{C}\setminus\{0\}$. Suppose that the function $f \in \mathcal{A}_p(\xi)$ such that*

$$\frac{D_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p} \neq 0$$

and

$$\left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^\eta \in \mathcal{H}_\xi(1,1)\cap\mathcal{Q}_\xi.$$

Suppose also that $\Delta(z,\xi)$ defined in (13) is univalent in $\mathbb{U}$ for all $\xi \in \overline{\mathbb{U}}$. If

$$\frac{1+A_1\xi z}{1+B_1\xi z}+\frac{(A_1-B_1)\xi z}{(1+A_1\xi z)(1+B_1\xi z)} \prec\prec \Delta(z,\xi) \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z}+\frac{(A_2-B_2)\xi z}{(1+A_2\xi z)(1+B_2\xi z)},$$

then

$$\frac{1+A_1\xi z}{1+B_1\xi z} \prec\prec \left(\frac{\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{z^p}\right)^\eta \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z},$$

where

$$\frac{1+A_1\xi z}{1+B_1\xi z} \quad \textit{and} \quad \frac{1+A_2\xi z}{1+B_2\xi z}$$

are the best subordinant and the best dominant, respectively.

Finally, by combining Theorem 4 with Theorem 8, we have the following sandwich-type results.

Theorem 11. *Let $\alpha,\eta \in \mathbb{C}\setminus\{0\}$ and $\nu,\mu \in \mathbb{C}$ with $\nu+\mu \neq 0$. Also let the functions $s_1(z,\xi)$ and $s_2(z,\xi)$ be convex univalent in $\mathbb{U}\times\overline{\mathbb{U}}$ with*

$$s_1(0,\xi)=s_2(0,\xi)=1 \quad \textit{and} \quad \Re\left(\frac{\alpha}{\eta}\right)>0.$$

Suppose that the function $s_2(z,\xi)$ satisfies the following condition:

$$\Re\left(1+\frac{zs_2''(z,\xi)}{s_2'(z,\xi)}\right) > \max\{0,-\Re(\alpha)\}.$$

Suppose that the function $f \in \mathcal{A}_p(\xi)$ *such that*

$$\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\neq 0$$

and

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta \in \mathcal{H}_\xi(1,1)\cap \mathcal{Q}_\xi.$$

Suppose also that $\nabla(z,\xi)$ *defined in* (16) *is univalent in* $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$. *If*

$$\alpha s_1(z,\xi)+zs_1'(z,\xi) \prec\prec \nabla(z,\xi) \prec\prec s_2(z,\xi)+\frac{zs_2'(z,\xi)}{s_2(z,\xi)},$$

then

$$s_1(z,\xi) \prec\prec \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\right)^\eta \prec\prec s_2(z,\xi),$$

where $s_1(z,\xi)$ *and* $s_2(z,\xi)$ *are the best subordinant and the best dominant, respectively.*

Setting

$$s_1(z,\xi)=\frac{1+A_1\xi z}{1+B_1\xi z} \quad \text{and} \quad s_1(z,\xi)=\frac{1+A_2\xi z}{1+B_2\xi z} \quad (-1 \leqq B_2 < B_1 < A_1 \leqq A_2 \leqq 1)$$

in Theorem 11, we arrive at the following corollary.

Corollary 13. *Let* $\eta \in \mathbb{C}\setminus\{0\}$. *Suppose that the function* $f \in \mathcal{A}_p(\xi)$ *such that*

$$\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\neq 0$$

and

$$\left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\neq 0\right)^\eta \in \mathcal{H}_\xi(1,1)\cap \mathcal{Q}_\xi.$$

Suppose also that the function $\nabla(z,\xi)$ *defined in* (16) *is univalent in* $\mathbb{U}$ *for all* $\xi \in \overline{\mathbb{U}}$. *If*

$$\frac{1+A_1\xi z}{1+B_1\xi z}+\frac{(A_1-B_1)\xi z}{(1+A_1\xi z)(1+B_1\xi z)} \prec\prec \nabla(z,\xi) \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z}+\frac{(A_2-B_2)\xi z}{(1+A_2\xi z)(1+B_2\xi z)},$$

then

$$\frac{1+A_1\xi z}{1+B_1\xi z} \prec\prec \left(\frac{\nu\mathcal{D}_p^{\lambda,t}(n+q-1,m-1)\Omega_z^{(\lambda,p)}f(z,\xi)+\mu\mathcal{D}_p^{\lambda,t}(n+q,m)f(z,\xi)}{(\nu+\mu)z^p}\neq 0\right)^\eta \prec\prec \frac{1+A_2\xi z}{1+B_2\xi z},$$

where

$$\frac{1+A_1\xi z}{1+B_1\xi z} \quad \textit{and} \quad \frac{1+A_2\xi z}{1+B_2\xi z}$$

are the best subordinant and the best dominant, respectively.

