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Topologically invariant φ-means and uniformly continuous functionals on the dual of the Lebesgue-Fourier algebra related to coset spaces

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Abstract. Let K be a compact subgroup of a locally compact group G. We investigate topologically invariant φ-means (with norm one) over the dual of the Lebesgue-Fourier algebra related to coset spaces G/K, where φ is a nonzero character of the Lebesgue-Fourier algebra on G/K. We prove that the set of all topologically invariant φ-means over dual of the Fourier algebra of G/K and the set of all topologically invariant φ-means over the dual of the Lebesgue-Fourier algebra of G/K have the same cardinality. Furthermore, we introduce and study the spaces weakly almost periodic functionals and uniformly continuous functionals over the Lebesgue-Fourier algebra of G/K.

1. Introduction

Let G be a locally compact group with the left Haar measure λ_G and identity e. The Fourier-Stieltjes algebra $B(G)$ is the linear span of continuous positive definite complex-valued functions on G; it is the dual of the group C^*-algebra $C^*(G)$ which with pointwise multiplication and the norm defined by duality is a commutative Banach algebra. The Fourier algebra $A(G)$ is the closed ideal of $B(G)$ generated by all functions in $B(G)$ with compact support. The algebra $A(G)$ is a commutative, semisimple, regular Banach algebra, whose Gelfand structure space coincides with G. The Banach space dual of $A(G)$ is the group von Neumann algebra $VN(G)$; the weakly closed $*$-subalgebra generated by range of the left regular representation λ of G on $L^2(G)$. Any function $f \in A(G)$ has a form $f(\cdot) = \langle \lambda(\cdot)h, g\rangle$ with $h, g \in L^2(G)$ and $\|f\|_{A(G)} = \|h\|_2\|g\|_2$. The Fourier algebra $A(G)$ was introduced by Eymard in [4].

Ghahramani and Lau studied the Lebesgue-Fourier algebra of group G in [9, 10]. The Lebesgue-Fourier algebra with convolution product is a Segal algebra and also, it with pointwise multiplication is an abstract Segal algebra with respect to $A(G)$.

Let K be a compact subgroup of G equipped with the normalized Haar measure λ_K, meaning $\lambda_K(K) = 1$. The quotient space G/K forms a homogeneous space. We denote the left coset xK in G/K by $\dot{x}$ and define the canonical map $\varphi_K : G \to G/K$ by $\varphi_K(x) = \dot{x}$ for all $x \in G$. Moreover, a G-invariant Radon measure μ exists on G/K, satisfying $\mu(x\dot{E}) = \mu(\dot{E})$ for all Borel subsets E of G and $x \in G$, where $\dot{E}$ represents the image $\varphi_K(E)$. Similarly, we denote the space of right cosets of K by $G\backslash K$. Define the map $P_K : A(G) \to A(G)$ by

$$P_K(u)(x) = \int_K u(xk)\, d\lambda_K(k)$$

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for all $u \in A(G)$. The image of $A(G)$ under P_K consists of functions in $A(G)$ that remain constant on the left cosets of K in G. The map P_K is a contractive projection [5]. Further, define the map M_K on the image of $A(G)$ under P_K via

$$M_K(u)(\dot{x}) = u(x),$$

for all $u \in A(G)$. This map is an injective homomorphism, and its range is denoted as $A(G/K)$, which we refer to as the Fourier algebra of the coset space G/K. The authors in [8] introduce a norm for $A(G/K)$ such that M_K is an isometry, ensuring that the inverse map

$$M_K^{-1} : A(G/K) \to A(G)$$

is an isometric homomorphism. The linear map Γ_K is defined as

$$\Gamma_K := M_K \circ P_K : A(G) \to A(G/K),$$

which is surjective and contractive (see [8]). This leads to

$$A(G/K) = \{\Gamma_K(u) : \ u \in A(G)\},$$

and thus we can express

$$\Gamma_K(u)(\dot{x}) = \int_K u(xk)\, d\lambda_K(k)$$

for all $u \in A(G)$. The Fourier algebra $A(G/K)$ is a regular, commutative, semisimple Banach algebra with Gelfand structure space G/K. The Fourier algebra $A(G/K)$ of the coset space G/K was defined and studied by Forrest [5], he extended pointwise multiplication of $A(G)$ to $A(G/K)$ and investigated some properties of the algebra.
Any function in $A(G/K)$ has a representation of the form

$$\Gamma_K(u)(\cdot) = \langle \lambda(\cdot)\Phi, g\rangle \quad (u \in A(G))$$

with $\Phi \in L^2(G\backslash K), g \in L^2(G)$. The dual of $A(G/K)$ is $VN(G/K)$, which is the weak*-closure of $\{\lambda(\Phi) : \Phi \in L^1(G/K)\}$ in $VN(G)$. For $x \in G$ and $k \in K$ the restriction of $\lambda(xk)$ to $L^2(G\backslash K)$ is independent of k. We denote this the restriction as $\lambda(\dot{x}) \in B(L^2(G\backslash K), L^2(G))$. Thus $VN(G/K)$ can be considerred as the smallest subspace of $B(L^2(G\backslash K), L^2(G))$ containing all the $\lambda(\dot{x})$ that is closed in the weak operator topology; see [20].

The algebra $L^1(G/K)$ of the coset space G/K was introduced by Reiter and Stegeman [22]; they transferred convolution multiplication of $L^1(G)$ to $L^1G/K)$ and studied its properties.

For a compact subgroup K of a locally compact group G, we investigated in [3] the Lebesgue-Fourier algebra $S^1A(G/K) = L^1(G/K) \cap A(G/K)$ on the coset space G/K with the norm

$$|||\cdot||| = \|\cdot\|_{L^1(G/K)} + \|\cdot\|_{A(G/K)}.$$

We proved that $S^1A(G/K)$ with pointwise multiplication is a commutative, semisimple, regular Banach algebra with Gelfand structure space G/K. In case $K = \{e\}$ it is the Lebesgue-Fourier algebra $S^1A(G)$ with pointwise multiplication. We examined key properties of $S^1A(G/K)$ and investigated its relationships with $A(G/K)$ and G. Specifically, in [3], we outlined the necessary conditions for $S^1A(G/K)$ to be approximately amenable, based on the algebraic and topological characteristics of G and K.

I am quoting the contents of this paragraph from [21]. For a compact subgroup K of G, a continuous function φ from G/K into the circle group $\mathbb{T}$ is called a character of G/K if $\varphi(x\dot{y}) = \varphi(\dot{x})\varphi(\dot{y})$ for each $x, y \in G$. The set of all characters of G/K is denoted by $\widehat{G/K}$. Every element of $\widehat{G/K}$ determines an element of $\Delta(M(G/K))$ whose restriction to $L^1(G/K)$ belongs to $\Delta(L^1(G/K))$ [21].

