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Duality for second order nondifferentiable multiobjective fractional variational problems

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Abstract. In the present work, we construct a second order symmetric dual pair with multiobjective and nondifferentiable settings over variational problems and explore weak, strong, and converse duality theorems with the help of second order $(\mathcal{F}, \alpha, \rho, d)$-convexity. First, a parametric method is used to transform the problem into an equivalent non-fractional form. In order to determine the bound on the optimal value of the primal problem and build the theoretical framework for strong duality, we then deduce the weak duality theorem for the designed problems. The strong duality demonstrated in the paper shows that a symmetric relationship exists between the primal and dual problems. The static case is additionally addressed by dropping the time component. The solutions in our work may be applied to a broader class of problems that arise in modeling mechanical engineering problems. The existence of the problem as required in the discussion is demonstrated by constructed examples.

1. Introduction

The relationship between primal and dual problems in nonlinear programming problems is of great importance from a theoretical and computational viewpoint. First, dual problems may be easier to solve, and the optimal solution to the primal problem can be found easily when the formal inquiry to discover the optimal solution of the dual problems has been finished. Moreover, a good estimate of the optimal primal solution can be obtained once we estimate the optimal dual solution. Various formulations have been proposed over the last few decades to solve problems arising in engineering, economics, and other related subjects.

In the study of mathematical programming, symmetric duality was introduced by Dorn [6], who established the fact that the dual of the companion dual problem of the associated primal is the primal itself. Dantzig [5] extended these results and discussed symmetric duality in nonlinear programming under the convexity and concavity assumptions. It was Mangasarian [13] who showed that the higher order dual gives less relaxation on the interval where the objective can lie. Chandra *et al.* [4] was the first to discuss symmetric dual formulations in nonlinear fractional programming. Mond *et al.* [16] focused on

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variational problems and derived duality theorems under invexity. Mond and Schechter [15] discussed a nondifferentiable symmetric dual problem where the presence of support in the objective function was the root cause of nondifferentiability. Subsequently, Mond and Schechter [15] and Yang *et al.* [21] studied symmetric dual nonlinear fractional problems and used pseudo-convexity and pseudo-concavity to arrive at duality results. Nahak and Nanda [17] established duality theorems for symmetric variational problems with constraints defined over cones. Singh *et al.* [20] investigates duality for multiobjective variational problems under second order (ϕ, ρ)-invexity. Saini and Gulati [19] reached duality theorems for Wolfe-type nondifferentiable multiobjective second order symmetric dual programs over arbitrary cones. Ahmad and Sharma [2] focused on a pair of multiobjective fractional variational symmetric dual problems over cones. Mishra [14] considered second order symmetric duality in mathematical programming under F-convexity.

Recently, Kailey and Gupta [11] have studied nondifferentiable variational programming problems in which constraints were imposed over cones and the objective function was nondifferentiable because of support functions, whereas Kang *et al.* [12] studied symmetric duality for nondifferentiable multiobjective fractional variational problems involving cones. Jayswal [8] studied a pair of multiobjective second order symmetric variational control programs over cone constraints under $\mathcal{F}$-convexity. Recently, Prasad *et al.* [18] worked on a variational problem where the objective function was supposed to be second order nondifferentiable symmetric fractional in nature and derived various duality theorems. The present work is an extension of this work to a multiobjective case.

In this paper, we especially focus on second order symmetric multiobjective nondifferentiable fractional variational problems under $(\mathcal{F}, \alpha, \rho, d)$ convexity and derive weak, strong, and converse duality theorems. The development of the paper can be seen as follows: In Section 2, we define a few basic concepts and recall the definition of second order $(\mathcal{F}, \alpha, \rho, d)$ functions. In Section 3, we construct a pair of second order symmetric multiobjective nondifferentiable fractional variational problems, and in Section 4, we derive suitable duality theorems. We construct a static case of the problem considered in our paper in Section 5 and finally come to conclusions in Section 6.

2. Preliminaries

Consider a nonlinear variational problem of the form

$$\text{(VP)} \qquad \text{Minimize} \int_a^b \zeta(t, v, \dot{v})\, dt$$

$$\text{subject to} \quad v(a) = \alpha, \;\; v(b) = \beta,$$

$$h(t, v, \dot{v}) \leqq 0, \;\; t \in I = [a, b],$$

where ζ is taken to be a smooth real-valued function defined on $I \times R^n \times R^n$ and h is vector m-valued function defined on $I \times R^n \times R^n$. The derivative of $v(t)$ w.r.t. t is denoted by $\dot{v}(t)$ or simply by $\dot{v}$. We reserve the symbol ζ_v to denote the derivative of ζ with respect to $v = (v^1, v^2, ..., v^n)^T$ as defined by

$$\zeta_v = \left(\frac{\partial \zeta}{\partial v^1}, \frac{\partial \zeta}{\partial v^2}, ..., \frac{\partial \zeta}{\partial v^n}\right)^T.$$

Similarly,

$$\zeta_{\dot{v}} = \left(\frac{\partial \zeta}{\partial \dot{v}^1}, \frac{\partial \zeta}{\partial \dot{v}^2}, ..., \frac{\partial \zeta}{\partial \dot{v}^n}\right)^T.$$

Moreover, ζ_{vv} denotes the $n \times n$ Hessian matrix of w.r.t. v. Let $M(t, v, \dot{v}) = \zeta_{vv} - 2D\zeta_{v\dot{v}} + D^2\zeta_{\dot{v}\dot{v}} - D^3\zeta_{\ddot{v}\dot{v}}, \; t \in I$. The norm of $v \in C(I, R)$ can be taken as

$$\| v \| = \| v \|_\infty + \| Dv \|_\infty,$$

where the symbol D is given by

$$\Omega = Dv \Leftrightarrow v(t) = \eta + \int_0^t \Omega(s)\, ds,$$

for a specified boundary value η.

Definition 2.1 A subset C of R^n is called a cone if it is not empty and $\lambda v \in C$ for any non-negative real number λ. In addition, if C is a convex set, then it is termed a convex cone.

Definition 2.2 The polar cone C^* for a cone C is described mathematically as

$$C^* = \{w \in R^n : v^T w \leq 0 \ \ \forall\, v \in C\}.$$

Definition 2.3 A functional $\mathcal{F} : I \times X \times X \times X \times X \times R^n \to R$, where $X \subseteq R^n$, is called sublinear corresponding to the last argument provided the following two properties hold for any $v, \dot{v}, u, \dot{u} \in X$:

(*i*) $\mathcal{F}(t, v, \dot{v}, u, \dot{u}; \theta_1 + \theta_2) \leqq \mathcal{F}(t, v, \dot{v}, u, \dot{u}; \theta_1) + \mathcal{F}(t, v, \dot{v}, u, \dot{u}; \theta_2),\ \forall\, \theta_1,\ \theta_2 \in R^n$,

(*ii*) $\mathcal{F}(t, v, \dot{v}, u, \dot{u}; p\theta) = p\mathcal{F}(t, v, \dot{v}, u, \dot{u}; \theta),\ \forall\, p \geqq 0,\ \forall\, \theta \in R^n$.

If we take $p = 0$ in conditon (*ii*), we get $\mathcal{F}(t, v, \dot{v}, u, \dot{u}; 0) = 0$. We represent $\mathcal{F}(t, v, \dot{v}, u, \dot{u}; \theta)$ by $\mathcal{F}(t, v, u; \theta)$ in order to keep notation simpler.

Definition 2.4 Under the assumptions that $C \subseteq R^n$ is compact and convex, the support can be described by

$$s(v|C) = \max\{v^T u : u \in C\}.$$

It is to be noted that a support function has a subdifferential, *i.e.*, $\exists\, w \in R^n$ satisfying

$$s(u|C) \geqq s(v|C) + w^T(u - v),\ \forall u \in C.$$

The subdifferential of a support function, *i.e.*, $s(v|C)$ is given by

$$\partial s(v|C) = \{w \in C; w^T v = s(v|C)\}.$$

The normal cone Q corresponding to any point $v \in Q$ is mathematically defined as

$$N_Q(v) = \{u \in R^n : u^T(w - v) \leqq 0,\ \ \forall w \in Q\}.$$

Observe that, $u \in N_C(v) \ \Leftrightarrow\ s(u|C) = v^T u$.

