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Existence of solutions of an infinite system of mixed Volterra-Fredholm type integral equation in sequence space

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Abstract. Applying the fixed point theorem, we offer the existence of solutions of an infinite system of mixed Volterra-Fredholm type of nonlinear integral equations in the sequence space c_0. To further demonstrate the given existence result, we provided examples.

1. Introduction

The term Volterra-Fredholm integral equations comes from problems of parabolic boundary value, mathematical modeling based on the spatio-temporal development of an epidemic, from a variety of physical and biological models. There are two versions of the mixed Volterra-Fredholm integral equations, namely:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt \tag{1}$$

and

$$u(x) = f(x) + \lambda \int_a^x \int_a^b k(r,t)u(t)dtdr. \tag{2}$$

Equation (1) contains disjoint Volterra and Fredholm integrals, whereas equation (2) contains mixed Volterra and Fredholm integrals. Detailed discussions of these types of integral equations can be found in [29]. Applications of Volterra-Fredholm integral equations typically occur in the fields related to physics, fluid dynamics, electrodynamics and biology. Numerous publications have recently been published that focus on understanding these equations and their properties. Many researchers have shown immense interests on this issue, and many generalizations of the same have been given by a lot of researchers. The work has been developed using the collocation method in [7, 12, 15, 16, 27], CAS wavelets method [14], Taylor expansion methods [8], block-pulse functions [26], linear programming [16], spectral methods [9], etc. Numerous approaches have been used for nonlinear computation of the two dimensional Volterra-Fredholm integral equations such as the matrix based method [18], the homotopy perturbation method [13], the modified

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homotopy perturbation method [14], the spline collocation method [7] and the iterative method [28]. In the articles [17, 24] we found the solvability of linear mixed Volterra-Fredholm integral equations by applying Banach fixed point theorem. Recently, [26] studied mixed nonlinear Volterra-Fredholm equation by using generalized Banach fixed point theorem.

Kuratowski [23] was the one who first proposed the idea of MNC in 1930. An entirely new FPT was created by Darbo [9] connecting the concept of MNC. Later it referred by Darbo FPT. By generalization of Banach principle of contraction [5], Meir and Keeler [22] proved a new FPT, which is very interesting. There are lots of works have been done by using fixed point theorems in the field of integral equations. We have referred some recent works on application of fixed point theorems in integral equations for better understanding [3, 11, 19–21].

We found that there are lots of works have been done in sequence spaces by many authors. Mursaleen and Mohiuddine [25] proved existence theorems in l_p space for infinite system of differential equations by using Meir Keeler FPT. Later, Arab et al. [4] proved existence of solutions of system of integral equations in two variables. The existence of solutions of an infinite system of integral equations in sequence spaces c_0 and l_1 discussed by Das et al. in [10].

The infinite system of mixed Volterra-Fredholm integral equations is a new research topic which is not studied yet. This fact motivate us to approach this study to fill this research gap. Here in this paper, we try to show the existence of solutions of an infinite system of mixed Volterra-Fredholm integral equations by using Meir Keeler FPT in the sequence space c_0 and verified our result with the help of suitable examples. The paper is systematically organized as: In Section 2 we have stated some notations and auxiliary facts, which will be helpful for our main finding. In Section 3 we prove our main results. Next we have given examples to support of our main result and also verified the examples.

Let us consider the followings as:
$\mathcal{E}$= Banach space with norm $\|.\|$,
$\mathcal{B}(y_0, d)$= Closed ball(centre y_0,radius d) defined in $\mathcal{E}$.
If $\mathcal{X} \neq \emptyset$ and $\mathcal{X} \subset \mathcal{E}$, then we define:-
$\overline{\mathcal{X}}$= Closure of $\mathcal{X}$,
$Conv\mathcal{X}$=Convex closure of $\mathcal{X}$,
$\mathcal{M}_{\mathcal{E}}$= Family of all nonempty and bounded subset of $\mathcal{E}$,
$\mathcal{N}_{\mathcal{E}}$= Subfamily consist of all relatively compact sets.
The following definition of MNC considered from [6].

