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Special solutions to a matrix equation over the split quaternions

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Abstract. Split quaternions, as an extension of classical quaternions, exhibit distinct algebraic properties and offer valuable applications across various fields. This paper investigates solutions to the split quaternion matrix equation $\sum_{i=1}^{l} A_iX_iB_i = C$, providing necessary and sufficient conditions for its solvability and expressions for various types of solutions. Solvability criteria and explicit solution forms are derived for general, pure imaginary, and real solutions. Additionally, corresponding conditions and expressions are presented for (skew-)centro-Hermitian solutions. Finally, numerical examples and algorithms are provided to validate the accuracy of the obtained results.

1. Introduction

A quaternion was first introduced by Hamilton in 1843 [1]. Quaternions and quaternion matrices have been extensively studied and applied in various fields due to their significant roles in image processing, quantum physics, robotics, and signal processing [2–4]. The set of quaternions is defined as

$$\mathbf{H} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbf{R}\},$$

where $\mathbf{R}$ is the real number field, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Six years after Hamilton's discovery, James Cockle expanded the concept by presenting split quaternions [5]. The set of all split quaternions is denoted by

$$\mathbf{SH} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbf{R}\},$$

where three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = -1,\ \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1.$$

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For a split quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, its conjugate, real part, imaginary part, and module are defined as $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$, $Re(q) = q_0, Im(q) = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, and $\|q\| = \sqrt{|q\bar{q}|} = \sqrt{|q_0^2 + q_1^2 - q_2^2 - q_3^2|}$, respectively. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements, and nontrivial idempotents [6, 7]. Due to these complicated characteristics, studying split quaternions is more challenging than quaternions.

Quaternions and split quaternions are essential in areas such as spatial geometry, quantum mechanics, and electromagnetism. Quaternions effectively describe 3D rotations and reflections [8], while split quaternions provide powerful tools for understanding structures in Minkowski space, including classifications of mutual planes and Lorentzian electromagnetic phenomena [9]. They also play a critical role in space-time reflection symmetric systems and have advanced applications in solving problems like the Schrödinger equation and least squares analysis [10–14].

Quaternion matrix equations find wide applications in several fields, including mathematics, engineering, system and control theory, data analysis, color image processing, and optimal control [15–20]. The problem of solving these matrix equations holds significant practical value and has attracted considerable attention from researchers. As a result, various solutions to matrix equations, like least squares solutions, (anti-)symmetric solutions, Hermitian solutions, and η-Hermitian solutions, have been investigated (see, e.g., [21–39]). These studies mainly focus on the existence, uniqueness, and properties of solutions in quaternion matrices.

In contrast, some special solutions, such as pure imaginary solutions, real solutions, and (skew-) centro-Hermitian solutions, have received limited attention in existing research. For instance, Au-Yeung and Cheng [40] examined pure imaginary quaternionic solutions to the Hurwitz matrix equations. Wang et al. [41] studied the quaternion matrix equation $AXB = C$ and established solvability conditions for both real and pure imaginary solutions. Further, the authors [42, 43] derived formulas for the least squares solutions corresponding to pure imaginary and real solutions over quaternions. In addition, Şimşek et al. [44] considered (skew-)centro-Hermitian solutions and provided expressions for these solutions to the quaternion matrix equation $(AXB, DXE) = (C, F)$. On the other hand, Zhang et al. [45] utilized real representation to investigate these solutions and compared the efficiency of their approach with that of [44]. However, despite these efforts, there has been no research focused on these special solutions within the split quaternion field.

Motivated by the aforementioned discussions, this paper delves into a special class of solutions to the following split quaternion matrix equation

$$\sum_{i=1}^{l} A_i X_i B_i = C. \tag{1}$$

This equation holds substantial potential for deriving general, pure imaginary, real, and (skew-)centro-Hermitian solutions. To further explore its properties, we investigate these solutions under specific constraints, advancing our understanding of split quaternion matrix equations and their applications.

For convenience, throughout this paper, we denote the sets of all $m \times n$ complex matrices, real matrices, split quaternion matrices, split quaternion centro-Hermitian matrices, and split quaternion skew-centro-Hermitian matrices by $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$, $\mathbf{SH}^{m\times n}$, $\mathbf{SH}_{CH}^{m\times n}$, and $\mathbf{SH}_{SCH}^{m\times n}$. For $A \in \mathbf{C}^{m\times n}$, $Re(A)$ and $Im(A)$ represent the real and imaginary parts of matrix A. The symbols $A^{\mathbf{T}}$, $\bar{A}$, $A^{\mathbf{H}}$, $A^{\dagger}$ correspond to the transpose, conjugate, conjugate transpose, and Moore-Penrose inverse of A, respectively. The greatest integer function of any real number s is denoted by $\lfloor s \rfloor$. The symbol $\|\cdot\|$ stands for the 2-norm of matrices. Additionally, $X_{\sigma_j} = [X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}]$ represents the set of solutions to the split quaternion matrix equation (1), including the general, pure imaginary, and real solutions, while $\Omega_j = [\Omega_1, \Omega_2]$ refers to the centro-Hermitian solution and skew-centro-Hermitian solution.

The remainder of this paper is organized as follows. In Section 2, we introduce the preliminaries necessary for addressing Equation (1), including the complex representation and the vec-operator for split quaternion matrices. We also provide definitions and results related to (skew-)centro-Hermitian matrices and the vec$_r$-operator, which is specifically designed for these types of matrices. In Section 3, we derive necessary and sufficient conditions for the solvability of Equation (1) and provide expressions for its general, pure imaginary, and real solutions. Section 4 establishes necessary and sufficient conditions for Equation (1) to admit (skew-)centro-Hermitian solutions. In Section 5, we present algorithms and numerical examples to illustrate the main results of this paper. Section 6 concludes with some remarks.

2. Preliminary

2.1. A complex representation of split quaternion matrix

In this subsection, we revisit the complex representation of split quaternion matrices and provide the relevant tools for solving Equation (1).

Any split quaternion matrix $A = A_0 + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ can be uniquely expressed as $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in \mathbf{C}^{m\times n}$. A map $\mathcal{G}$ from $\mathbf{SH}^{m\times n}$ to $\mathbf{C}^{2m\times 2n}$ is defined as

$$\mathcal{G} : A = A_1 + A_2\mathbf{j} \mapsto \mathcal{G}(A) := \begin{bmatrix} A_1 & A_2 \\ \bar{A}_2 & \bar{A}_1 \end{bmatrix},$$

where $\mathcal{G}(A)$ is called the complex representation of split quaternion matrix A. It is easy to verify that the following statements are true.

Proposition 2.1 ([46]). *For $A,\ B \in \boldsymbol{SH}^{m\times n}$ and $k_1,\ k_2 \in \boldsymbol{R}$, we have the following:*
(1) $A = B$ *if and only if* $\mathcal{G}(A) = \mathcal{G}(B)$;
(2) $\mathcal{G}(AB) = \mathcal{G}(A)\mathcal{G}(B)$;
(3) $\mathcal{G}(k_1A + k_2B) = k_1\mathcal{G}(A) + k_2\mathcal{G}(B)$;
(4) $\mathcal{G}(I_n) = I_{2n}$, *where I_n is an identity matrix with order n.*

For simplicity of expressions, we define

$$\Phi_A = [A_1, A_2]$$

and

$$\Psi_A = [Re(A_1), Im(A_1), Re(A_2), Im(A_2)].$$

Furthermore, it is noteworthy that the following proposition holds for operators Φ_A and Ψ_A.

Proposition 2.2 ([47]). *If $A = A_1 + A_2\mathbf{j} \in \boldsymbol{SH}^{m\times n}$, then*
(1) $A = A_1 + A_2\mathbf{j} \cong \Phi_A = \begin{bmatrix} I_n \\ \mathbf{j}I_n \end{bmatrix}$;
(2) $A = Re(A_1) + Im(A_1)\mathbf{i} + Re(A_2)\mathbf{j} + Im(A_2)\mathbf{k} \cong \Psi_A \begin{bmatrix} I_n \\ \mathbf{i}I_n \\ \mathbf{j}I_n \\ \mathbf{k}I_n \end{bmatrix}$.

Next, we present some properties related to Φ_A as follows.

Lemma 2.3 ([46]). *Suppose that $A = A_1 + A_2\mathbf{j} \in \boldsymbol{SH}^{m\times n}$, $B = B_1 + B_2\mathbf{j} \in \boldsymbol{SH}^{n\times t}$, $k \in \boldsymbol{R}$, then*
(1) $A = B$ *if and only if* $\Phi_A = \Phi_B$;
(2) $\Phi_{A+B} = \Phi_A + \Phi_B$, $\Phi_{kA} = k\Phi_A$;
(3) $\Phi_{AB} = \Phi_A\mathcal{G}(B)$.

The definition and relevant properties of the vec-operator over split quaternion matrices can be further introduced. For $A = (a_{ik}) \in \mathbf{SH}^{m\times n}$, let $a_k = (a_{1k},\ a_{2k},\ \cdots,\ a_{mk})$, $k = 1, 2, \ldots, n$. The vec-operator of A, denoted by vec(A), is defined as

$$\text{vec} : \mathbf{SH}^{m\times n} \to \mathbf{SH}^{mn},$$

$$A \mapsto \mathrm{vec}(A) = (a_1^{\mathbf{T}},\ a_2^{\mathbf{T}},\ \cdots,\ a_n^{\mathbf{T}})^{\mathbf{T}}.$$

Since $\Psi_A = [Re(A_1), Im(A_1), Re(A_2), Im(A_2)]$, we have

$$\mathrm{vec}(\Psi_A) = \begin{bmatrix} \mathrm{vec}(Re(A_1)) \\ \mathrm{vec}(Im(A_1)) \\ \mathrm{vec}(Re(A_2)) \\ \mathrm{vec}(Im(A_2)) \end{bmatrix}.$$

Proposition 2.4 ([46, 47]). *For $A, B \in \boldsymbol{SH}^{m\times n}$, $k \in \boldsymbol{R}$, then the following statements hold.*

(1) $\mathrm{vec}(A) = \mathrm{vec}(B)$ *if and only if* $A = B$;

(2) $\mathrm{vec}(A + kB) = \mathrm{vec}(A) + k\mathrm{vec}(B)$;

(2) $\mathrm{vec}([A_1, A_2]) = \mathrm{vec}(\Phi_A) = \begin{bmatrix} \mathrm{vec}(A_1) \\ \mathrm{vec}(A_2) \end{bmatrix}$.

