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B^p-almost periodic solutions in finite-dimensional distributions to semilinear stochastic differential equations

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Abstract. In this paper, we consider a class of semilinear stochastic differential equations in real separable Hilbert spaces. Based on the theory of evolutionary operator family, Banach fixed point theorem and inequality technique, we obtain the existence and uniqueness of p-th Besicovitch almost periodic (B^p-almost periodic) solutions in finite-dimensional distributions of this class of semilinear stochastic differential equations. Finally, we provide an example to demonstrate the effectiveness of our results.

1. Introduction

Stochastic differential equations have been developed for more than 80 years [15, 16, 21, 24, 27–29]. Since the Japanese mathematician Kiyoshi Itô established the theory of stochastic calculus in the 1940s, the theory of stochastic differential equations has developed rapidly and has been widely used in economics, biology, physics, automation and other fields. In these applications, the dynamics of stochastic differential equations plays a very important role. Therefore, it is of great theoretical and practical significance to study and reveal the dynamics of stochastic differential equations.

The concept of almost periodic functions was first proposed by H. Bohr [8–10], a famous Danish mathematician, in 1924-1926, and has developed rapidly in the following decades, and the concept of almost periodic functions in various senses has been constantly proposed. Such as the concepts of Stepanov almost periodic, Weyl almost periodic and Besicovitch almost periodic functions [6, 12, 13]. In a sense, the concept of Besicovitch almost periodic functions is the most generalized and complex concept [4]. At the same time, since the almost periodic function theory was proposed, the existence of almost periodic equations in various senses of differential equations has become one of the important objects of qualitative research of differential equations [14, 18]. The same is true for stochastic differential equations [7, 30–32]. At present, there are many results about almost periodic solutions of stochastic differential equations. However, most

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of the results about almost periodic solutions of stochastic differential equations are about almost periodic solutions in the sense of p-th mean. It is worth mentioning that, as pointed out in [17, 25], it is more reasonable to study almost periodic solutions in distribution for stochastic differential equations. Although there are some results about almost periodic solutions of stochastic differential equations in distribution, these results are almost results in the sense of one-dimensional distribution [19, 20]. As is known to all, the one-dimensional distribution of random processes cannot reflect their behavior well. However, random processes can be completely determined by their finite-dimensional distributions. In addition, up to date, the results of Besicovitch almost periodic solutions of stochastic differential equations are few, and the results of Besicovitch almost periodic solutions in finite-dimensional distributions of stochastic differential equations in infinite dimensional Hilbert space have not been reported.

Inspired by the above discussion, the main purpose of this paper is to study the existence and uniqueness of B^p-almost periodic solutions of the following semilinear stochastic equation driven by Brownian motion in a separable Hilbert space $\mathbb{H}$:

$$dX(t) = A(t)X(t)dt + F(t, X(t))dt + G(t, X(t))d\omega(t),\ t \in \mathbb{R}, \tag{1}$$

where $A(t) : Dom(A(t)) \subset L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ is a family of densely defined closed linear operator satisfying the so-called "Acquistapace-Terreni" conditions, functions $F : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ and $G : \mathbb{R} \times \mathbb{H} \to \mathbb{L}_2^0$, where $\mathbb{L}_2^0 = \mathbb{L}_2^0(\mathbb{H}, \mathbb{H})$ is a separable Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_{\mathbb{L}_2^0}$ satisfying some additional conditions will be stated lader, and $\omega(t)$ is a two-sided standard one-dimensional Brownian motion with values in $\mathbb{H}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\mathcal{F}_t = \sigma\{\omega(s) - \omega(\tau); s, \tau \leq t\}$.

The rest of the paper is organized as follows. In Section 2, we introduce some definitions and preliminary lemmas. In Section 3, we state and prove the existence and uniqueness of B^p-almost periodic solutions in finite-dimensional distributions of system (1). In Section 4, an example is given to demonstrate our results.

2. Preliminaries

For a random variable $X : (\Omega, \mathcal{F}, P) \to \mathbb{H}$, let $law(X) := P \circ X^{-1}$ be its distribution and $E(X)$ its expectation. Denote by $\mathcal{L}^p(\Omega, \mathbb{H})$ the space of all measurable, $\mathbb{H}$-valued random variables with $E(\|X\|^p) = \int_\Omega \|X\|^p dP < \infty$.

Let $(\mathbb{E}, d)$ indicate a metric space. Consider the metric space $(\mathbb{E}^m, d_m)$, where $d_m(u, v) = \max\limits_{1\leq i\leq m}\{d(x_i, y_i)\}$ for $u = (x_1, x_2, \ldots, x_m) \in \mathbb{E}^m$. Denote by $\mathcal{P}(\mathbb{E}^m)$ the collection of Borel probability measures on $\mathbb{E}^m$. Let $BC(\mathbb{E}^m, \mathbb{R})$ stand for the space of all bounded and continuous functions from $\mathbb{E}^m$ to $\mathbb{R}$ with the norm $\|f\|_0 := \sup\limits_{u\in\mathbb{E}^m} |f(u)| < \infty$.