Topologically invariant means and uniformly continuous functionals play an important role in harmonic analysis and Banach algebras, especially in the study of locally compact groups. These concepts help us understand the behavior of function spaces and their connections to group structures.

In 1974 [12], Edmond E. Granirer studied weakly almost periodic and uniformly continuous functionals on the Fourier algebra of a locally compact group. He carefully defined these function spaces and explored their properties. One of his key findings was that, for certain locally compact groups, the set of uniformly continuous functionals is the same as the set of weakly almost periodic functionals.

Subsequently, in [6], Brian Forrest and Tianxuan Miao studied topologically invariant means on $A_M(G)$ and $A_{cb}(G)$, which are closures of $A(G)$ in spaces of bounded and completely bounded multipliers. They examined the uniformly continuous functionals associated with key group algebras, such as $A(G)$ and $A_M(G)$. An interesting result they established was a direct link between left invariant means on $VN(G)$ and $A_M(G)^*$, offering new insights into weakly almost periodic functionals and Arens regularity. The concepts explored in our paper are influenced by the work of [6], with some results and proofs bearing similarities to those presented in that study.

More recently, the article [7] examines the Arens regularity of ideals in Fourier algebras and their related multiplier spaces. The authors establish conditions under which an ideal in $A(G)$, $A_{cb}(G)$, or $A_M(G)$ is Arens regular, particularly demonstrating that such regularity often implies discreteness of the underlying group G. Additionally, the paper explores connections between approximate identities and the Arens regularity of ideals, showing that an ideal with a bounded approximate identity is Arens regular if and only if it has finite dimension.

Many researchers have studied invariant means. For important and useful works on this topic, see Ilie's study [13], Kumar's researches [16, 17], and his joint work with Lal [19].

In this paper, after the introduction, we define and study topologically invariant $\varphi_{\dot{x}}$-means (with norm one) on the dual of the Lebesgue-Fourier algebra related to coset spaces for every $\dot{x} \in G/K$ in the next section. Additionally, we prove that the number of topologically invariant $\varphi_{\dot{x}}$-means (with norm one) on $VN(G/K)$ is equal to the number of topologically invariant $\varphi_{\dot{x}}$-means (with norm one) on $S^1A(G/K)^*$ for every $\dot{x} \in G/K$. As an application of this result, we show that $S^1A(G/K)$ admits a unique topological invariant $\varphi_{\dot{e}}$-mean with norm one if and only if K is open. We also prove that $A(G/K)$ is $\varphi_{\dot{x}}$-amenable and so $S^1A(G/K)$ is $\varphi_{\dot{x}}$-amenable for every $\dot{x} \in G/K$. It implies that there exists a unique topologically invariant $\varphi_{\dot{x}}$-mean on the space weakly almost periodic functionals over the dual of $S^1A(G/K)$ for every $\dot{x} \in G/K$. Finally, we focus in the relationship between the space weakly almost periodic functionals and the space uniformly continuous functionals over the Lebesgue-Fourier algebra of G/K.

2. Topologically invariant φ-means and uniformly continuous functionals

In this section, we present results for $S^1A(G/K)$ that are analogous to those obtained by Brian Forrest and Tianxuan Miao ([6]) for $A_M(G)$. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then $\Delta(\mathcal{B}) = \{\varphi|_{\mathcal{B}} : \varphi \in \Delta(\mathcal{A})\}$, where $\Delta(\mathcal{A})$ is consisting of all nonzero characters on $\mathcal{A}$ [2].

Definition 2.1. Let K be a compact subgroup of a locally compact group G and $\dot{x} \in G/K$. A linear functional m on $S^1A(G/K)^*$ is called a topologically invariant $\varphi_{\dot{x}}$-mean if

$$m(\varphi_{\dot{x}}) = 1 \quad \text{and} \quad \langle m, \Gamma_K(v) \cdot T\rangle = \Gamma_K(v)(\dot{x})\langle m, T\rangle$$

for $T \in S^1A(G/K)^*$, $v \in S^1A(G)$, where $\varphi_{\dot{x}} \in \Delta(S^1A(G/K))$. In addition to, if $\|m\| = 1$, then m is called a topologically invariant $\varphi_{\dot{x}}$-mean with norm one. We denote by $TIM_{\dot{x}}(S^1A(G/K))$ and $TIM_{\dot{x}}(A(G/K))$ the set of all topologically invariant $\varphi_{\dot{x}}$-means with norm one on $S^1A(G/K)^*$ and $VN(G/K)$.

We define the uniformly continuous functionals on the dual of the $S^1A(G/K)$ the following:

$$UCB_s(\widehat{G/K}) = \overline{\operatorname{span}\left\{S^1A(G/K) \cdot S^1A(G/K)^*\right\}}^{\|\cdot\|_{S^1A(G/K)^*}}$$

and the space weakly almost periodic functionals is contains of $T \in S^1A(G/K)^*$ such that

$$\left\{\Gamma_K(v) \mapsto \Gamma_K(v) \cdot T : S^1A(G/K) \to S^1A(G/K)^* \text{ with } |||\Gamma_K(v)||| \leq 1\right\}$$

is relatively weakly compact and we denote it with $WAP_s(\widehat{G/K})$. Also, the uniformly continuous functionals on $VN(G/K)$ were defined as follows:

$$UCB(\widehat{G/K}) = \overline{\text{span}\Big\{A(G/K) \cdot VN(G/K)\Big\}}^{\|\cdot\|_{VN(G/K)}}$$

and the space weakly almost periodic functionals is contains of $T \in VN(G/K)$ such that

$$\Big\{\Gamma_K(v) \mapsto \Gamma_K(v) \cdot T : A(G/K) \to VN(G/K) \text{ with } \|\Gamma_K(v)\| \leq 1\Big\}$$

is relatively weakly compact and we denote it with $WAP(\widehat{G/K})$, (see [13]). Now, we consider the inclusion map
$i : S^1A(G/K) \to A(G/K)$ and its adjoints. Since $S^1A(G/K)$ is dense in $A(G/K)$, i^* is injective. It is easily verified the map i^* is restriction. Furthermore, $S^1A(G/K)^*$ is a Banach $A(G/K)$-bimodule with the following module actions:

$$\langle T \cdot \Gamma_K(v), \Gamma_K(w)\rangle = \langle T, \Gamma_K(v)\Gamma_K(w)\rangle, \quad \langle \Gamma_K(v) \cdot T, \Gamma_K(w)\rangle = \langle T, \Gamma_K(w)\Gamma_K(v)\rangle$$

for each $v \in A(G), w \in S^1A(G)$ and $T \in S^1A(G/K)^*$.