Now, we consider the following definition of second order $(\mathcal{F}, \alpha, \rho, d)$-convex function.

Definition 2.5 The functional $\int_a^b f(t, v, \dot{v})dt$ is called second order $(\mathcal{F}, \alpha, \rho, d)$-convex at $u(t) \in R^n$ if

$$\int_a^b f(t, v, \dot{v})\, dt - \int_a^b f(t, u, \dot{u})\, dt + \frac{1}{2}\int_a^b q(t)^T M q(t)\, dt$$
$$\geqq \int_a^b \mathcal{F}(t, v, \dot{v}; \alpha(v, u)(f_v(t, u, \dot{u}) - D f_{\dot{v}}(t, u, \dot{u}) + Mq(t)))\, dt + \rho \int_a^b d^2(t, v, \dot{v}, u, \dot{u})\, dt,$$

for all $v(t),\ q(t) \in R^n,\ t \in I$ and for any sublinear function $\mathcal{F}$ as defined above.

Remark 2.1

(*i*) If we substitute $d = 0$ and $\alpha = 1$ in the Definition 2.5, second order $\mathcal{F}$-convex function discussed in Prasad *et al.* [18] can be obtained.

(*ii*) If $M(t, x, \dot{x}) = 0$, then the above expression shortens to that discussed in ([1],[17]). Moreover, if $d = 0$ and $\alpha = 1$, then we get the definition of invexity proposed by Mond *et al.* [16].

(*iii*) If $\mathcal{F}(t, v, u; a) = \phi(t, v, u)^T a$, then our proposed Definition 2.5 coincides with that of Ahmed *et al.* [7].

The example constructed below will assure the presence of second order $(\mathcal{F}, \alpha, \rho, d)$-convex functions.
Example 2.1 Let I=[0,1]. Define $f : I \times R \times R \mapsto R$ by $f(t, v, \dot{v}) = v^2(t) - 2$ and $\alpha : R \times R \mapsto R$ by $\alpha(v, u) = v + 2$. Consider the functional $\mathcal{F} : I \times R \times R \times R \times R \times R \mapsto R$ given by $\mathcal{F}(t, v, \dot{v}, u, \dot{u}; a) = -\left|\frac{a}{v+2}\right|$.
Let $d : R \times R \mapsto R$ be given by $d(v, u) = \sqrt{v^2 + u^2}$ and $\rho = -2$. Then $\int_0^1 \phi(t, v, \dot{v})dt$ is second order $(\mathcal{F}, \alpha, \rho, d)$-convex at $u(t) = 0$, since

$$\int_0^1 f(t, v, \dot{v})dt - \int_0^1 f(t, u, \dot{u})\,dt + \frac{1}{2}\int_0^1 q(t)^T M q(t)\,dt$$

$$\begin{aligned} &= \int_0^1 (v^2(t) - 2)\,dt - \int_0^1 (u^2(t) - 2)\,dt + \frac{1}{2}\int_0^1 2q^2(t)\,dt \\ &= \int_0^1 (v^2(t) + q^2(t))dt, \end{aligned}$$

whereas

$$\int_0^1 \mathcal{F}(t, v, \dot{v}; \alpha(v, u)(f_v(t, u, \dot{u}) - Df_{\dot{v}}(t, u, \dot{u}) + Mq(t)))dt \;\; + \;\; \rho\int_0^1 (d^2(t, v, \dot{v}, u, \dot{u}))dt$$

$$\begin{aligned} &= \int_0^1 \mathcal{F}(t, v, \dot{v}, (v + 2)(2u(t) + 2q(t))dt \; + \; (-2)\int_0^1 (v(t)^2 + u(t)^2)dt \\ &= -2\int_0^1 |q(t)|v^2(t)dt. \end{aligned}$$

From what has been done, it follows that

$$\int_0^1 \phi(t, v, \dot{v})dt - \int_0^1 \phi(t, u, \dot{u})\,dt + \frac{1}{2}\int_0^1 q(t)^T M q(t)\,dt \geqq \int_0^1 \mathcal{F}(t, v, u; \phi_v(t, u, \dot{u}) - D\phi_{\dot{v}}(t, u, \dot{u}) + Mq(t))\,dt.$$

Hence $\int_0^1 \phi(t, v, \dot{v})dt$ is second order $(\mathcal{F}, \alpha, \rho, d)$-convex at $u(t) = 0$.
Now,

$$\int_0^1 f(t, v, \dot{v})dt - \int_0^1 f(t, u, \dot{u})\,dt$$

$$\begin{aligned} &= \int_0^1 (v^2(t) - 2)\,dt - \int_0^1 (u^2(t) - 2)\,dt \\ &= \int_0^1 (t^2 - 2 + 2)\,dt \\ &= \Big[\frac{t^3}{3}\Big]_0^1 \\ &= 0.33. \end{aligned}$$

and

$$\int_0^1 \mathcal{F}(t, v, u; f_v(t, u, \dot{u}) - Df_{\dot{v}}(t, u, \dot{u}))\,dt$$

$$
\begin{aligned}
&= \int_0^1 2t^2\, dt \\
&= \Big[\frac{2t^3}{3}\Big]_0^1 \\
&= 0.66.
\end{aligned}
$$

Therefore,

$$\int_0^1 \phi(t,v,\dot{v})dt - \int_0^1 \phi(t,u,\dot{u})\, dt \ngeqq \int_0^1 \mathcal{F}(t,v,u;\phi_v(t,u,\dot{u}) - D\phi_{\dot{v}}(t,u,\dot{u}).$$

Hence, $\int_0^1 \phi(t,v,\dot{v})dt$ which is not $\mathcal{F}$-convex at $u(t) = 0$.
Let C_1 and C_2 to be closed convex cones with nonempty interiors in R^n and , R^m respectively.

3. Second Order Multiobjective Nondifferentiable Symmetric Duality

In this section, we introduce the following second order symmetric dual multiobjective nondifferentiable fractional variational programs over cone constraints:
Primal (PP)

$$\text{Minimize} \quad \Bigg(\frac{\int_a^b (f^1(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^1(t)^T A_1 p^1(t) + s(w|E_1) - x^T z_1)\, dt}{\int_a^b (g^1(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^1(t)^T B_1 p^1(t) - s(w|F_1) + x^T r_1)\, dt}, \ldots,$$

$$\frac{\int_a^b (f^k(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^k(t)^T A_k p^k(t) + s(w|E_k) - x^T z_k)\, dt}{\int_a^b (g^k(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^k(t)^T B_k p^k(t) - s(w|F_k) + x^T r_k)\, dt}\Bigg)$$

subject to

$$w(a) = \alpha_1, \quad w(b) = \alpha_2, \quad \dot{w}(a) = \alpha_3, \quad \dot{w}(b) = \alpha_4,$$
$$x(a) = \beta_1, \quad x(b) = \beta_2, \quad \dot{x}(a) = \beta_3, \quad \dot{x}(b) = \beta_4,$$

$$
\begin{aligned}
&\sum_{i=1}^{k} \lambda_i[(f_x^i - Df_{\dot{x}}^i + A_i p^i(t) - z_i) \\
&\quad - \left(\frac{\int_a^b (f^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T A_i p^i(t) + s(w|E_i) - x^T z_i)\, dt}{\int_a^b (g^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T B_i p^i(t) - s(w|F_i) + x^T r_i)\, dt}\right) \\
&\quad \Big(g_x^i - Dg_{\dot{x}_i} + B_i p^i(t) + r_i\Big)] \in C_2^*,
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&x^T \sum_{i=1}^{k} \lambda_i[(f_x^i - Df_{\dot{x}}^i + A_i p^i(t) - z_i) \\
&\quad - \left(\frac{\int_a^b (f^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T A_i p^i(t) + s(w|E_i) - x^T z_i)\, dt}{\int_a^b (g^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T B_i p^i(t) - s(w|F_i) + x^T r_i)\, dt}\right) \\
&\quad \Big(g_x^i - Dg_{\dot{x}_i} + B_i p^i(t) + r_i\Big)] \geqq 0,
\end{aligned}
\tag{2}
$$

