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Some Tauberian theorems for the statistical limit and statistical summability of $\ell^{(k)}$ mean of functions

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Abstract. If a function s has a limit ξ at ∞, then the statistical limit of function s exists and equals ξ. However, the converse implication is not always true.

In this paper, we present Tauberian conditions on general logarithmic control modulo that allow us to obtain the existence of ordinary limit of a function that has statistical limit ξ.

Additionally, we introduce the statistical summability of $\ell^{(k)}$ mean of functions for each nonnegative integer k and provide conditions for Tauberian theorems. Our theorems generalize some well-known Tauberian theorems in the literature.

1. Introduction

Given a real- or complex-valued function f on $[1,\infty)$ which is integrable in Lebesgue's sense over interval $[1,\infty)$. In this case we write $f \in L^1_{loc}[1,\infty)$. We set

$$s(x) = \int_1^x f(t)dt, \tag{1}$$

for $x > 1$. The logarithmic mean of function s in (1) is denifed by

$$\ell(x) = \frac{1}{logx}\int_1^x \frac{s(t)}{t}dt. \tag{2}$$

The function s is said to be logarithmic summable at ∞, or equivalently, the integral $\int_1^\infty f(t)dt$ is said to be logarithmic summable (in symbolically summable $(L,1)$), if the finite limit of (2)

$$\lim_{x\to\infty} \ell(x) = \xi \tag{3}$$

exists. The logarithmic summability method is regular. So the existence of limit

$$\lim_{x\to\infty} s(x) = \xi \tag{4}$$

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implies the existence of limit of (3).

On the other hand, for a real- or complex-valued function $f \in L^1_{loc}(\mathbb{R}_+)$, the $(C,1)$ mean of a function $s(x) = \int_0^x f(t)dt$ for $x > 0$ is denoted by $\sigma(x)$ and defined by

$$\sigma(x) = \frac{1}{x}\int_0^x s(t)dt. \tag{5}$$

The function s is said to be $(C,1)$ summable at ∞, or in other words, the integral $\int_0^\infty f(t)dt$ is said to be $(C,1)$ summable, if the limit of (5)

$$\lim_{x\to\infty} \sigma(x) = \xi \tag{6}$$

exists. When we research the relationship between the logarithmic summability method and the $(C,1)$ summability method, we see that the logarithmic summability method is more effective than the $(C,1)$ summability method [5].

If we take the logarithmic mean of s by k times for each nonnegative integer k, we obtain the $\ell^{(k)}$ mean of function s and it is defined by

$$\ell^{(k)}(x) = \frac{1}{logx}\int_1^x \frac{\ell^{(k-1)}}{t}dt. \tag{7}$$

Taking $k = 1$ in (7), we get $\ell(x)$ and for $k = 0$, we assume that $\ell^{(0)}(x) = s(x)$. The function s is said to be summable $\ell^{(k)}$ at ∞, or equivalently, the integral $\displaystyle\int_1^\infty f(t)dt$ is said to be summable $\ell^{(k)}$, if the finite limit

$$\lim_{x\to\infty} \ell^{(k)}(x) = \xi \tag{8}$$

exists.

It is known that $\ell^{(k)}$ summability method is regular too. So the existence of limit of (4) implies the existence of limit (8). Moreover it is clear that the existence of limit (3) implies the existence of limit (8). The converse implications of all these inclusions are generally not true. For example the integral $\displaystyle\int_1^\infty (2\cos x + logx(\cos x - x\sin x))dx$ is not convergent. Furthermore this integral is not logarithmic summable. However it is $\ell^{(k)}$ summable to $-\sin 1$ for $k = 2$. Thus, it is seen that the $\ell^{(k)}$ summability method is more effective.

For a function s, we have the following identity which is known as the Kronecker identity in the sense of logarithmic summability method:

$$s(x) - \ell(x) = \tau_\ell(x), \tag{9}$$

where $\tau_\ell(x) = \frac{1}{\log x}\int_1^x f(t)\log t dt$. For nonnegative integer k, we have the following identity:

$$\ell^{(k)}(x) - \ell^{(k+1)}(x) = \tau_\ell^{(k)}(x), \tag{10}$$

where

$$\tau_\ell^{(k)}(x) = \begin{cases} \frac{1}{\log x}\int_1^x \frac{\tau_\ell^{(k-1)}(t)}{t}dt, & k \geq 1 \\ \tau_\ell(x), & k = 0. \end{cases}$$

The identy (10) is obtained by taking the $\ell^{(k)}$ mean of both sides of identiy (9).