5. Concluding Remarks and Observations

In our present investigation, we have considered a number of applications of the principles of strong differential subordination and strong differential superordination in Geometric Function Theory of Complex Analysis. We have made use of a general multiplier transformation in order to obtain several new strong differential subordination results and several new strong differential superordination results. In each of our results, we have the best subordinant and best dominant. By using using such general multiplier transformations as those that we have applied in this paper in the analysis of strong differential subordination and strong differential superordination, one can obtain many different properties of other subclasses of analytic function and univalent functions.

For the purpose to mainly motivate and significantly prepare the interested researchers on the subjects dealt with in this paper, we choose to conclude this presentation by referring them to several related recent developments (see, for example, [16], [24], [29], [31], [36], [38], [42], [44] and [45]) on differential subordinations and differential superordinations, multiplier transformations, as well as the associated sandwich-type results.

References

[1] E. H. Abd and W. G. Atshan, Strong subordination for p-valent functions involving a linear operator, *J. Phys. Conf. Ser.* **1818** (2021), Article ID 012113, 1–8.

[2] R. Aghalary and P. Arjomandinia, On a first order strong differential subordination and application to univalent functions, *Commun. Korean Math. Soc.* **37** (2022), 445–454.

[3] E. E. Ali, H. M. Srivastava, R. M. El-Ashwah and A. M. Albalahi, Differential subordination and differential superordination for classes of admissible multivalent functions associated with a linear operator, *Mathematics* **10** (2022), Article ID 4690, 1–20.

[4] S. M. Amsheri and V. Zharkova, Some strong differential subordinations obtained by fractional derivative operator, *Internat. J. Math. Anal.* **6** (2012), 2159–2172.

[5] L. Andrei and M. Choban, Some strong differential subordinations using a differential operator, *Carpathian J. Math.* **31** (2015), 143–156.

[6] J. A. Antonino, Strong differential subordination and applications to univalency conditions, *J. Korean Math. Soc.*, **43** (2006), 311–322.

[7] J. A. Antonino and S. Romaguera, Strong differential subordination to Briot-Bouquet differential equations, *J. Differ. Equ.* **114** (1994), 101–105.

[8] N. E. Cho, Strong differential subordination properties for analytic functions involving the Komatu integral operator, *Boundary Value Probl.* **2013** (2013), Article ID 44, 1–10.

[9] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion and argument properties of certain subclasses of meromorphic functions associated with a family of multiplier transformations, *J. Math. Anal. Appl.* **300** (2004), 505–520.

[10] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations, *Integral Transforms Spec. Funct.* **16** (2005), 647–658.

[11] N. E. Cho, O. S. Kwon and H. M. Srivastava, Strong differential subordination and superordination for multivalently meromorphic functions involving the Liu-Srivastava operator, *Integral Transforms Spec. Funct.* **21** (2010), 589–601.

[12] N. E. Cho and H. M. Srivastava, Subordinations by η-convex functions for a class of nonlinear integral operators, *Bull. Sci. Math.* **187** (2023), Article ID 103304, 1–16.

[13] M. P. Jeyaraman and T. K. Suresh, Strong differential subordination and superordination of analytic functions, *J. Math. Anal. Appl.* **385** (2012), 854–864.

[14] M. P. Jeyaraman, T. K. Suresh and E. K. Reddy, Strong differential subordination and superordination of analytic functions associated with Komatu operator, *Internat. J. Nonlinear Anal. Appl.* **4** (2013), 26–44.

[15] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.* **176** (1993), 138–147.

[16] H. Kaur, R. Brar and S. S. Billing, Certain theorems involving differential superordination and sandwich-type results, *Stud. Univ. Babeş-Bolyai Math.* **69** (2024), 535–552.

[17] A. Alb Lupaş, Certain strong differential subordinations using Sălăgean and Ruscheweyh operators, *Adv. Appl. Math. Anal.* **6** (2011), 27–34.

[18] A. Alb Lupaş, On special strong differential subordinations using multiplier transformation, *Appl. Math. Lett.* **25** (2012), 624–630.

[19] A. Alb Lupaş, On special strong differential superordinations using Sălăgean and Ruscheweyh operators, *J. Adv. Appl. Comput. Math.* **1** (2014), 28–34.