Lemma 2.2. *Let K be a compact subgroup of a locally compact group G. Then*
(a) $A(G/K) \cdot S^1A(G/K)^* \subseteq UCB_s(\widehat{G/K})$;
(b) $i^*(A(G/K) \cdot VN(G/K)) = A(G/K) \cdot i^*(VN(G/K))$;
(c) $i^*(UCB(\widehat{G/K})) \subseteq UCB_s(\widehat{G/K})$;
(d) *If G is amenable, then* $A(G/K) \cdot S^1A(G/K)^* = UCB_s(\widehat{G/K})$;
(e) $S^1A(G/K) \cdot S^1A(G/K)^* \subseteq i^*(VN(G/K))$;
(f) $A_c(G/K) \cdot S^1A(G/K)^* \subseteq i^*(UCB(\widehat{G/K}))$;
(g) *If G is compact, then* $S^1A(G/K) \cdot S^1A(G/K)^* = UCB(\widehat{G/K})$.

Proof. (a) Suppose that $v \in A(G)$ and $T \in S^1A(G/K)^*$. Then there exists $(v_n)_{n\in\mathbb{N}}$ in $S^1A(G)$ such that $\|\Gamma_K(v_n) - \Gamma_K(v)\| \to 0$. Thus for each $u \in S^1A(G)$

$$\begin{aligned} |\langle \Gamma_K(v_n) \cdot T - \Gamma_K(v) \cdot T, \Gamma_K(u)\rangle| &= |\langle (\Gamma_K(v_n) - \Gamma_K(v)) \cdot T, \Gamma_K(u)\rangle| \\ &= |\langle T, \Gamma_K(u)(\Gamma_K(v_n) - \Gamma_K(v))\rangle| \\ &\leq \|T\| \, \|\Gamma_K(v_n) - \Gamma_K(v)\| \, \||\Gamma_K(u)\|| \\ &\to 0. \end{aligned}$$

Hence

$$\begin{aligned} \Gamma_K(v) \cdot T &\in \overline{S^1A(G/K) \cdot S^1A(G/K)^*}^{\|\cdot\|_{S^1A(G/K)^*}} \\ &\subseteq \overline{\text{span}\{S^1A(G/K) \cdot S^1A(G/K)^*\}}^{\|\cdot\|_{S^1A(G/K)^*}} \\ &= UCB_s(\widehat{G/K}). \end{aligned}$$

(b) Suppose that $v \in A(G)$ and $T \in VN(G/K)$. Then for each u in $S^1A(G)$

$$\begin{aligned} \langle i^*(\Gamma_K(v) \cdot T), \Gamma_K(u)\rangle &= \langle \Gamma_K(v) \cdot T, i(\Gamma_K(u))\rangle \\ &= \langle T, i(\Gamma_K(u))\Gamma_K(v)\rangle \\ &= \langle T, i(\Gamma_K(u)\Gamma_K(v))\rangle \\ &= \langle i^*(T), \Gamma_K(u)\Gamma_K(v)\rangle \\ &= \langle \Gamma_K(v) \cdot i^*(T), \Gamma_K(u)\rangle. \end{aligned}$$

Hence $i^*(A(G/K) \cdot VN(G/K)) = A(G/K) \cdot i^*(VN(G/K))$.

(c) For each $v \in A(G)$ and $T \in VN(G/K)$, we have

$$i^*(\Gamma_K(v) \cdot T) = \Gamma_K(v) \cdot i^*(T) \in A(G/K) \cdot S^1A(G/K)^* \subseteq UCB_s(\widehat{G/K}).$$

(d) Since G is amenable, $A(G/K)$ has a bounded approximate identity; see [5, Theorem 4.2]. Thus from Cohen's factorization theorem and $S^1A(G/K)^*$ is a $A(G/K)$-module and from (a), it follows that $A(G/K) \cdot S^1A(G/K)^*$ is a closed subspace of $UCB_s(\widehat{G/K})$. Therefore, the result follows.

(e) Assume that $u \in S^1A(G)$ and $T \in S^1A(G/K)^*$. Then we define $\varphi : A(G/K) \to \mathbb{C}$ with $\varphi(\Gamma_K(v)) = \langle T, \Gamma_K(v)\Gamma_K(u)\rangle$ for all $v \in A(G)$. Therefore

$$\begin{aligned} \|\varphi\| &= \sup\{|\varphi(\Gamma_K(v))| : \ v \in A(G) \text{ and } \|\Gamma_K(v)\| \le 1\} \\ &= \sup\{|\langle T, \Gamma_K(v)\Gamma_K(u)\rangle| : \ v \in A(G) \text{ and } \|\Gamma_K(v)\| \le 1\} \\ &\le \sup\{\|T\| \, |||\Gamma_K(v)\Gamma_K(u)||| : \ v \in A(G) \text{ and } \|\Gamma_K(v)\| \le 1\} \\ &\le \|T\| \, |||\Gamma_K(u)|||. \end{aligned}$$

Thus

$$\Gamma_K(u) \cdot T = \varphi|_{S^1A(G/K)} = i^*(\varphi) \in i^*(VN(G/K)).$$

(f) We know that $\Gamma_K(u) \cdot T \in i^*(VN(G/K))$ for all $u \in S^1A(G)$ and $T \in S^1A(G/K)^*$. We show that $\Gamma_K(u) \cdot T \in i^*(UCB(\widehat{G/K}))$. Assume that $u_0 \in A(G)$ with compact support and $T \in S^1A(G/K)^*$. By the regularity of $S^1A(G/K)$, there exists a function $v \in S^1A(G) \subseteq A(G)$ such that

$$\Gamma_K(v)|_{\mathrm{supp}\Gamma_K(u_0)} = 1 \quad \text{and} \quad \Gamma_K(v)\Gamma_K(u_0) = \Gamma_K(u_0).$$

Thus

$$\Gamma_K(u_0) \cdot T = (\Gamma_K(v)\Gamma_K(u_0)) \cdot T = \Gamma_K(v) \cdot (\Gamma_K(u_0) \cdot T) \in A(G/K) \cdot i^*(VN(G/K)) \subseteq i^*(UCB(\widehat{G/K})).$$

(g) If G is compact, then Proposition 2.5 from [3] implies that $S^1A(G/K) = A(G/K)$ and so it follows. □

Remark 2.3. The inclusion

$$S^1A(G/K) \cdot S^1A(G/K)^* \subseteq i^*(UCB(\widehat{G/K}))$$

is not valid in general. The operator i^* has a closed range if and only if the mapping i itself possesses a closed range, which occurs precisely when

$$S^1A(G/K) = A(G/K),$$

or equivalently, by Proposition 2.5 in [3], when G is compact.