$$w(t) \in C_1, \quad t \in I,$$

$$z_i \in G, \quad r_i \in H.$$

Dual (DP)

$$\text{Maximize} \quad \Bigg(\frac{\int_a^b (f^1(t,v,\dot v,u,\dot u) - \frac{1}{2}q^1(t)^T K_1 q^1(t) - s(u|G_1) + v^T\omega_1)\,dt}{\int_a^b (g^1(t,v,\dot v,u,\dot u) - \frac{1}{2}q^1(t)^T L_1 q^1(t) + s(u|H_1) - v^T n_1)\,dt}, \ldots,$$

$$\frac{\int_a^b (f^k(t,v,\dot v,u,\dot u) - \frac{1}{2}q^k(t)^T K_k q^k(t) - s(u|G_k) + v^T\omega_k)\,dt}{\int_a^b (g^k(t,v,\dot v,u,\dot u) - \frac{1}{2}q^k(t)^T L_k q^k(t) + s(u|H_k) - v^T n_k)\,dt}\Bigg)$$

subject to

$$v(a) = \gamma_1, \quad v(b) = \gamma_2, \quad \dot v(a) = \gamma_3, \quad \dot v(b) = \gamma_4,$$

$$u(a) = \delta_1, \quad u(b) = \delta_2, \quad \dot u(a) = \delta_3, \quad \dot u(b) = \delta_4,$$

$$-\sum_{i=1}^{k} \lambda_i[(f^i_w - Df^i_{\dot w} + K_i q^i(t) + \omega_i) - \left(\frac{\int_a^b (f^i(t,v,\dot v,u,\dot u) - \frac{1}{2}q^i(t)^T K_i q^i(t) - s(u|G_i) + v^T\omega_i)\,dt}{\int_a^b (g^i(t,v,\dot v,u,\dot u) - \frac{1}{2}q^i(t)^T L_i q^i(t) + s(u|H_i) - v^T n_i)\,dt}\right) \left(g^i_w - Dg^i_{\dot w} + L_i q^i(t) - n_i\right)] \in C_1^*, \tag{3}$$

$$v^T\sum_{i=1}^{k} \lambda_i[(f^i_w - Df^i_{\dot w} + K_i q^i(t) + \omega_i) - \left(\frac{\int_a^b (f^i(t,v,\dot v,u,\dot u) - \frac{1}{2}q^i(t)^T K_i q^i(t) - s(u|G_i) + v^T\omega_i)\,dt}{\int_a^b (g^i(t,v,\dot v,u,\dot u) - \frac{1}{2}q^i(t)^T L_i q^i(t) + s(u|H_i) - v^T n_i)\,dt}\right) \left(g^i_w - Dg^i_{\dot w} + L_i q^i(t) - n_i\right)] \leq 0, \tag{4}$$

$$u(t) \in C_2, \quad t \in I,$$

$$\omega_i \in E, \quad n_i \in F,$$

where
(i) $f^i : I \times C_1 \times C_1 \times C_2 \times C_2 \to R_+$, and $g^i : I \times C_1 \times C_1 \times C_2 \times C_2 \to R_+ \setminus \{0\}$,
(ii) $A_i(t,w,\dot w,x,\dot x) = f^i_{xx} - 2Df^i_{x\dot x} + D^2 f^i_{\dot x\dot x} - D^3 f^i_{\dot x\ddot x}, \ t \in I$,
(iii) $B_i(t,w,\dot w,x,\dot x) = g^i_{xx} - 2Dg^i_{x\dot x} + D^2 g^i_{\dot x\dot x} - D^3 g^i_{\dot x\ddot x}, \ t \in I$,
(iv) $K_i(t,w,\dot w,x,\dot x) = f^i_{ww} - 2Df^i_{w\dot w} + D^2 f^i_{\dot w\dot w} - D^3 f^i_{\dot w\ddot w}, \ t \in I$,
(v) $L_i(t,w,\dot w,x,\dot x) = g^i_{ww} - 2Dg^i_{w\dot w} + D^2 g^i_{\dot w\dot w} - D^3 g^i_{\dot w\ddot w}, \ t \in I$,
(vi) $p^i : I \to R^m, \ q^i : I \to R^n$,
(vii) $F(w,x) = \int_a^b (f^i - \frac{1}{2}p^i(t)^T A_i p^i(t) + s(w|E_i) - x^T z_i)\,dt$,
(viii) $G(w,x) = \int_a^b (g^i - \frac{1}{2}p^i(t)^T B_i p^i(t) - s(w|F_i) + x^T r_i)\,dt$,
(ix) E and F are taken as compact convex sets in R^n and
(x) G and H are taken as compact convex sets in R^m.

In primal and dual problems defined above, numerators are bound to be nonnegative, whereas denominators are bound to be positive. First of all, we transform our problem to parametric form by introducing l and m defined by

$$l_i = \frac{\int_a^b (f^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T A_i p^i(t) + s(w|E_i) - x^T z_i)\,dt}{\int_a^b (g^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T B_i p^i(t) - s(w|F_i) + x^T r_i)\,dt},$$

$$m_i = \frac{\int_a^b (f^i(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q^i(t)^T K_i q^i(t) - s(u|G_i) + v^T \omega_i)\,dt}{\int_a^b (g^i(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q^i(t)^T L_i q^i(t) + s(u|H_i) - v^T n_i)\,dt}.$$

Equivalently, the above problems can be stated as
Primal (PP$'$)

$$\text{Minimize} \qquad l = (l_1, l_2, l_3, \ldots.., l_k)$$

subject to

$$w(a) = \alpha_1, \quad w(b) = \alpha_2, \quad \dot{w}(a) = \alpha_3, \quad \dot{w}(b) = \alpha_4,$$
$$x(a) = \beta_1, \quad x(b) = \beta_2, \quad \dot{x}(a) = \beta_3, \quad \dot{x}(b) = \beta_4,$$

$$\int_a^b (f^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T A_i p^i(t) + s(w|E_i) - x^T z_i)\,dt$$
$$-l_i \int_a^b (g^i(t,w,\dot{w},x,\dot{x}) - \frac{1}{2}p^i(t)^T B_i p^i(t) - s(w|F_i) + x^T r_i)\,dt = 0, \tag{5}$$

$$\sum_{i=1}^{k} \lambda_i[(f_x^i - Df_{\dot{x}}^i + A_i p^i(t) - z_i) - l_i\left(g_x^i - Dg_{\dot{x}}^i + B_i p^i(t) + r_i\right)] \in C_2^*, \tag{6}$$

$$x^T \sum_{i=1}^{k} \lambda_i[(f_x^i - Df_{\dot{x}}^i + A_i p^i(t) - z_i) - l_i\left(g_x^i - Dg_{\dot{x}}^i + B_i p^i(t) + r_i\right)] \geq 0, \tag{7}$$

$$w(t) \in C_1, \quad t \in I,$$
$$z_i \in G, \quad r_i \in H.$$

Dual (DP$'$)

$$\text{Maximize} \qquad m = (m_1, m_2, m_3, \ldots.., m_k)$$

subject to

$$v(a) = \gamma_1, \quad v(b) = \gamma_2, \quad \dot{v}(a) = \gamma_3, \quad \dot{v}(b) = \gamma_4,$$
$$u(a) = \delta_1, \quad u(b) = \delta_2, \quad \dot{u}(a) = \delta_3, \quad \dot{u}(b) = \delta_4,$$