For $f \in BC(\mathbb{E}^m, \mathbb{R}), \zeta, \eta \in \mathcal{P}(\mathbb{E}^m)$, we define

$$\|f\|_{Lip} = \sup_{u\neq v} \frac{|f(u) - f(v)|}{d_m(u, v)}, \quad \|f\|_A = \max\{\|f\|_0, \|f\|_{Lip}\}, \quad d_B(\zeta, \eta) = \sup_{\|f\|_A\leq 1} \left| \int_{\mathbb{E}^m} f \mathrm{d}(\zeta - \eta) \right|.$$

According to [1], the space $(\mathcal{P}(\mathbb{E}^m), d_B(\cdot, \cdot))$ is a Polish space.

Definition 2.1. *[22] A stochastic process $X : \mathbb{R} \to \mathcal{L}^p(\Omega, \mathbb{H})$ is said to be $\mathcal{L}^p$-continuous if for any $s \in \mathbb{R}$,*

$$\lim_{t\to s} E\|X(t) - X(s)\|^p = 0.$$

It is $\mathcal{L}^p$-bounded if $\sup\limits_{t\in\mathbb{R}} E\|X(t)\|^p < \infty$.

The definition of the Besicovitch almost periodic stochastic process in p-th mean is as follows:

Definition 2.2. *[20] A stochastic process $X \in \mathcal{L}^p_{loc}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ is said to be Besicovitch almost periodic in p-th mean if for every $\varepsilon > 0$, there exists a positive number ℓ such that every interval of length ℓ contains a number τ such that*

$$\limsup_{l\to\infty} \left(\frac{1}{2l} \int_{-l}^{l} E\|X(t + \tau) - X(t)\|^p dt \right)^{\frac{1}{p}} < \varepsilon.$$

We give the following definition of a Besicovitch almost periodic stochastic process in p-th that depends on a parameter:

Definition 2.3. *A process $f : \mathbb{R}\times\mathbb{H} \to \mathbb{H}$ with $f(\cdot, x) \in \mathcal{L}^p_{loc}(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ is said to be Besicovitch almost periodic in p-th mean in $t \in \mathbb{R}$ uniformly with respect to $x \in \mathbb{H}$ if for every $\varepsilon > 0$ and each compact subset $\mathbb{K}$ of $\mathbb{H}$, there exists a positive number $\ell(\varepsilon, \mathbb{K})$ such that every interval of length $\ell(\varepsilon, \mathbb{K})$ contains a number τ satisfying*

$$\limsup_{l\to\infty}\left(\frac{1}{2l}\int_{-l}^{l} E\|f(t+\tau, x) - f(t, x)\|^p dt\right)^{\frac{1}{p}} < \varepsilon,\ \ t \in \mathbb{R}, x \in \mathbb{K}.$$

The following is the definition of p-th Besicovitch almost periodic stochastic processes in finite-dimensional distributions.

Definition 2.4. *A stochastic process $X : \mathbb{R} \to \mathcal{L}^p(\Omega, \mathbb{H})$ is said to be B^p-almost periodic in finite-dimensional distributions if for any $\varepsilon > 0$ and every finite points $t_1, t_2, \dots, t_m \in \mathbb{R}$, there exists an $\ell > 0$ such that every interval with length ℓ contains a τ satisfying*

$$\limsup_{l\to+\infty}\left(\frac{1}{2l}\int_{-l}^{l} d_B^p(\mathcal{D}_X(t+\tau), \mathcal{D}_X(t))dt\right)^{\frac{1}{p}} < \varepsilon,$$

where the mapping $\mathcal{D}_X : \mathbb{R} \to (\mathcal{P}(\mathbb{X}^m)$ is defined by

$$\mathcal{D}_X(t) = law(X(t+t_1), X(t+t_2), \dots, X(t+t_m)).$$

By the definition of d_B and the fact that integrals can be computed in the original domain or in the image domain, one can readily get that

Lemma 2.5. *Let $X : \mathbb{R} \to \mathcal{L}^p(\Omega, \mathbb{H})$ be a stochastic process. Then for any $\tau \in \mathbb{R}$ and every finite points $t_1, t_2, \dots, t_m \in \mathbb{R}$, we have*

$$d_B^p(\mathcal{D}_X(t+\tau), \mathcal{D}_X(t)) \le \max_{1\le i\le m}\left\{E\|X(t_i + t + \tau) - X(t_i + t)\|^p\right\},$$

where $\mathcal{D}_X(t)$ is defined in Definition 2.4.

Lemma 2.6. *[17] Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for every $t \in \mathbb{R}$,*

$$0 \le h(t) \le a + b\int_{-\infty}^{t} e^{-c(t-s)}h(s)ds,$$

where $a, b, c \ge 0$ are constants and $c > b$. Then,

$$h(t) \le a\frac{\gamma}{c-b}.$$

Lemma 2.7. *[7] (Burkholder-Davis-Gundy inequality) For arbitrary $\mathbb{L}_2^0$−valued predictable process $h(\cdot)$ and for any $p \ge 2$, one has*

$$E\left(\sup_{s\in[0,t]}\left\|\int_0^s h(s)\mathrm{d}\omega(s)\right\|^p\right) \le C_p E\left(\int_0^t \|h(s)\|^2_{\mathbb{L}_2^0}\mathrm{d}s\right)^{\frac{p}{2}}$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$.