Theorem 2.4. *Let K be a compact subgroup of a locally compact group G. Then*

$$i^{**} : TIM_{\dot{x}}(S^1A(G/K)) \longrightarrow TIM_{\dot{x}}(A(G/K))$$

is a bijection for every $\dot{x} \in G/K$.

Proof. Let $\dot{x} \in G/K$. We first prove that $i^{**}(TIM_{\dot{x}}(S^1A(G/K))) \subseteq TIM_{\dot{x}}(A(G/K))$. Let $m \in TIM_{\dot{x}}(S^1A(G/K))$. If $v \in A(G)$ and $T \in VN(G/K)$, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $S^1A(G)$ such that $\|\Gamma_K(u_n) - \Gamma_K(v)\| \to 0$. Since $\|\cdot\|_{L^\infty(G/K)} \le \|\cdot\|_{A(G/K)}$, we have $\Gamma_K(u_n)(\dot{x}) \to \Gamma_K(v)(\dot{x})$. Also, from the proof of the above theorem, it follows that $\|\Gamma_K(u_n) \cdot T - \Gamma_K(v) \cdot T\| \to 0$. Therefore

$$\begin{aligned} \langle i^{**}(m), \Gamma_K(v) \cdot T\rangle &= \lim_{n\to\infty} \langle i^{**}(m), \Gamma_K(u_n) \cdot T\rangle \\ &= \lim_{n\to\infty} \langle m, i^*(\Gamma_K(u_n) \cdot T)\rangle \\ &= \lim_{n\to\infty} \langle m, \Gamma_K(u_n) \cdot i^*(T)\rangle \\ &= \lim_{n\to\infty} \Gamma_K(u_n)(\dot{x})\langle m, i^*(T)\rangle \\ &= \Gamma_K(v)(\dot{x})\langle i^{**}(m), T\rangle. \end{aligned}$$

Also, $\|i^{**}(m)\| \leq \|m\| = 1$ and $\langle i^{**}(m), \varphi_{\dot{x}}\rangle = \langle m, i^{*}(\varphi_{\dot{x}})\rangle = 1$. Hence $\|i^{**}(m)\| = 1$ and so $i^{**}(m) \in TIM_{\dot{x}}(A(G/K))$.

Now, we show that i^{**} is injective. For this, let m_1, m_2 be elements in $TIM_{\dot{x}}(S^1A(G/K))$ with $m_1 \neq m_2$. Thus there exists $T \in S^1A(G/K)^*$ such that $\langle m_1, T\rangle \neq \langle m_2, T\rangle$. We can choose a function u_0 in $S^1A(G)$ such that $\Gamma_K(u_0)(\dot{x}) = 1$. Then

$$\langle m_1, \Gamma_K(u_0) \cdot T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, \Gamma_K(u_0) \cdot T\rangle.$$

Since $\Gamma_K(u_0) \cdot T \in VN(G/K)$, we conclude that

$$\begin{aligned}
\langle i^{**}(m_1), \Gamma_K(u_0) \cdot T\rangle &= \langle m_1, i^{*}(\Gamma_K(u_0) \cdot T)\rangle \\
&= \langle m_1, \Gamma_K(u_0) \cdot T\rangle \\
&\neq \langle m_2, \Gamma_K(u_0) \cdot T\rangle \\
&= \langle m_2, i^{*}(\Gamma_K(u_0) \cdot T)\rangle \\
&= \langle i^{**}(m_2), \Gamma_K(u_0) \cdot T\rangle.
\end{aligned}$$

Hence $i^{**}(m_1) \neq i^{**}(m_2)$.

Now, we show that i^{**} is surjective. To see this, we let M be an element in $TIM_{\dot{x}}(A(G/K))$. We can choose a function u_0 in $S^1A(G)$ such that $\Gamma_K(u_0)(\dot{x}) = 1$. If $T \in S^1A(G/K)^*$, then $\Gamma_K(u_0) \cdot T$ belong to $VN(G/K)$. We define $m \in S^1A(G/K)^{**}$ such that

$$\langle m, T\rangle := \langle M, \Gamma_K(u_0) \cdot T\rangle$$

for $T \in S^1A(G/K)^*$. Hence, if $v \in S^1A(G)$, then

$$\begin{aligned}
\langle m, \Gamma_K(v) \cdot T\rangle &= \langle M, \Gamma_K(u_0) \cdot (\Gamma_K(v) \cdot T)\rangle \\
&= \langle M, \Gamma_K(v) \cdot (\Gamma_K(u_0) \cdot T)\rangle \\
&= \Gamma_K(v)(\dot{x})\langle M, \Gamma_K(u_0) \cdot T\rangle \\
&= \Gamma_K(v)(\dot{x})\langle m, T\rangle.
\end{aligned}$$

Also,

$$\langle m, \varphi_{\dot{x}}\rangle = \langle M, \Gamma_K(u_0) \cdot \varphi_{\dot{x}}\rangle = \Gamma_K(u_0)(\dot{x})\langle M, \varphi_{\dot{x}}\rangle = \langle M, \varphi_{\dot{x}}\rangle = 1.$$

Thus $\|m\| \geq 1$. If $\|m\| > 1$, then there exists $T \in S^1A(G/K)^*$ with $\|T\| \leq 1$ and $1 < \langle m, T\rangle = \langle M, \Gamma_K(u_0) \cdot T\rangle$. Hence $\|M\| > 1$ and this is contradiction. Thus $\|m\| = 1$. If $T \in VN(G/K)$, then

$$\begin{aligned}
\langle i^{**}(m), T\rangle &= \langle m, i^{*}(T)\rangle \\
&= \langle M, \Gamma_K(u_0) \cdot i^{*}(T)\rangle \\
&= \langle M, i^{*}(T)\rangle \\
&= \langle M, T\rangle.
\end{aligned}$$

Therefore $i^{**}(m) = M$. □

Corollary 2.5. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) *$S^1A(G/K)$ admits a unique topological invariant $\varphi_{\dot{e}}$-mean with norm one;*

(b) *K is open.*

Proof. Since the cardinality of $TIM_{\dot{e}}(S^1A(G/K))$ is equal to the cardinality of $TIM_{\dot{e}}(A(G/K))$ and also $A(G/K)$ admits a unique topological invariant $\varphi_{\dot{e}}$-mean with norm one if and only if K is open, it follows; see [17, Corollary 1.9]. □

Proposition 2.6. *Let K be a compact subgroup of a locally compact group G. Then the restriction map $r : TIM_{\dot{x}}(S^1A(G/K)) \to TIM_{\dot{x}}(UCB_s(\widehat{G/K}))$ is a bijection for every $\dot{x} \in G/K$.*

Proof. We show that r is injective. For this, let $m_1, m_2 \in TIM_{\dot{x}}(S^1A(G/K))$ with $m_1 \neq m_2$. Thus there exists $T \in S^1A(G/K)^*$ such that $\langle m_1, T\rangle \neq \langle m_2, T\rangle$. We can choose a function u_0 in $S^1A(G)$ such that $\Gamma_K(u_0)(\dot{x}) = 1$. Then

$$\langle m_1, \Gamma_K(u_0)\cdot T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, \Gamma_K(u_0)\cdot T\rangle.$$

Since $\Gamma_K(u_0)\cdot T \in UCB_s(\widehat{G/K})$, we conclude that $r(m_1) \neq r(m_2)$.