At the end of this subsection, we recall the definitions of the Kronecker product and the Moore-Penrose inverse, and also provide an expression of $\mathrm{vec}(\Phi_{AXB})$. If $A = (a_{ij}) \in \mathbf{SH}^{m\times n}$, $B \in \mathbf{SH}^{k\times t}$, then the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

The Moore-Penrose generalized inverse of $A \in \mathbf{C}^{m\times n}$, denoted by $A^{\dagger}$ is a unique matrix X that satisfies the Penrose equations

$$AXA = A, \quad XAX = X, \quad (AX)^{\mathbf{H}} = AX, \quad (XA)^{\mathbf{H}} = XA.$$

Lemma 2.5 ([46]). *Let $A = A_1 + A_2\mathbf{j} \in \boldsymbol{SH}^{m\times n}$, $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $B = B_1 + B_2\mathbf{j} \in \boldsymbol{SH}^{n\times t}$, where $A_1, A_2 \in \boldsymbol{C}^{m\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$, and $B_1, B_2 \in \boldsymbol{C}^{n\times t}$. Then*

$$\mathrm{vec}(\Phi_{AXB}) = [\mathcal{G}(B)^{T} \otimes A_1, \mathcal{G}(B\mathbf{j})^{H} \otimes A_2] \begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathrm{j}X\mathrm{j}}) \end{bmatrix}.$$

2.2. (skew-)centro-Hermitian matrices

This subsection introduces the relevant definitions and results necessary to derive the (skew-)centro-Hermitian solutions of Equation (1).

The $m \times n$ complex matrix $C = (c_{ik})$, where $1 \leqslant i \leqslant m$ and $1 \leqslant k \leqslant n$, is called centro-Hermitian if

$$c_{ik} = \bar{c}_{m-i+1,n-k+1}.$$

It is termed skew-centro-Hermitian if

$$c_{ik} = -\bar{c}_{m-i+1,n-k+1},$$

where $\bar{c}$ denoted the complex conjugate of c [44]. In the given definition, if $C \in \mathbf{SH}^{m\times n}$, then the terms split quaternion centro-Hermitian and split quaternion skew-centro-Hermitian are used instead of centro-Hermitian and skew-centro-Hermitian, respectively. If $C \in \mathbf{SH}^{m\times n}$ is a centro-Hermitian matrix then it is equivalent to $J_m C J_n = \bar{C}$. Similarly, if it is a skew-centro-Hermitian matrix, it is equivalent to $J_m C J_n = -\bar{C}$, where $J_k = (e_k, e_{k-1}, \dots, e_1)$ with e_u being the u^{th} column of the identity matrix I of order k. For the matrix $A = (a_{ij}) \in \mathbf{SH}^{m\times n}$, let $a_k = (a_{1k}, a_{2k}, \dots, a_{mk})^{\mathbf{T}}$, where $k = 1, 2, \dots, r$ and $r = \lfloor \frac{n+1}{2} \rfloor$. We define the vec_r-operator of A as

$$\mathrm{vec}_r : \mathbf{SH}^{m\times n} \to \mathbf{SH}^{mn},$$

$$A \mapsto \mathrm{vec}_r(A) = (a_1^{\mathbf{T}},\ a_2^{\mathbf{T}},\ \cdots,\ a_r^{\mathbf{T}})^{\mathbf{T}}.$$

We now introduce the following lemma, which is a criterion for determining whether a given split quaternion matrix is (skew-)centro-Hermitian.

Lemma 2.6 ([44]). *Let $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{p\times q}$ with $X_1, X_2 \in \boldsymbol{C}^{p\times q}$. Then*

(1) $X \in \boldsymbol{SH}_{CH}^{p\times q} \iff \mathrm{vec}(\Psi_X) = K^{+}\mathrm{vec}_r(Re(X_1)) + K^{-}\mathrm{vec}_r(Im(X_1))\mathbf{i} + K^{-}\mathrm{vec}_r(Re(X_2))\mathbf{j} + K^{-}\mathrm{vec}_r(Im(X_2))\mathbf{k}$;

(2) $X \in \boldsymbol{SH}_{SCH}^{n\times n} \iff \mathrm{vec}(\Psi_X) = K^{-}\mathrm{vec}_r(Re(X_1)) + K^{+}\mathrm{vec}_r(Im(X_1))\mathbf{i} + K^{+}\mathrm{vec}_r r(Re(X_2))\mathbf{j} + K^{+}\mathrm{vec}_r(Im(X_2))\mathbf{k}$,

where $K^{\pm} = [K_1^{\pm}, K_2^{\pm}, \ldots, K_r^{\pm}]$, such that $K_j^{i\pm}$ represents the i-th column of the $pq \times p$ matrix $K_j^{\pm}$, with $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, r$. $K_j^{i\pm}$ is defined as follows:

- *If $j < r$, $K_j^{i\pm} = e_{i+(j-1)p} \pm e_{pq-(i-1)-(j-1)p}$;*
- *If $j = r$,*

$$K_r^{i\pm} = \begin{cases} e_{i+(r-1)p} \pm e_{pq-(i-1)-(r-1)p}, & \text{if } q \text{ is an even number and } p \text{ is an even or odd number;} \\ \frac{1}{2}(e_{i+(r-1)p} \pm e_{pq-(i-1)-(r-1)p}), & \text{if } q \text{ is an odd number and } p \text{ is an even number;} \\ \frac{1}{2}(e_{i+(r-1)p} \pm e_{pq-(i-1)-(r-1)p}), & \text{if both } q \text{ and } p \text{ are odd numbers and } i \neq \left\lfloor \frac{p+1}{2} \right\rfloor; \\ e_r, & \text{if both } q \text{ and } p \text{ are odd numbers and } i = \left\lfloor \frac{p+1}{2} \right\rfloor. \end{cases}$$

Note that the symbol $\pm$ represents a plus sign (or a minus sign) when the matrix X is centro-Hermitian (or skew-centro-Hermitian). If only the plus sign is used, then all signs are the same. For example, when considering centro-Hermitian matrices, $K^{+} = [K_1^{+}, K_2^{+}, \ldots, K_r^{+}]$.

Example 2.7. *Consider a matrix $X \in \boldsymbol{SH}^{p\times q}$, where $p = q = 4$, and $r = 2$(i.e., $r = \lfloor \frac{q+1}{2} \rfloor$), and let $i = 1, 2, 3, 4$, $j = 1, 2$. For centro-Hermitian matrix X , the matrix $K^{+} = [K_1^{+}, K_2^{+}] \in \boldsymbol{C}^{16\times 8}$ is formed from the blocks $K_1^{+} \in \boldsymbol{C}^{16\times 4}$ and $K_2^{+} \in \boldsymbol{C}^{16\times 4}$, where the individual block matrices are further decompose as follows:*

$$K_1^{+} = [K_1^{1^+}, K_1^{2^+}, K_1^{3^+}, K_1^{4^+}], \quad K_2^{+} = [K_2^{1^+}, K_2^{2^+}, K_2^{3^+}, K_2^{4^+}],$$

where each block $K_j^{i^+}$ is defined by specific entries of standard basis vectors $e_k \in \boldsymbol{R}^{16\times 1}$. For the first block K_1^{+} , the elements are defined as

$$K_1^{1^+} = e_1 + e_{16}, \quad K_1^{2^+} = e_2 + e_{15}, \quad K_1^{3^+} = e_3 + e_{14}, \quad K_1^{4^+} = e_4 + e_{13}.$$

Similarly, for the second block K_2^{+}, we have

$$K_2^{1^+} = e_5 + e_{12}, \quad K_2^{2^+} = e_6 + e_{11}, \quad K_2^{3^+} = e_7 + e_{10}, \quad K_2^{4^+} = e_8 + e_9.$$

Thus, the complete matrix $K^{+} \in \boldsymbol{C}^{16\times 8}$ is given by

$$K^{+} = [e_1 + e_{16}, e_2 + e_{15}, e_3 + e_{14}, e_4 + e_{13}, e_5 + e_{12}, e_6 + e_{11}, e_7 + e_{10}, e_8 + e_9].$$

To investigate the special solutions to Equation (1), we present the following lemma.

Lemma 2.8 ([47]). *The matrix equation $Ax = b$, where $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^n$, has a solution $x \in \mathbf{R}^n$ if and only if*

$$AA^{\dagger}b = b.$$

In this case, the general solution can be expressed as

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y, \tag{2}$$

where $y \in \mathbf{R}^n$ is an arbitrary vector. When the matrix equation $Ax = b$ is inconsistent, Equation (2) represents the least squares solution. When the matrix equation is consistent, the unique solution with the minimal norm is given by

$$x = A^{\dagger}b. \tag{3}$$

Moreover, if $A^{\dagger}A = I_n$ or $rank(A) = n$, this minimal norm solution is also the unique solution. In cases where the matrix equation $Ax = b$ is not consistent, Equation (3) stand as the unique minimal norm least squares solution.

3. The general, pure imaginary, and real solutions of Equation (1)

In this section, we present the general solutions, the pure imaginary solutiosn, and the real solutions to the split quaternion matrix equation (1). If X_i is pure imaginary solution, then $Re(X_{i1}) = 0$, implying $\Psi_{X_i} = [O, Im(X_{i1}), Re(X_{i2}), Im(X_{i2})]$. In the case where X_j is a real solution, $Im(X_{j1}) = Re(X_{j2}) = Im(X_{j2}) = 0$, resulting $\Psi_{X_j} = [Re(X_{j1}), O, O, O]$. To investigate these solutions, we begin with the following lemma.