The terms of slow decrease and slow oscillation for $(C,1)$ summability method of sequences were introduced by Schmidt [13] and Hardy [4] respectively. The definitions of slowly decreasing and slowly oscillating function with respect to logarithmic summability method were introduced by Móricz [5] based on the definitions of slow decrease and slow oscillation for $(C,1)$ summability method respectively. Additionally, in [5], the relationships between one-sided boundedness and boundedness of $x \log x f(x)$ with these concepts was presented.

A function $s : [1,\infty) \to \mathbb{R}$ is said to be slowly decreasing with respect to logarithmic summability method if for every $\epsilon > 0$ there exist $x_0 = x_0(\epsilon) > 1$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$s(t) - s(x) \geq -\epsilon \quad \text{whenever} \quad x_0 \leq x < t \leq x^{\lambda}. \tag{11}$$

Moreover a function s is slowly decreasing with respect to logarithmic summability method if and only if

$$\lim_{\lambda\to 1^+} \liminf_{x\to\infty} \inf_{x<t\leq x^{\lambda}} (s(t) - s(x)) \geq 0. \tag{12}$$

A function $s : [1,\infty) \to \mathbb{C}$ is said to be slowly oscillating with respect to logarithmic summability method if for every $\epsilon > 0$ there exist $x_0 = x_0(\epsilon) > 1$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$|s(t) - s(x)| \leq \epsilon \quad \text{whenever} \quad x_0 \leq x < t \leq x^{\lambda}. \tag{13}$$

Furthermore a function s is slowly oscillating with respect to logarithmic summability method if and only if

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \sup_{x<t\leq x^{\lambda}} |s(t) - s(x)| = 0. \tag{14}$$

The concepts of the classical and general control modulo for sequences were introduced by Dik [1]. These concepts were introduced by Okur and Totur [8] for the logarithmic summability method of integrals.

The classical logarithmic control modulo for the functions and general logarithmic control modulo of integer order $p \geq 1$ of s are denoted by

$$\omega_{\ell}^{(0)}(x) = x \log x f(x) \tag{15}$$

$$\omega_{\ell}^{(p)}(x) = \omega_{\ell}^{(p-1)}(x) - \ell\left(\omega^{(p-1)}(x)\right) \tag{16}$$

respectively.

The idea of statistical convergence was introduced by Fast [2]. The definition of statistical limit for a function is given in [6] as follows.

Let s be a real- or complex-valued function which is measurable in Lebesgue's sense on some interval (x_0,∞), where $x_0 \geq 0$. A function s has statistical limit at ∞, if there exists a number ξ such that for every $\epsilon > 0$,

$$\lim_{v\to\infty} \frac{1}{v-u} |\{x \in (u,v) : |s(x) - \xi| > \epsilon\}| = 0, \tag{17}$$

where the notion $|\{\cdot\}|$ indicates the Lebesgue measure of the set $\{\cdot\}$. If (17) exists, then we write

$$st - \lim_{x\to\infty} s(x) = \xi. \tag{18}$$

Additionaly the following limits are weaker conditions than (18).

$$st-\lim_{x\to\infty}\ell(x)=\xi, \tag{19}$$

$$st-\lim_{x\to\infty}\ell^{(k)}(x)=\xi \tag{20}$$

and

$$st-\lim_{x\to\infty}\sigma(x)=\xi. \tag{21}$$

It is evident that the implication of "(4) ⇒ (18)" is true. However the converse of implication is not general true.

The converse implications "(3) ⇒ (4)", "(8) ⇒ (4)", "(18) ⇒ (4)", "(19) ⇒ (4)", "(20) ⇒ (4)" and "(21) ⇒ (4)" can be made true with the help of some additional conditions. In the literature, these additional conditions are known as 'Tauberian conditions', and the resulting theorems are known as 'Tauberian theorems' after A. Tauber [14], who was the first to prove such theorems.

In Tauberian theory, logarithmic summability method for the sequences [9, 10, 17] and functions [3] has been studied by many authors. In [5], the following two results are given.