Now, we show that r is surjective. To see this, we let M be an element in $TIM_{\dot{x}}(UCB_s(\widehat{G/K}))$. We can choose a function u_0 in $S^1A(G)$ such that $\Gamma_K(u_0)(\dot{x}) = 1$. Thus, $\Gamma_K(u_0)\cdot T \in UCB_s(\widehat{G/K})$ for $T \in S^1A(G/K)^*$. Define $m \in S^1A(G/K)^{**}$ by

$$\langle m, T\rangle := \langle M, \Gamma_K(u_0)\cdot T\rangle$$

for $T \in S^1A(G/K)^*$. Moreover,

$$\langle m, \varphi_{\dot{x}}\rangle = \langle M, \Gamma_K(u_0)\cdot \varphi_{\dot{x}}\rangle = \Gamma_K(u_0)(\dot{x})\langle M, \varphi_{\dot{x}}\rangle = \langle M, \varphi_{\dot{x}}\rangle = 1.$$

Hence $\|m\| = 1$. Now, if $v \in S^1A(G)$ and $T \in S^1A(G/K)^*$, then

$$\begin{aligned}\langle m, \Gamma_K(v)\cdot T\rangle &= \langle M, \Gamma_K(u_0)\cdot(\Gamma_K(v)\cdot T)\rangle\\ &= \langle M, \Gamma_K(v)\cdot(\Gamma_K(u_0)\cdot T)\rangle\\ &= \Gamma_K(v)(\dot{x})\langle M, \Gamma_K(u_0)\cdot T\rangle\\ &= \Gamma_K(v)(\dot{x})\langle m, T\rangle.\end{aligned}$$

Thus $m \in TIM_{\dot{x}}(S^1A(G/K))$. Also, if $T \in UCB_s(\widehat{G/K})$, then

$$\langle r(m), T\rangle = \langle m, T\rangle = \langle M, \Gamma_K(u_0)\cdot T\rangle = \langle M, T\rangle.$$

Therefore $r(m) = M$. □

Corollary 2.7. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) *There exists a unique topological invariant $\varphi_{\dot{e}}$-mean with norm one on $UCB_s(\widehat{G/K})$;*
(b) *K is open.*

Proof. Since the cardinality of $TIM_{\dot{e}}(S^1A(G/K))$ is equal to the cardinality of $TIM_{\dot{e}}(UCB_s(\widehat{G/K}))$, it follows from Corollary 2.5. □

Remark 2.8. All the aforementioned results remain valid for all $\varphi_{\dot{x}}$-means and $\varphi_{\dot{e}}$-means, regardless of whether they have norm one.

A Banach algebra $\mathcal{A}$ is called φ-amenable if there is a bounded linear functional m on $\mathcal{A}^*$ such that

$$m(\varphi) = 1 \quad\text{and}\quad m(f\cdot a) = \varphi(a)m(f)$$

for $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$. In other words, $\mathcal{A}$ is called φ-amenable if there exists a topologically invariant φ-mean on $\mathcal{A}^*$. These notions was introducted and studied by Kaniuth, Lau and Pym in [14, 15]. The analogue of the following theorem for locally compact groups is presented as an example in [14].

Theorem 2.9. *Let K be a compact subgroup of a locally compact group G. Then $A(G/K)$ is $\varphi_{\dot{x}}$-amenable for every $\dot{x} \in G/K$.*

Proof. Let $\dot{x} \in G/K$ and $\mathcal{U}$ be a neighbourhood basis of e in G. We put $V := KU$ for $U \in \mathcal{U}$. Hence $0 < \lambda_G(V) \leq \lambda_G(K\overline{U}) < \infty$ and $e \in V$. Now, we set for each $U \in \mathcal{U}$,

$$u_V(s) := \frac{1}{\lambda_G(V)}(\chi_{xV} * \check{\chi}_V)(s) = \frac{\lambda_G(xV \cap sV)}{\lambda_G(V)} \qquad (s \in G).$$

Thus $u_V \in A(G)$ and $\| u_V \| = 1$. Thus $\Gamma_K(u_V)(\dot{x}) = 1$ and $\| \Gamma_K(u_V) \| = 1$. Suppose that m is weak*-cluster point of $(\Gamma_K(u_V))_V$ in $A(G/K)^{**}$. Then

$$m(\varphi_{\dot{x}}) = \lim_V \langle \Gamma_K(u_V), \varphi_{\dot{x}} \rangle = \lim_V \Gamma_K(u_V)(\dot{x}) = 1.$$

Also, we have

$$\begin{aligned} m(\lambda(\dot{s})) &= \lim_V \langle \Gamma_K(u_V), \lambda(\dot{s}) \rangle \\ &= \lim_V \Gamma_K(u_V)(\dot{s}) \\ &= \lim_V \int_K u_V(sk) d\lambda_K(k) \\ &= \lim_V \frac{\lambda_G(xV \cap sV)}{\lambda_G(V)}. \end{aligned}$$

If $\dot{s} \neq \dot{x}$, then $s \neq x$ and so $m(\lambda(\dot{s})) = 0$. Hence if $\dot{s} = \dot{x}$, then for $\dot{s} \in G/K$

$$\begin{aligned} m\Big(\lambda(\dot{s}) \cdot \Gamma_K(u)\Big) &= \lim_V \langle \Gamma_K(u_V), \lambda(\dot{s}) \cdot \Gamma_K(u) \rangle \\ &= \lim_V \langle \lambda(\dot{s}), \Gamma_K(u_V)\Gamma_K(u) \rangle \\ &= \lim_V \Gamma_K(u_V)\Gamma_K(u)(\dot{s}) \\ &= \Gamma_K(u)(\dot{x}) m(\lambda(\dot{s})). \end{aligned}$$

Since $\lambda(\dot{s})$ generate $VN(G/K)$, we conclude that $m(T \cdot \Gamma_K(u)) = \Gamma_K(u)(\dot{x}) m(T)$ for all $T \in VN(G/K)$ and $u \in A(G)$. □