Lemma 3.1. *Suppose that* $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$. *Then the following conditions hold.*
(1) *For* $X \in X_{\sigma_1}$, *it follows that*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = M_\sigma^{(n)}\mathrm{vec}(\Psi_X).$$

(2) *For* $X \in X_{\sigma_2}$, *we obtain*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = K_\sigma^{(n)}\mathrm{vec}(\Psi_{X_i}).$$

(3) *For* $X \in X_{\sigma_3}$, *there exists*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = I_\sigma^{(n)}\mathrm{vec}(\Psi_{X_j}),$$

where the matrices $M_\sigma^{(n)}$, $K_\sigma^{(n)}$, *and* $I_\sigma^{(n)}$ *are defined as*

$$M_\sigma^{(n)} = \begin{bmatrix} I_{n^2} & \boldsymbol{i}I_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & \boldsymbol{i}I_{n^2} \\ I_{n^2} & -\boldsymbol{i}I_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -\boldsymbol{i}I_{n^2} \end{bmatrix}, \quad K_\sigma^{(n)} = \begin{bmatrix} \boldsymbol{i}I_{n^2} & 0 & 0 \\ 0 & I_{n^2} & \boldsymbol{i}I_{n^2} \\ -\boldsymbol{i}I_{n^2} & 0 & 0 \\ 0 & I_{n^2} & -\boldsymbol{i}I_{n^2} \end{bmatrix}, \quad I_\sigma^{(n)} = \begin{bmatrix} \boldsymbol{i}I_{n^2} \\ 0 \\ \boldsymbol{i}I_{n^2} \\ 0 \end{bmatrix}.$$

Proof. We only consider (1), (2) and (3) can be similarly proved. If $X = X_1 + X_2\mathbf{j} \in \mathbf{SH}^{n\times n}$, it follows that

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(\bar{X}_1) \\ \mathrm{vec}(\bar{X}_2) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(Re(X_1) + \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_2) + \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_1) - \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_2) - \mathbf{i}Im(X_2)) \end{bmatrix} = \begin{bmatrix} I_{n^2} & \mathbf{i}I_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & \mathbf{i}I_{n^2} \\ I_{n^2} & -\mathbf{i}I_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -\mathbf{i}I_{n^2} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(Re(X_1)) \\ \mathrm{vec}(Im(X_1)) \\ \mathrm{vec}(Re(X_2)) \\ \mathrm{vec}(Im(X_2)) \end{bmatrix} = M_\sigma^{(n)}\mathrm{vec}(\Psi_X).$$

□

By Lemmas 2.5 and 3.1, we can obtain the following results.

Lemma 3.2. *Let* $A = A_1 + A_2\mathbf{j} \in \boldsymbol{SH}^{m\times n}$, $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $B = B_1 + B_2\mathbf{j} \in \boldsymbol{SH}^{n\times t}$, *where* $A_1, A_2 \in \boldsymbol{C}^{m\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$, *and* $B_1, B_2 \in \boldsymbol{C}^{n\times t}$.
(1) *If* $X \in X_{\sigma_1}$, *it follows that*

$$\mathrm{vec}(\Phi_{AXB}) = [\mathcal{G}(B)^{\boldsymbol{T}} \otimes A_1, \mathcal{G}(B\mathbf{j})^{\boldsymbol{H}} \otimes A_2]M_\sigma^{(n)}\mathrm{vec}(\Psi_X). \quad (4)$$

(2) *For* $X \in X_{\sigma_2}$, *we obtain*

$$\mathrm{vec}(\Phi_{AXB}) = [\mathcal{G}(B)^{\boldsymbol{T}} \otimes A_1, \mathcal{G}(B\mathbf{j})^{\boldsymbol{H}} \otimes A_2]K_\sigma^{(n)}\mathrm{vec}(\Psi_{X_i}). \quad (5)$$

(3) *If* $X \in X_{\sigma_3}$, *we have*

$$\mathrm{vec}(\Phi_{AXB}) = [\mathcal{G}(B)^{\boldsymbol{T}} \otimes A_1, \mathcal{G}(B\mathbf{j})^{\boldsymbol{H}} \otimes A_2]I_\sigma^{(n)}\mathrm{vec}(\Psi_{X_j}).$$

Based on the aforementioned context, we now turn our attention to the general, imaginary, and real solutions of Equation (1), and the following notations are necessary. Let $A_i = A_{i1} + A_{i2}\mathbf{j} \in \mathbf{SH}^{m\times n}$, $B_i = B_{i1} + B_{i2}\mathbf{j} \in \mathbf{SH}^{n\times t}$, $C = C_{11} + C_{12}\mathbf{j} \in \mathbf{SH}^{m\times t}$. For $i = \overline{1,l}$, set

$$T_i = \begin{bmatrix}\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} & \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\end{bmatrix} M_\sigma^{(n)}, \quad \widetilde{T} = \begin{bmatrix} Re[T_1, T_2, \dots T_l] \\ Im[T_1, T_2, \dots T_l]\end{bmatrix},$$

$$L_i = \begin{bmatrix}\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} & \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\end{bmatrix} K_\sigma^{(n)}, \quad \widetilde{L} = \begin{bmatrix} Re[L_1, L_2, \dots L_l] \\ Im[L_1, L_2, \dots L_l]\end{bmatrix},$$

$$Q_i = \begin{bmatrix}\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} & \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\end{bmatrix} I_\sigma^{(n)}, \quad \widetilde{Q} = \begin{bmatrix} Re[Q_1, Q_2, \dots Q_l] \\ Im[Q_1, Q_2, \dots Q_l]\end{bmatrix},$$

$$\widetilde{t} = I_{4ln^2} - \widetilde{T}^\dagger \widetilde{T}, \quad \widetilde{l} = I_{3ln^2} - \widetilde{L}^\dagger \widetilde{L}, \quad \widetilde{q} = I_{ln^2} - \widetilde{Q}^\dagger \widetilde{Q}, \quad e = \begin{bmatrix}\mathrm{vec}(Re(\Phi_C)) \\ \mathrm{vec}(Im(\Phi_C))\end{bmatrix}.$$

Theorem 3.3. *Consider the general solution of Equation* (1)*. Then, the following descriptions hold.*
(1) *Equation* (1) *is solvable if and only if*

$$\widetilde{T}\widetilde{T}^\dagger e = e. \tag{6}$$

(2) *If Equation* (1) *has a solution, then general solution is given by*

$$X_{\sigma_1} = \left\{ [X_1, X_2, X_3, \dots, X_l] \,\middle|\, \begin{bmatrix}\mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l})\end{bmatrix} = \widetilde{T}^\dagger e + (I_{4ln^2} - \widetilde{T}^\dagger \widetilde{T}) y \right\}, \tag{7}$$

where y is an arbitraty vector with the appropriate order. Otherwise, (7) *is the least squares solution.*
(3) *When* (7) *is the general solution of Equation* (1)*, then*

$$\begin{bmatrix}\mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l})\end{bmatrix} = \widetilde{T}^\dagger e \tag{8}$$

is the minimal norm solution of Equation (1)*. This solution is unique when $\widetilde{t} = 0$ or rank $(\widetilde{T}) = 4ln^2$. Otherwise,* (8) *is the minimal norm least squares solution in the case of inconsistency.*

Proof. By Lemma 2.3 and Equation (4), it follows that

$$\begin{aligned}
\textstyle\sum_{i=1}^{l} A_i X_i B_i = C &\iff \textstyle\sum_{i=1}^{l} \Phi_{A_i X_i B_i} = \Phi_C, \\
&\iff \textstyle\sum_{i=1}^{l} \mathrm{vec}(\Phi_{A_i X_i B_i}) = \mathrm{vec}(\Phi_C), \\
&\iff \textstyle\sum_{i=1}^{l} \begin{bmatrix}\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} & \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\end{bmatrix} M_\sigma^{(n)} \mathrm{vec}(\Psi_X) = \mathrm{vec}(\Phi_C), \\
&\iff \textstyle\sum_{i=1}^{n} T_i \mathrm{vec}(\Psi_X) = \mathrm{vec}(\Phi_C), \\
&\iff \begin{bmatrix} Re[T_1, T_2, \dots T_l] \\ Im[T_1, T_2, \dots T_l]\end{bmatrix} \begin{bmatrix}\mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l})\end{bmatrix} = \begin{bmatrix} Re(\mathrm{vec}(\Phi_C)) \\ Im(\mathrm{vec}(\Phi_C))\end{bmatrix}, \\
&\iff \widetilde{T} \begin{bmatrix}\mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l})\end{bmatrix} = e.
\end{aligned}$$

By Lemma 2.8, Equation (1) has a solution $[X_1, X_2, X_3, \ldots, X_l] \in X_{\sigma_1}$ if and only if

$$\widetilde{T}\widetilde{T}^{\dagger}e = e,$$

which implies (6) holds. In this case,

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{T}^{\dagger}e + (I_{4ln^2} - \widetilde{T}^{\dagger}\widetilde{T})y = \widetilde{T}^{\dagger}e + \widetilde{t}e,$$

which implies that Equation (7) holds. Moreover, when Equation (1) is consistent and $\widetilde{t} = 0$ (i.e rank $(\widetilde{T}) = 4ln^2$), Equation (8) represents the unique solution. Otherwise, Equation (8) represents the minimal norm least squares solution to Equation (1). □

Theorem 3.4. *For the pure imaginary solution of Equation* (1)*, the following descriptions hold.*
(1) *Equation* (1) *is solvable if and only if*

$$\widetilde{L}\widetilde{L}^{\dagger}e = e. \tag{9}$$

(2) *If Equation* (1) *has a pure imaginary solution, then*

$$X_{\sigma_2} = \left\{ [X_1, X_2, X_3, \ldots, X_l] \,\middle|\, \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{L}^{\dagger}e + (I_{3ln^2} - \widetilde{L}^{\dagger}\widetilde{L})w \right\} \tag{10}$$

is the general pure imagiary solution, where w is an arbitrary vector with the appropriate size. Otherwise, Equation (10) is the least squares pure imaginary solution.
(3) *When* (10) *represents the general pure imaginary solution of Equation* (1)*, then*

