Filomat 39:14 (2025), 4657–4665
https://doi.org/10.2298/FIL2514657S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterization of generalized strong convergence

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Abstract. In this paper, introducing the concept of $\mathcal{B}-$uniform integrability of sequences, we prove that a sequence is $\mathcal{B}-$strongly convergent if and only if it is $\mathcal{B}-$statistically convergent and $\mathcal{B}-$uniformly integrable where $\mathcal{B} = \left(B_j\right) = (b_{nk}(j))$ is a sequence of nonnegative regular summability matrices. We also give a criterion for $\mathcal{B}-$statistical convergence.

1. Introduction and Preliminaries

The main theme of using summability methods is to make a non-convergent sequence to converge. This leads us to study generalized strong convergence ([14]) and generalized statistical convergence ([16]). We recall that the generalization is made by means of a sequence $\mathcal{B} = \left(B_j\right)$ of nonnegative regular matrices $B_j = (b_{nk}(j))$. Kolk [12] proved some inclusion results between generalized strong convergence and generalized statistical convergence by using modulus functions under some restrictions. In the present paper, introducing the concept of $\mathcal{B}-$uniform integrability of sequences, we prove that the sequence $x = (x_k)$ is $\mathcal{B}-$strongly convergent if and only if it is $\mathcal{B}-$statistically convergent and $\mathcal{B}-$ uniformly integrable. This result is an extension of those of Kolk [12], Connor [2] and [3] and, Khan and Orhan [11]. In order to prove our main result we first characterize $\mathcal{B}-$uniform integrability. In the final section we give a criterion for $\mathcal{B}-$statistical convergence which is an analog of Shoenberg's result [20]. We also refine our result by using the idea of Salat [19], (see, also [5], [9]).

2. $\mathcal{B}-$Strong Convergence and $\mathcal{B}-$Uniform Integrability

Let $\mathcal{B} = \left(B_j\right)$ be a sequence of infinite matrices $B_j = (b_{nk}(j))$. A sequence $x = (x_k)$ is called $\mathcal{B}-$summable to L, briefly $\mathcal{B}-\lim x = L$ ([21]) if $\lim_n \sum b_{nk}(j)\, x_k = L$, uniformly in j. Of course we assume here that the series is convergent for each n and j. Sometimes such a method $\mathcal{B}$ is called a sequential method of summability. A sequential method of summability $\mathcal{B}$ is called regular if every convergent sequence $x = (x_k)$ is $\mathcal{B}-$summable and $\mathcal{B}-\lim x = \lim x$. It is well-known ([1], [21]) that the method $\mathcal{B} = \left(B_j\right)$ with $B_j = (b_{nk}(j))$ is regular if and only if

i) $\lim_n b_{nk}(j) = 0$ for all k, uniformly in j,

2020 *Mathematics Subject Classification*. Primary xxxxx mandatory; Secondary xxxxx, xxxxx optionally.
Keywords. Statistical convergence, Uniform integrability, Strong convergence.
Received: 06 February 2025; Revised: 10 March 2025; Accepted: 11 March 2025
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ii) $\lim_n \sum_k b_{nk}(j) = 1$, uniformly in j,

iii) $\sum_k |b_{nk}(j)| < \infty$ for all n and j; and there exists an integer m such that $\sup_{j\in\mathbb{N},\, n\geq m} \sum_k |b_{nk}(j)| < \infty$.

Throughout the paper we write

$$\|\mathcal{A}\| := \sup_{n,j} \sum_k \left| b_{nk}^{(j)} \right| < \infty$$

to mean that, there exists a constant M such that $\sum_k |b_{nk}(j)| \leq M$ (for all n, for all j) and the series $\sum_k b_{nk}(j)$ converges for each n, uniformly in j.We note in passing that $\mathcal{B}-$summability is closely related to almost convergence [13] and ergodic theory [18].

By $\mathcal{R}^+$ we denote the set of all regular sequential methods $\mathcal{B}$ with $b_{nk}(j) \geq 0$ for all n, k, j. Throughout the paper we assume that $\mathcal{B} \in \mathcal{R}^+$. By an index set we mean a set $K = \{k_j\} \subset \mathbb{N}$, where $k_j < k_{j+1}$ for all j. An index set K has $\mathcal{B}$ density provided that $\mathcal{B} \in \mathcal{R}^+$ and

$$\lim_n \sum_{k\in K} b_{nk}(j) = d, \text{ uniformly in } j$$

(see Kolk [12], for details). In this case we write $\delta_{\mathcal{B}}(K) = d$.