Definition 1.1. A function $\mu : \mathcal{M}_{\mathcal{E}} \to [0, \infty)$ is called a MNC if the following conditions hold:

(i) the family $\ker \mu = \{\mathcal{X} \in \mathcal{M}_{\mathcal{E}} : \mu(\mathcal{X}) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_{\mathcal{E}}$.
(ii) $\mathcal{X} \subset \mathcal{Y} \implies \mu(\mathcal{X}) \leq \mu(\mathcal{Y})$.
(iii) $\mu(\overline{\mathcal{X}}) = \mu(\mathcal{X})$.
(iv) $\mu(\text{Conv}\,\mathcal{X}) = \mu(\mathcal{X})$.
(v) $\mu(\lambda\mathcal{X} + (1-\lambda)\mathcal{Y}) \leq \lambda\mu(\mathcal{X}) + (1-\lambda)\mu(\mathcal{Y})$ for $\lambda \in [0, 1]$.
(vi) if $\mathcal{X}_i$ is a sequence of closed sets from $\mathcal{M}_{\mathcal{E}}$, such that $\mathcal{X}_{i+1} \subset \mathcal{X}_i$ for $i = 1, 2, 3, ...$and if $\lim_{i\to\infty} \mu(\mathcal{X}_i) = 0$, then $\mathcal{X}_\infty := \bigcap_{i=1}^{\infty} \mathcal{X}_i$ is nonempty.

The family $\ker \mu$ is known as kernel of measure μ.
Measures μ to be sublinear, if it holds the conditions

(i) $\mu_n(\lambda\mathcal{X}) = |\lambda|\mu_n(\mathcal{X})$, for $\lambda \in \mathbb{R}, n \in \mathbb{N}$.
(ii) $\mu_n(\mathcal{X} + \mathcal{Y}) \leq \mu_n(\mathcal{X}) + \mu_n(\mathcal{Y})$.

A sublinear MNC μ holds the condition

$$\mu(\mathcal{X} \cup \mathcal{Y}) = \max\{\mu(\mathcal{X}), \mu(\mathcal{Y})\}$$

and such that $\ker \mu = \mathcal{N}_{\mathcal{E}}$ is said to be regular. For a bounded subset $\mathcal{S}$ of a metric space $\mathcal{X}$, the Kuratowski MNC is stated as

$$\alpha(\mathcal{S}) = \inf\{\delta > 0 : \mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i, \text{diam}\mathcal{S}_i \leq \delta \text{ for } 1 \leq i \leq n \leq \infty\},$$

where diam$\mathcal{S}$ denotes the diameter of $\mathcal{S}_i$, means

$$\text{diam}\mathcal{S}_i = \sup\{d(x, y) : x, y \in \mathcal{S}_i\}.$$

The Hausdorff MNC for a bounded set $\mathcal{S}$ is denoted as

$$\chi(\mathcal{S}) = \inf\{\epsilon > 0 : \mathcal{S} \text{ has finite } \epsilon - \text{net in } \mathcal{X}\}.$$

Let us recall some basic assets of Hausdorff MNC. Assume $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded subsets of the metric space $(\mathcal{X}, d)$. Then we have

(i) $\chi(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is totally bounded;
(ii) $\chi(\mathcal{F}) = \chi(\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ denotes closure of $\mathcal{F}$;
(iii) $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \chi(\mathcal{F}_1) \leq \chi(\mathcal{F}_2)$;
(iv) $\chi(\mathcal{F}_1 \cup \mathcal{F}_2) = \max\{\chi(\mathcal{F}_1), \chi(\mathcal{F}_2)\}$;
(v) $\chi(\mathcal{F}_1 \cap \mathcal{F}_2) \leq \min\{\chi(\mathcal{F}_1), \chi(\mathcal{F}_2)\}$;

In case of a normed space $(X, \|.\|)$, the function χ has some additional properties connected with the linear structure. For example, we have

(i) $\chi(\mathcal{F}_1 + \mathcal{F}_2) \leq \chi(\mathcal{F}_1) + \chi(\mathcal{F}_2)$,
(ii) $\chi(\mathcal{F} + x) = \chi(\mathcal{F})$ for all $x \in \mathcal{X}$,
(iii) $\chi(\alpha\mathcal{F}) = |\alpha|\chi(\mathcal{F})$ for all $\alpha \in \mathbb{C}$.