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{L}^{\dagger}e \tag{11}$$

is the minimal norm pure imaginary solution of Equation (1)*. In this case, if $\widetilde{l} = 0$ or rank $(\widetilde{L}) = 3ln^2$, then* (11) *is the unique pure imaginary solution. Otherwise, if Equation* (1) *is inconsistent,* (11) *is the minimal norm least squares pure imaginary solution.*

Proof. By Lemma 2.3 and Equation (5), it follows that

$$\begin{aligned}
\textstyle\sum_{i=1}^{l} A_iX_iB_i = C \iff & \textstyle\sum_{i=1}^{l} \Phi_{A_iX_iB_i} = \Phi_C, \\
\iff & \textstyle\sum_{i=1}^{l} \mathrm{vec}(\Phi_{A_iX_iB_i}) = \mathrm{vec}(\Phi_C), \\
\iff & \textstyle\sum_{i=1}^{l} \left[\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} \;\; \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2} \right] K_{\sigma}^{(n)} \mathrm{vec}(\Psi_{X_i}) = \mathrm{vec}(\Phi_C), \\
\iff & \textstyle\sum_{i=1}^{l} L_i \mathrm{vec}(\Psi_{X_i}) = \mathrm{vec}(\Phi_C), \\
\iff & \begin{bmatrix} Re[L_1, L_2, \ldots L_l] \\ Im[L_1, L_2, \ldots L_l] \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \begin{bmatrix} Re(\mathrm{vec}(\Phi_C)) \\ Im(\mathrm{vec}(\Phi_C)) \end{bmatrix}, \\
\iff & \widetilde{L} \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = e.
\end{aligned}$$

By Lemma 2.8, Equation (1) has a pure imaginary solution $[X_1, X_2, X_3, \ldots, X_l] \in X_{\sigma_2}$ if and only if

$$\widetilde{L}\widetilde{L}^{\dagger}e = e.$$

In this case,

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{L}^{\dagger}e + (I_{3ln^2} - \widetilde{L}^{\dagger}L)W = \widetilde{L}^{\dagger}e + \widetilde{l}w,$$

which implies that (10) holds. In particular, when Equation (1) is consistent and $\widetilde{l} = 0$ or rank $(\widetilde{L}) = 3n^3$, (11) represents the unique pure imaginary solution. Otherwise, when Equation (1) is inconsistent, (11) is the minimal norm least squares pure imaginary solution. □

We will now study the real solution of Equation (1). Since the method is similar to those of Theorems 3.3 and 3.4, we will only provide the results and omit the detailed proof for simplicity.

Theorem 3.5. *Consider the real solution of Equation* (1)*. Then, the following descriptions hold.*

(1) *Equation* (1) *is solvable if and only if*

$$\widetilde{Q}\widetilde{Q}^{\dagger}e = e. \tag{12}$$

(2) *If Equation* (1) *has a real solution, then*

$$X_{\sigma_3} = \left\{ [X_1, X_2, X_3, \ldots, X_l] \;\middle|\; \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{Q}^{\dagger}e + (I_{ln^2} - \widetilde{Q}^{\dagger}\widetilde{Q})v \right\} \tag{13}$$

is the general real solution, where v is an arbitrary vector with the appropriate size. Otherwise, (13) *is the least squares real solution.*

(3) *When* (13) *is the general real solution of Equation* (1)*, then minimal norm real solution is given by*

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = \widetilde{Q}^{\dagger}e. \tag{14}$$

This solution is unique when $\widetilde{q} = 0$ or rank $(\widetilde{Q}) = ln^2$. If Equation (1) *is inconsistent, then* (14) *is the minimal norm least squares real solution.*

In this section, we have discussed the general solution, pure imaginary solution, and real solution of split quaternion matrix equation (1), providing a solid foundation for understanding their fundamental properties. In the following section, we will shift our attention to more specialized solutions, including the centro-Hermitian and skew-centro-Hermitian solutions of Equation (1).

4. The (skew-)centro-Hermitian solutions of Equation (1)

To analyze the (skew-)centro-Hermitian solutions, the following lemma provides the expression for the structure of $\mathrm{vec}(\Phi_{AXB})$ in terms of the vec_{Ω} operator, where

$$\mathrm{vec}_{\Omega}(\Psi_A) = \begin{bmatrix} \mathrm{vec}_r(Re(A_1)) \\ \mathrm{vec}_r(Im(A_1)) \\ \mathrm{vec}_r(Re(A_2)) \\ \mathrm{vec}_r(Im(A_2)) \end{bmatrix}.$$

Lemma 4.1. *Suppose that $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$.*
(1) *If X is a centro-Hermitian matrix, it yields that*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = W_\Omega^{(n)+}\mathrm{vec}_\Omega(\Psi_X),$$

where

$$W_\Omega^{(n)+} = \begin{bmatrix} K^+ & \boldsymbol{i}K^- & 0 & 0 \\ 0 & 0 & K^- & \boldsymbol{i}K^- \\ K^+ & -\boldsymbol{i}K^- & 0 & 0 \\ 0 & 0 & K^- & -\boldsymbol{i}K^- \end{bmatrix}.$$

(2) *If X is a skew-centro-Hermitian matrix, it follows that*

$$\begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} = W_\Omega^{(n)-}\mathrm{vec}_\Omega(\Psi_X),$$

where

$$W_\Omega^{(n)-} = \begin{bmatrix} K^- & \boldsymbol{i}K^+ & 0 & 0 \\ 0 & 0 & K^+ & \boldsymbol{i}K^+ \\ K^- & -\boldsymbol{i}K^+ & 0 & 0 \\ 0 & 0 & K^+ & -\boldsymbol{i}K^+ \end{bmatrix}.$$

Proof. For skew-centro-Hermitian matrix $X = X_1 + X_2\mathbf{j} \in \mathbf{SH}^{n\times n}$, it follows that

$$\begin{aligned} \begin{bmatrix} \mathrm{vec}(\Phi_X) \\ \mathrm{vec}(\Phi_{\mathbf{j}X\mathbf{j}}) \end{bmatrix} &= \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(\bar{X}_1) \\ \mathrm{vec}(\bar{X}_2) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(Re(X_1) + \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_2) + \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_1) - \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_2) - \mathbf{i}Im(X_2)) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathrm{vec}(Re(X_1) + \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_1) - \mathbf{i}Im(X_1)) \\ \mathrm{vec}(Re(X_2) + \mathbf{i}Im(X_2)) \\ \mathrm{vec}(Re(X_2) - \mathbf{i}Im(X_2)) \end{bmatrix} \\ &= \begin{bmatrix} K^- & \mathbf{i}K^+ & 0 & 0 \\ 0 & 0 & K^+ & \mathbf{i}K^+ \\ K^- & -\mathbf{i}K^+ & 0 & 0 \\ 0 & 0 & K^+ & -\mathbf{i}K^+ \end{bmatrix} \begin{bmatrix} \mathrm{vec}_r(Re(X_1)) \\ \mathrm{vec}_r(Im(X_1)) \\ \mathrm{vec}_r(Re(X_2)) \\ \mathrm{vec}_r(Im(X_2)) \end{bmatrix} = W_\Omega^{(n)-}\mathrm{vec}_\Omega(\Psi_X). \end{aligned}$$

For centro-Hermitian matrix $X \in \mathbf{SH}^{n\times n}$, the proof is similar, so it is omitted. □

In the framework of Lemmas 2.5 and 4.1, we can get the desired results.

Lemma 4.2. *Suppose that $A = A_1 + A_2\mathbf{j} \in \boldsymbol{SH}^{m\times n}$, $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $B = B_1 + B_2\mathbf{j} \in \boldsymbol{SH}^{n\times t}$, where $A_1, A_2 \in \boldsymbol{C}^{m\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$, and $B_1, B_2 \in \boldsymbol{C}^{n\times t}$.*
(1) *If X is a centro-Hermitian matrix, it follows that*

$$\mathrm{vec}(\Phi_{AXB}) = \left[\mathcal{G}(B)^{\mathbf{T}} \otimes A_1,\ \mathcal{G}(B_{\mathbf{j}})^{\mathbf{H}} \otimes A_2\right] W_\Omega^{(n)+}\mathrm{vec}_\Omega(\Psi_X). \tag{15}$$

(2) *If X is a skew-centro-Hermitian matrix, then*

$$\mathrm{vec}(\Phi_{AXB}) = \left[\mathcal{G}(B)^{\mathbf{T}} \otimes A_1,\ \mathcal{G}(B_{\mathbf{j}})^{\mathbf{H}} \otimes A_2\right] W_\Omega^{(n)-}\mathrm{vec}_\Omega(\Psi_X). \tag{16}$$

Lemma 4.3. *Suppose that $X = X_1 + X_2\mathbf{j} \in \boldsymbol{SH}^{n\times n}$, $X_1, X_2 \in \boldsymbol{C}^{n\times n}$, then the following descriptions hold.*
(1) *For a centro-Hermitian matrix X, it follows that*

$$\mathrm{vec}(\Psi_X) = V_\Omega^{(n)+}\mathrm{vec}_\Omega(\Psi_X), \tag{17}$$

where $V_{\Omega}^{(n)+} = diag(K^{+}, K^{-}, K^{-}, K^{-})$.

(2) *For a skew-centro-Hermitian matrix X, we have*

$$\operatorname{vec}(\Psi_X) = V_{\Omega}^{(n)-}\operatorname{vec}_{\Omega}(\Psi_X) \tag{18}$$

where $V_{\Omega}^{(n)-} = diag(K^{-}, K^{+}, K^{+}, K^{+})$.