Let $\mathcal{B} \in \mathcal{R}^+$. A sequence $x = (x_k)$ is called $\mathcal{B}-$statistically convergent to number L, if for every $\varepsilon > 0$

$$\delta_{\mathcal{B}}(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

(see, e.g., Kolk [12], Mursaleen and Edely [16]). For statistical convergence see, e.g. Miller [15], Fridy [6], Fridy and Orhan [7], [8], Connor [2] and [3].

Maddox [14], Mursaleen [17] introduced the concept of $\mathcal{B}-$strong summability: a sequence $x = (x_k)$ is called $\mathcal{B}-$strongly summable to a number L if

$$\lim_n \sum_k b_{nk}(j) |x_k - L| = 0, \text{uniformly in } j.$$

In this section we assume that $\mathcal{B} \in \mathcal{R}^+$ and $\sum_k b_{nk}(j) = 1$ for all n and j. Such class of sequential methods will be denoted by $\mathcal{M}^+$.

We now present a definition concerning $\mathcal{B}-$uniform integrability which is an analog of Khan and Orhan's definition [11].

Definition 2.1. *Let $\mathcal{B} \in \mathcal{M}^+$. A sequence $x = (x_k)$ is called $\mathcal{B}-$uniformly integrable if*

$$\lim_{t\to\infty} \sup_{n,j} \sum_{k:|x_k|>t} b_{nk}(j) |x_k| = 0, \textit{uniformly in } j.$$

By $\mathcal{U}_{\mathcal{B}}$ we denote the space of all $\mathcal{B}-$uniformly integrable sequences. Observe that any bounded sequence is $\mathcal{B}-$uniformly integrable.

Next we give some conditions which are equivalent to $\mathcal{B}-$uniform integrability.

We note that the following result is motivated by the Summer Seminar lectures given by M. K. Khan [10].

Theorem 2.2. *Let $\mathcal{B} \in \mathcal{M}^+$. A sequence $x = (x_k)$ is $\mathcal{B}-$uniformly integrable if and only if*

i) $\sup_{n,j} \sum_{k=1}^{\infty} b_{nk}(j) |x_k| < \infty$, *ii*) *given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *so that for any subset* E *of positive integers for which*

$$\sup_{n,j} \sum_{k \in E} b_{nk}(j) < \delta$$

we have

$$\sup_{n,j} \sum_{k} b_{nk}(j) |x_k| < \varepsilon.$$

Proof. *Let* $x \in \mathcal{U}_{\mathcal{B}}$. *Then for* $\varepsilon > 0$, *choose a positive real number* t_0 *such that for every* $t \geq t_0$,

$$\sup_{n,j} \sum_{k:|x_k|>t} b_{nk}(j) |x_k| < \frac{\varepsilon}{2}. \tag{2.1}$$

Hence we have

$$\sum_{k} b_{nk}(j) |x_k| \leq \sum_{k:|x_k|\leq t_0} b_{nk}(j) |x_k| + \sum_{k:|x_k|>t_0} b_{nk}(j) |x_k|. \tag{2.2}$$

Now using (2.1) *and* (2.2) *we see that*

$$\sup_{n,j} \sum_{k} b_{nk}(j) |x_k| \leq t_0 \sup_{n,j} \sum_{k} b_{nk}(j) + \frac{\varepsilon}{2} < \infty.$$

This proves (*i*). *We now prove* (*ii*). *To get this, take* $\delta = \dfrac{\varepsilon}{2t_0}$, $(t_0 \neq 0)$, *and for any subset* $E \subseteq \mathbb{N}$, *we let*

$$\sup_{n,j} \sum_{k \in E} b_{nk}(j) < \delta. \tag{2.3}$$

Observe now that

$$\sum_{k \in E} b_{nk}(j) |x_k| \leq \sum_{k \in E:|x_k|>t_0} b_{nk}(j) |x_k| + \sum_{k \in E:|x_k|\leq t_0} b_{nk}(j) |x_k|$$