Corollary 2.10. *Let K be a compact subgroup of a locally compact group G. Then $S^1A(G/K)$ is $\varphi_{\dot{x}}$-amenable for every $\dot{x} \in G/K$.*

Proof. Recall from [2, Corollary 2.4] that if $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be an abstract Segal algebra with respect to $\mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$, then $\mathcal{A}$ is φ-amenable if and only if $\mathcal{B}$ is $\varphi |_{\mathcal{B}}$-amenable. □

Theorem 2.11. *Let K be a compact subgroup of a locally compact group G. Then there exists a unique topologically invariant $\varphi_{\dot{x}}$-mean on $WAP_s(\widehat{G/K})$ for every $\dot{x} \in G/K$.*

Proof. According to the previous corollary, there exists a topologically invariant $\varphi_{\dot{x}}$-mean M on $S^1A(G/K)^*$. Then the restriction of M to $WAP_s(\widehat{G/K})$ is clearly a topologically invariant $\varphi_{\dot{x}}$-mean on $WAP_s(\widehat{G/K})$. By Goldstine's theorem there exists a net $(u_\alpha)_{\alpha \in \Lambda}$ in $S^1A(G)$ such that $\pi(\Gamma_K(u_\alpha)) \to M$ in the weak*-topology of $S^1A(G/K)^{**}$, where π is the canonical injection of $S^1A(G/K)$ into $S^1A(G/K)^{**}$. Also, $\pi(\Gamma_K(u_\alpha))|_{WAP_s(\widehat{G/K})} \to M$ in the weak*-topology of $WAP_s(\widehat{G/K})^*$. Thus $\varphi_{\dot{x}}(\Gamma_K(u_\alpha)) \to M(\varphi_{\dot{x}})$ and so $\Gamma_K(u_\alpha)(\dot{x}) \to 1$. Suppose that m be any topologically invariant $\varphi_{\dot{x}}$-mean on $WAP_s(\widehat{G/K})$ and also $T \in WAP_s(\widehat{G/K})$, $u \in S^1A(G)$. Then

$$\begin{aligned} \langle \Gamma_K(u_\alpha) \cdot T, \Gamma_K(u) \rangle &= \langle T, \Gamma_K(u)\Gamma_K(u_\alpha) \rangle \\ &= \langle T, \Gamma_K(u_\alpha)\Gamma_K(u) \rangle \\ &= \langle \Gamma_K(u) \cdot T, \Gamma_K(u_\alpha) \rangle \\ &\to \langle M, \Gamma_K(u) \cdot T \rangle \\ &= \Gamma_K(u)(\dot{x}) \langle M, T \rangle. \end{aligned}$$

Thus $\Gamma_K(u_\alpha) \cdot T \to \langle M, T \rangle \varphi_{\dot{x}}$ in the weak*-topology of $S^1A(G/K)^{**}$. Also, we have $\Gamma_K(u_\alpha) \cdot T \to \langle M, T \rangle \varphi_{\dot{x}}$ in weak-topology, since T is weakly almost periodic. Therefore there exists a sequence $(v_n)_{n \in \mathbb{N}}$ of convex combinations of the $\Gamma_K(u_\alpha)$'s such that $\varphi_{\dot{x}}(\Gamma_K(v_n)) \to 1$ and $\Gamma_K(v_n) \cdot T \to \langle M, T\rangle \varphi_{\dot{x}}$ in norm. Thus

$$\begin{aligned}\langle m, T\rangle &= \lim_{n\to\infty} \varphi_{\dot{x}}(\Gamma_K(v_n))\langle m, T\rangle \\ &= \lim_{n\to\infty} \langle m, \Gamma_K(v_n) \cdot T\rangle \\ &= \langle m, \langle M, T\rangle \varphi_{\dot{x}}\rangle \\ &= \langle M, T\rangle.\end{aligned}$$

Hence invariant $\varphi_{\dot{x}}$-mean on $WAP_s(\widehat{G/K})$ is unique. □

For every function $u : G \to \mathbb{C}$ and $x \in G$, the left translation is defined by

$$L_x u(y) = u(x^{-1}y) \qquad (\forall y \in G),$$

also, the right translation is defined by

$$R_x u(y) = u(yx) \qquad (\forall y \in G).$$

For every $\Phi \in L^\infty(G/K)$ and $x \in G$, the left translation is defined by

$$L_x\Phi(\dot{y}) = \Phi(x^{-1}\dot{y}) \qquad (\mu\text{-locally almost all } \dot{y} \in G/K),$$

where the left translation is induced of the natural action of G on the left coset space G/K and

$$L^\infty(G/K) = \{T_K^\infty(u) \ : \ u \in L^\infty(G)\},$$

$T_K^\infty : L^\infty(G) \to L^\infty(G/K)$ is defined by

$$T_K^\infty(u)(\dot{x}) = \int_K u(xk)d\lambda_K(k)$$

for μ-almost all $\dot{x} \in G/K$ and $u \in L^\infty(G)$; it is a surjective norm decreasing linear map; see [21].
For every $v \in L^1(G)$ and $x \in G$, the left translation is defined by

$$L_x T_K^1(v)(\dot{y}) = T_K^1(v)(x^{-1}\dot{y}) \qquad (\mu\text{-locally almost all } \dot{y} \in G/K)$$

where the left translation is induced of the natural action of G on the left coset space G/K and

$$L^1(G/K) = \{T_K^1(u) \ : \ u \in L^1(G)\},$$

$T_K^1 : L^1(G) \to L^1(G/K)$ is defined by

$$T_K^1(u)(\dot{x}) = \int_K u(xk)d\lambda_K(k)$$

for μ-locally almost all $\dot{x} \in G/K$ and $u \in L^1(G)$; it is a surjective norm decreasing linear map; see [11].

Lemma 2.12. *Let K be a compact subgroup of a locally compact group G. Then*

(a) *$A(G/K)$ is closed under the left translation;*

(b) *The map $x \mapsto L_x\Gamma_K(u) : G \to A(G/K)$ is continuous for all $u \in A(G)$;*

(c) $\| L_x\Gamma_K(u) \| = \| \Gamma_K(u) \|$ *for all $u \in A(G)$ and $x \in G$.*

Proof. (a). For each $u \in A(G)$, and $x \in G$,

$$\| L_x\Gamma_K(u) \| = \| \Gamma_K(L_x u) \| \leq \| \Gamma_K \| \| u \| < \infty.$$

(b). Let $x_\alpha \to e$ in G. Then for each $u \in A(G)$,

$$\| L_{x_\alpha}\Gamma_K(u) - \Gamma_K(u) \| = \| \Gamma_K(L_{x_\alpha}u) - \Gamma_K(u) \| \leq \| \Gamma_K \| \| L_{x_\alpha}u - u \| \to 0.$$