Definition 1.2. *[2] Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach spaces and let μ_1 and μ_2 be arbitrary MNC on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. An operator $\mathcal{T}$ from $\mathcal{E}_1$ to $\mathcal{E}_2$ is called a (μ_1, μ_2)-condensing operator if it is continuous and $\mu_2(\mathcal{T}(\mathcal{D})) < \mu_1(\mathcal{D})$ for every set $\mathcal{D} \subset \mathcal{E}_1$ with compact closure.*

Remark 1.3. *If $\mathcal{E}_1 = \mathcal{E}_2$ and $\mu_1 = \mu_2 = \mu$, then $\mathcal{T}$ is called μ-condensing operator.*

Theorem 1.4. *[4] Let $\mathcal{E}$ be a Banach space and C be a nonempty, closed, bounded and convex subset of $\mathcal{E}$. For the continuous mapping $\mathcal{T} : C \to C$, a constant $k \in [0, 1)$ such that $\mu_2(\mathcal{T}(C)) < k\mu_1(\mathcal{T}(C))$. Then $\mathcal{T}$ has a fixed point in C.*

Definition 1.5. *[22] For a metric space $(\mathcal{X}, d)$, the mapping $\mathcal{T}$ on $\mathcal{X}$ is called Meir-Keeler contraction if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that*

$$\epsilon \leq d(x, y) < \epsilon + \delta \implies d(\mathcal{T}x, \mathcal{T}y) < \epsilon, \ \forall\, x, y \in \mathcal{X}.$$

Theorem 1.6. *[22] For a complete metric space $(\mathcal{X}, d)$, if the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Meir-Keeler contraction; then $\mathcal{T}$ has a unique fixed point.*

Definition 1.7. *[1] Consider the Banach space $\mathcal{E}$. Let μ be an arbitrary MNC on $\mathcal{E}$ and C be a nonempty subset of $\mathcal{E}$. The operator $\mathcal{T} : C \to C$ is a Meir-Keeler condensing operator if for any $\epsilon > 0$, $\exists\, \delta > 0$ such that*

$$\epsilon \leq \mu(\mathcal{X}) < \epsilon + \delta \implies \mu(\mathcal{T}(\mathcal{X})) < \epsilon$$

holds for any bounded subset $\mathcal{X}$ of C.

Theorem 1.8. *[1] Consider $\mathcal{E}$ is a Banach space and C is a nonempty, bounded, closed and convex subset of $\mathcal{E}$. Let μ be an arbitrary MNC on $\mathcal{E}$. If $\mathcal{T} : C \to C$ is a continuous and Meir-Keeler condensing operator, then $\mathcal{T}$ has at least one fixed point and the set of all fixed points of $\mathcal{T}$ in C is compact.*

2. Main results

In this section, we are discussed the existence of solutions of mixed Volterra-Fredholm infinite systems of integral equations using the Meir-Keeler condensing operators in the sequence space c_0.

In the Banach space $(c_0, \|.\|)$, the Hausdorff MNC χ can be expressed as

$$\chi(\mathcal{D}) = \lim_{n\to\infty}[\sup_{w(t)\in\mathcal{D}}(\max_{k\geq n}|w_k(t)|)],$$

where $w(t) = (w_i(t))_i^\infty \in c_0$ for each $t \in \mathbb{R}_+$ and $\mathcal{D} \in \mathcal{M}_{c_0}$.
Let us assume the infinite system of mixed Volterra-Fredholm type integral equations

$$w_n(t) = f_n(t, \int_0^x \int_0^a g_n(t, v, w(v))dvdt, w(t)) \tag{3}$$

where $w(t) = (w_i(t))_i^\infty$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $(w_i(t)) \in C(\mathbb{R}_+, \mathbb{R})$ for all $i \in \mathbb{N}$.