Lemma 2.8. *[26] (Hölder's inequality) Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then*

$$\int_\Omega f(x)g(x)dx \le \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |g(x)|^q dx\right)^{\frac{1}{q}}.$$

The following condition is called Acquistapace-Terreni condition [3]:

(AT) There exist constants $\lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$ and $\alpha, \beta \in (0,1]$ with $\alpha + \beta > 1$ such that

$$\sum_{\theta} \cup \{0\} \subset \rho(A(t) - \lambda_0),\ \ \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|}$$

and

$$\|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^{\alpha}|\lambda|^{\beta}$$

for $t, s \in \mathbb{R}, \lambda \in \sum_{\theta} := \{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \leq \theta\}$.

Lemma 2.9. *[2] If condition (AT) is fulfilled, then there exists a unique evolution family $\{U(t, s) : -\infty < s \leq t < +\infty\}$ on $L^p(\Omega, \mathbb{H})$, which governs the linear part of the equation* (1)*.*

Throughout the rest of this paper, we make the following assumptions:

(H_1) $A(t) : Dom(A(t)) \subset L^p(\Omega, \mathbb{H}) \to L^p(\Omega, \mathbb{H})$ generates a uniformly exponentially stable evolution family $\{U(t, s) : -\infty < s \leq t < +\infty\}$, that is, there exist constants $M > 0$ and $\delta > 0$ such that

$$\|U(t, s)\| \leq Me^{-\delta(t-s)},\ \ -\infty < s \leq t < +\infty,$$

where $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

(H_2) For any $\varepsilon > 0$, there exists $\ell = \ell(\varepsilon) > 0$ such that every interval of length ℓ contains at least a number τ with the property that

$$\|U(t + \tau, s + \tau) - U(t, s)\| \leq \varepsilon e^{-\frac{\delta}{2}(t-s)}$$

for all $t, s \in r$ with $t \geq s$.

(H_3) For all $x, y \in \mathbb{H}$ and $t \in \mathbb{R}$, there exist constants $L_1^F, L_1^G, L_2^F, L_2^G$ such that

$$\|F(t, x)\| \leq L_1^F(1 + \|x\|),\ \ \|G(t, x)\|_{\mathbb{L}_2^0} \leq L_1^G(1 + \|x\|),$$

$$\|F(t, x) - F(t, y)\| \leq L_2^F\|x - y\|,\ \ \|G(t, x) - G(t, y)\|_{\mathbb{L}_2^0} \leq L_2^G\|x - y\|.$$

(H_4) Let

$$r^1 := 2^{p-1}\frac{1}{p\delta}\Big[M^{\frac{q+p}{q}}(L_2^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}q + 2C_pM^p(L_2^G)^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\Big] < 1,\ \ (p > 2),$$
$$r^2 := \frac{M^2}{\delta}\Big[\frac{2}{\delta}(L_2^F)^2 + (L_2^G)^2\Big] < 1,\ \ (p = 2);$$

$$\rho^1 := 6^{p-1}M\Big[M^{\frac{p}{q}}(L_2^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}} + M^{p-1}C_p(L_2^G)^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\Big] < \frac{p\delta}{2},$$
$$\rho^2 := 6M^2\Big[(L_2^F)^2\frac{1}{\delta} + (L_2^G)^2\Big] < \delta.$$

(H_5) The mappings F and G are B^p-almost periodic in p-th mean in $t \in \mathbb{R}$ uniformly with respect to x in $\mathbb{H}$.

Remark 2.10. *Condition (H_1) is a critical condition that ensures both the convergence of the integrals involved in the integral representation of the solution to system* (1) *and the contraction property of the operator Φ to be defined in the net section. Condition (H_2) requires the evolution family to satisfy a certain type of almost periodicity. Condition (H_3) imposes that the nonlinear functions F and G fulfill Lipschitz conditions and linear growth conditions. Condition (H_4) is a technical condition. Condition (H_5) guarantees that system* (1) *is a Besicovitch almost periodic system.*

3. Main results

We denote by $CB(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ the space of all bounded and continuous functions from $\mathbb{R}$ to $\mathcal{L}^p(\Omega, \mathbb{H})$. Let $\mathbb{X} = CB(\mathbb{R}, \mathcal{L}^p(\Omega, \mathbb{H}))$ with the norm $\|\phi\|_{\mathbb{X}} = \sup_{t\in\mathbb{R}}\{E\|\phi(t)\|^p\}^{\frac{1}{p}}$, where $\phi \in \mathbb{X}$. Then $\mathbb{X}$ is a Banach space.

Definition 3.1. *An $\mathcal{F}_t$-progressively measurable stochastic process $X(t)$ is called a solution of system* (1)*, if $X(t)$ satisfies the following stochastic integral equation:*

$$X(t) = U(t,a)X(a) + \int_a^t U(t,s)F(s,X(s))ds + \int_a^t U(t,s)G(s,X(s))d\omega(s)$$

for all $t \geq a$ and each $a \in \mathbb{R}$.