Based on the above discussion, we now focus on the (skew-)centro-Hermitian solutions of Equation (1). Let $A_i = A_{i1} + A_{i2}\mathbf{j} \in \mathbf{SH}^{m\times n}$, $B_i = B_{i1} + B_{i2}\mathbf{j} \in \mathbf{SH}^{n\times t}$, $C = C_{11} + C_{12}\mathbf{j} \in \mathbf{SH}^{m\times t}$. For $(i = \overline{1,l})$, the following notations are necessary.

$$\Gamma_i = \left[\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} \;\; \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\right] W_{\Omega}^{(n)+}, \quad \widetilde{\Gamma} = \begin{bmatrix} Re(\Gamma_1, \Gamma_2, \ldots, \Gamma_l) \\ Im(\Gamma_1, \Gamma_2, \ldots, \Gamma_l) \end{bmatrix},$$

$$\Lambda_i = \left[\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} \;\; \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\right] W_{\Omega}^{(n)-}, \quad \widetilde{\Lambda} = \begin{bmatrix} Re(\Lambda_1, \Lambda_2, \ldots, \Lambda_l) \\ Im(\Lambda_1, \Lambda_2, \ldots, \Lambda_l) \end{bmatrix},$$

$$\widetilde{\gamma} = I_{\hat{n}} - \widetilde{\Gamma}^{\dagger}\widetilde{\Gamma}, \quad \widetilde{\lambda} = I_{\hat{n}} - \widetilde{\Lambda}^{\dagger}\widetilde{\Lambda}, \quad e = \begin{bmatrix} \operatorname{vec}(Re(\Phi_C)) \\ \operatorname{vec}(Im(\Phi_C)) \end{bmatrix},$$

where

$$\hat{n} = \begin{cases} 2ln^2, & \text{if n is even;} \\ 2l(n^2+n), & \text{if n is odd.} \end{cases}$$

Theorem 4.4. *Let us consider the centro-Hermitian solution of Equation* (1). *The following descriptions are valid.*

(1) *Equation* (1) *is solvable if and only if*

$$\widetilde{\Gamma}\widetilde{\Gamma}^{\dagger}e = e. \tag{19}$$

(2) *If Equation* (1) *has the centro-Hermitian solution, then*

$$\Omega_1 = \left\{ [X_1, X_2, X_3, \ldots, X_l] \,\middle|\, \begin{bmatrix} \operatorname{vec}(\Psi_{X_1}) \\ \operatorname{vec}(\Psi_{X_2}) \\ \vdots \\ \operatorname{vec}(\Psi_{X_l}) \end{bmatrix} = V_{\Omega}^{(n)+}\widetilde{\Gamma}^{\dagger}e + V_{\Omega}^{(n)+}(I_{\hat{n}} - \widetilde{\Gamma}^{\dagger}\widetilde{\Gamma})\alpha \right\} \tag{20}$$

is the general centro-Hermitian solution, where α *is an arbitraty vector with the appropriate order. Otherwise,* (20) *is the least squares centro-Hermitian solution.*

(3) *When* (20) *is the general centro-Hermitian solution of Equation* (1), *then*

$$\begin{bmatrix} \operatorname{vec}(\Psi_{X_1}) \\ \operatorname{vec}(\Psi_{X_2}) \\ \vdots \\ \operatorname{vec}(\Psi_{X_l}) \end{bmatrix} = V_{\Omega}^{(n)+}\widetilde{\Gamma}^{\dagger}e \tag{21}$$

is the minimal norm centro-Hermitian solution of Equation (1). *At this time, if* $\widetilde{\gamma} = 0$ *or rank* $(\widetilde{\Gamma}) = \hat{n}$, *then* (21) *is the unique centro-Hermitian solution. Otherwise,* (21) *represents the minimal norm least squares centro-Hermitian solution in the case of inconsistency.*

Proof. By Lemma 2.3 and Equation (15), it follows that

$$\begin{aligned}
\sum_{i=1}^{l} A_iX_iB_i = C &\iff \sum_{i=1}^{n} \Phi_{A_iX_iB_i} = \Phi_C,\\
&\iff \sum_{i=1}^{l} \mathrm{vec}(\Phi_{A_iX_iB_i}) = \mathrm{vec}(\Phi_C),\\
&\iff \sum_{i=1}^{l} \left[\mathcal{G}(B_i)^{\mathbf{T}} \otimes A_{i1} \ \ \mathcal{G}(B_i\mathbf{j})^{\mathbf{H}} \otimes A_{i2}\right] W_{\Omega}^{(n)+}\mathrm{vec}_{\Omega}(\Psi_X) = \mathrm{vec}(\Phi_C),\\
&\iff \sum_{i=1}^{l} \Gamma_i \mathrm{vec}_{\Omega}(\Psi_X) = \mathrm{vec}(\Phi_C),\\
&\iff \begin{bmatrix} Re[\Gamma_1, \Gamma_2, \ldots \Gamma_l] \\ Im[\Gamma_1, \Gamma_2, \ldots \Gamma_l] \end{bmatrix} \begin{bmatrix} \mathrm{vec}_{\Omega}(\Psi_{X_1}) \\ \mathrm{vec}_{\Omega}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}_{\Omega}(\Psi_{X_l}) \end{bmatrix} = \begin{bmatrix} Re(\mathrm{vec}(\Phi_C)) \\ Im(\mathrm{vec}(\Phi_C)) \end{bmatrix},\\
&\iff \widetilde{\Gamma} \begin{bmatrix} \mathrm{vec}_{\Omega}(\Psi_{X_1}) \\ \mathrm{vec}_{\Omega}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}_{\Omega}(\Psi_{X_l}) \end{bmatrix} = e.
\end{aligned}$$

By Lemma 2.8, Equation (1) has a centro-Hermitian solution $[X_1, X_2, X_3, \ldots, X_l] \in \Omega_1$ if and only if

$$\widetilde{\Gamma}\widetilde{\Gamma}^{\dagger}e = e.$$

Hence,

$$\begin{bmatrix} \mathrm{vec}_{\Omega}(\Psi_{X_1}) \\ \mathrm{vec}_{\Omega}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}_{\Omega}(\Psi_{X_l}) \end{bmatrix} = \widetilde{\Gamma}^{\dagger}e + (I_{\hat{n}} - \widetilde{\Gamma}^{\dagger}\Gamma)\alpha = \widetilde{\Gamma}^{\dagger}e + \widetilde{\gamma}\alpha \tag{22}$$

is the general centro-Hermitian solution or the least squares centro-Hermitian solution, where α is an arbitrary vector with the appropriate order. By (22), we have

$$\begin{bmatrix} \mathrm{vec}_{\Omega}(\Psi_{X_1}) \\ \mathrm{vec}_{\Omega}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}_{\Omega}(\Psi_{X_l}) \end{bmatrix} = \widetilde{\Gamma}^{\dagger}e + \widetilde{\gamma}\alpha.$$

According to Equation (17), we obtain that

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = V_{\Omega}^{(n)+}(\widetilde{\Gamma}^{\dagger}e + \widetilde{\gamma}\alpha).$$

Specially, when the system (1) is consistent and $\widetilde{\gamma} = 0$ (i.e rank $(\widetilde{\Gamma}) = \hat{n}$), (21) is the unique centro-Hermitian solution. Otherwise, (21) is the minimal norm least squares centro-Hermitian solution. □

Theorem 4.5. *Consider the skew-centro-Hermitian solution of Equation* (1)*.*

(1) *Equation* (1) *has a solution if and only if*

$$\widetilde{\Lambda}\widetilde{\Lambda}^{\dagger}e = e. \tag{23}$$

(2) *If Equation* (1) *has the skew-centro-Hermitian solution, then*

$$\Omega_2 = \left\{[X_1, X_2, X_3, \ldots, X_l] \,\middle|\, \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = V_{\Omega}^{(n)-}\widetilde{\Lambda}^{\dagger}e + V_{\Omega}^{(n)-}(I_{\hat{n}} - \widetilde{\Lambda}^{\dagger}\widetilde{\Lambda})\beta\right\} \tag{24}$$

is the general skew-centro-Hermitian solution, where β is an arbitraty vector with the appropriate order. Otherwise, (24) *is the least squares skew-centro-Hermitian solution.*
(3) *When* (24) *is the general skew-centro-Hermitian solution of Equation* (1)*, then*

$$\begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \vdots \\ \mathrm{vec}(\Psi_{X_l}) \end{bmatrix} = V_{\Omega}^{(n)-} \widetilde{\Lambda}^{\dagger} e. \tag{25}$$

This gives the minimal norm skew-centro-Hermitian solution of Equation (1)*. In the case where $\widetilde{\lambda} = 0$ or rank* ($\widetilde{\Lambda}$) $= \hat{n}$, (25) *becomes the unique skew-centro-Hermitian solution. If Equation* (1) *is inconsistent, then* (25) *provides the minimal norm least squares skew-centro-Hermitian solution.*

The proof of Theorem 4.5 follows similarly to that of Theorem 4.4, and is therefore omitted.

5. Numerical Examples

In the previous sections, we have established the necessary and sufficient conditions for the existence of some special types of solutions to Equation (1). Here, we focus on the specific forms of these solutions by presenting algorithms tailored for their computation. To demonstrate the effectiveness of these algorithms, we also provide three numerical examples. In particular, Algorithm 1 addresses the cases for the general, imaginary, and real solutions, while Algorithm 2 focuses on (skew-)centro-Hermitian solutions.

Algorithm 1

1. For i from 1 to l, input $A_i = A_{i1} + A_{i2}\mathbf{j} \in \mathbf{SH}^{m\times n}$, $B_i = B_{i1} + B_{i2}\mathbf{j} \in \mathbf{SH}^{n\times t}$, and $C = C_1 + C_2\mathbf{j} \in \mathbf{SH}^{m\times t}$.
2. Compute T_i, $\widetilde{T}$, L_i, $\widetilde{L}$, Q_i, $\widetilde{Q}$, $\widetilde{t}$, $\widetilde{l}$, $\widetilde{q}$, and e.
3. For the unique solutions, compute $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_j}$ using the following cases;
   (i) If (6) hold and rank ($\widetilde{T}$) = $4ln^2$ then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_1}$ according to (8).
   (ii) If (9) hold and rank ($\widetilde{L}$) = $3ln^2$ then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_2}$ according to (11).
   (iii) If (12) hold and rank ($\widetilde{Q}$) = ln^2 then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_3}$ according to (14).
4. For the non-unique solutions, calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_j}$ under the following conditions;
   (i) If only (6) hold then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_1}$ according to (7).
   (ii) If only(9) hold then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_2}$ according to (10).
   (iii) If only (12) hold then calculate $[X_1,\ X_2, \ldots, X_l] \in X_{\sigma_3}$ according to (13).
5. If the system (1) is inconsistent then calculate the minimal norm least squares solutions according to (8), (11), (14).