$$\int_a^b (f^i(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q^i(t)^T K_i q^i(t) - s(u|G_i) + v^T \omega_i)\,dt$$
$$-m_i \int_a^b (g^i(t,v,\dot{v},u,\dot{u}) - \frac{1}{2}q^i(t)^T L_i q^i(t) + s(u|H_i) - v^T n_i)\,dt = 0, \tag{8}$$

$$-\sum_{i=1}^{k} \lambda_i[(f_w^i - Df_{\dot{w}}^i + K_i q^i(t) + \omega_i) - m_i\left(g_w^i - Dg_{\dot{w}}^i + L_i q^i(t) - n_i\right)] \in C_1^*, \tag{9}$$

$$v^T \sum_{i=1}^{k} \lambda_i[(f_w^i - Df_{\dot{w}}^i + K_i q^i(t) + \omega_i) - m_i\left(g_w^i - Dg_{\dot{w}}^i + L_i q^i(t) - n_i\right)] \leq 0, \tag{10}$$

$$u(t) \in C_2, \quad t \in I,$$

$$\omega_i \in E, \quad n_i \in F.$$

Remark 3.1

(i) The problems (PP) and (DP) will settle down to the problems considered by Prasad *et al.* [18] on taking k=1.

(ii) The problems (PP) and (DP) will get down to the problems examined by Kailey and Gupta [11], if $A = B = K = L = \{0\}$.

(iii) The problems (PP) and (DP) will get down to the problems investigated by Jayswal and Jha [10], if we take $E = F = G = H = 0$,

(iv) In addition to (iii) above, if $A = B = K = L = \{0\}$, then we will get the problems studied by Ahmad *et al.* [3].

4. Duality Theorems

Now, we intend to derive the appropriate duality theorems for the primal-dual pair (PP′) and (DP′) which are equally applicable to the primal-dual pair (PP) and (DP).

Theorem 4.1. *(Weak duality). Let* $(w, x, l_i, \lambda_i, p^i, z_i, r_i)$ *and* $(v, u, m_i, q^i, \lambda_i, \omega_i, n_i)$ *be feasible solutions to primal* (PP') *and dual* (DP')*, respectively. Further, assume that*

(a) $\sum_{i=1}^{k} \lambda_i(\int_a^b (f^i(t, ., ., u(t), \dot{u}(t)) + (.)^T \omega_i - m_i(g^i(t, ., ., u(t), \dot{u}(t)) - (.)^T n_i))\, dt$ *is second order* $(\mathcal{F}, \alpha_i^1, \rho_i^1, d_i^1)$*-convex in* $w(t)$ *and* $\dot{w}(t)$*,*

(b) $\sum_{i=1}^{k} \lambda_i(\int_a^b (f^i(t, w(t), \dot{w}(t), ., .) - (.)^T z_i - l_i(g^i(t, w(t), \dot{w}(t), ., .) - (.)^T r_i))\, dt$ *is second order* $(\mathcal{G}, \alpha_i^2, \rho_i^2, d_i^2)$*-convex in* $x(t)$ *and* $\dot{x}(t)$*,*

(c) $\mathcal{F}(t, w, v; \alpha_i(w, v)\xi) + v^T \xi \geqq 0, \ \forall w, v \in C_1, \ -\xi \in C_1^*, \ t \in I,$

(d) $\mathcal{G}(t, u, x; \alpha_i(u, x)\zeta) + x^T \zeta \geqq 0, \ \forall u, x \in C_2, \ -\zeta \in C_2^*, \ t \in I,$

(e) $\int_a^b (g^i(t, w, \dot{w}, u, \dot{u}) + u^T r_i - w^T n_i)\, dt \geqq 0$ *and*

(f) $\rho_i^1 \int_a^b \{d_i^1(t, w, \dot{w}, v, \dot{v})\}^2 dt + \rho_i^2 \int_a^b \{d_i^2(t, u, \dot{u}, x, \dot{x})\}^2 dt \geq 0$ *with* $\rho_i^1 \geqq 0$, $\rho_i^2 \geqq 0$.

Then $l_i \geqq m_i$.

Proof. By constraint (9) and the assumption (c) , we get

$$\mathcal{F}(t, w, v; \alpha_i(w, v) \sum_{i=1}^{k} \lambda_i[(f_w^i - Df_{\dot{w}}^i + K_i q^i(t) + \omega_i) - m_i(g_w^i - Dg_{\dot{w}}^i + L_i q^i(t) - n_i)]$$

$$+ v^T \sum_{i=1}^{k} \lambda_i[(f_w^i - Df_{\dot{w}}^i + K_i q^i(t) + \omega_i) - m_i(g_w^i - Dg_{\dot{w}}^i + L_i q^i(t) - n_i)] \geqq 0,$$

which on using inequality (10) yields

$$\mathcal{F}(t, w, v; \alpha_i(w, v) \sum_{i=1}^{k} \lambda_i[(f_w^i - Df_{\dot{w}}^i + K_i q^i(t) + \omega_i)$$

$$-m_i(g^i_w - Dg^i_{\dot{w}} + L_iq^i(t) - n_i))] \geqq 0. \tag{11}$$

Since $\sum_{i=1}^k \lambda_i(\int_a^b (f^i(t,.,.,u(t),\dot{u}(t)) + (.)^T\omega_i - m_i(g^i(t,.,.,u(t),\dot{u}(t)) - (.)^T n_i))\,dt$ is second order $(\mathcal{F},\alpha_i^1,\rho_i^1,d_i^1)$-convex at $v(t)$ for fixed $u(t)$ *and* $\dot{u}(t)$ we have

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i \int_a^b \Big((f^i(t,w,\dot{w},u,\dot{u}) + w^T\omega_i - f^i(t,v,\dot{v},u,\dot{u}) + \frac{1}{2}q^i(t)^T K_i q^i(t) - v^T\omega_i) \\
&\quad -m_i(g^i(t,w,\dot{w},u,\dot{u}) - w^T n_i - g^i(t,v,\dot{v},u,\dot{u}) + v^T n_i + \frac{1}{2}q^i(t)^T L_i q^i(t))\Big)\,dt \\
&\quad \geqq \sum_{i=1}^k \lambda_i \int_a^b \mathcal{F}(t,w,v;\alpha_i(w,v)(f^i_w - Df^i_{\dot{w}} + K_iq^i(t) + \omega_i) \\
&\quad -m_i(g^i_w - Dg^i_{\dot{w}} + L_iq^i(t) - n_i)) + \rho_i^1 \int_a^b \{d_i^1(t,w,\dot{w},v,\dot{v})\}^2 dt,
\end{aligned}$$

which due to (11) reduces to

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i \int_a^b [(f^i(t,w,\dot{w},u,\dot{u}) + w^T\omega_i - f^i(t,v,\dot{v},u,\dot{u}) + \frac{1}{2}q^i(t)^T K_i q^i(t) - v^T\omega_i) \\
&\quad -m_i(g^i(t,w,\dot{w},u,\dot{u}) - w^T n_i - g^i(t,v,\dot{v},u,\dot{u}) + v^T n_i + \frac{1}{2}q^i(t)^T L_i q^i(t))]\,dt \\
&\quad \geqq \sum_{i=1}^k \lambda_i\ \rho_i^1 \int_a^b \{d_i^1(t,w,\dot{w},v,\dot{v})\}^2 dt.
\end{aligned}$$

This can be formulated as,

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i \int_a^b [(f^i(t,w,\dot{w},u,\dot{u}) + w^T\omega_i - f^i(t,v,\dot{v},u,\dot{u}) + \frac{1}{2}q^i(t)^T K_i q^i(t) - v^T\omega_i) \\
&\quad +m_i(g^i(t,v,\dot{v},u,\dot{u}) + u^T r_i - v^T n_i - \frac{1}{2}q^i(t)^T L_i q^i(t)) - m_i(g^i(t,w,\dot{w},u,\dot{u}) \\
&\quad +u^T r_i - w^T n_i)] \geqq \sum_{i=1}^k \lambda_i\ \rho_i^1 \int_a^b \{d_i^1(t,w,\dot{w},v,\dot{v})\}^2 dt.
\end{aligned}$$