Theorem 3.2. *If $(H_1) - (H_5)$ and (AT) hold. Then system* (1) *has a unique B^p-almost periodic solution in finite-dimensional distributions in the space $\mathbb{X}$.*

Proof. From Definition 3.1 and assumption (H_1), letting $a \to -\infty$ yields the following stochastic integral equation

$$X(t) = \int_{-\infty}^t U(t,s)F(s,X(s))ds + \int_{-\infty}^t U(t,s)G(s,X(s))d\omega(s). \tag{2}$$

We define a nonlinear operator $\Phi : \mathbb{X} \to \mathbb{X}$ by setting

$$\begin{aligned}(\Phi\phi)(t) =& \int_{-\infty}^t U(t,s)F(s,\phi(s))ds + \int_{-\infty}^t U(t,s)G(s,\phi(s))d\omega(s)\\ :=&(\Phi_1\phi)(t) + (\Phi_2\phi)(t),\end{aligned}$$

where $\phi \in \mathbb{X}, t \in \mathbb{R}$.

Firstly, we will show that Φ is a self-mapping from $\mathbb{X}$ to $\mathbb{X}$. For any $\phi \in \mathbb{X}$, one finds

$$\begin{aligned}E\|(\Phi\phi)(t)\|^p \leq& 2^{p-1}E\Big\|\int_{-\infty}^t U(t,s)F(s,\phi(s))ds\Big\|^p + 2^{p-1}E\Big\|\int_{-\infty}^t U(t,s)G(s,\phi(s))d\omega(s)\Big\|^p\\ :=&M_1 + M_2.\end{aligned} \tag{3}$$

By the Hölder inequality, (H_1) and (H_3), one derives that

$$\begin{aligned}M_1 \leq& 2^{p-1}E\Big\{\Big[\int_{-\infty}^t \|U(t,s)\|^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^t \|U(t,s)\|^{\frac{p}{q}}\|F(s,\phi(s))\|^p ds\Big]\Big\}\\ \leq& 2^{p-1}M^{\frac{q+p}{q}}\Big[\int_{-\infty}^t e^{-\frac{q}{p}\delta(t-s)}ds\Big]^{\frac{p}{q}}E\Big[\int_{-\infty}^t e^{-\frac{p}{q}\delta(t-s)}(L_1^F)^p(1+\|\phi(s)\|)^p ds\Big]\\ \leq& 2^{p-1}M^{\frac{q+p}{q}}(L_1^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\int_{-\infty}^t e^{-\frac{p}{q}\delta(t-s)}(1+\|\phi\|_{\mathbb{X}})^p ds\\ =& 2^{p-1}M^{\frac{q+p}{q}}(L_1^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\frac{q}{p\delta}(1+\|\phi\|_{\mathbb{X}})^p < +\infty.\end{aligned} \tag{4}$$

By Lemma 2.7, (H_1) and (H_3), when $p > 2$, one gets

$$M_2 \leq 2^{p-1}C_pE\Big(\int_{-\infty}^t \|U(t,s)G(s,\phi(s))\|^2 ds\Big)^{\frac{p}{2}}$$

$$\le 2^{p-1}C_p\Big[\int_{-\infty}^{t}\|U(t,s)\|^{\frac{p}{p-2}}ds\Big]^{\frac{p-2}{p}\times\frac{p}{2}}E\Big[\int_{-\infty}^{t}\|U(t,s)\|^{\frac{p}{2}}\|G(s,\phi(s))\|_{\mathbb{L}_2^0}^{p}ds\Big]$$

$$\le 2^{p-1}C_pM^p\Big[\int_{-\infty}^{t}e^{-\frac{p}{p-2}\delta(t-s)}ds\Big]^{\frac{p-2}{2}}E\Big[\int_{-\infty}^{t}e^{-\frac{p}{2}\delta(t-s)}(L_1^G)^p(1+\|\phi(s)\|)^pds\Big]$$

$$\le 2^{p-1}C_pM^p(L_1^G)^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\frac{2}{p\delta}(1+\|\phi\|_{\mathbb{X}})^p<+\infty \tag{5}$$

and when $p=2$, since $C_2=1$, one has

$$M_2\le 2C_2E\Big[\int_{-\infty}^{t}\|U(t,s)\|^2\|G(s,\phi(s))\|_{\mathbb{L}_2^0}^2ds\Big]$$

$$\le 2M^2E\Big[\int_{-\infty}^{t}e^{-2\delta(t-s)}(L_1^G)^2(1+\|\phi(s)\|)^2ds\Big]$$

$$\le 2M^2(L_1^G)^2\frac{1}{2\delta}(1+\|\phi\|_{\mathbb{X}})^2<+\infty. \tag{6}$$

Substituting (4)–(6) into (3), we obtain $\|\Phi\|_{\mathbb{X}}=\sup_{t\in\mathbb{R}}\{E\|\Phi(t)\|^p\}^{\frac{1}{p}}<+\infty$ which implies that Φ is bounded. The continuity of $(\Phi_1\phi)(t)$ can be shown as in [5]. The continuity of $(\Phi_2\phi)(t)$ follows from Property 7.3 in [28]. Therefore, we have $\Phi(\mathbb{X})\subset\mathbb{X}$.