Corollary 1.1. *Suppose that a real-valued function* $s \in L^1_{loc}[1,\infty)$ *is slowly decreasing with respect to summability* $(L,1)$*, then the implication* (3) ⇒ (4) *holds true.*

Corollary 1.2. *Suppose that a complex-valued function* $s \in L^1_{loc}[1,\infty)$ *is slowly oscillating with respect to summability* $(L,1)$*, then the implication* (3) ⇒ (4) *holds true.*

In the first result, the existence of the ordinary limit of real-valued function s was obtained from its logarithmic summability under condition of slow decrease. In the second result, the existence of the ordinary limit of complex-valued function s was obtained from its logarithmic summability under condition of slow oscillation.

In [18] and [8], the ordinary limit of function s was achieved by further weakening the conditions. In these works, the authors obtained the ordinary limit of function s by the help of logarithmic Kronecker identity and conditions on general logarithmic control modulo.

With the introduction of statistical convergence, many mathematicians turned their attention in this direction, and several Tauberian theorems have been obtained using this method for sequences [11, 12, 15, 16, 19]. When we investigate the situation in terms of functions, we see the work [6] and [7]. In these works, relationship between statistical limit and $(C,1)$ mean is examined. Also statistical extension of classical Tauberian theorems for the logarithmic summability method was introduced and following Tauberian theorems are proven.

Theorem 1.3. *If the statistical limit of s exists and equals* ξ*, and s is bounded, then s is* $(C,1)$ *summable.*

Theorem 1.4. *If a real-valued function s is measurable and slowly decreasing with respect to summability* $(L,1)$*, then the implication* (18) ⇒ (4) *holds true.*

Theorem 1.5. *If a complex-valued function s is measurable and slowly oscillating with respect to summability* $(L,1)$*, then the implication* (18) ⇒ (4) *holds true.*

Theorem 1.6. *If a real-valued function* $s \in L^1_{loc}[1,\infty)$ *is slowly decreasing with respect to summability* $(L,1)$*, then the implication* (19) ⇒ (4) *holds true.*

Theorem 1.7. *If a complex-valued function* $s \in L^1_{loc}[1,\infty)$ *is slowly oscillating with respect to summability* $(L,1)$*, then the implication* (19) ⇒ (4) *holds true.*

In our work, we prove some Tauberian theorems for real and complex-valued functions. In our theorems we obtain the ordinary limit of function s from existence of statistical limit of $\ell^{(k)}$ mean of s. Also we achieve the ordinary limit by the help of the condition on general logarithmic control modulo. In the paper, weaker conditions than those used in existing theorems in [6] are provided. Therefore, our theorems generalize some Tauberian theorems in [6]. If we want to classify our main results in terms of the implications we mentioned earlier, we can say that we shall obtain Tauberian conditions for the implications "(18) ⇒ (4)" and "(20) ⇒ (4)" in our theorems.

2. Auxiliary results

The following two results are given in [6].

Lemma 2.1. *If a complex-valued function s is such that the condition (13) is satisfied only for $\epsilon := 1$, where $x_0 > 1$ and $\lambda > 1$, there exist a constant B_2 such that*

$$\frac{1}{\log t}\int_{x_0}^{t}\frac{|s(t)-s(x)|}{x}dx \leq B_2 \quad whenever \quad t > x_0^{\lambda}. \tag{22}$$

Lemma 2.2. *If the real-valued function s is slowly decreasing, then so is its logarithmic mean $\ell(x)$.*

Motivated by this results we can establish the following lemma.

Lemma 2.3. *If the complex-valued function s is slowly oscillating, then so is its logarithmic mean $\ell(x)$.*

Proof. Let $0 < \epsilon < 1$ and $x_0 \leq x < t \leq x^{\lambda}$, where $x_0 = x_0(\epsilon)$ and $\lambda = \lambda(\epsilon)$ occur in (13) and let