2.1. Solvability of the system (3)

Consider the assumptions:

1. $f_n : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ $(n \in \mathbb{N})$ are continuous with

$$K_n = \sup_n\{|f_n(t, 0, w^0(t)| : t \in \mathbb{R}_+\} < \infty,$$

where $w^0(t) = (w_n^0(t))_{n=1}^\infty \in \mathbb{R}^\infty$ and $w_n^0(t) = 0, \forall\, n \in \mathbb{N}, t \in \mathbb{R}_+$. Also, there exist $u_n, m_n : \mathbb{R}_+ \to \mathbb{R}_+ (n \in \mathbb{N})$ are continuous functions such that

$$|f_n(t, p(t), w(t)) - f_n(t, q(t), \overline{w}(t))| \leq u_n(t)\max_{i\geq n}|w_i(t) - \overline{w_i}(t)| + m_n(t)|p(t) - q(t)|,$$

where $w(t) = (w_i(t))_{i=1}^\infty, \overline{w}(t) = (\overline{w_i}(t))_{i=1}^\infty \in \mathbb{R}^\infty$

2. $g_n : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ $(n \in \mathbb{N})$ are continuous and there exists a constant

$$G_n = \sup_n\{m_n(t)|\int_0^x \int_0^a g_n(t, v, w(v))dvdt| : t, v \in \mathbb{R}_+\}.$$

Also

$$\lim_{t\to\infty}|m_n(t)\int_0^x \int_0^a [g_n(t, v, w(v)) - g_n(t, v, \overline{w}(v))]dvdt| = 0.$$

3. Let us assume an operator W from $\mathbb{R}_+ \times c_0$ to c_0 as follows
$(t, w(t)) \to (Ww)(t)$, where

$$(Ww)(t) = (f_1(t, v_1, w(t)), f_2(t, v_2, w(t)), f_3(t, v_3, w(t)), ...),$$

and

$$v_n(w) = \int_0^x \int_0^a g_n(t, v, w(v))dvdt.$$

4. For $n \to \infty$, $K_n \to 0$ and $G_n \to 0$. Also, $\sup K_n = K$, $\sup G_n = G$ and $\sup\{u_n(t) : t \in \mathbb{R}_+\} = U < \infty$ such that $0 < U < 1$.

Theorem 2.1. *Under the conditions* (1) − (4)*, the system* (3) *has at least one solution* $w(t) = (w_i(t))_{i=0}^\infty \in c_0$*,* $t \in \mathbb{R}_+$*,* $n \in \mathbb{N}$ *and* $(w_i(t)) \in C(\mathbb{R}_+, \mathbb{R})$ *for all* $i \in \mathbb{N}$*.*

Proof. We have

$$
\begin{aligned}
\|w(t)\|_{c_0} &= \max_{n\geq 1} |f_n(t, \int_0^x \int_0^a g_n(t,v,w(v))dvdt, w(t))| \\
&= \max_{n\geq 1} |f_n(t, \int_0^x \int_0^a g_n(t,v,w(v))dvdt, w(t)) - f_n(t,0,w^0(t)) + f_n(t,0,w^0(t))| \\
&\leq \max_{n\geq 1} |f_n(t, \int_0^x \int_0^a g_n(t,v,w(v))dvdt, w(t)) - f_n(t,0,w^0(t))| + \max_{n\geq 1} |f_n(t,0,w^0(t))| \\
&\leq \max_{n\geq 1} u_n(t) \max_{n\geq 1} |w(t) - w^0(t)| + m_n(t)| \int_0^x \int_0^a g_n(t,v,w(v))dvdt| + K \\
&\leq U\|w(t)\|_{c_0} + G + K \\
&\implies \|w(t)\|_{c_0} \leq U\|w(t)\|_{c_0} + G + K \\
&\implies (1-U)\|w(t)\|_{c_0} \leq G + K \\
&\implies \|w(t)\|_{c_0} \leq \frac{G+K}{1-U} = d(say).
\end{aligned}
$$