Secondly, we will prove that Φ is a contraction mapping. For any $\varphi,\psi\in\mathbb{X}$, similar to (4)–(5), for $p>2$, we have

$$E\|(\Phi\varphi)(t)-(\Phi\psi)(t)\|^p$$

$$\le 2^{p-1}E\Big\|\int_{-\infty}^{t}U(t,s)[F(s,\varphi(s))-F(s,\psi(s))]ds\Big\|^p+2^{p-1}E\int_{-\infty}^{t}U(t,s)[G(s,\varphi(s))-G(s,\psi(s))]d\omega(s)\Big\|^p$$

$$\le 2^{p-1}M^{\frac{q+p}{q}}\Big[\int_{-\infty}^{t}e^{-\frac{q}{p}\delta(t-s)}ds\Big]^{\frac{p}{q}}E\Big[\int_{-\infty}^{t}e^{-\frac{p}{q}\delta(t-s)}(L_2^F)^p\|\varphi(s)-\psi(s)\|^pds\Big]+2^{p-1}C_pM^p\Big[\int_{-\infty}^{t}e^{-\frac{p}{(p-2)}\delta(t-s)}ds\Big]^{\frac{p-2}{2}}$$

$$\times E\Big[\int_{-\infty}^{t}e^{-\frac{p}{2}\delta(t-s)}(L_2^G)^p\|\varphi(s)-\psi(s)\|^pds\Big]$$

$$\le 2^{p-1}\frac{1}{p\delta}\Big[M^{\frac{q+p}{q}}(L_2^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}q+2C_pM^p(L_2^G)^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\Big]\|\varphi-\psi\|_{\mathbb{X}}^p.$$

Similar to (6), for $p=2$, we obtain

$$\|\Phi\varphi-\Phi\psi\|_{\mathbb{X}}^2\le\frac{M^2}{\delta}\Big[\frac{2}{\delta}(L_2^F)^2+(L_2^G)^2\Big]\|\varphi-\psi\|_{\mathbb{X}}^2.$$

Therefore, we obtain

$$\|\Phi\varphi-\Phi\psi\|_{\mathbb{X}}\le\sqrt[p]{r^1}\|\varphi-\psi\|_{\mathbb{X}},\ (p>2),$$

$$\|\Phi\varphi-\Phi\psi\|_{\mathbb{X}}\le\sqrt{r^2}\|\varphi-\psi\|_{\mathbb{X}},\ (p=2).$$

Hence, according to (H_4), Φ is a contraction mapping. So, Φ has a unique fixed point x in $\in\mathbb{X}$, that is (1) has a unique solution x in $\mathbb{X}$.

Finally, let us show that x is a B^p-almost periodic solution in finite-dimensional distributions. Since $x\in CB(\mathbb{R},\mathcal{L}^p(\Omega,\mathbb{H}))$, we know that x is bounded. From (H_2) and (H_5), for any $\varepsilon>0$, there exists a positive number ℓ and a compact subset $\mathbb{K}$ that contains x such that every interval of length ℓ contains a number τ such that

$$\big\|F(\cdot+\tau,y)-F(\cdot,y)\big\|_{B^p}<\varepsilon\ \text{ and }\ \big\|G(t+\tau,y)-G(t,y)\big\|_{B^p}<\varepsilon \tag{7}$$

for all $y \in \mathbb{K}$ and

$$\|U(t+\tau, s+\tau) - U(t,s)\| \le \varepsilon e^{-\frac{\delta}{2}(t-s)} \tag{8}$$

for all $t, s \in r$ with $t \ge s$.

According to (2), for any $t_i \in \mathbb{R}$, we have

$$\begin{aligned}
x(t_i+t+\tau) &= \int_{-\infty}^{t_i+t+\tau} U(t_i+t+\tau, s)F(s, x(s))ds + \int_{-\infty}^{t_i+t+\tau} U(t_i+t+\tau, s)G(s, x(s))d\omega(s) \\
&= \int_{-\infty}^{t_i+t} U(t_i+t+\tau, s+\tau)F(s+\tau, x(s+\tau))ds \\
&\quad + \int_{-\infty}^{t_i+t} U(t_i+t+\tau, s+\tau)G(s+\tau, x(s+\tau))d\Big[\omega(s+\tau) - \omega(\tau)\Big],
\end{aligned}$$

where $\omega(s+\tau) - \omega(\tau)$ is a Brownian motion with the same distribution as $\omega(s)$.

We consider

$$\begin{aligned}
x(t_i+t+\tau) &= \int_{-\infty}^{t_i+t} U(t_i+t+\tau, s+\tau)F(s+\tau, x(s+\tau))ds \\
&\quad + \int_{-\infty}^{t_i+t} U(t_i+t+\tau, s+\tau)G(s+\tau, x(s+\tau))d\omega(s).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{1}{2l}\int_{-l}^{l} E\|x(t_i+t+\tau) - x(t_i+t)\|^p dt \\
=&\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} [U(t_i+t+\tau, s+\tau) - U(t_i+t, s)]F(s+\tau, x(s+\tau))ds \\
&+ \int_{-\infty}^{t_i+t} U(t_i+t, s)[F(s+\tau, x(s+\tau)) - F(s, x(s+\tau))]ds \\
&+ \int_{-\infty}^{t_i+t} U(t_i+t, s)[F(s, x(s+\tau)) - F(s, x(s))]ds \\
&+ \int_{-\infty}^{t_i+t} [U(t_i+t+\tau, s+\tau) - U(t_i+t, s)]G(s+\tau, x(s+\tau))d\omega(s) \\
&+ \int_{-\infty}^{t_i+t} U(t_i+t, s)[G(s+\tau, x(s+\tau)) - G(s, x(s+\tau))]d\omega(s) \\
&+ \int_{-\infty}^{t_i+t} U(t_i+t, s)[G(s, x(s+\tau)) - G(s, x(s))]d\omega(s)\Big\|^p dt \\
\le& 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} [U(t_i+t+\tau, s+\tau) - U(t_i+t, s)]F(s+\tau, x(s+\tau))ds\Big\|^p dt \\
&+ 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} U(t_i+t, s)[F(s+\tau, x(s+\tau)) - F(s, x(s+\tau))]ds\Big\|^p dt \\
&+ 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} U(t_i+t, s)[F(s, x(s+\tau)) - F(s, x(s))]ds\Big\|^p dt \\
&+ 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} [U(t_i+t+\tau, s+\tau) - U(t_i+t, s)]G(s+\tau, x(s+\tau))d\omega(s)\Big\|^p dt \\
&+ 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} U(t_i+t, s)[G(s+\tau, x(s+\tau)) - G(s, x(s+\tau))]d\omega(s)\Big\|^p dt
\end{aligned}$$