$$1 < \lambda \leq 1 + \frac{\epsilon}{\max\{1, B_2\}} \tag{23}$$

where B_2 from Lemma 2.1. By (2) we have

$$\begin{aligned}
|\ell(t)-\ell(x)| &= \left|\frac{1}{\log t}\int_1^t \frac{s(v)}{v}dv - \frac{1}{\log x}\int_1^x \frac{s(v)}{v}dv\right| \\
&= \left|\left(\frac{1}{\log x}-\frac{1}{\log t}\right)\int_1^x \frac{s(x)-s(v)}{v}du + \frac{1}{\log t}\int_x^t \frac{s(v)-s(x)}{v}dv\right| \\
&\leq \left|\left(\frac{1}{\log x}-\frac{1}{\log t}\right)\left\{\int_1^{x_0}+\int_{x_0}^{x}\right\}\frac{s(x)-s(v)}{v}dv\right| \\
&\quad + \left|\frac{1}{\log t}\int_x^t \frac{s(v)-s(x)}{v}du\right| \\
&\leq \left(\frac{1}{\log x}-\frac{1}{\log t}\right)\log x_0|s(x)| + \left(\frac{1}{\log x}-\frac{1}{\log t}\right)\int_1^{x_0}\left|\frac{s(v)}{v}\right|dv \\
&\quad + \left(\frac{1}{\log x}-\frac{1}{\log t}\right)\int_{x_0}^{x}\left|\frac{s(x)-s(v)}{v}dv\right| + \frac{1}{\log t}\int_x^t \left|\frac{s(v)-s(x)}{v}\right|dv.
\end{aligned}$$

By (13) we obtain

$$\lim_{x\to\infty}\left(\frac{1}{\log x}-\frac{1}{\log t}\right)\log x_0|s(x)| = 0. \tag{24}$$

From $s \in L_{loc}[1,\infty)$, we have

$$\lim_{x\to\infty}\left(\frac{1}{\log x}-\frac{1}{\log t}\right)\int_1^{x_0}\left|\frac{s(v)}{v}\right|dv = 0. \tag{25}$$

By $x < t \le x^\lambda$

$$\frac{\log t - \log x}{\log t} \le \log x(\lambda - 1). \tag{26}$$

Using (23), (26) and Lemma 2.1 we obtain that

$$\left(\frac{1}{\log x} - \frac{1}{\log t}\right)\int_{x_0}^{x}\left|\frac{s(x)-s(v)}{v}dv\right| \le (\lambda-1)B_2 \le \epsilon. \tag{27}$$

Using (13), (23) and (26) we concluded that

$$\frac{1}{\log t}\int_{x}^{t}\left|\frac{s(v)-s(x)}{v}\right|dv \le \epsilon. \tag{28}$$

Combining (24), (25), (27) and (28), we obtain that

$$|\ell(t) - \ell(x)| \le 4\epsilon,$$

where $x_0 \le x < t \le x^\lambda$ and large enough x. Therefore, it is proven that the slow oscillation of $\ell(x)$ occurs. □

In the next lemma, we examined the relationship between statistical limit and logarithmic mean by Theorem 1.3.

Lemma 2.4. *If the statistical limit of s exists and equals ξ and if s is bounded, then statistical limit of $\ell(x)$ exists and equals ξ.*

The following lemmas are proved in [8].

Lemma 2.5. *For each nonnegative integer k,*

$$x\log x\frac{d}{dx}\ell^{(k+1)}(x) = \tau_\ell^{(k)}(x).$$

Lemma 2.6. *For each nonnegative integer k,*

$$\tau_\ell^{(k)}(x) - \tau_\ell^{(k+1)}(x) = x\log x\frac{d}{dx}\tau_\ell^{(k+1)}(x).$$

Lemma 2.7. *For each positive integer p,*

$$\omega_\ell^{(p)}(x) = \left(x\log x\frac{d}{dx}\right)_p \tau_\ell^{(p-1)}(x).$$

3. Main results

Theorem 3.1. *Let $s(x) \in L^1_{loc}[1,\infty)$ be a real-valued function. If $s(x)$ is slowly decreasing with respect to summability $(L,1)$, then the implication (20) ⇒ (4) holds true.*

Proof. We have slow decrease of $\ell^{(k)}(x)$ with respect to summability $(L,1)$ for each integer $k \ge 1$ from the slow decrease of $s(x)$ with respect to summability $(L,1)$. Using Theorem 1.4, we get

$$\lim_{x\to\infty}\ell^{(k)}(x) = \xi. \tag{29}$$

Applying Corollary 1.1, we obtain

$$\lim_{x\to\infty} \ell^{(k-1)}(x) = \xi$$

by (29). If we continue in this vein, we get that

$$\lim_{x\to\infty} \ell^{(1)}(x) = \xi$$

Thus we conclude that $s(x)$ converges to ξ at ∞ by Corollary 1.1. □

Remark 3.2. *Theorem 3.1 generalizes Theorem 1.6. Indeed, if we take $k = 1$ in the statistical limit of $\ell^{(k)}(x)$, Theorem 1.6 is obtained.*