Algorithm 2

1. For i from 1 to l, input $A_i = A_{i1} + A_{i2}\mathbf{j} \in \mathbf{SH}^{m\times n}$, $B_i = B_{i1} + B_{i2}\mathbf{j} \in \mathbf{SH}^{n\times t}$, and $C = C_1 + C_2\mathbf{j} \in \mathbf{SH}^{m\times t}$.
2. Compute $\Gamma_i, \widetilde{\Gamma}, \Lambda_i, \widetilde{\Lambda}, \gamma, \lambda$, and e.
3. For the unique solutions, compute $[X_1, X_2, \ldots, X_l] \in \Omega_j$ using the following cases;
   (i) If (19) hold and rank $(\widetilde{\Gamma}) = \hat{n}$ then calculate $[X_1, X_2, \ldots, X_l] \in \Omega_1$ according to (21).
   (ii) If (23) hold and rank $(\widetilde{\Lambda}) = \hat{n}$ then calculate $[X_1, X_2, \ldots, X_l] \in \Omega_2$ according to (25).
4. For the non-unique solutions, compute $[X_1, X_2, \ldots, X_l] \in \Omega_j$ under the following conditions;
   (i) If only (19) hold then calculate $[X_1, X_2, \ldots, X_l] \in \Omega_1$ according to (20).
   (ii) If only (23) hold then calculate $[X_1, X_2, \ldots, X_l] \in \Omega_2$ according to (24).
5. If the system (1) is inconsistent then calculate the minimal norm least squares solutions according to (21), (25).

We apply Algorithms 1 and 2 to solve Equation (1). Examples 5.1 and 5.2 will address the general solution and the pure imaginary solution of Equation (1), while in Example 5.3, we focus on the skew-centro-Hermitian solution. If the Equation (1) is consistent then the norms of $\widetilde{T}\widetilde{T}^{\dagger}e - e$, $\widetilde{L}\widetilde{L}^{\dagger}e - e$, and $\widetilde{\Lambda}\widetilde{\Lambda}^{\dagger}e - e$ should be small.

Example 5.1. *Consider the general solution of Equation* (1). *Let* $m = 3$, $n = 3$, $t = 3$, $l = 3$.

$$A_1 = A_{11} + A_{12}\mathbf{j},\ B_1 = B_{11} + B_{12}\mathbf{j},\ A_2 = A_{21} + A_{22}\mathbf{j},\ B_2 = B_{21} + B_{22}\mathbf{j},\ A_3 = A_{31} + A_{32}\mathbf{j},\ B_3 = B_{31} + B_{32}\mathbf{j},$$

$$\widehat{X_1} = X_{11} + X_{12}\mathbf{j},\ \widehat{X_2} = X_{21} + X_{22}\mathbf{j},\ \widehat{X_3} = X_{31} + X_{32}\mathbf{j},$$

where

$$A_1 = \begin{bmatrix} 1+i+8j+k & i+j+k & i+6j+k \\ i+3j+k & 1+i+5j+k & i+7j+k \\ i+4j+k & i+9j+k & 1+i+2j+k \end{bmatrix},\ A_2 = \begin{bmatrix} 1+8i+8j+k & i+j & 6i+6j \\ 3i+3j & 5i+5j+k & 7i+7j \\ 4i+4j & 9i+9j & 1+2i+2j+k \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1+i+j+8k & j+k & j+6k \\ j+3k & 1+i+j+5k & j+7k \\ j+4k & j+9k & 1+i+j+2k \end{bmatrix},\ B_1 = \begin{bmatrix} 1+i+j+8k & 1+k & 1+6k \\ 1+3k & 1+i+5k & 1+7k \\ 1+4k & 1+9k & 1+i+j+2k \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1+i+j+k & 1+i+j & 1+i+j \\ 1+i+j & 1+i+j+k & 1+i+j \\ 1+i+j & 1+i+j & 1+i+j+k \end{bmatrix},\ B_3 = \begin{bmatrix} -1+i+j-8k & i-k & i-6k \\ i-3k & -1+i+j-5k & i-7k \\ i-4k & i-9k & -1+i+j-2k \end{bmatrix},$$

$$\widehat{X_1}(:,1) = \begin{bmatrix} 0.0430 - 1.3617i - 0.0679j + 0.3786k \\ 0.1690 + 0.4550i - 0.1952j + 0.8116k \\ 0.6491 - 0.8487i - 0.2176j + 0.5328k \end{bmatrix},\ \widehat{X_1}(:,2) = \begin{bmatrix} 0.7317 - 0.3349i - 0.3031j + 0.3507k \\ 0.6477 + 0.5528i + 0.0230j + 0.9390k \\ 0.4509 + 1.0391i + 0.0513j + 0.8759k \end{bmatrix},$$

$$\widehat{X_1}(:,3) = \begin{bmatrix} 0.5470 - 1.1176i + 0.8261j + 0.5502k \\ 0.2963 + 1.2607i + 1.5270j + 0.6225k \\ 0.7447 + 0.6601i + 0.4669j + 0.5870k \end{bmatrix},\ \widehat{X_2}(:,1) = \begin{bmatrix} 0.2077 - 0.1922i - 0.4446j + 0.3188k \\ 0.3012 - 0.2741i - 0.1559j + 0.4242k \\ 0.4709 + 1.5301i + 0.2761j + 0.5079k \end{bmatrix},$$

$$\widehat{X_2}(:,2) = \begin{bmatrix} 0.2305 - 0.2490i - 0.2612j + 0.0855k \\ 0.8443 - 1.0642i + 0.4434j + 0.2625k \\ 0.1948 + 1.6035i + 0.3919j + 0.8010k \end{bmatrix},\ \widehat{X_2}(:,3) = \begin{bmatrix} 0.2259 + 1.2347i - 1.2507j + 0.0292k \\ 0.1707 - 0.2296i - 0.9480j + 0.9289k \\ 0.2277 - 1.5062i - 0.7411j + 0.7303k \end{bmatrix},$$

$$\widehat{X_3}(:,1) = \begin{bmatrix} 0.4886 + 0.2323i - 1.6642j + 0.7150k \\ 0.5785 + 0.4264i - 0.5900j + 0.9037k \\ 0.2373 - 0.3728i - 0.2781j + 0.8909k \end{bmatrix},\ \widehat{X_3}(:,2) = \begin{bmatrix} 0.4588 - 0.2365i + 0.4227j + 0.3342k \\ 0.9631 + 2.0237i - 1.6702j + 0.6987k \\ 0.5468 - 2.2584i + 0.4716j + 0.1978k \end{bmatrix},$$

$$\widehat{X_3}(:,3) = \begin{bmatrix} 0.5211 + 2.2294i - 1.2128j + 0.0305k \\ 0.2316 + 0.3376i + 0.0662j + 0.7441k \\ 0.4889 + 1.0001i + 0.6524j + 0.5000k \end{bmatrix}.$$

Let

$$\Psi_{A_1} = (A_{11}, A_{12}),\ \Psi_{B_1} = (B_{11}, B_{12}),\ \Psi_{A_2} = (A_{21}, A_{22}),\ \Psi_{B_2} = (B_{21}, B_{22}),\ \Psi_{A_3} = (A_{31}, A_{32}),\ \Psi_{B_3} = (B_{31}, B_{32}),$$

$$\Psi_{\widehat{X_1}} = (X_{11}, X_{12}),\ \Psi_{\widehat{X_2}} = (X_{21}, X_{22}),\ \Psi_{\widehat{X_3}} = (X_{31}, X_{32}),$$

$$\Psi_C = \Psi_{A_1}\mathcal{G}(\widehat{X_1})\mathcal{G}(B_1) + \Psi_{A_2}\mathcal{G}(\widehat{X_2})\mathcal{G}(B_2) + \Psi_{A_3}\mathcal{G}(\widehat{X_3})\mathcal{G}(B_3).$$

*By using **MATLAB**, we have*

$$\left\| [\widetilde{T_1}\widetilde{T_1}^{\dagger} e - e] + [\widetilde{T_2}\widetilde{T_2}^{\dagger} e - e] + [\widetilde{T_3}\widetilde{T_3}^{\dagger} e - e] \right\| = 1.4959 \times 10^{-12},$$

and

$$rank(\widetilde{T_1}) = rank(\widetilde{T_2}) = rank(\widetilde{T_3}) = 108 = 4ln^2.$$

*According to **Algorithm** 1 and Theorem* 3.3*, we can see that Equation* (1) *is consistent and has a unique solution with the least norm* $[X_1, X_2, X_3] \in X_{\sigma_1}$*, and we get*

$$\left\| \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \mathrm{vec}(\Psi_{X_3}) \end{bmatrix} - \begin{bmatrix} \mathrm{vec}(\Psi_{\widehat{X_1}}) \\ \mathrm{vec}(\Psi_{\widehat{X_2}}) \\ \mathrm{vec}(\Psi_{\widehat{X_3}}) \end{bmatrix} \right\| = 1.4959 \times 10^{-14}.$$

Example 5.2. *Suppose* A_1*,*A_2*,* A_3*,* B_1*,* B_2*,* B_3*,* $\widehat{X_1}$*,* $\widehat{X_2}$*,* $\widehat{X_3}$*,* Ψ_{A_1}*,* Ψ_{A_2}*,* Ψ_{A_3}*,* Ψ_{B_1}*,* Ψ_{B_1}*,* Ψ_{B_3}*,* $\Psi_{\widehat{X_1}}$*,* $\Psi_{\widehat{X_2}}$*,* $\Psi_{\widehat{X_3}}$ *are the same as in Example* 5.1*. By using **MATLAB**, we can obtain*

$$\left\| [\widetilde{L_1}\widetilde{L_1}^{\dagger} e - e] + [\widetilde{L_2}\widetilde{L_2}^{\dagger} e - e] + [\widetilde{L_3}\widetilde{L_3}^{\dagger} e - e] \right\| = 1.9152 \times 10^{-13}.$$