Using (8) together with $u^T r_i \leqq s(v|H_i)$ in the above inequality, we have

$$\begin{aligned}
&\sum_{i=1}^k \lambda_i \int_a^b [(f^i(t,w,\dot{w},u,\dot{u}) + w^T\omega_i - s(u|G_i)) - m_i(g^i(t,w,\dot{w},u,\dot{u}) + u^T r_i - w^T n_i)]\,dt \\
&\quad \geqq \sum_{i=1}^k \lambda_i\ \rho_i^1 \int_a^b \{d_i^1(t,w,\dot{w},v,\dot{v})\}^2 dt.
\end{aligned} \tag{12}$$

Similarly, the assumption $\sum_{i=1}^k \lambda_i \int_a^b (f^i(t,w(t),\dot{w}(t),.,.) - (.)^T z_i - l_i(g^i(t,w(t),\dot{w}(t),.,.) - (.)^T r_i))\,dt$ is second order $(\mathcal{G},\alpha_i^2,\rho_i^2,d_i^2)$-convex at $x(t)$ for fixed $w(t)$ *and* $\dot{w}(t)$ yields,

$$\sum_{i=1}^k \lambda_i \int_a^b [(-f^i(t,w,\dot{w},u,\dot{u}) + u^T z_i - s(w|E_i)) + l_i(g^i(t,w,\dot{w},u,\dot{u}) - w^T n_i + u^T r_i))]\,dt$$

$$\geqq \sum_{i=1}^{k} \lambda_i \, \rho_i^2 \int_a^b \{d_i^2(t, w, \dot{w}, v, \dot{v})\}^2 dt. \tag{13}$$

On adding (12) and (13) and using assumption (f), we get

$$\sum_{i=1}^{k} \lambda_i \int_a^b ((w^T\omega_i - s(u|G_i) + u^T r_i - s(w|E_i)) + (l_i - m_i)(g^i(t, w, \dot{w}, u, \dot{u}) + u^T z_i - w^T n_i))\, dt \geqq 0.$$

Since $u^T z_i \leqq s(v|G_i)$ *and* $w^T\omega_i \leqq s(w|E_i)$, the above inequality yields

$$\sum_{i=1}^{k} \int_a^b (l_i - m_i)(g^i(t, w, \dot{w}, u, \dot{u}) + u^T r_i - w^T n_i)\, dt \geqq 0,$$

which due to (e) gives

$$l_i \geqq m_i.$$

Hence proved. □

Theorem 4.2. *(Strong Duality). Under the assumptions that*

(i) $(\bar{w}, \bar{x}, \bar{1}_i, \bar{p}, \bar{z}_i, \bar{r}_i, \bar{\lambda}_i)$ *is an optimal solution of (PP′),*

(ii) matrices $(A_i - \bar{l}_i B_i)$, $i \in k$ *are considered to be nonsingular,*

(iii) $(f_x^i - \bar{z}_i) - \bar{l}_i(g_x^i + \bar{r}_i) - D(f_{\dot{x}}^i - \bar{l}_i g_{\dot{x}}^i) + (A_i - \bar{l}_i B_i)\bar{p^i}(t) \neq 0$ *and*

(iv) the matrix given by

$$\Big((A_i\bar{p^i}(t))_x - \bar{l}_i(B_i\bar{p^i}(t))_x - D(A_i\bar{p^i}(t))_{\dot{x}} + \bar{l}_i D(B_i\bar{p^i}(t))_{\dot{x}} + D^2(A_i\bar{p^i}(t))_{\ddot{x}}$$
$$-\bar{l}_i D^2(B_i\bar{p^i}(t))_{\ddot{x}} - D^3(A_i\bar{p^i}(t))_{\dddot{x}} + \bar{l}_i D^3(B_i\bar{p^i}(t))_{\dddot{x}} + D^4(A_i\bar{p^i}(t))_{\ddddot{x}} - \bar{l}_i D^4(B_i\bar{p^i}(t))_{\ddddot{x}}\Big)$$

is positive or negative definite.

Then, there exist $\bar{\omega}_i \in E$, $\bar{n}_i \in F$ *that make* $(\bar{w}, \bar{x}, \bar{l}_i, \bar{p}^i, \bar{\omega}_i, \bar{n}_i, \bar{\lambda}_i)$ *a solution of (DP′). Furthermore,* $(\bar{w}, \bar{x}, \bar{l}_i, \bar{p}^i = 0, \bar{\omega}_i, \bar{n}_i, \bar{\lambda}_i)$ *becomes an optimal solution of (DP′) under additional assumptions stated in Theorem 4.1.*

Proof. Since $(\bar{w}, \bar{x}, \bar{l}_i, \bar{p^i}, \bar{z}_i, \bar{r}_i)$ be an optimal solution of (PP′), $\exists$ $\alpha_i \in R$, $\beta_i \in R$, $\gamma_i \in C_2$ and $\xi_i \in R$ fulfilling the following Fritz John optimality conditions at the point $(\bar{w}(t), \bar{x}(t), \bar{l}_i, \bar{p^i}(t))$:

$$\Big[\sum_{i=1}^{k} \beta_i\Big((f_w^i + \bar{\omega}_i) - \bar{l}_i(g_w^i - \bar{n}_i) - D(f_{\dot{w}}^i - \bar{l}_i g_{\dot{w}}^i) - \frac{1}{2}(\bar{p^i}(t)^T A_i\bar{p^i}(t))_w + \frac{\bar{l}_i}{2}(\bar{p^i}(t)^T B_i\bar{p^i}(t))_w + \frac{1}{2}D(\bar{p^i}(t)^T A_i\bar{p^i}(t))_{\dot{w}}$$
$$-\frac{\bar{l}_i}{2}D(\bar{p^i}(t)^T B_i\bar{p^i}(t))_{\dot{w}} - \frac{1}{2}D^2(\bar{p^i}(t)^T A_i\bar{p^i}(t))_{\ddot{w}} + \frac{\bar{l}_i}{2}D^2(\bar{p^i}(t)^T B_i\bar{p^i}(t))_{\ddot{w}} + \frac{1}{2}D^3(\bar{p^i}(t)^T A_i\bar{p^i}(t))_{\dddot{w}}$$
$$-\frac{\bar{l}_i}{2}D^3(\bar{p^i}(t)^T B_i\bar{p^i}(t))_{\dddot{w}} - \frac{1}{2}D^4(\bar{p^i}(t)^T A_i\bar{p^i}(t))_{\ddddot{w}} + \frac{\bar{l}_i}{2}D^4(\bar{p^i}(t)^T B_i\bar{p^i}(t))_{\ddddot{w}}\Big)$$
$$+\sum_{i=1}^{k} \lambda_i(\gamma - \xi\bar{x})^T\Big(f_{xw}^i - \bar{l}_i g_{xw}^i - D(f_{x\dot{w}}^i - \bar{l}_i g_{x\dot{w}}^i) - D(f_{\dot{x}w}^i - \bar{l}_i g_{\dot{x}w}^i) + D^2(f_{\dot{x}\dot{w}}^i - \bar{l}_i g_{\dot{x}\dot{w}}^i)$$
$$-D^3(f_{\dot{x}\ddot{w}}^i - \bar{l}_i g_{\dot{x}\ddot{w}}^i) + (A_i\bar{p^i}(t))_w - \bar{l}_i(B_i\bar{p^i}(t))_w - D((A_i\bar{p^i}(t))_{\dot{w}} - \bar{l}_i(B_i\bar{p^i}(t))_{\dot{w}}) + D^2((A_i\bar{p^i}(t))_{\ddot{w}}$$
$$-\bar{l}_i(B_i\bar{p^i}(t))_{\ddot{w}}) - D^3((A_i\bar{p^i}(t))_{\dddot{w}} - \bar{l}_i(B_i\bar{p^i}(t))_{\dddot{w}}) + D^4((A_i\bar{p^i}(t))_{\ddddot{w}} - \bar{l}_i(B_i\bar{p^i}(t))_{\ddddot{w}})\Big)\Big](w(t) - \bar{w}(t)) \geqq 0, \tag{14}$$