[20] A. Alb Lupaş, Applications of a multiplier transformation and Ruscheweyh derivative for obtaining new strong differential subordinations, *Symmetry* **13** (2021), Article ID 1312, 1–10.

[21] A. Alb Lupaş, New results on a fractional integral of extended Dziok-Srivastava operator regarding strong subordinations and superordinations, *Symmetry* **15** (2023), Article ID 1544, 1–19.

[22] A. Alb Lupaş and G. I. Oros, Strong differential superordination results involving extended Sălăgean and Ruscheweyh operators, *Mathematics* **9** (2021), Article ID 2487, 1–12.
[23] S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.
[24] A. A. Mostafa, T. Bulboacă and M. K. Aouf, Sandwich results for multivalent functions defined by generalized Srivastava-Attiya operator, *J. Fract. Calc. Appl.* **14** (2) (2023), Article ID 2, 1–11.
[25] G. I. Oros, Strong differential superordination, *Acta Univ. Apulensis Math. Inform.* **19** (2009), 101–106.
[26] G. Oros, Briot-Bouquet strong differential superordinations and sandwich theorems, *Math. Rep.* **12** (62) (2010), 277–283.
[27] G. I. Oros, On a new strong differential subordination, *Acta Univ. Apulensis Math. Inform.*, **32** (2012), 243–250.
[28] G. I. Oros and G. Oros, Strong differential subordination, *Turkish J. Math.* **33** (2009), 249–257.
[29] G. I. Oros, G. Oros and H. Ö. Güney, Introduction in third-order fuzzy differential subordination, *Hacet. J. Math. Statist.* **53** (2024), 1627–1641.
[30] A. Oshah and M. Darus, Strong differential subordination and superordination of new generalized derivative operator, *Korean J. Math.* **23** (2015), 503–519.
[31] A. Oshah and M. Darus, Subordination and superordination for certain analytic functions associated with Ruscheweyh derivative and a new generalised multiplier transformation, *Bul. Acad. Ştiinţe Repub. Mold. Mat.* **2022 (98)** (1) (2022) 22–34.
[32] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (1987), 1057–1077.
[33] R. Şendruţiu, Strong differential subordinations obtained by Ruscheweyh operator, *J. Comput. Anal. Appl.* **14** (2012), 328–340.
[34] T. N. Shanmugam, S. Sivasubramanian and H. M. Srivastava, Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations, *Integral Transforms Spec. Funct.* **17** (2006), 889–899.
[35] M. M. Soren, Sandwich results for analytic functions involving with iterations of the Owa-Srivastava operator and its combination, *Asian-Eur. J. Math.* **7** (2014), Article ID 1450063, 1–21.
[36] M. M. Soren and L.-I. Cotîrlă, Fuzzy differential subordination and superordination results for the Mittag-Leffler type Pascal distribution, *AIMS Math.* **9** (2024), 21053–21078.
[37] H. M. Srivastava, A. Prajapati and P. Gochhayat, Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator, *Appl. Math. Inform. Sci.* **12** (2018), 469–481.
[38] H. M. Srivastava, P. O. Sabir, K. I. Abdullah, N. H. Mohammed, N. Chorfi and P. O. Mohammed, A comprehensive subclass of bi-univalent functions defined by a linear combination and satisfying subordination conditions, *AIMS Math.* **8** (2023), 29975–29994.
[39] H. M. Srivastava and A. K. Wanas, Strong differential sandwich results of λ-pseudo-starlike functions with respect to symmetrical points, *Math. Moravica* **23** (2) (2019), 45–58.
[40] S. R. Swamy, Some strong differential subordinations using a new generalized multiplier transformation, *Acta Univ. Apulensis Math. Inform.* **34** (2013), 285–291.
[41] A. O. Tăut, Some strong differential subordinations obtained by Sălăgean differential operator, *Stud. Univ. Babeş-Bolyai Math.* **55** (2010), 221–228.
[42] S. Tan, A sufficient condition for second-order differential subordination of harmonic mappings, *Bull. Malays. Math. Sci. Soc.* **4** (2025), Article ID 61, 1–13.
[43] A. K. Wanas and A. H. Majeed, New strong differential subordination and superordination of meromorphic multivalent quasi-convex functions, *Kragujev. J. Math.* **44** (2020), 27–39.
[44] A. K. Wanas and H. M. Srivastava, Differential sandwich theorems for Bazilevič function defined by convolution structure, *Turkish J. Inequal.* **4** (2) (2020), 10–21.
[45] X.-Y. Wang and Y.-J. Liu, Subordination and superordination results for a certain of integral operator involving generalized Mittag-Leffler functions, *Chinese Quart. J. Math.* **38** (2023), 379–391.