Assume that $\overline{\mathcal{B}} = \overline{\mathcal{B}}(w^0(t), d)$ is a radius d closed ball with center $w^0(t)$, therefore $\overline{\mathcal{B}}$ is nonempty, closed, convex subset of c_0. Consider the operator $W = (W_i)$ on $C(\mathbb{R}_+, \overline{\mathcal{B}})$ stated as below. For all $t \in \mathbb{R}_+$

$$(Ww)(t) = (W_i w)(t) = \{f_i(t, v_i(w), w(t)\},$$

where $w(t) = (w_i(t)) \in \overline{\mathcal{B}}$ and $w_i(t) \in C(R_+ \times \mathbb{R}_+, \mathbb{R})$, $\forall i \in \mathbb{N}$.
Since each $t \in \mathbb{R}_+$, we have by pre-defined condition (3)

$$\lim_{i\to\infty} (W_i w)(t) = \lim_{i\to\infty} f_i(t, v_i(w), w(t)) = 0.$$

Therefore $(Ww)(t) \in c_0$. It follows from the fact $\|(Ww)(t) - w^0(t)\|_{c_0} \leq d$, W is a self mapping on $\overline{\mathcal{B}}$. Now we show that W is continuous on $C(\mathbb{R}_+, \overline{\mathcal{B}})$.
Let $\epsilon > 0$ and arbitrary $c(t) = (c_j(t))_{j=1}^{\infty}$, $e(t) = (e_j(t))_{j=1}^{\infty} \in c_0$ be such that

$$\|c - e\|_{c_0} < \frac{\epsilon}{2U}.$$

For $t \in \mathbb{R}_+$, we have

$$
\begin{aligned}
&|(w_n c)(t) - (w_n e)(t)| \\
&= |f_n(t, v_n(c), c(t)) - f_n(t, v_n(e), e(t))| \\
&\leq u_n(t) \max_{i\geq n} |c_i(t) - e_i(t)| + m_n(t)|v_n(c) - v_n(e)| \\
&\leq U \max_{i\geq n} |c_i(t) - e_i(t)| + m_n(t)|v_n(c) - v_n(e)| \\
&< \frac{\epsilon}{2} + m_n(t)| \int_0^x \int_0^a [g_n(t,v,c(v)) - g_n(t,v,e(v))]dvdt|.
\end{aligned}
$$

By using pre-defined condition (2), we can choose $T_1 > 0$ for which $\max(t) > T_1$

$$|m_n(t) \int_0^x \int_0^a [g_n(t,v,c(v)) - g_n(t,v,e(v))]dvdt| < \frac{\epsilon}{2}.$$

Hence $|(W_n c)(t) - (W_n e)(t)| < \epsilon$.
For $t \in [0, T_1]$, let $X = \sup\{x(t) : t \in [0, T_1]\}$; $M = \sup\{m_n(t) : t \in [0, T_1]\}$ and

$g = \sup_n\{|g_n(t,v,c(v)) - g_n(t,v,e(v))| : t \in [0,T_1], v \in [0,X]\}$.
Then $|(W_nc)(t) - (W_ne)(t)| < \epsilon + MgXa$.
Since g_n is continuous on $[0,T_1] \times [0,X] \times c_0$, we have $g_n \to 0$ as $\epsilon \to 0$. Therefore, $|(W_nc)(t) - (W_ne)(t)| \to 0$ as $\|c(t) - e(t)\|_{c_0} \to 0$. Thus W is continuous on $\overline{\mathcal{B}} \subset c_0$.
Next we show that W is a condensing operator of Meir-Keeler type. For $\epsilon > 0$ we find a $\delta > 0$ such that $\epsilon \le \chi(\overline{\mathcal{B}}) < \epsilon + \delta \implies \chi(W(\overline{\mathcal{B}})) < \epsilon$.
We have

$$\begin{aligned}
\chi(W(\overline{\mathcal{B}})) &= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathcal{B}}}\{\max|f_n(t,v_n(w),w(t))|\}] \\
&= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathcal{B}}}\{\max|f_n(t,v_n(w),w(t)) + f_n(t,0,w^0) - f_n(t,0,w^0)|\}] \\
&= \lim_{n\to\infty}[\sup_{w(t)\in\overline{\mathcal{B}}} \max(u_n(t)\max|w_i(t)| + m_n(t)|\int_0^x\int_0^a g_n(t,v,w(v))dvdt| + K_n \\
&\le U\chi(\overline{\mathcal{B}}) + \lim_{n\to\infty}(G_n + K_n) \\
&\le U\chi(\overline{\mathcal{B}}).
\end{aligned}$$

Observe that

$$\chi(W(\overline{\mathcal{B}})) \le U\chi(\overline{\mathcal{B}}) < \epsilon \implies \chi(\overline{\mathcal{B}}) < \frac{\epsilon}{U}.$$