$$\sum_{i=1}^{k}(\beta_i-\xi\lambda_i)((f_x^i-Df_{\dot{x}}^i-z_i)-\bar{l}_i(g_x^i-Dg_{\dot{x}}^i+r_i))+\sum_{i=1}^{k}\beta_i\Big(-\frac{1}{2}(\bar{p^i}(t)^TA_ip^i(t))_x+\frac{\bar{l}_i}{2}(\bar{p^i}(t)^TB_ip^i(t))_x+\frac{1}{2}D(\bar{p^i}(t)^TA_i\bar{p^i}(t))_{\dot{x}}$$
$$-\frac{\bar{l}_i}{2}D(\bar{p^i}^T(t)B_i\bar{p^i}(t))_{\dot{x}}-\frac{1}{2}D^2(\bar{p^i}(t)^TA_i\bar{p^i}(t))_{\ddot{x}}+\frac{\bar{l}_i}{2}D^2(\bar{p^i}(t)^TB_i\bar{p^i}(t))_{\ddot{x}}+\frac{1}{2}D^3(\bar{p^i}(t)^TA_i\bar{p^i}(t))_{\dddot{x}}$$
$$-\frac{\bar{l}_i}{2}D^3(\bar{p^i}(t)^TB_i\bar{p^i}(t))_{\dddot{x}}-\frac{1}{2}D^4(\bar{p^i}(t)^TA_i\bar{p^i}(t))_{\ddddot{x}}+\frac{\bar{l}_i}{2}D^4(\bar{p^i}(t)^TB_i\bar{p^i}(t))_{\ddddot{x}}\Big)$$
$$+\sum_{i=1}^{k}\lambda_i(\gamma-\xi\bar{y})^T\Big(A_i-\bar{l}_iB_i+(A_i\bar{p^i}(t))_x-\bar{l}_i(B_i\bar{p^i}(t))_x-D(A_i\bar{p^i}(t))_{\dot{x}}+\bar{l}_iD(B_i\bar{p^i}(t))_{\dot{x}}+D^2(A_i\bar{p^i}(t))_{\ddot{x}}$$
$$+\bar{l}_iD^2(B_i\bar{p^i}(t))_{\ddot{x}}-D^3(A_i\bar{p^i}(t))_{\dddot{x}}+\bar{l}_iD^3(B_i\bar{p^i}(t))_{\dddot{x}}+D^4(A_i\bar{p^i}(t))_{\ddddot{x}}-\bar{l}_iD^4(B_i\bar{p^i}(t))_{\ddddot{x}}\Big)$$
$$-\xi\sum_{i=1}^{k}\lambda_i(A_i\bar{p^i}(t)-\bar{l}_iB_i\bar{p^i}(t))=0, \tag{15}$$

$$\sum_{i=1}^{k}\alpha_i+\sum_{i=1}^{k}\beta_i(g^i-\frac{1}{2}\bar{p^i}^T(t)B_ip^i(t)-s(\bar{w}|F_i)+y^T\bar{z}_i)+\sum_{i=1}^{k}\lambda_i(\gamma-\xi\bar{x}(t))^T(-g_x^i+Dg_{\dot{x}}^i-B_i\bar{p^i}(t)+\bar{r}_i)=0, \tag{16}$$

$$\sum_{i=1}^{k}(-\beta_i(A_i\bar{p^i}(t)-\bar{l}_iB_i\bar{p^i}(t))+(\gamma-\xi\bar{y}(t))^T\lambda_i(A_i-\bar{l}_iB_i))=0, \tag{17}$$

$$\gamma\sum_{i=1}^{k}\lambda_i((f_x^i-\bar{z}_i)-\bar{l}_i(g_x^i+\bar{r}_i)-D(f_{\dot{x}}^i-\bar{l}g_{\dot{x}}^i)+A_ip^i(t)-\bar{l}_iB_ip^i(t))=0, \tag{18}$$

$$\xi\bar{x}(t)((f_x^i-\bar{z}_i)-\bar{l}_i(g_x^i+\bar{r}_i)-D(f_{\dot{x}}^i-\bar{l}_ig_{\dot{x}}^i)+A_ip^i(t)-\bar{l}_iB_ip^i(t))=0, \tag{19}$$

$$(\gamma-\xi\bar{x}(t))^T\sum_{i=1}^{k}[(f_x^i-\bar{z}_i-Df_{\dot{x}}^i+A_ip^i(t))-l_i(g_x^i+\bar{r}_i-Dg_{\dot{x}}^i+B_ip^i(t))]-\delta=0, \tag{20}$$

$$s(\bar{w}|E_i)=\bar{w}^T\bar{\omega}_i,\ \bar{\omega}_i\in E_i, \tag{21}$$
$$s(\bar{w}|F_i)=\bar{w}^T\bar{n}_i,\ \bar{n}_i\in F_i, \tag{22}$$
$$\beta_i\bar{x}^T+(\gamma-\xi\bar{x})\in N_J(z_i), \tag{23}$$
$$\bar{l}_i[\beta_i\bar{x}^T+(\gamma-\xi\bar{x})]\in N_K(r_i), \tag{24}$$
$$(\alpha_i,\beta_i(t),\gamma,\xi)\neq 0,\ t\in I, \tag{25}$$
$$(\alpha_i,\beta_i(t),\gamma,\xi)\geqq 0,\ t\in I, \tag{26}$$
$$\delta^T\bar{\lambda}_i=0. \tag{27}$$

Using assumption (*ii*), equation (17) yields

$$\sum_{i=1}^{k}(\gamma-\xi\bar{y})^T=\sum_{i=1}^{k}\beta_i\bar{p^i}(t). \tag{28}$$

Converting (15) into a suitable form, we get

$$\sum_{i=1}^{k}(\beta_i-\xi(t)\lambda_i)((f_x^i-\bar{z}_i)-\bar{l}_i(g_x^i+\bar{r}_i)-D(f_{\dot{x}}^i-\bar{l}_ig_{\dot{x}}^i))+\sum_{i=1}^{k}\lambda_i(A_i-\bar{l}_iB_i)$$

$$(\gamma - \xi\bar{x}(t) - \xi\bar{p^i}(t)) + \Big((A_i\bar{p^i}(t))_x - \bar{l}_i(B_i\bar{p^i}(t))_x - D(A_i\bar{p^i}(t))_{\dot{x}} + \bar{l}_iD(B_i\bar{p^i}(t))_{\dot{x}}$$
$$+D^2(A_i\bar{p}^i(t))_{\ddot{x}} - \bar{l}D^2(B_i\bar{p}^i(t))_{\ddot{x}} - D^3(A_i\bar{p^i}(t))_{\dddot{x}} + \bar{l}_iD^3(B_i\bar{p}^i(t))_{\dddot{x}}$$
$$+D^4(A_i\bar{p}^i(t))_{\ddddot{x}} - \bar{l}_iD^4(B_i\bar{p}^i(t))_{\ddddot{x}}\Big)(\gamma - \xi\bar{x}(t) - \frac{1}{2}\beta_i\bar{p}^i(t)) = 0. \quad (29)$$

Since $\lambda > 0$, equation (27) implies $\delta = 0$. Therefore, from equation (20), we get

$$(\gamma - \xi\bar{x}(t))^T\sum_{i=1}^{k}[(f_x^i - \bar{z}_i - Df_{\dot{x}}^i + A_ip^i(t)) - l_i(g_x^i + \bar{r}_i - Dg_{\dot{x}}^i + B_ip^i(t))] = 0 \quad (30)$$