$$+ 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\| \int_{-\infty}^{t_i+t} U(t_i+t,s)[G(s,x(s+\tau))-G(s,x(s))]d\omega(s)\Big\|^p dt$$

$$:= \sum_{i=1}^{6} N_i. \tag{9}$$

In view of (8), (H_3) and the Hölder inequality, we obtain

$$\begin{aligned}
N_1 \le& 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\{\Big[\int_{-\infty}^{t_i+t}\|U(t_i+t+\tau,s+\tau)-U(t_i+t,s)\|^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\\
&\times\Big[\int_{-\infty}^{t_i+t}\|U(t_i+t+\tau,s+\tau)-U(t_i+t,s)\|^{\frac{p}{q}}\|F(s+\tau,x(s+\tau))\|^p ds\Big]\Big\}dt\\
\le& 6^{p-1}\varepsilon^{\frac{q+p}{q}}\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t_i+t}e^{-\frac{q}{2p}\delta(t-s)}ds\Big]^{\frac{p}{q}}E\Big[\int_{-\infty}^{t_i+t}e^{-\frac{p}{2q}\delta(t-s)}(L_1^F)^p(1+\|x(s+\tau)\|)^p ds\Big]dt\\
\le& 6^{p-1}\varepsilon^{\frac{q+p}{q}}(L_1^F)^p\Big(\frac{2p}{q\delta}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t_i+t}e^{-\frac{p}{2q}\delta(t-s)}(1+\|x\|_{\mathbb{X}})^p ds dt\\
\le& 6^{p-1}\varepsilon^{\frac{q+p}{q}}(L_1^F)^p\Big(\frac{2p}{q\delta}\Big)^{\frac{p}{q}}\frac{2q}{p\delta}(1+\|x\|_{\mathbb{X}})^p.
\end{aligned}\tag{10}$$

According to (7) and the Hölder inequality, we get

$$\begin{aligned}
N_2 \le& 6^{p-1}\frac{1}{2l}\int_{-l}^{l} E\Big\{\Big[\int_{-\infty}^{t_i+t}\|U(t_i+t,s)\|^{\frac{q}{p}}ds\Big]^{\frac{p}{q}}\Big[\int_{-\infty}^{t_i+t}\|U(t_i+t,s)\|^{\frac{p}{q}}\|F(s+\tau,x(s+\tau))-F(s,x(s+\tau))\|^p ds\Big]\Big\}dt\\
\le& 6^{p-1}M^{\frac{q+p}{q}}\frac{1}{2l}\int_{-l}^{l}\Big[\int_{-\infty}^{t_i+t}e^{-\frac{q}{p}\delta(t-s)}ds\Big]^{\frac{p}{q}}\int_{-\infty}^{t_i+t}e^{-\frac{p}{q}\delta(t-s)}E\|F(s+\tau,x(s+\tau))-F(s,x(s+\tau))\|^p ds dt\\
\le& 6^{p-1}M^{\frac{q+p}{q}}\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\int_{-\infty}^{0}e^{\frac{p}{q}\delta s}\frac{1}{2l}\int_{-l}^{l}E\|F(t_i+t+s+\tau,x(t_i+t+s+\tau))-F(t_i+t+s,x(t_i+t+s+\tau))\|^p dt ds\\
\le& 6^{p-1}M^{\frac{q+p}{q}}\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\int_{-\infty}^{0}e^{\frac{p}{q}\delta s}\frac{1}{2l}\int_{s-l}^{s+l}E\|F(t_i+t+\tau,x(t_i+t+\tau))-F(t_i+t,x(t_i+t+\tau))\|^p dt ds\\
\le& 6^{p-1}M^{\frac{q+p}{q}}\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\int_{-\infty}^{0}e^{\frac{p}{q}\delta s}\frac{2(|s|+l)}{2l}\frac{1}{2(|s|+l)}\\
&\times\int_{-|s|-l}^{|s|+l}E\|F(t_i+t+\tau,x(t_i+t+\tau))-F(t_i+t,x(t_i+t+\tau))\|^p dt ds\\
\le& 6^{p-1}M^{\frac{q+p}{q}}\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\frac{q}{p\delta}\varepsilon^p.
\end{aligned}\tag{11}$$