Theorem 3.3. *Let $s(x)$ be a real-valued and measurable function. If $\ell^{(1)}(\omega^{(p)}(x))$ is slowly decreasing with respect to summability $(L, 1)$ for some nonnegative integer p and $s(x)$ bounded, then the implication (18) $\Rightarrow$ (4) holds true.*

Proof. From Lemma 2.4, we obtain

$$st - \lim_{x\to\infty} \ell^{(1)}(\omega^{(r)}(x)) = 0. \tag{30}$$

for nonnegative integer r. If Theorem 1.4 is applied to the slow decrease of $\ell^{(1)}(\omega^{(p)}(x))$ with respect to summability $(L, 1)$ and (30) for $r = p$, the result

$$\lim_{x\to\infty} \ell^{(1)}(\omega^{(p)}(x)) = 0$$

is obtained. Therefore from the identity

$$\ell^{(1)}(\omega^{(p)}(x)) = x \log x \frac{d}{dx} \ell^{(2)}(\omega^{(p-1)}(x)),$$

which is obtained from Lemma 2.5 and Lemma 2.7, we obtain that

$$x \log x \frac{d}{dx} \ell^{(2)}(\omega^{(p-1)}(x)) \geq -H, \text{ for some } H > 0, \text{ large enough } x.$$

Hence we conclude that $\ell^{(2)}(\omega^{(p-1)}(x))$ is slowly decreasing with respect to summability $(L, 1)$. Now using (16) with the slow decrease of $\ell^{(1)}(\omega^{(p)}(x))$ with respect to summability $(L, 1)$, we get $\ell^{(1)}(\omega^{(p-1)}(x))$ is slowly decreasing with respect to summability $(L, 1)$. If these operations are repeated in this way $p - 2$ more times, we have the slow decrease of $\ell^{(1)}(\omega^{(0)}(x))$ with respect to summability $(L, 1)$. Now taking $r = 0$ in (30), we find

$$st - \lim_{x\to\infty} \ell^{(1)}(\omega^{(0)}(x)) = 0. \tag{31}$$

Then Theorem 1.4 yields $\lim_{x\to\infty} \ell^{(1)}(\omega^{(0)}(x)) = 0$ by (31). So, this result gives that $\ell^{(1)}(x)$ is slowly decreasing with respect to summability $(L, 1)$. Using Lemma 2.4, we conclude that

$$st - \lim_{x\to\infty} \ell^{(1)}(x) = \xi. \tag{32}$$

Combining the slow decrease of $\ell^{(1)}(x)$ with respect to summability $(L, 1)$ and (32) gives

$$\lim_{x\to\infty} \ell^{(1)}(x) = \xi. \tag{33}$$

Now it is easy to say that $\lim_{x\to\infty} s(x) = \xi$ with the Kronecker identity and (33). □

Corollary 3.4. *Let $s(x)$ be a real-valued and measurable function. If $\omega^{(p)}(x) \geq -H$ for some nonnegative integer p, some $H > 0$ and sufficiently large x and $s(x)$ bounded, then the implication* (18) $\Rightarrow$ (4) *holds true.*

Proof. By benefiting from the identity

$$\omega^{(p)}(x) = x \log x \frac{d}{dx} \ell^{(1)}(\omega^{(p-1)}(x)),$$

we get that $\ell^{(1)}(\omega^{(p-1)}(x))$ is slowly decreasing with respect to summability $(L, 1)$. Hence proof is completed. □

Theorem 3.5. *Let $s(x) \in L^1_{loc}[1, \infty)$ be a complex-valued function.If $s(x)$ is slowly oscillating with respect to $(L, 1)$ then the implication* (20) $\Rightarrow$ (4) *holds true.*