In the light of (28), equation (29) becomes

$$\sum_{i=1}^{k}(\beta_i - \xi(t)\lambda_i)((f_x^i - \bar{z}_i) - \bar{l}_i(g_x^i + \bar{r}_i) - D(f_{\dot{x}}^i - \bar{l}_ig_{\dot{x}}^i)) + (A_i - \bar{l}_iB_i)\bar{p^i}(t))$$
$$+\frac{1}{2}(\gamma - \xi\bar{x}(t))^T((A_i\bar{p}^i(t))_x - \bar{l}_i(B_i\bar{p}^i(t))_x - D(A_i\bar{p}^i(t))_{\dot{x}} + \bar{l}_iD(B_i\bar{p}^i(t))_{\dot{x}}$$
$$+D^2(A_i\bar{p}^i(t))_{\ddot{x}} - \bar{l}_iD^2(B_i\bar{p}^i(t))_{\ddot{x}} - D^3(A_i\bar{p}^i(t))_{\dddot{x}} + \bar{l}_iD^3(B_i\bar{p}^i(t))_{\dddot{x}}$$
$$+D^4(A_i\bar{p}^i(t))_{\ddddot{x}} - \bar{l}_iD^4(B_i\bar{p}^i(t))_{\ddddot{x}}) = 0. \quad (31)$$

Multiplying $(\gamma - \xi\bar{x}(t))$ to both sides of above equation and using (30), the above equation give

$$\frac{1}{2}((\gamma - \xi\bar{x}(t))^T)^2\Big((A_i\bar{p}^i(t)_x - \bar{l}_i(B_i\bar{p}^i(t))_x - D(A_i\bar{p}^i(t))_{\dot{x}} + \bar{l}_iD(B_i\bar{p}^i(t))_{\dot{x}}$$
$$+D^2(A_i\bar{p}^i(t))_{\ddot{x}} - \bar{l}_iD^2(B_i\bar{p}^i(t))_{\ddot{x}} - D^3(A_i\bar{p}^i(t))_{\dddot{x}} + \bar{l}_iD^3(B_i\bar{p}^i(t))_{\dddot{x}}$$
$$+D^4(A_i\bar{p}^i(t))_{\ddddot{x}} - \bar{l}_iD^4(B_i\bar{p}^i(t))_{\ddddot{x}}\Big) = 0,$$

which due to hypothesis (*iv*) provides

$$\gamma = (\xi\bar{x}(t))^T. \quad (32)$$

On substituting (32) in (31), we obtain

$$\sum_{i=1}^{k}(\beta_i - \xi(t)\lambda_i)((f_x^i - \bar{z}_i) - \bar{l}_i(g_x^i + \bar{r}_i) - D(f_{\dot{x}}^i - \bar{l}_ig_{\dot{x}}^i)) + (A_i - \bar{l}_iB_i)\bar{p}^t(t) = 0, \quad (33)$$

which due to hypothesis (*iii*) leads to

$$\sum_{i=1}^{k}\beta_i = \sum_{i=1}^{k}\lambda_i\xi(t). \quad (34)$$

Now, if we substitute $\xi(t) = 0$ in (34), we get $\beta_i = 0$ which leads to $\gamma = 0$ on using (32). Moreover, we use (16) to get $\sum_{i=1}^{k}\alpha_i = 0$. Finally, we get $(\alpha_i, \beta_i(t), \gamma, \xi) \neq 0,\ t \in I$ contradicting (21). Therefore, we take $\xi(t) > 0,\ t \in I$ and thus $\beta_i > 0$. The fact that $\xi(t) > 0,\ t \in I$ along with (32) will yield

$$\bar{x}(t) = \frac{\gamma(t)}{\xi(t)} \in C_2,\ t \in I.$$

Using the relation (32) and (34) in (14), we obtain

$$\sum_{i=1}^{k} \beta_i((f^i_w + \bar{\omega}_i) - \bar{l}_i(g^i_w - \bar{n}_i) - D(f^i_{\dot{w}} - \bar{l}_i g^i_{\dot{w}}))(w(t) - \bar{w}(t)) \geqq 0,\ t \in I. \tag{35}$$

Suppose $w(t) \in C_1$ so that $w(t) + \bar{w}(t) \in C_1$. Replacing $w(t) + \bar{w}(t)$ in place of $w(t)$ in (35), we get

$$w(t)^T \sum_{i=1}^{k} \lambda_i((f^i_w + \bar{\omega}_i) - \bar{l}_i(g^i_w - \bar{n}_i) - D(f^i_{\dot{w}} - \bar{l}_i g^i_{\dot{w}}))(w(t) - \bar{w}(t)) \geqq 0,\ t \in I.$$

From the property of polar cone, we have

$$-\sum_{i=1}^{k} \lambda_i((f^i_w + \bar{\omega}_1) - \bar{l}_i(g^i_w - \bar{n}_i) - D(f^i_{\dot{w}} - \bar{l}_i g^i_{\dot{w}}))(w(t) - \bar{w}(t)) \in C_1^*,\ t \in I.$$

Again, if we take $w(t) = 0$ and $w(t) = 2\bar{w}(t)$ simultaneously in equation (35), we have

$$\bar{w}(t) \sum_{i=1}^{k} \lambda_i((f^i_w + \bar{\omega}_i) - \bar{l}_i(g^i_w - \bar{n}_i) - D(f^i_{\dot{w}} - \bar{l}_i g^i_{\dot{w}}))(w(t) - \bar{w}(t)) = 0,\ t \in I.$$

Thus, it becomes clear that $(\bar{w}(t), \bar{x}(t), \bar{l}_i, \bar{p}^i(t), \lambda_i, \bar{\omega}_i, \bar{n}_i)$ be a feasible solution to (DP′).

Further, with the help of (23), (32) and (34), we have $\bar{x} \in N_{G_i}(\bar{z}_i)$ and since G_i is a compact convex set in R^m one can conclude $\bar{x}^T \bar{r}_i = s(\bar{x}|G_i)$. Similarly, $\bar{x}^T \bar{n}_i = s(\bar{x}|H_i)$. So, (PP′) and (DP′) have equal objective function values. The optimality for (DP′) can be seen in the light of the weak duality theorem. □

Theorem 4.3. *(Converse Duality). Under assumptions that*

(i) $(\bar{v}, \bar{u}, \bar{m}_i, \bar{q}^i(t), \bar{\omega}_i, \bar{n}_i)$ *is an optimal solution of (DP′),*

(ii) matrices $K_i - \bar{m}_i L_i$ *are considered to be nonsingular,*

(iii) $f^i_w - \bar{\omega}_i - \bar{m}_i(g^i_w + \bar{n}_i) - D(f^i_{\dot{w}} - \bar{m}_i g^i_{\dot{w}}) + (K_i - \bar{m}_i L_i)\bar{q}^i(t) \neq 0$, *and*

(iv) the matrix given by

$$\Big((K_i\bar{q}^i(t)_w - \bar{m}_i(L_i\bar{q}^i(t))_w - D(B_i\bar{q}^i(t))_{\dot{w}} + \bar{m}_i D(L_i\bar{q}^i(t))_{\dot{w}} + D^2(K_i\bar{q}^i(t))_{\ddot{w}}$$

$$-\bar{m}_i D^2(L_i\bar{q}^i(t))_{\ddot{w}} - D^3(K_i\bar{q}^i(t))_{\dddot{w}} + \bar{m}_i D^3(L_i\bar{q}^i(t))_{\dddot{w}} + D^4(K_i\bar{q}^i(t))_{\ddddot{w}} - \bar{m}_i D^4(L_i\bar{q}^i(t))_{\ddddot{w}}\Big)$$

is positive or negative definite.