(c). For each $u \in A(G)$ and $x \in G$, we can obtain

$$\begin{aligned} \| L_x \| &= \sup\{\| L_x(\Gamma_K(u)) \| \ : \ u \in A(G) \text{ and } \| \Gamma_K(u) \| \leq 1\} \\ &= \sup\{\| \Gamma_K(L_x(u)) \| \ : \ u \in A(G) \text{ and } \| \Gamma_K(u) \circ \varphi_K \| \leq 1\} \\ &\leq \sup\{\| \Gamma_K(L_x(u)) \| \ : \ u \in A(G) \text{ and } \| u \| \leq 1\} \\ &\leq \sup\{\| L_x(u) \| \ : \ u \in A(G) \text{ and } \| u \| \leq 1\} \\ &= \sup\{\| u \| \ : \ u \in A(G) \text{ and } \| u \| \leq 1\} \\ &= 1. \end{aligned}$$

Therefore for each $u \in A(G)$ and $x \in G$ we have

$$\| L_x(\Gamma_K(u)) \| \leq \| L_x \| \| \Gamma_K(u) \| \leq \| \Gamma_K(u) \| .$$

Since L_x is invertible with ${L_x}^{-1} = L_{x^{-1}}$ and we have the same as above $\| L_{x^{-1}} \| \leq 1$, so for each $u \in A(G)$ and $x \in G$

$$\begin{aligned} \| \Gamma_K(u) \| &= \| L_{x^{-1}}L_x(\Gamma_K(u)) \| \\ &\leq \| L_{x^{-1}} \| \| L_x(\Gamma_K(u)) \| \\ &\leq \| L_x(\Gamma_K(u)) \| . \end{aligned}$$

It follows that $\| L_x(\Gamma_K(u)) \| = \| \Gamma_K(u) \|$. □

Lemma 2.13. *Let K be a compact subgroup of a locally compact group G. Then*

(a) *$S^1A(G/K)$ is closed under the left translation;*

(b) *The map $x \mapsto L_x\Gamma_K(u) : G \to S^1A(G/K)$ is continuous for all $u \in S^1A(G)$;*

(c) *$||| L_x\Gamma_K(u) ||| = ||| \Gamma_K(u) |||$ for all $u \in S^1A(G)$ and $x \in G$.*

Proof. (a). For each $u \in S^1A(G)$ and $x \in G$,

$$||| L_x\Gamma_K(u) ||| = ||| \Gamma_K(L_x u) ||| \leq \| \Gamma_K \| \, ||| u ||| < \infty.$$

(b). Let $x_\alpha \to e$ in G. Then for each $u \in S^1A(G)$,

$$||| L_{x_\alpha}\Gamma_K(u) - \Gamma_K(u) ||| = ||| \Gamma_K(L_{x_\alpha}u) - \Gamma_K(u) ||| \leq \| \Gamma_K \| \, ||| L_{x_\alpha}u - u ||| \to 0.$$

(c). For each $u \in S^1A(G)$ and $x \in G$,

$$\begin{aligned} \| L_x\Gamma_K(u) \|_{L^1(G/K)} &= \int_{G/K} | L_x\Gamma_K(u)(\dot{y}) | \, d\mu(\dot{y}) \\ &= \int_{G/K} | \Gamma_K(u)(x^{-1}\dot{y}) | \, d\mu(\dot{y}) \\ &= \| \Gamma_K(u) \|_{L^1(G/K)} . \end{aligned}$$

So, we only need to say that by Lemma 2.12 the map $L_x : A(G/K) \to A(G/K)$ is isometry. □

The proof of a part of the following corollary is similar to Proposition 10 in [13].

Corollary 2.14. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*

(a) $UCB_s(\widehat{G/K}) \subseteq WAP_s(\widehat{G/K})$;

(b) *K is open.*

Proof. Assume that $UCB_s(\widehat{G/K}) \subseteq WAP_s(\widehat{G/K})$. Since there exists a unique topologically invariant $\varphi_{\dot{e}}$-mean M on $WAP_s(\widehat{G/K})$, then the restriction M to $UCB_s(\widehat{G/K})$ is a topologically invariant $\varphi_{\dot{e}}$-mean on $UCB_s(\widehat{G/K})$. We can choose a function w in $S^1A(G)$ such that $\Gamma_K(w)(\dot{e}) = 1$. Now, consider $M_1, M_2 \in TIM_{\dot{e}}(S^1A(G/K))$ such that $M_1 \neq M_2$. Suppose there exists $T \in S^1A(G/K)^*$ such that

$$\langle M_1, T\rangle \neq \langle M_2, T\rangle.$$

As a result, $\Gamma_K(w) \cdot T \in UCB_s(\widehat{G/K}) \subseteq WAP_s(\widehat{G/K})$, and we have

$$\langle M_1, \Gamma_K(w) \cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, \Gamma_K(w) \cdot T\rangle,$$

which contradicts Theorem 2.11. Hence, $UCB_s(\widehat{G/K})$ has a unique topologically invariant mean at $\varphi_{\dot{e}}$. In particular, the number of invariant $\varphi_{\dot{e}}$-means on $UCB_s(\widehat{G/K})$ is equal to the number of invariant $\varphi_{\dot{e}}$-means on $S^1A(G/K)^*$. Hence $S^1A(G/K)$ has a unique topologically invariant $\varphi_{\dot{e}}$-mean and so by Corollary 2.5, K is open.

Conversely, let $x \in G$. Then $\chi_{\dot{x}}$ is the characteristic function of $\{\dot{x}\}$. Since K is open, $\chi_{\dot{x}} \in S^1A(G/K)$. Therefore, for $u \in S^1A(G)$ and $T \in S^1A(G/K)^*$, we have

$$\begin{aligned}
\langle \Gamma_K(u)\chi_{\dot{x}} \cdot T, \Gamma_K(v)\rangle &= \langle \chi_{\dot{x}} \cdot T, \Gamma_K(v)\Gamma_K(u)\rangle \\
&= \langle T, \Gamma_K(v)(\dot{x})\Gamma_K(u)(\dot{x})\chi_{\dot{x}}\rangle \\
&= \Gamma_K(u)(\dot{x})T(\chi_{\dot{x}})\Gamma_K(v)(\dot{x}) \\
&= \Gamma_K(u)(\dot{x})T(\chi_{\dot{x}})\varphi_{\dot{x}}(\Gamma_K(v)).
\end{aligned}$$