Applying the operator "$\sup_{n,j}$*" on both sides and considering* (2.3) *we get*

$$\begin{aligned} \sup_{n,j} \sum_{k \in E} b_{nk}(j) |x_k| &\leq \frac{\varepsilon}{2} + t_0 \sup_{n,j} \sum_{k \in E} b_{nk} \\ &\leq \frac{\varepsilon}{2} + t_0 \delta \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

from which (*ii*) *follows.*
We now prove sufficiency. By (*i*) *we let*

$$M := \sup_{n,j} \sum_{k} b_{nk}(j) |x_k| < \infty.$$

Now we have

$$\sup_{n,j} \sum_{k:|x_k|>t} b_{nk}(j) |x_k| \leq \sup_{n,j} \sum_{k} b_{nk}(j) |x_k| = M.$$

Hence we get boundedness in t. *By* (ii), *for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\sup_{n,j} \sum_{k \in E} b_{nk}(j) < \delta \text{ implies } \sup_{n,j} \sum_{k \in E} b_{nk}(j) |x_k| < \varepsilon.$$

So for this $\varepsilon > 0$, *take* $t_0 = \frac{M}{\delta}$. *Now define the set* $E(t) := \{k : |x_k| \geq t\}$. *Observe that for any* $t \geq t_0$ *we have*

$$\begin{aligned} \sup_{n,j} \sum_{k \in E(t)} b_{nk}(j) &\leq \frac{1}{t} \sup_{n,j} \sum_{k} b_{nk}(j) |x_k| \\ &\leq \frac{M}{t} < \frac{M}{t_0} = \delta. \end{aligned}$$

Hence (ii) *holds for this choice of* $E(t)$ *and we conclude that*

$$\sup_{n,j} \sum_{k \in E(t)} b_{nk}(j) |x_k| < \varepsilon, \text{ for } t \geq t_0.$$

This yields that $x \in \mathcal{U}_{\mathcal{B}}$. □

We now characterize $\mathcal{B}-$strong convergence via $\mathcal{B}-$uniform integrability and $\mathcal{B}-$statistical convergence. It also extends Kolk's result [12] (see also, [9], [11]).

Theorem 2.3. *Let* $\mathcal{B} \in \mathcal{M}^+$ *and let* $x = (x_k)$ *be a sequence of real numbers. The sequence* x *is* $\mathcal{B}-$*strongly convergent to zero if and only if* $st_{\mathcal{B}} - \lim x = 0$ *and* $x \in \mathcal{U}_{\mathcal{B}}$.

Proof. *Assume* $st_{\mathcal{B}} - \lim x = 0$ *and* $x \in \mathcal{U}_{\mathcal{B}}$. *For every* $\varepsilon > 0$ *and* $t > 0$ *we have*

$$\begin{aligned} \sum_{k:|x_k| \leq t} b_{nk}(j) |x_k| &\leq \sum_{k:\varepsilon < |x_k| \leq t} b_{nk}(j) |x_k| + \sum_{k:|x_k| \leq \min(t,\varepsilon)} b_{nk}(j) |x_k| \\ &\leq t \sum_{k:|x_k| > \varepsilon} b_{nk}(j) + \varepsilon \sum_{k=1}^{\infty} b_{nk}(j) \\ &\leq t \sum_{k:|x_k| > \varepsilon} b_{nk}(j) + \varepsilon \end{aligned}$$

Applying the operator "$\limsup_n \sup_j$*" on both sides we conclude that*

$$\limsup_n \sup_j \sum_{k:|x_k| \leq t} b_{nk}(j) |x_k| \leq 0 + \varepsilon. \tag{2.4}$$

On the other hand one can have

$$\sum_{k} b_{nk}(j) |x_k| \leq \sum_{k:|x_k| > t} b_{nk}(j) |x_k| + \sum_{k:|x_k| \leq t} b_{nk}(j) |x_k|.$$