Similarly, we can obtain

$$N_5 \le 6^{p-1}M^p\Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\frac{2}{p\delta}\varepsilon^p,\ \ (p>2), \tag{12}$$

$$N_5 \le \frac{3M^2}{\delta}\varepsilon^2,\ \ (p=2). \tag{13}$$

Similar to (4), one can get

$$N_3 \le 6^{p-1}M^{\frac{q+p}{q}}(L_2^F)^p\Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}}\frac{1}{2l}\int_{-l}^{l}\int_{-\infty}^{t_i+t}e^{-\frac{p\delta}{q}(t-s)}E\|x(s+\tau)-x(s)\|^p ds dt.$$

By a change of variables and Fubini's theorem, we have

$$N_3 \leq 6^{p-1} M^{\frac{q+p}{q}} (L_2^F)^p \Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}} \int_{-\infty}^{l} e^{-\frac{p\delta}{q}(l-s)} \Big(\frac{1}{2l}\int_{s-2l}^{s} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt\Big) ds. \tag{14}$$

Likewise, when $p > 2$, we can get

$$N_6 \leq 6^{p-1} M^p C_p (L_2^G)^p \Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}} \int_{-\infty}^{l} e^{-\frac{p\delta}{2}(l-s)} \Big(\frac{1}{2l}\int_{s-2l}^{s} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt\Big) ds, \tag{15}$$

and when $p = 2$, we have

$$N_6 \leq 6M^2 (L_2^G)^2 \int_{-\infty}^{l} e^{-2\delta(l-s)} \Big(\frac{1}{2l}\int_{s-2l}^{s} E\|x(t_i+t+\tau)-x(t_i+t)\|^2 dt\Big) ds. \tag{16}$$

By Lemma 2.7, (8), (H_3) and the Hölder inequality, we have

$$\begin{aligned} N_4 \leq & 6^{p-1} C_p \frac{1}{2l}\int_{-l}^{l} E\Big[\int_{-\infty}^{t_i+t} \|[U(t_i+t+\tau, s+\tau) - U(t_i+t, s)]G(s+\tau, x(s+\tau))\|^2 ds\Big]^{\frac{p}{2}} dt, \\ & \leq 6^{p-1} C_p \varepsilon^p (L_1^G)^p \Big(\frac{2(p-2)}{p\delta}\Big)^{\frac{p-2}{2}} \frac{4}{p\delta}(1+\|x\|_{\mathbb{X}})^p, \ (p>2) \end{aligned} \tag{17}$$

and

$$\begin{aligned} N_4 \leq & 6C_2 \frac{1}{2l}\int_{-l}^{l} E\Big[\int_{-\infty}^{t_i+t} \|U(t_i+t+\tau, s+\tau) - U(t_i+t, s)\|^2 \|G(s+\tau, x(s+\tau))\|_{\mathbb{L}_2^0}^2 ds\Big] dt \\ \leq & 6\varepsilon^2 \frac{1}{2l}\int_{-l}^{l} E\Big[\int_{-\infty}^{t_i+t} e^{-\delta(t-s)} (L_1^G)^2 (1+\|x(s+\tau)\|)^2 ds\Big] dt \\ \leq & \frac{6}{\delta}\varepsilon^2 (L_1^G)^2 (1+\|x\|_{\mathbb{X}})^2, \ (p=2). \end{aligned} \tag{18}$$

Substituting (10)–(18) into (9), when $p > 2$, we obtain

$$\frac{1}{2l}\int_{-l}^{l} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt \leq \Delta^1 \varepsilon + \rho^1 \int_{-\infty}^{l} e^{-\frac{p\delta}{2}(l-s)} \Big(\frac{1}{2l}\int_{s-2l}^{s} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt\Big) ds,$$

where

$$\begin{aligned} \Delta^1 = & 6^{p-1}\varepsilon \frac{2}{p\delta}\Big[\varepsilon^{\frac{p}{q}} (L_1^F)^p \Big(\frac{2p}{q\delta}\Big)^{\frac{p}{q}} q + 2C_p \varepsilon^{p-1} (L_1^G)^p \Big(\frac{2(p-2)}{p\delta}\Big)^{\frac{p-2}{2}}\Big](1+\|x\|_{\mathbb{X}})^p \\ & + 6^{p-1}\varepsilon^p \frac{1}{p\delta}\Big[M^{\frac{q+p}{q}} \Big(\frac{p}{q\delta}\Big)^{\frac{p}{q}} q + 2M^p \Big(\frac{p-2}{p\delta}\Big)^{\frac{p-2}{2}}\Big]. \end{aligned}$$

By (H_4), we know $\rho^1 < \frac{p\delta}{2}$. Thus, we conclude by Lemma 2.6 that

$$\frac{1}{2l}\int_{-l}^{l} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt < \Delta^1 \varepsilon \frac{p\delta}{p\delta - 2\rho^1}.$$

Hence, according to Lemma 2.5, we have

$$\frac{1}{2l}\int_{-l}^{l} d_B^p(\mathcal{D}_x(t+\tau), \mathcal{D}_x(t)) dt \leq \max_{1\leq i\leq m}\Big\{\frac{1}{2l}\int_{-l}^{l} E\|x(t_i+t+\tau)-x(t_i+t)\|^p dt\Big\} < \Delta^1 \epsilon \frac{p\delta}{p\delta - 2\rho^1}. \tag{19}$$