Now taking $\delta = \dfrac{\epsilon(1-U)}{U}$, we get $\epsilon \le \chi(\overline{\mathcal{B}}) \le \dfrac{\epsilon}{U} = \epsilon + \delta$.
Therefore W is a Meir-Keeler condensing operator which is defined on the set $\overline{\mathcal{B}} \subset c_0$. Hence W satisfies all the conditions of the Theorem 1.8, which implies W has a fixed point in $\overline{\mathcal{B}}$. Thus the system (3) has a solution in c_0. □

Example 2.2. *Consider the infinite system of integral equations*

$$w_n(t) = \frac{1}{t+n^2} + \sum_{i=n}^{\infty}\frac{|w_i(t)|}{3i^2} + \frac{1}{n^3e^t}\int_0^x\int_0^\pi \frac{\sin(w_i(v))}{2+\cos(\sum_{i=1}^{\infty} w_i(v))}dvdt. \tag{4}$$

Show that the system (4) *has a solution in* c_0.

<u>Solution</u>: Here $f_n(t,v_n(w(t)),w(t)) = \dfrac{1}{t+n^2} + \sum_{i=n}^{\infty}\dfrac{|w_i(t)|}{3i^2} + \dfrac{1}{n^3e^t}v_n(w(t))$, where $v_n(w(t)) = \int_0^x\int_0^\pi g_n(t,v,w(v))dvdt$ and $g_n = \dfrac{\sin(w_i(v))}{2+\cos(\sum_{i=1}^{\infty} w_i(v))}$.
If $w(t) \in c_0$ then $f_n \in c_0$. Now if $y(t) = (y_i(t)) \in c_0$ then we have

$$\begin{aligned}
&|f_n(t,v_n(w(t)),w(t)) - f_n(t,v_n(y(t)),y(t))| \\
&\le \sum_{i=n}^{\infty}\frac{1}{3i^2}|w_i(t) - y_i(t)| + \frac{1}{n^3e^t}|v_n(w(t)) - v_n(y(t))| \\
&\le (\sum_{i=n}^{\infty}\frac{1}{3i^2})\max|w_i(t) - y_i(t)| + \frac{1}{n^3e^t}|v_n(w(t)) - v_n(y(t))| \\
&\le \frac{\pi^2}{18}\max_{i\ge n}|w_i(t) - y_i(t)| + \frac{1}{n^3e^t}|v_n(w(t)) - v_n(y(t))|.
\end{aligned}$$

Here $u_n(t) = \dfrac{\pi^2}{18}$, $m_n(t) = \dfrac{1}{n^3 e^t}$.
We have, $0 < U < 1$ and $K_n = \sup |\dfrac{1}{t+n^2}| \leq 1$; i.e., $K_n \to 0$ as $n \to \infty$.
Also

$$G_n = \sup \frac{1}{n^3 e^t} \left| \int_0^x \int_0^\pi \frac{\sin(w_i(v))}{2 + \cos(\sum_{i=1}^{\infty} w_i(v))} dv dt \right|.$$

Since

$$\left| \int_0^x \int_0^\pi \frac{\sin(w_i(v))}{2 + \cos(\sum_{i=1}^{\infty} w_i(v))} dv dt \right| \leq 2 \left| \int_0^x \int_0^\pi dv dw \right| = 2\pi x.$$

Therefore $G_n = \sup \dfrac{2\pi x}{n^3 e^t} = \dfrac{2\pi}{n^3}$ and $G_n \to 0$ as $n \to \infty$.
We have $t \to \infty$

$$\left| \frac{1}{n^3 e^t} \int_0^x \int_0^\pi [g_n(t, v, w(v)) - g_n(t, v, \overline{w}(v))] dv dt \right| \to 0.$$

Moreover f_n and g_n are continuous functions. Hence the equation (4) satisfies all the assumptions (1)-(4). Hence the system (4) has a solution in c_0.

Example 2.3. *Consider the infinite system of integral equations*

$$w_n(t) = \frac{1}{n+e^t} + \sum_{i=n}^{\infty} \frac{\sin t \cos t w_i(t)}{2i^2} + \frac{1}{n^2 e^t} \int_0^x \int_0^\pi \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dv dt. \tag{5}$$

Show that the system (5) *has a solution in* c_0.