By (2.4) we have

$$\begin{aligned} \limsup_n \sup_j \sum_{k} b_{nk}(j) |x_k| &\leq \varepsilon + \limsup_n \sup_j \sum_{k:|x_k| > t} b_{nk}(j) |x_k| \\ &\leq \varepsilon + \sup_{n,j} \sum_{k:|x_k| > t} b_{nk}(j) |x_k| \\ &\leq \varepsilon + \lim_{t \to \infty} \sup_{n,j} \sum_{k:|x_k| > t} b_{nk}(j) |x_k| \\ &= \varepsilon + 0 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we conclude that

$$\lim_n \sup_j \sum_k b_{nk}(j)|x_k| = 0,$$

i.e., x is $\mathcal{B}-$strongly convergent to zero.
To prove the converse, assume that

$$\lim_n \sup_j \sum_k b_{nk}(j)|x_k| = 0. \tag{2.5}$$

Hence given $\varepsilon > 0$, we immediately have

$$\sum_{k:|x_k|>\varepsilon} b_{nk}(j) = \sum_{k:\frac{|x_k|}{\varepsilon}>1} b_{nk}(j) \leq \frac{1}{\varepsilon}\sum_{k=1}^{\infty} b_{nk}(j)|x_k|$$

It follows from (2.5) that

$$\lim_n \sup_j \sum_{k:|x_k|>\varepsilon} b_{nk}(j) = 0,$$

i.e.,

$$st_{\mathcal{B}} - \lim x = 0.$$

By (2.5), given $\varepsilon > 0$ get a natural number $N = N(\varepsilon)$ so that,

$$\sup_j \sum_k b_{nk}(j)|x_k| < \varepsilon, \text{ for all } n \geq N.$$

Since $\sup_j \sum_k b_{nk}(j)|x_k| < \infty$ for each $n = 1, 2, ..., N-1$, choose a positive integer K large enough so that $\sup_j \sum_{k>K} b_{nk}(j)|x_k| < \varepsilon$ for all $n < N$. Hence,

$$\sup_{n,j} \sum_{k:|x_k|>t} b_{nk}(j)|x_k| < \varepsilon$$

provided that $t > \max\{|x_1|, ..., |x_k|\}$. This proves that x is $\mathcal{B}-$uniformly integrable. □

3. A Criterion for $\mathcal{B}-$statistical convergence

Schoenberg [20] proved a criterion for ordinary statistical convergence (see, also [5], [9], [22]).In the same vein this section presents a criterion for $\mathcal{B}-$statistical convergence. We will also improve this result later on.

The next result uses the same technique as in [20], so we omit its proof (see, also [4]).

Lemma 3.1. *Let $\mathcal{B} \in \mathcal{R}^+$ and let $x = (x_k)$ be a sequence of real numbers and $st_{\mathcal{B}} - \lim x = L$. If the function $g : \mathbb{R} \to \mathbb{R}$ is continuous at $y = L$, then $st_{\mathcal{B}} - \lim g(x) = g(L)$.*

Below we give an analog of Schoenberg's criterion. The main tool in proving this result is the Fourier transformation.

Theorem 3.2. *Let $\mathcal{B} \in \mathcal{R}^{+}$ and let $x = (x_k)$ be a sequence. Then $st_{\mathcal{B}} - \lim x = L$ if and only if*

$$\lim_n \sup_j \sum_{k=1}^{\infty} b_{nk}(j)\, e^{itx_k} = e^{itL}, \text{ for every real } t. \tag{3.1}$$

Proof. Let $st_{\mathcal{B}} - \lim x = L$, and let $g(x) = e^{itx}$ for a fixed $t \in \mathbb{R}$. Notice that g is a continuous function of x. then we have by Lemma 3.1 that $st_{\mathcal{B}} - \lim e^{itx_k} = e^{itL}$. Since $\left(e^{itx_k}\right)$ is a bounded sequence, it follows from Theorem 2.2 that $\left(e^{itx_k}\right)$ is $\mathcal{B}$–strongly summable. As the sequence $\mathcal{B} = (b_{nk}(j))$ satisfies the conditions of Maddox [14] we conclude that $\left(e^{itx_k}\right)$ is $\mathcal{B}$–summable , i.e.,

$$\lim_n \sup_j \sum_k b_{nk}(j)\, e^{itx_k} = e^{itL}.$$

Conversely assume that (3.1) holds. Following [20] we define a continuous function M by

$$M(y) = \begin{cases} 0 & ; \text{ if } \ y \le -1 \\ 1 + y & ; \text{ if } \ -1 < y < 0 \\ 1 - y & ; \text{ if } \ 0 \le y < 1 \\ 0 & ; \text{ if } \ y \ge 1. \end{cases}$$