Then, there exist $\bar{z}_i \in G_i$, $\bar{r}_i \in H_i$ *which make* $(\bar{w}, \bar{x}, \bar{m}_i, \bar{q}^i, \bar{z}_i, \bar{r}_i)$ *a solution of (PP′). Furthermore,* $(\bar{w}, \bar{x}, \bar{m}_i, \bar{q}^i = 0, \bar{z}_i)$ *becomes an optimal solution to (PP′) under additional conditions stated in Theorem 4.1.*

5. Static Formulation

If we discard the time factor in the problems (PP) and (DP) then our problems transform into the second order fractional symmetric dual programs over cones given below:

Primal Problem (SPP)

$$\text{Min}\ \left(\frac{(f^1(w,x) - \frac{1}{2}p^{1^T}\nabla_{xx}f^1(w,x)p^1 + s(x|E_1) - x^T z_1)}{(g^1(w,x) - \frac{1}{2}p^{1^T}\nabla_{xx}g^1(w,x)p^1 - s(w|F_1) + x^T r_1)}, \ldots, \frac{(f^k(w,x) - \frac{1}{2}p^{k^T}\nabla_{xx}f^k(w,x)p^k + s(w|E_k) - x^T z_k)}{(g^k(w,x) - \frac{1}{2}p^{k^T}\nabla_{xx}g^k(w,x)p^k - s(w|F_k) + x^T r_k)}\right)$$

subject to

$$\sum_{i=1}^{k}\lambda_i[(f_x^i-\nabla_x f^i(w,x)+\nabla_{xx}f^i(w,x)p^i-z_i)$$

$$-\left(\frac{(f^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}f^i(w,x)p^i+s(w|E_i)-x^Tz_i)}{(g^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}g^i(w,x)p^i-s(w|F_i)+x^Tr_i)}\right)\left(g_x^i-\nabla_x g^i(w,x)+\nabla_{xx}g^i(w,x)p^i+r_i\right)]\in C_2^*,$$

$$x^T\sum_{i=1}^{k}\lambda_i[(f_x^i-\nabla_x f^i(w,x)+\nabla_{xx}f^i(w,x)p^i-z_i)$$

$$-\left(\frac{(f^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}f^i(w,x)p^i+s(w|E_i)-x^Tz_i)}{(g^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}g^i(w,x)p^i-s(w|F_i)+x^Tr_i)}\right)\left(g_x^i-\nabla_x g^i(w,x)+\nabla_{xx}g^i(w,x)p^i+r_i\right)]\geqq 0,$$

$$w\in C_1,$$

$$z_i\in G,\quad r_i\in H.$$

Dual Problem (SDP)

$$\text{Max}\quad\left(\frac{(f^1(v,u)-\frac{1}{2}q^{1^T}\nabla_{ww}f^1(v,u)q^1-s(v|G_1)+v^T\omega_1)}{(g^1(v,u)-\frac{1}{2}q^{1^T}\nabla_{ww}g^1(v,u)q^1+s(v|H_1)-v^Tn_1)},\ldots,\frac{(f^k(v,u)-\frac{1}{2}q^{k^T}\nabla_{ww}f^k(v,u)q^k-s(v|G_k)+v^T\omega_k)}{(g^k(v,u)-\frac{1}{2}q^{k^T}\nabla_{ww}g^k(v,u)q^k+s(v|H_k)-v^Tn_k)}\right)$$

$$-\sum_{i=1}^{k}\lambda_i[(f_w^i-\nabla_w f^i(v,u)+\nabla_{ww}f^i(v,u)q^i+\omega_i)$$

$$-\left(\frac{(f^i(v,u)-\frac{1}{2}q^{i^T}\nabla_{ww}f^i(v,u)q^i-s(v|G_i)+v^T\omega_i}{(g^i(v,u)-\frac{1}{2}q^{i^T}\nabla_{ww}g^i(v,u)q^i+s(v|H_i)-v^Tr_i)}\right)\left(g_w^i-\nabla_w g^i(v,u)+\nabla_{ww}g^i(v,u)q^i-n_i\right)]\in C_1^*,$$

$$v^T\sum_{i=1}^{k}\lambda_i[(f_w^i-\nabla_w f^i(v,u)+\nabla_{ww}f^i(v,u)q^i+\omega_i)$$

$$-\left(\frac{(f^i(v,u)-\frac{1}{2}q^{i^T}\nabla_{ww}f^i(v,u)q^i-s(v|G_i)+v^T\omega_i)}{(g^i(v,u)-\frac{1}{2}q^{i^T}\nabla_{ww}g^i(v,u)q^i+s(v|H_i)-v^Tr_i)}\right)\left(g_w^i-\nabla_w g^i(v,u)+\nabla_{ww}g^i(v,u)q^i-n_i\right)]\leq 0,$$

$$u\in C_2,$$

$$\omega_i\in E,\ \ n_i\in F,$$

Equivalent formulations can be done as
Primal Problem (SPP$'$)

$$\text{Minimize}\qquad l=(l_1,l_2,l_3,.....,l_k)$$

subject to

$$(f^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}f^i(w,x)p^i+s(w|E_i)-x^Tz_i)-l_i(g^i(w,x)-\frac{1}{2}p^{i^T}\nabla_{xx}g^i(w,x)p^i-s(w|F_i)+x^Tr_i)=0,$$

$$\sum_{i=1}^{k} \lambda_i[(f_x^i - \nabla_x f^i(w,x) + \nabla_{xx} f^i(w,x)p^i - z_i) - l_i\left(g_x^i - \nabla_x g^i(w,x) + \nabla_{xx} g^i(w,x)p^i + r_i\right)] \in C_2^*,$$

$$x^T \sum_{i=1}^{k} \lambda_i[(f_x^i - \nabla_x f^i(w,x) + \nabla_{xx} f^i(w,x)p^i - z_i) - l_i\left(g_x^i - \nabla_x g^i(w,x) + \nabla_{xx} g^i(w,x)p^i + r_i\right)] \geq 0,$$

$$w \in C_1,$$

$$z_i \in G, \quad r_i \in H.$$

Dual Problem (SDP$'$)

$$\text{Maximize} \qquad m = (m_1, m_2, m_3, \ldots.., m_k)$$

subject to

$$(f^i(v,u) - \frac{1}{2}{q^i}^T \nabla_{ww} f^i(v,u)q^i - s(v|G_i) + v^T\omega_i) - m_i(g^i(v,u) - \frac{1}{2}{q^i}^T \nabla_{ww} g^i(v,u)q^i + s(v|H_i) - v^T n_i) = 0$$

$$-\sum_{i=1}^{k} \lambda_i[(f_w^i - \nabla_w f^i(w,x) + \nabla_{ww} f^i(v,u)q^i + \omega_i) - m_i\left(g_w^i - \nabla_w g^i(w,x) + \nabla_{ww} g^i(v,u)q^i - n_i\right)] \in C_1^*,$$

$$u^T \sum_{i=1}^{k} \lambda_i[(f_w^i - \nabla_w f^i(w,x) + \nabla_{ww} f^i(v,u)q^i + \omega_i) - m_i\left(g_w^i - \nabla_w g^i(w,x) + \nabla_{ww} g^i(v,u)q^i - n_i\right)] \leq 0,$$

$$u \in C_2,$$

$$\omega_i \in E, \quad n_i \in F.$$

We can easily establish weak and strong duality results. One can refer to the work of Jayswal and Prasad [9] for detailed investigation.

6. Conclusions

In the present paper, we have derived a weak duality theorem for a pair of second order multiobjective symmetric nondifferentiable fractional variational problems. Strong and converse duality theorems are also derived using Fritz-John optimality conditions. Finally, appropriate duality theorems are discussed for static symmetric dual problems by dropping down the time coordinate in our considered problems. As stated in the present research paper, a number of recently published articles become special instances of our study. The present work can be extended to higher order cases, which gives a significantly stricter bound. Moreover, we can also weaken the convexity requirements to apply the results of this paper to a more sophisticated class of problems. Still, there is a possibility to get potential extensions and generalizations of the current work that require the results to hold for broader classes of functionals or spaces. Advanced mathematical principles, as well as other related areas of mathematics, such as differential inclusions, non-smooth analysis, and set-valued analysis, have the potential to address such challenging problems.

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