When $p = q = 2$, we get

$$\frac{1}{2l}\int_{-l}^{l} E\|x(t_i + t + \tau) - x(t_i + t)\|^2 dt \le \Delta^2\varepsilon + \rho^2 \int_{-\infty}^{l} e^{-\delta(l-s)}\Big(\frac{1}{2l}\int_{s-2l}^{s} E\|x(t_i + t + \tau) - x(t_i + t)\|^2 dt\Big)ds,$$

where

$$\Delta^2 = \frac{6}{\delta}\varepsilon^2\Big(\frac{4}{\delta}(L_1^F)^2 + (L_1^G)^2\Big)(1 + \|x\|_{\mathbb{X}})^2 + \frac{3M^2}{\delta}\varepsilon^2\Big(\frac{2}{\delta} + 1\Big).$$

By (H_4), we know $\rho^2 < \delta$. Thus, we conclude by Lemma 2.6 that

$$\frac{1}{2l}\int_{-l}^{l} E\|x(t_i + t + \tau) - x(t_i + t)\|^2 dt < \Delta^2\varepsilon\frac{\delta}{\delta - \rho^2}.$$

Hence,

$$\frac{1}{2l}\int_{-l}^{l} d_B^2(\mathcal{D}_x(t+\tau), \mathcal{D}_x(t))dt \le \max_{1\le i\le m}\Big\{\frac{1}{2l}\int_{-l}^{l} E\|x(t_i + t + \tau) - x(t_i + t)\|^2 dt\Big\} < \Delta^2\epsilon\frac{\delta}{\delta - \rho^2}. \tag{20}$$

From (19) and (20) it follows that $x(t)$ is B^p-almost periodic in infinite-dimensional distributions. The proof is completed.

□

4. Example

Example 4.1. *We consider the following stochastic differential equation:*

$$\begin{cases} \frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2}{\partial \xi^2}u(t,\xi) - 2u(t,\xi) + \frac{1}{2}(\sin t + \cos\sqrt{3})u(t,\xi) \\ \qquad +\frac{1}{4}\left[\cos t + \sin\sqrt{3}t + \frac{2}{1+t^2}\right]\sin u(t,\xi) \\ \qquad +\frac{1}{3}[\cos u(t,x) + \sin 5t]\frac{d\omega(t)}{dt},\ t \in \mathbb{R},\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}. \end{cases} \tag{21}$$

Take $\mathbb{H} = L^2[0,\pi]$ with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)_2$. Define operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by setting

$$Ax = \frac{\partial^2 x(\xi)}{\partial \xi^2} - 2x$$

with domain

$$D(A) = \{x(\cdot) \in \mathbb{H} : x'' \in \mathbb{H}, x' \in \mathbb{H} \text{ are absolutely continuous on } [0,\pi],\ x(0) = x(\pi) = 0\}.$$

According to [23], we know that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$ on $\mathbb{H}$ satisfying

$$\|T(t)\| \le e^{-3t} \quad \text{for} \quad t > 0.$$

In addition,

$$T(t)x = \sum_{n=1}^{+\infty} e^{(-n^2+2)t}(x, y_n)_2\, y_n,\ t \ge 0,\ x \in \mathbb{H},$$

where $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Define a family of linear operators $A(t) : D(A(t)) = D(A) \to \mathbb{H}$ by

$$A(t)x(\xi) = \Big(A + \frac{1}{2}(\sin t + \cos\sqrt{3}t)\Big)x(\xi),\ \forall\xi \in [0,\pi],\ x \in D(A).$$

Then, the system

$$\begin{cases} x'(t) = A(t)x(t),\ t > s, \\ x(s) = x \in \mathbb{H} \end{cases}$$

has an associated evolution family $\{U(t,s)\}_{t\geq s}$ on $\mathbb{H}$, which can be explicitly written as

$$U(t,s)x = \Big(T(t-s)e^{\int_s^t \frac{1}{2}(\sin\zeta+\cos\sqrt{3}\zeta)d\zeta}\Big)x.$$

Obviously, for any $\tau \in \mathbb{R}$, one has

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq e^{-(t-s)}\Big|\frac{1}{2}(\sin(\mu+\tau) + \cos\sqrt{3}(\mu+\tau)) - \frac{1}{2}(\sin\mu + \cos\sqrt{3}\mu)\Big|,$$

where $s \leq \mu \leq t$. Since the function $\frac{1}{2}(\sin t + \cos\sqrt{3}t)$ is almost periodic, (H_2) is fulfilled. Moreover,

$$\|U(t,s)\| \leq e^{-2(t-s)} \quad \text{for} \quad t \geq s.$$

Hence, condition (H_1) is verified. In addition, in virtue of [11], we see that $A(t)$ satisfies condition (AT).

Let

$$F(t, x(\xi)) = \frac{1}{4}\Big[\cos t + \sin\sqrt{3}t + \frac{2}{1+t^2}\Big]\sin u(t,\xi)$$

and

$$G(t, x(\xi)) = \frac{1}{3}[\cos u(t,x) + \sin 5t].$$

Then (21) can be transformed into the abstract equation (1).

Moreover, for $p = 2$, it is easy to see that (H_3)-(H_5) are also satisfied with $M = 1$, $\delta = 2$, $L_1^F = L_2^F = 1$, $L_1^G = L_2^G = \frac{1}{3}$.

Therefore, by Theorem 3.2, system (21) has a unique B^2-almost periodic solution in finite-dimensional distributions.

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