Also,

$$rank(\widetilde{L_1}) = rank(\widetilde{L_2}) = rank(\widetilde{L_3}) = 36 \neq 3ln^2.$$

*Based on **Algorithm 1** and Theorem* 3.4*, it follows that Equation* (1) *is consistent and has multiple non-unique pure imaginary solutions, one of which imaginary solution can be expressed as* $X_3 = ImX_{31}\boldsymbol{i} + ReX_{32}\boldsymbol{j} + ImX_{32}\boldsymbol{k}$*, where*

$$X_1 = \begin{bmatrix} -1.7979i - 1.0507j + 0.7119k & -1.0554i - 1.4293j + 0.4426k & -1.1400i - 2.0698j + 0.5278k \\ -1.2607i - 0.8523j + 1.0887k & -2.2868i - 0.9142j + 0.2471k & -2.0206i - 1.6469j + 0.4073k \\ -1.1739i - 2.3916j + 0.9156k & -0.9054i - 1.0347j + 1.8997k & -1.3428i + 0.0871j - 0.4737k \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 1.4473i - 4.3855j + 7.9704k & -0.6400i - 1.8808j + 3.8693k & 0.0021i - 3.1471j + 4.3973k \\ -0.9636i - 1.9033j + 3.5651k & 1.9088i - 3.2173j + 5.1148k & 1.0637i - 4.8741j + 9.2563k \\ 0.4036i - 3.5856j + 5.81481k & -0.1731i - 3.9007j + 7.7430k & -0.63081i - 0.33042j + 1.9215k \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 4.4466i + 4.4040j + 0.0336k & 1.1515i + 4.4040j - 0.7651k & 1.8264i + 1.9771j - 0.4514k \\ 0.4923i - 0.1554j - 0.0682k & 3.5183i + 3.2705j - 0.0290k & 5.5026i + 5.2822j + 0.1329k \\ 3.3195i + 2.8505j - 1.6470k & 3.8590i + 3.5737j - 0.1457k & 2.0129i + 1.5698j - 0.0978k \end{bmatrix}.$$

In addition,

$$\left\| \begin{bmatrix} \mathrm{vec}(\Psi_{X_1}) \\ \mathrm{vec}(\Psi_{X_2}) \\ \mathrm{vec}(\Psi_{X_3}) \end{bmatrix} - \begin{bmatrix} \mathrm{vec}(\Psi_{\widehat{X_1}}) \\ \mathrm{vec}(\Psi_{\widehat{X_2}}) \\ \mathrm{vec}(\Psi_{\widehat{X_3}}) \end{bmatrix} \right\| = 8.0519.$$

Example 5.3. *Consider the skew-centro-Hermitian solution of Equation* (1). *Let* $m = 3$, $n = 4$, $t = 5$, $l = 2$.

$$A_1 = A_{11} + A_{12}\mathbf{j},\ B_1 = B_{11} + B_{12}\mathbf{j},\ A_2 = A_{21} + A_{22}\mathbf{j},\ B_2 = B_{21} + B_{22}\mathbf{j}, \widehat{X_1} = X_{11} + X_{12}\mathbf{j},\ \widehat{X_2} = X_{21} + X_{22}\mathbf{j},$$

where

$$A_1 = \begin{bmatrix} 4+5i-5j & -5j-3k & -5-3k & 2+i \\ 1+5j+2k & 1+i+3j+2k & 2 & 1-4i+4j-4k \\ -3-5i-j+4k & 3+i-4j+3k & -1-4i+2j-4k & 1+3i-5j+3k \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 3-j-2k & -1+2i & -1-3i+3j+2k & -3j-4k \\ -1-2i-3j+k & -1+3i+3j+2k & 4+5i+3k & -1-i-j \\ -2-2i+2j & 2i+2k & 4+i+j+2k & -2-4i+j-2k \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1-2i+3k & 2-2i-2j+2k & 3-i-3j+4k & 5-3i+j-4k & 2i+3k \\ -4+2i-j-3k & 3+i+3j+k & 5+3k & -3-3i-2j+2k & -3+2i+4k \\ 2+i+j & 4+4i-3j+3k & 2+3i-2j-4k & -3+5i-2j & -1+i-4j-3k \\ -1+4i+j-3k & 2+3i-j+2k & 1-3i-3j-2k & 5-2k & -3-4i-4j-2k \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -2-2i+3j+4k & 1-i-j-k & -2-i-4j & -4-4i+j-2k & i+2k \\ 2-4i+2j-4k & 2+i+2j+3k & 2-2i+3j+3k & 4-4j+k & j-3k \\ -1+4j+4k & 2i+j-3k & -4+i-j+3k & 4-3j & 2+i \\ -2i-k & -2-4i-2k & 2+i-j-2k & -i-2j-3k & 3i-3j-k \end{bmatrix},$$

$$\widehat{X_1} = \begin{bmatrix} 1-2i-j & -3-4i-2j-3k & -2-4i-3j-2k & -5-i-j+k \\ 1+5i+3j+k & k & -2+3j+3k & 1+3i-2k \\ -1+3i-2k & 2-3j+3k & k & -1+5i+3j+k \\ 5-i-j+k & 2-4i-3j-2k & 3-4i-2j-3k & -1-2i-j \end{bmatrix},$$

$$\widehat{X_2} = \begin{bmatrix} -2-2i+3j+4k & 1-i-5j-5k & -2-i-4j & -4-4i+j-2k \\ 2-4i+2j-4k & 2+i+2j+3k & 2-5i+3j+3k & 4+5i-4j+k \\ -4+5i-4j+k & -2-5i+3j+3k & -2+i+2j+3k & -2-4i+2j-4k \\ 4-4i+j-2k & 2-i-4j & -1-i-5j-5k & 2-2i+3j+4k \end{bmatrix}.$$

Let $\Psi_{A_1} = (A_{11}, A_{12})$, $\Psi_{B_1} = (B_{11}, B_{12})$, $\Psi_{A_2} = (A_{21}, A_{22})$, $\Psi_{B_2} = (B_{21}, B_{22})$, $\Psi_{\widehat{X_1}} = (X_{11}, X_{12})$, $\Psi_{\widehat{X_2}} = (X_{21}, X_{22})$, $\Psi_C = \Psi_{A_1}\mathcal{G}(\widehat{X_1})\mathcal{G}(B_1) + \Psi_{A_2}\mathcal{G}(\widehat{X_2})\mathcal{G}(B_2)$. *By using* ***MATLAB*** *and* ***Algorithm*** 2, *we have*

$$\left\|[\widetilde{\Lambda_1}\widetilde{\Lambda_1}^{\dagger}e - e] + [\widetilde{\Lambda_2}\widetilde{\Lambda_2}^{\dagger}e - e]\right\| = 4.917 \times 10^{-13}.$$

Moreover,

$$rank(\widetilde{\Lambda_1}) = rank(\widetilde{\Lambda_2}) = 64 = 2ln^2.$$

According to Theorem 4.5, *we can see that Equation* (1) *is inconsistent and does not have skew-centro-Hermitian solution. Thus, we provide the unique minimal norm least squares skew-centro-Hermitian solution, where*

$$X_1(:,1) = \begin{bmatrix} -0.1254-0.3676i+2.8420j+1.2907k \\ 1.9655-0.1506i+2.3295j-1.250k \\ -0.4316+0.9738i+0.3533j+0.5307k \\ -0.7752-1.9930i+2.5822j-1.7494k \end{bmatrix},\ X_1(:,2) = \begin{bmatrix} 0.1834-0.1614i-0.7958j-0.8259k \\ 2.8327+0.1078i+2.6491j+0.8313k \\ -0.3636+0.2224i+1.5533j+1.6246k \\ 2.9027-0.0784i+0.1544j-0.9836k \end{bmatrix},$$

$$X_1(:,3) = \begin{bmatrix} -2.9027-0.0784i+0.1544j-0.9836k \\ 0.3636+0.2224i+1.5533j+1.6246k \\ -2.8327+0.1078i+2.6491j+0.8313k \\ -0.1834-0.1614i-0.7958j-0.8259k \end{bmatrix},\ X_1(:,4) = \begin{bmatrix} 0.7752-1.9930i+2.5822j-1.7494k \\ 0.4316+0.9738i+0.3533j+0.5307k \\ 1.9655-0.1506i+2.3295j-1.250k \\ 0.1254-0.3676i+2.8420j+1.2907k \end{bmatrix},$$

$$X_2(:,1) = \begin{bmatrix} -2.7774 + 2.5662i + 0.9548j - 0.9905k \\ 0.6892 + 1.6115i - 0.0751j - 1.3507k \\ 2.9198 + 1.9035i + 0.6812j + 0.0703k \\ -2.6204 + 0.2309i - 0.0196j + 0.1724k \end{bmatrix}, \ X_2(:,2) = \begin{bmatrix} -1.0544 - 1.5422i + 1.5419j - 0.4044k \\ 0.5795 + 0.0897i - 0.7620j + 2.0775k \\ 0.3433 + 1.5146i - 0.3461j + 4.0611k \\ 0.2949 + 0.4979i - 0.9458j + 1.8150k \end{bmatrix},$$

$$X_2(:,3) = \begin{bmatrix} -0.2949 + 0.4979i - 0.9458j + 1.8150k \\ -0.3433 + 1.5146i - 0.3461j + 4.0611k \\ -0.5795 + 0.0897i - 0.7620j + 2.0775k \\ 1.0544 - 1.5422i + 1.5419j - 0.4044k \end{bmatrix}, \ X_2(:,4) = \begin{bmatrix} 2.6204 + 0.2309i - 0.0196j + 0.1724k \\ -2.9198 + 1.9035i + 0.6812j + 0.0703k \\ -0.6892 + 1.6115i - 0.0751j - 1.3507k \\ 2.7774 + 2.5662i + 0.9548j - 0.9905k \end{bmatrix}.$$

6. Conclusion

This paper offers a comprehensive study of the solutions to the split quaternion matrix equation (1). We derived the necessary and sufficient conditions for the solvability of Equation (1) and provided expressions for the general, pure imaginary, and real solutions, as well as (skew-)centro-Hermitian solutions. In cases where the matrix equation (1) is inconsistent, we offered expressions for the least squares solutions. To illustrate the practical applicability of our results, we included numerical examples and algorithms. This work contributes to a deeper understanding of split quaternion matrix equations.