Since the function M is a Lebesgue integrable function, its Fourier transformation is given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} M(y)\, e^{-ity} dy, \quad -\infty < t < \infty,$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin\left(\frac{t}{2}\right)}{\frac{t}{2}} \right).$$

Furthermore, inverse Fourier transformation of the function f is

$$M(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ity} dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{t}{2}\right)}{\frac{t}{2}} \right)^2 e^{ity} dt. \tag{3.2}$$

In order to complete the proof, we need to show that $st_{\mathcal{B}} - \lim x = L$ in which $L = 0$. It is enough to consider the case $K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k| \ge \varepsilon\}$. Substituting $\frac{t}{\varepsilon} = u$, we get

$$M\left(\frac{y}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{\varepsilon t}{2}\right)}{\frac{\varepsilon t}{2}} \right)^2 e^{ity} dt.$$

Hence

$$\sum_k b_{nk}(j)\, M\left(\frac{x_k}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{\varepsilon t}{2}\right)}{\frac{\varepsilon t}{2}} \right)^2 \left(\sum_k b_{nk}(j)\, e^{itx_k} \right) dt.$$

Observe that (3.2) is an absolutely convergent integral. Now the Lebesgue dominated convergence theorem yields that

$$\begin{aligned}
&\lim_n \sup_j \sum_k b_{nk}(j) M\left(\frac{x_k}{\varepsilon}\right) \\
&= \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{\varepsilon t}{2}\right)}{\frac{\varepsilon t}{2}} \right)^2 \left(\lim_n \sup_j \sum_k b_{nk}(j) e^{itx_k} \right) dt \\
&= \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{\varepsilon t}{2}\right)}{\frac{\varepsilon t}{2}} \right)^2 e^{itL} dt \\
&= M(0) \\
&= 1. \qquad (3.3)
\end{aligned}$$

On the other hand by the definition of M, we have

$$\begin{aligned}
\sum_k b_{nk}(j) M\left(\frac{x_k}{\varepsilon}\right) &= \sum_{k-1<\frac{x_k}{\varepsilon}<0} b_{nk}(j) M\left(\frac{x_k}{\varepsilon}\right) + \sum_{k:0\leq\frac{x_k}{\varepsilon}<1} b_{nk}(j) M\left(\frac{x_k}{\varepsilon}\right) \\
&\leq \sum_{k\in\mathbb{N}} b_{nk}(j) - \sum_{k\in K} b_{nk}(j).
\end{aligned}$$

Using the regularity of $\mathcal{B}$ and (3.3) we conclude that

$$\lim_n \sup_j \sum_{k\in K} b_{nk}(j) = 0,$$

this proves the theorem. □

We now show that the condition (3.1) can be weakened when x is in the class

$$S^*_{\mathcal{B}} := \left\{ x = (x_k) : \left(\sup_{n,j} \sum_k b_{nk}(j) |x_k| \right) \in l^\infty \right\}.$$

The next result is an analogue of Salat's result [19] (see also, [5], [9]).

Theorem 3.3. *Let $\mathcal{B} \in \mathcal{R}^+$ and let $x \in S^*_{\mathcal{B}}$. Then $st_{\mathcal{B}} - \lim x = L$ if and only if*

$$\lim_n \sup_j \sum_k b_{nk}(j) e^{itx_k} = e^{itL} \qquad (3.4)$$

for each rational number t.

Proof. *By Theorem 3.2 we get the necessity. Assume now that (3.4) holds for each rational number t. Let t_0 be an arbitrary real number. We are going to prove that*

$$\lim_n \sup_j \sum_k b_{nk}(j) e^{it_0 x_k} = e^{it_0 L}. \qquad (3.5)$$

Let

$$C_{nj}(t_0, t) = \sum_k b_{nk}(j) e^{it_0 x_k} - \sum_k b_{nk}(j) e^{itx_k}.$$

Observe that

$$\left|C_{nj}(t_0,t)\right| \leq \sum_k b_{nk}(j)\ \sqrt{(\cos t_0 x_k - \cos t x_k)^2 + (\sin t_0 x_k - \sin t x_k)^2}.$$