Since

$$\Gamma_K(v)(\dot{x}) = L_x^*(\varphi_{\dot{e}})(\Gamma_K(v)) \quad (v \in S^1A(G)),$$

therefore for $u \in S^1A(G)$ and $T \in S^1A(G/K)^*$, we have $\Gamma_K(u)\chi_{\dot{x}} \cdot T = \Big(\Gamma_K(u)(\dot{x})T(\chi_{\dot{x}})\Big)L_x^*(\varphi_{\dot{e}})$. Now, if $|||\Gamma_K(u)||| \leq 1$, then

$$\Gamma_K(u)\chi_{\dot{x}} \cdot T \in \Big\{\alpha L_x^*(\varphi_{\dot{e}}) \ : |\alpha| \leq |T(\chi_{\dot{x}})|\Big\} = \Big\{\alpha L_x \ : |\alpha| \leq |T(\chi_{\dot{x}})|\Big\}$$

and so the last set is compact in $S^1A(G/K)^*$. Therefore, the set $\Big\{\Gamma_K(u)\chi_{\dot{x}} \cdot T : |||\Gamma_K(u)||| \leq 1,\ u \in S^1A(G)\Big\}$ is compact in $S^1A(G/K)^*$. Hence, $\chi_{\dot{x}} \cdot T \in WAP_s(\widehat{G/K})$. Suppose $v \in S^1A(G)$ and $T \in S^1A(G/K)^*$. Let $\epsilon > 0$. Since $S^1A(G/K)$ is Tauberian, there exists a function $w \in S^1A(G)$ such that $|||\Gamma_K(w) - \Gamma_K(v)||| < \epsilon$, where $\Gamma_K(w)$ has compact support. Since K is open, the support of $\Gamma_K(w)$ is finite, meaning that $\Gamma_K(w)$ can be expressed as a finite linear combination of characteristic functions. Consequently, $\Gamma_K(w) \cdot T$ is a finite linear combination of compact operators, making it a compact operator. Now, for any $u \in S^1A(G)$ with $|||\Gamma_K(u)||| \leq 1$, we have

$$\begin{aligned}
|\langle \Gamma_K(w) \cdot T - \Gamma_K(v) \cdot T, \Gamma_K(u)\rangle| &= |\langle (\Gamma_K(w) - \Gamma_K(v)) \cdot T, \Gamma_K(u)\rangle| \\
&= |\langle T, \Gamma_K(u)(\Gamma_K(w) - \Gamma_K(v))\rangle| \\
&\leq \|T\|\ |||\Gamma_K(w) - \Gamma_K(v)|||\ |||\Gamma_K(u)|||.
\end{aligned}$$

This shows that $\Gamma_K(w) \cdot T$ approximates $\Gamma_K(v) \cdot T$ in the operator norm. Thus, $\Gamma_K(v) \cdot T$ is a compact operator, which implies

$$S^1A(G/K) \cdot S^1A(G/K)^* \subseteq WAP_s(\widehat{G/K}).$$

Therefore, we conclude

$$UCB_s(\widehat{G/K}) \subseteq WAP_s(\widehat{G/K}).$$

Hence, the proof is complete. □

Remark 2.15. Let K be a normal compact subgroup of a compact group G. Then $S^1A(G/K) = A(G/K)$ and hence $WAP_s(\widehat{G/K}) \subseteq UCB_s(\widehat{G/K})$; see [13, Ramark 9(i), Ramark11(ii), Proposition 9].

The Arens product, introduced by Richard Arens in [1], extends the multiplication operation of a Banach algebra $\mathcal{A}$ to its second dual $\mathcal{A}^{**}$ in two distinct ways. These extensions are known as the first and second Arens products. Generally, these two products are not the same. The first and second Arens products on $\mathcal{A}^{**}$ is defined as follows:
For $M, N \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$, we can define $a \cdot f, f \cdot a \in \mathcal{A}^*$, $f \cdot M, N \cdot f \in \mathcal{A}^*$, by

$$\begin{aligned} \langle f \cdot a, b \rangle &= \langle f, ab \rangle \\ \langle a \cdot f, b \rangle &= \langle f, ba \rangle \\ \langle N \cdot f, a \rangle &= \langle N, f \cdot a \rangle \\ \langle f \cdot M, a \rangle &= \langle M, a \cdot f \rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle M \odot N, f \rangle &= \langle M, N \cdot f \rangle \\ \langle M \boxdot N, f \rangle &= \langle N, f \cdot M \rangle. \end{aligned}$$

A Banach algebra $\mathcal{A}$ is Arens regular if, for every $M, N \in \mathcal{A}^{**}$, the two Arens products coincide:

$$M \odot N = M \boxdot N.$$

Moreover, $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}) = \mathcal{A}^*$.

Corollary 2.16. *Let K be a compact subgroup of a locally compact group G. If $S^1A(G/K)$ is Arens regular, then K is open.*

Proof. If $S^1A(G/K)$ is Arens regular, then $WAP_s(\widehat{G/K}) = S^1A(G/K)^*$. Thus by theorem 2.11, there exists a unique topologically invariant $\varphi_{\dot{e}}$-mean on $S^1A(G/K)^*$ and so by Corollary 2.5, K is open. □

Corollary 2.17. *Let K be a compact subgroup of a locally compact group G. Then the following statements are equivalent.*
(a) *$S^1A(G/K)$ is a C^*-algebra;*
(b) *G/K is finite.*

Proof. Since $S^1A(G/K)$ is a C^*-algebra, it is amenable and Arens regular. Thus $S^1A(G/K)$ has a bounded approximate identity and so by Proposition 2.5 from [3], G/K is compact. Also, by Corrollary 2.16, G/K is discrete. Hence G/K is finite. Now, if G/K is finite, then $S^1A(G/K) = \mathbb{C}^{|G/K|} = C_0(G/K)$. Thus $S^1A(G/K)$ is a C^*-algebra. □

3. Further remarks and open problems

We end the work by some remarks and open problems. The following problems are of interest to us.

(1) Let K be a compact subgroup of a locally compact group G. We wish to see whether $UCB_s(\widehat{G/K}) = WAP_s(\widehat{G/K})$ if and only if G/K is finite.

(2) Let K be a compact subgroup of a locally compact group G. If G is a compact group, then $S^1A(G/K) = A(G/K)$ and so $i^*(UCB(\widehat{G/K})) = UCB_s(\widehat{G/K})$. The converse is of interest to us; precisely, we like to see whether $i^*(UCB(\widehat{G/K})) = UCB_s(\widehat{G/K})$ if and only if G is compact.

(3) It would be interesting if the content could be generalized to a wider range of abstract Segal algebras.

(4) Is the converse statement of Corollary 2.16 necessarily true? This result has been proven for locally compact groups in the nice works [9] and [10]; the Lebesgue-Fourier algebra on G with pointwise multiplication, $S^1A(G)$, is Arens regular if and only if G is discrete.

(5) The results of this paper also apply to the Lebesgue-Fourier algebra associated with locally compact groups.

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