References

[1] W.R. Hamilton, *On quaternions, or on a new system of imaginaries in algebra*, Lond, **25(163)** (1844), 10–13.
[2] S.R. Adler, D.R. Finkelstein, *Quaternionic quantum mechanics and quantum fields*, Oxford University Press, USA. **88**(1995).
[3] B.J. Chen, Q.S. Liu, X.M. Sun, X. Li, H.Z. Shu, *Removing Gaussian noise for color images by quaternion representation and optimisation of weights in non-local means filter*, IEEE Image Process, **8(10)** (2014), 591–600.
[4] Q.G. Wang, *Quaternion transformation and its application to the displacement analysis of spatial mechanisms*, Acta Mech. Sin., **15** (1983), 54–61.
[5] J. Cockle, *On systems of algebra involving more than imaginary and on equations of the fifth degree*, Lond, **35(238)** (1849), 434–437.
[6] X. Liu, Y. Zhang, *Consistency of split quaternion matrix equations $AX^* - XB = CY + D$ and $X - AX^*B = CY + D$*, Adv. Appl. Clifford Algebras, **29**(2019), 1-20.
[7] A. Khalid, Q.W. Wang, Z.H. Gao, *The general solution to a system of real split quaternion matrix equations*, Filomat, **38**(2024), 8827–8840.
[8] İ. Öztürk, M. Ozdemir, *On geometric interpretations of split quaternions*, Math. Meth. Appl. Sci., **46** (2022), 408–422.
[9] Z. Özdemir, O.O. Tuncer, I. Gök, *Kinematic equations of Lorentzian magnetic flux tubes based on split quaternion algebra*, Eur. Phys. J. Plus., **136** (2021), 1-18.
[10] L.M. Xie, Q.W. Wang, Z.H. He, *The generalized hand-eye calibration matrix equations $AX - YB = C$ over dual quaternions*, Comput. Appl. Math., **44**(2025), 137.
[11] T.S. Jiang, Z.Z. Zhang, Z.W. Jiang, *Algebraic techniques for eigenvalues and eigenvectors of a split quaternion matrix in split quaternionic mechanics*, Comput. Phys. Commun., **229** (2018), 1–7.
[12] Z.W. Guo, T.S. Jiang, V.I. Vasil'ev, G. Wang, *A novel algebraic approach for the Schrödinger equation in split quaternionic mechanics*, Appl. Math. Lett., **137** (2023), 108485.
[13] Z.R. Jia, Q.W. Wang, *The general solutions to a system of tensor equations over split quaternions algebra with applications*, Mathematics, **13**(2025), 644.
[14] L.J. Zhu, Q.W. Wang, Z.L. Kou, *The least-norm solution to a matrix equation over the dual quaternion algebra*, Symmetry, **17**(2025), 267.
[15] D.S. Cvetković-Ili, J.N. Radenković, Q.W. Wang, *Algebraic conditions for the solvability to some systems of matrix equations*, Linear Multilinear Algebra, **69** (2019), 1-31.
[16] F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl., **251**(1997), 21-57.
[17] L.Q. Yang, Q.W. Wang, Z. Kou, *A system of tensor equations over the dual split quaternion algebra with an application*, Mathematics, **12**(2024), 3571.
[18] L. Shi, Q.W. Wang, L.M. Xie, X.F. Zhang, *Solving the dual generalized commutative quaternion matrix equation $AXB = C$*, Symmetry, **16**(2014), 1359.
[19] S.Q. Duan, Q.W. Wang, X.F. Duan, *On Rayleigh quotient iteration for dual quaternion Hermitian eigenvalue problem*, Mathematics, **12**(2024), 4006.
[20] Q.W. Wang, L.M. Xie, Z.H. Gao, *A survey on solving the matrix equation $AXB = C$ with applications*, Mathematics, **13**(2025), 450.
[21] M. Dehghan, A. Shirilord, *The use of homotopy analysis method for solving generalized Sylvester matrix equation with applications*, Eng. Comput. Germany, 38(2022), 2699–2716.
[22] T. Li, Q.W. Wang, *Structure preserving quaternion biconjugate gradient method*, SIAM J. Matrix Anal. A., **45**(2024), 306-326.
[23] A. Rehman, Q.W. Wang, I. Ali, M. Akram, M.O. Ahmad, *A constraint system of generalized Sylvester quaternion matrix equations*, Adv. Appl. Clifford Algebras, **27**(2017), 3183-3196.
[24] X. Zhang, *The system of generalized Sylvester quaternion matrix equations and its applications*, Appl. Math. Comput., **273**(2016), 74-81.

[25] S.F. Yuan, A.P. Liao, Y. Lei, *Least squares Hermitian solution of the matrix equation* $(AXB, DXE) = (C, F)$ *with the least norm over the skew field of quaternions*, Math. Comput. Model, **48**(2008), 91–100.

[26] S.F. Yuan, Q.W. Wang, X. Zhang, *Least-squares problem for the quaternion matrix equation* $AXB + CYD = E$ *over different constrained matrices*, Int. J. Comput. Math., **90**(2013), 565-576.

[27] S.W. Yu, *The real and complex Hermitian solutions to a system of quaternion matrix equations with applications*, Int. J. Math. Math. Sci., **2012** (2012), 19.

[28] Q.W. Wang, H.S. Zhang, S.W. Yu, *On solutions to the quaternion matrix equation* $AXB + CYD = E$, Electron. J. Linear Algebra, **17**(2008), 343-358.

[29] Q.W. Wang, *Bisymmetric and centrosymmetric solutions to system of real quaternion matrix equation*, Comput. Math. Appl., **49**(2005), 641–650.

[30] C.Q. Zhang, Q.W. Wang, A. Dmytryshyn, Z.H. He, *Investigation of some Sylvester-type quaternion matrix equations with multiple unknowns*, Comput. Appl. Math., **43**(2024), 1-26.

[31] S.F. Yuan, A.P. Liao, P. Wang, *Least squares* η*-bihermitian problems of the quaternion matrix equation* $(AXB, DXE) = (C, F)$, Linear Multilinear Algebra, **63**(2015), 1849–1863.

[32] Z.H. He, Q.W. Wang, *A simultaneous decomposition for seven matrices with applications*, J. Comput. Appl. Math., **349**(2019), 93–113.

[33] A.B. Mamagani, M. Jafari, *On properties of generalized quaternion algebra*, J. Nov. Appl. Sci., **2**(2013), 683–689.

[34] X. Liu, Z.H. He, η*-Hermitian solution to a system of quaternion matrix equations*, Bull. Malays. Math. Sci. Soc., **43**(2020), 4007-4027.

[35] M. Dehghan, M. Hajarian, *Solving coupled matrix equations over generalized bisymmetric matrices*, Int. J. Control Autom. Syst., **10** (2012), 905-912.

[36] F. Zhang, M. Wei, Y. Li, J. Zhao. *An efficient real representation method for least squares problem of the quaternion constrained matrix equation* $AXB + CYD = E$, Int. J. Comput. Math., **98**(2021), 1408-1419.

[37] F. Zhang, M. Wei, Y. Li, J. Zhao, *An efficient method for least-squares problem of the quaternion matrix equation* $X - A\widehat{X}B = C$ *with applications*, Linear Multilinear Algebra, **70**(2022), 2569-2581.

[38] H.T. Zhang, L.N. Liu, H. Liu, Y.X. Yuan, *The solution of the matrix equation* $AXB = D$ *and the system of matrix equations* $AX = C, XB = D$ *with* $X^*X = I_p$, Appl. Math. Comput., **418** (2022), no. 126789.

[39] A.P. Liao, Z.Z. Bai, Y. Lei, *Best approximate solution of matrix equation* $AXB + CYD = E$, SIAM J. Matrix Anal. A., **27**(2005), 675-688.

[40] Y.H. Au-Yeung, C.M. Cheng, *On the pure imaginary quaternionic solutions of the Hurwitz matrix equations*, Linear Algebra Appl., **419**(2006), 630–642.

[41] Q.W. Wang, S.W. Yu, W. Xie, *Extreme ranks of real matrices in solution of the quaternion matrix equation* $AXB = C$ *with applications*, Algebra Colloquium, **17**(2010), 345–360.

[42] S.F. Yuan, Q.W. Wang, X.F. Duan, *On solutions of the quaternion matrix equation* $AX = B$ *and their applications in color image restoration*, Appl. Math. Comput., **221**(2013), 10–20.

[43] S.F. Yuan, *Least squares pure imaginary solution and real solution of the quaternion matrix equation* $AXB + CXD = E$ *with the least norm*, J. Appl. Math., **2014**(2014), 9.

[44] S. Şimşek, M. Sarduvan, H. Özdemir, *Centrohermitian and skew-centrohermitian solutions to the minimum residual and matrix nearness problems of the quaternion matrix equation* $(AXB, DXE) = (C, F)$, Adv. Appl. Clifford Algebras, **27**(2017), 2201-2214.

[45] F. Zhang, Y. Li, J. Zhao, *A real representation method for special least squares solutions of the quaternion matrix equation* $(AXB, DXE) = (C, F)$, AIMS Math., **7**(2022), 14595-14613.

[46] S.F. Yuan, Q.W. Wang, Y.B. Yu, Y. Tian, *On Hermitian solutions of the split quaternion matrix equation* $AXB + CXD = E$, Adv. Appl. Clifford Algebras, **27**(2017), 3235-3252.

[47] Z.H. Gao, Q.W. Wang, L.M. Xie, *The (anti-)*η*−Hermitian solution to a novel system of matrix equations over the split quaternion algebra*, Math. Meth. Appl. Sci., (2024), 1-18.