The mean value theorem yields that

$$\left|C_{nj}(t_0,t)\right| \leq |t - t_0| \sum_k b_{nk}(j) |x_k| .$$

As $x \in S^*_{\mathcal{B}}$, *there is* $M > 0$ *so that*

$$\left|C_{nj}(t_0,t)\right| \leq |t - t_0| M. \tag{3.6}$$

Hence we see that

$$\left|\sum_k b_{nk}(j) e^{it_0 x_k} - e^{it_0 L}\right| \leq \left|\sum_k b_{nk}(j) e^{itx_k} - e^{itL}\right| + \left|e^{itL} - e^{it_0 L}\right| + \left|C_{nj}(t_0,t)\right| .$$

Let $\varepsilon > 0$*. By the continuity of* $g(x) = e^{ixL}$*, we get a rational number* t *so that*

$$\left|e^{itL} - e^{it_0 L}\right| < \frac{\varepsilon}{3}, \tag{3.7}$$

and by (3.6) we have

$$\left|C_{nj}(t_0,t)\right| \leq \frac{\varepsilon}{3}. \tag{3.8}$$

It follows from (3.4), (3.7) and (3.8) that (3.5) holds. Since $t_0 \in \mathbb{R}$ *is arbitrary (3.4) holds for every real number* t*. So, by Theorem 3.2, we have that*

$$st_{\mathcal{B}} - \lim x = L.$$

□

References

[1] H. T. Bell, *Order summability and almost convergence*, Proceedings of the American Mathematical Society, 38 (3), 548-552, 1973.
[2] J. Connor, *The statistical and strong p−Cesàro convergence of sequences*, Analysis, 5, 301-313, 1988.
[3] J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canadian Mathematical Bulletin; 32, 194-198, 1989.
[4] J. Connor and K. G. Grosse-Erdmann, *Sequential definitons of continuity for real functions*, Rocky Mountain J. Math. 35, no.1, 93-121, 2003.
[5] K. Demirci, *A Criterion for A-statistical convergence*, Indian J. Pure and Appl. Math. 29 (5), 559-564, 1988.
[6] J. A. Fridy, *On statistical convergence*, Analysis, 5, 301-313, 1985.
[7] J. A. Fridy and C. Orhan, *Statistical limit superior and inferior*, Proceedings of the American Mathematical Society. 125 (2), 3625-3631, 1997.
[8] J. A. Fridy and C. Orhan, *Lacunary Statistical Convergence*, Pacific Journal of Mathematics 160, 43–51, 1993.
[9] M. Gülfırat, *A-Statistical Convergence with a Rate and Applications to Approximation*, Filomat, 36 (15), 5323-5335, 2022.
[10] M. K. Khan, *Summer Seminar talks on "Probabilisitic Methods in Summability Theory"*, Ankara University, 21 August-1 September 2006.
[11] M. K. Khan and C. Orhan, *Characterizations of strong and statistical convergences*, Publicationes Mathematicae Debrecen, 76 (1-2): 77-88, 2010.
[12] E. Kolk, *Inclusion relations between the statistical convergence and strong summability*, Acta et Comm. Univ. Tartu. Math. 2, 39-54, 1998.
[13] G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Mathematica, 80 (1), 167-190, 1948.

[14] I. J. Maddox,*On strong almost convergence*, Math. Proc. Camb. Phil. Soc., 83, 345–350, 1979.
[15] H. I. Miller,*A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. 347, 1811-1819, 1995.
[16] M. Mursaleen and O. H. H. Edely, *Generalized statistical convergence*, Information Sciences 162, 287-294, 2004.
[17] M. Mursaleen, *On Strongly $F_{\mathcal{B}}$-Summable Sequences*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., c. 28, 1979.
[18] G. Oğuz and C. Orhan, *Mean ergodic type theorems*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68 (2), 2019.
[19] T. Salat, *On statistically convergent sequences of real numbers*, Math Slovaca, 30, no.2, 139-140, 1980.
[20] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66 , 361-375, 1959.
[21] M. Stieglitz, *Eine verallgemeinerung des begriffs der fast konvergenz*, Math. Japon,18 (1), 53-70, 1973.
[22] N. Şahin Bayram, *Criteria for statistical convergence with respect to power series methods*, Positivity, 25 (3), 1097-1115, 2021.