Proof. By the help of the slow oscillation of $s(x)$ with respect to $(L, 1)$, we get slow oscillation of $\ell^{(k)}(x)$ with respect to $(L, 1)$ for each integer $k \geq 1$. Using Theorem 1.5, we get the limit (29). Applying Corollary 1.2, we obtain

$$\lim_{x \to \infty} \ell^{(k-1)}(x) = \xi$$

by (29). If we repeat these operations $k - 2$ times, we obtain the result

$$\lim_{x \to \infty} \ell^{(1)}(x) = \xi$$

Hence we conclude that $s(x)$ converges to ξ at ∞ by Corollary 1.2. □

Remark 3.6. *Theorem 3.5 generalizes Theorem 1.7. Indeed, if we take $k = 1$ in the statistical limit of $\ell^{(k)}(x)$, Theorem 1.7 is obtained.*

Theorem 3.7. *Let $s(x)$ be a complex-valued and measurable function. If $\ell^{(1)}(\omega^{(p)}(x))$ is slowly oscillating with respect to summability $(L, 1)$ for some nonnegative integer p and $s(x)$ bounded, then the implication* (18) $\Rightarrow$ (4) *holds true.*

Proof. From Lemma 2.4, we obtain (30) for nonnegative integer r. If Theorem 1.5 is applied to the slow oscillation of $\ell^{(1)}(\omega^{(p)}(x))$ with respect to summability $(L, 1)$ and (30) for $r = p$, the result

$$\lim_{x \to \infty} \ell^{(1)}(\omega^{(p)}(x)) = 0$$

is obtained. Therefore from the identity

$$\ell^{(1)}(\omega^{(p)}(x)) = x \log x \frac{d}{dx} \ell^{(2)}(\omega^{(p-1)}(x)),$$

we conclude that

$$\left| x \log x \frac{d}{dx} \ell^{(2)}(\omega^{(p-1)}(x)) \right| \leq K, \text{ for some } K > 0, \text{ large enough } x.$$

Then we may write that $\ell^{(2)}(\omega^{(p-1)}(x))$ is slowly oscillating with respect to summability $(L, 1)$. Now using (16) with the slow oscillation of $\ell^{(1)}(\omega^{(p)}(x))$ with respect to summability $(L, 1)$, we get $\ell^{(1)}(\omega^{(p-1)}(x))$ is slowly oscillating with respect to summability $(L, 1)$. Continuing in this way, we get the slow oscillation of $\ell^{(1)}(\omega^{(0)}(x))$ with respect to summability $(L, 1)$. Now taking $r = 0$ in (30) yields

$$st - \lim_{x \to \infty} \ell^{(1)}(\omega^{(0)}(x)) = 0. \tag{34}$$

Combining (34) and last result, we conclude $\lim_{x \to \infty} \ell^{(1)}(\omega^{(0)}(x)) = 0$ by Theorem 1.5. Then, this result gives that $\ell^{(1)}(x)$ is slowly oscillating with respect to summability $(L, 1)$. With Lemma 2.4, we have (32). Combining the slow oscillation of $\ell^{(1)}(x)$ with respect to summability $(L, 1)$ and (32) gives (33). Using the Kronecker identity and (33) gives $\lim_{x \to \infty} s(x) = \xi$. □

Remark 3.8. *Theorem 3.7 generalizes Theorem 1.5. Since* $s(x)$ *is slowly oscillating with respect to summability* $(L,1)$*, we obtain that* $\ell^{(1)}(\omega^{(0)}(x))$ *is slowly oscillating with respect to summability* $(L,1)$ *by (9). However the slow oscillation of* $\ell^{(1)}(\omega^{(0)}(x))$ *with respect to summability* $(L,1)$ *does not imply the* $s(x)$*'s.*

Corollary 3.9. *Let* $s(x)$ *be a complex-valued and measurable function. If* $\left|\omega^{(p)}(x)\right| \leq K$ *for some nonnegative integer* p*, some* $K > 0$ *and sufficiently large* x *and* $s(x)$ *bounded, then the implication* (18) $\Rightarrow$ (4) *holds true.*

Proof. By benefiting from the identity

$$\omega^{(p)}(x) = x \log x \frac{d}{dx} \ell^{(1)}(\omega^{(p-1)}(x)),$$

we conclude that $\ell^{(1)}(\omega^{(p-1)}(x))$ is slowly oscillating with respect to summability $(L,1)$. Hence proof is completed. □

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