Solution: From (5), we have

$$\begin{aligned} w_n(t) &= \frac{1}{n+e^t} + \sum_{i=n}^{\infty} \frac{\sin t \cos t w_i(t)}{2i^2} + \frac{1}{n^2 e^t} \int_0^x \int_0^\pi \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dv dt \\ &= \frac{1}{n+e^t} + \sum_{i=n}^{\infty} \frac{\sin(2t) w_i(t)}{4i^2} + \frac{1}{n^2 e^t} \int_0^x \int_0^\pi \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dv dt \end{aligned}$$

Here $f_n(t, v_n(w(t)), w(t)) = \dfrac{1}{n+e^t} + \sum_{i=n}^{\infty} \dfrac{\sin(2t) w_i(t)}{4i^2} + \dfrac{1}{n^2 e^t} v_n(w(t))$,

where $v_n(w(t)) = \int_0^x \int_0^\pi g_n(t, v, w(v)) dv dt$ and $g_n(t) = \dfrac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))}$.

If $w(t) \in c_0$ then $f_n \in c_0$. Now if $y(t) = (y_i(t)) \in c_0$ then we have

$$
\begin{aligned}
&|f_n(t, v_n(w(t)), w(t)) - f_n(t, v_n(y(t)), y(t))| \\
&\leq \sum_{i=n}^{\infty} \frac{|\sin 2t|}{4i^2} |w_i(t) - y_i(t)| + \frac{1}{n^2 e^t} |v_n(w(t)) - v_n(y(t))| \\
&\leq \sum_{i=n}^{\infty} \frac{\max |\sin 2t|}{4i^2} \max |w_i(t) - y_i(t)| + \frac{1}{n^2 e^t} |v_n(w(t)) - v_n(y(t))| \\
&\leq (\sum_{i=n}^{\infty} \frac{1}{4i^2}) \max |w_i(t) - y_i(t)| + \frac{1}{n^2 e^t} |v_n(w(t)) - v_n(y(t))| \\
&\leq \frac{\pi^2}{24} \max_{i \geq n} |w_i(t) - y_i(t)| + \frac{1}{n^2 e^t} |v_n(w(t)) - v_n(y(t))|.
\end{aligned}
$$

Here $u_n(t) = \dfrac{\pi^2}{24}$, $m_n(t) = \dfrac{1}{n^2 e^t}$. We have $0 < U < 1$ and $K_n = \sup |\dfrac{1}{n + e^t}| \leq 1$; i.e., $K_n \to 0$ as $n \to \infty$.
Also

$$
G_n = \sup \frac{1}{n^2 e^t} \left| \int_0^x \int_0^\pi \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dv dt \right|.
$$

Since

$$
\left| \int_0^x \int_0^\pi \frac{\cos(w_i(v)) + \sin(\sum_{i=1}^{\infty} w_i(v))}{3 + \sin(w_i(v))} dv dt \right| \leq 2 \left| \int_0^x \int_0^\pi dv dw \right| = 2\pi x.
$$

Therefore $G_n = \sup \dfrac{2\pi x}{n^2 e^t} = \dfrac{2\pi}{n^2 e}$ and $G_n \to 0$ as $n \to \infty$.
We have, $t \to \infty$

$$
\left| \frac{1}{n^2 e^t} \int_0^x \int_0^\pi [g_n(t, v, w(v)) - g_n(t, v, \overline{w}(v))] dv dt \right| \to 0.
$$

Moreover f_n and g_n are continuous functions. Hence the equation (5) satisfies all the assumptions (1)-(4). Hence the system (5) has a solution in c_0.

Conclusion

There are lots of works have been done in sequence spaces, but study of mixed Volterra-Fredholm integral equations in sequence spaces is still a research area where we can explore more results. In our present work, we have solved theoretically infinite system of mixed Volterra-Fredholm integral equation in the sequence space c_0. In our future work we can explore numerical methods and it's application by taking this type of infinite system of mixed Volterra-Fredholm integral equations. Another future work of our paper is that we can try to solve mixed integral equations in other sequence spaces.

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