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Exploring variable order fractional initial value problems: Existence results and weak topology insights

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Abstract. The aim of this paper is to fill an absence in the study of initial value problems with weak topology that use variable order fractional calculus. Using Krasnoselskii's type fixed point theorem under weak topology, we construct suitable conditions to assure the existence of weak solutions for various types of initial value problems involving variable order Riemann-Liouville fractional derivative. Some examples are provided to illustrate the results.

1. Introduction

Fractional calculus is an area of mathematical analysis that deals with non-integer order derivatives and integrals. Its origins can be traced back to pioneers such as Leibniz, Euler, and Liouville, but it gained major prominence in the late nineteenth and early twentieth centuries, thanks to contributions by mathematicians such as Riemann, Liouville, and Grünwald. Traditional fractional calculus uses fixed-order operators, which remain constant throughout the process, and was largely utilized as a theoretical tool with applications in physics, engineering, and biology. However, as our understanding of complex systems progressed, it became clear that fixed-order fractional calculus had limits in adequately describing some processes.

This recognition led to the evolution of variable-order fractional calculus, allowing the order of differentiation or integration operators to vary with respect to time, such as those of the Grünwald-Letnikov, Erdélyi-Kober, Riesz, Riemann-Liouville, Caputo, Hadamard, and Hilfer types [1, 2]. This development has been driven by the need to address real-world complexities more effectively, particularly in fields such as finance, signal processing, and control theory. By allowing the order of differentiation or integration to vary, researchers can capture intricate dynamics and phenomena that were previously challenging to model accurately. We point out a few papers [22–24].

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Likewise, the concept of weak topology is essential in functional analysis, offering a framework for understanding convergence and continuity in function spaces. Weak topology enables a more adaptable and abstract method for exploring mathematical structures (see [11, 13, 14, 16]). Combining variable-order fractional calculus with weak topology opens up a viable path for investigating the nuanced behavior of fractional operators in more realistic and dynamic mathematical settings.

Fractional Volterra and Fredholm integral equations are extensions of classical integral equations that use fractional derivatives to model memory and hereditary characteristics in many physical and engineering systems. The Volterra type usually involves integration over a variable upper limit, whereas Fredholm equations integrate over a set range. These equations can often be difficult to solve analytically; hence, approaches like fixed-point theory and numerical approximations are commonly used (for more details, see [5, 7, 9, 21]).

In [25], Zhang et al. provided a totally new notion of a continuous approximation solution to the initial value problem for variable-order differential equations with a variable-order Caputo fractional derivative in finite intervals:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\psi(t)}x(t) = F(t, x(t)), \quad t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where $0 < \psi(t) < 1$ is the variable-order function, $x_0 \in \mathbb{R}$, $F(t, x(t))$ is a given real-valued function, and ${}^{C}\mathcal{D}_{0^+}^{\psi(t)}$ denotes the variable-order Caputo fractional derivative.

In this paper, we are interested in the quantitative theory for the following initial value problem involving the Riemann-Liouville fractional derivative of variable order:

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\psi(t)}x(t) = F(t, x(t)), \qquad 0 \le t \le T < +\infty, \\ I_{0^+}^{1-\psi(t)}x(t)\Big|_{t=0} = x_0. \end{cases} \tag{IVP}$$

where $x_0 \in E$, $x \in L^1([0, T], E)$, $\psi : [0, T] \longrightarrow (0, 1]$ is a measurable function, and ${}^{RL}\mathcal{D}_{0^+}^{\psi(t)}$, $I_{0^+}^{1-\psi(t)}$ are the left-hand Riemann-Liouville derivative and integral of variable orders $\psi(t), 1 - \psi(t)$, respectively. The function F is nonlinear.

Several researchers have studied the existence, uniqueness, and stability of solutions to fractional differential equations using fixed-point theory with weak topology in various works for integral equations (see [3, 6, 10, 12] and the references therein). However, to the best of our knowledge, the application of fixed-point theorems under weak topology in the study of fractional differential equations of variable order has yet to be sufficiently extended.

By combining the theory of fixed points under weak topology with the De Blasi measure of weak noncompactness and the theory of variable-order fractional calculus, we offer sufficient conditions on the function F to establish that (IVP) has at least one integrable solution.

For this purpose, in Section 2, we provide some early notions and lemmas based on variable-order fractional calculus theory and weak topology. In Section 3, by using some important definitions and lemmas of fractional integral and derivative, we convert (IVP) into a Volterra-type integral equation. Following that, we offer our main results, which are based on a variation of Krasnoselskii's fixed-point theorem, and we end the paper with some examples to illustrate the obtained outcomes.

2. Preliminaries

This section introduces some essential concepts and lemmas that will be required for reaching our results in the next sections.

Definition 2.1. *Let $[0, T]$, $T > 0$, be a subset of $\mathbb{R}$, we denote by $L^1([0, T], E)$ the space of measurable functions $x : [0, T] \longrightarrow E$ which are Bochner integrable on $[0, T]$ with values in a finite dimensional Banach space $(E, \|\cdot\|)$,* *equipped with the norm*

$$\|x\|_{L^1} = \int_0^T \|x(s)\|\, ds.$$

Note that $L^1_+([0,T],E)$ stands for the positive cone of the space $L^1([0,T],E)$.

We denote by $\mathcal{D}(\mathcal{I})$ the domain of an operator $\mathcal{I}$, and $\mathcal{M}_E$ is the collection of all nonempty bounded subsets of E, while $\mathcal{W}_E$ stands for its subfamily consisting of all relatively weakly compact sets. Moreover, the symbol $\overline{\Omega}^{\,\omega}$ will stand for the weak closure of Ω.

Definition 2.2. *[2] For $-\infty < a < b < +\infty$, we consider the mapping $\psi : [a,b] \longrightarrow (0,+\infty)$. Then, the left hand Riemann-Liouville fractional integral of variable order $p(t)$ for function x is defined by*

$$I_{a^+}^{\psi(t)} x(t) = \int_a^t \frac{(t-s)^{\psi(t)-1}}{\Gamma(\psi(t))} x(s)ds, \quad t > a. \tag{1}$$

Definition 2.3. *[24] For $-\infty < a < b < +\infty$, we consider the mapping $\psi : [a,b] \longrightarrow (0,1)$. Then, the left hand Riemann-Liouville fractional derivative of variable order $\psi(t)$ for function x is defined by*

$${}^{RL}\mathcal{D}_{a^+}^{\psi(t)} x(t) = \left(\frac{d}{dt}\right) I_{a^+}^{1-\psi(t)} x(t) = \left(\frac{d}{dt}\right) \int_a^t \frac{(t-s)^{-\psi(t)}}{\Gamma(1-\psi(t))} x(s)ds, \quad t > a. \tag{2}$$

It is generally known that when the order is merely a constant, the Riemann-Liouville fractional integral and derivative are precisely the same as the constant order fractional integral and derivative. Thus, the semi-group property produces the following properties.

$$\begin{aligned} I_{0^+}^{\psi_1} I_{0^+}^{\psi_2} &= I_{0^+}^{\psi_2} I_{0^+}^{\psi_1} \\ &= I_{0^+}^{\psi_1+\psi_2}. \end{aligned}$$

With these qualities, the fractional order differential equation can be converted into an equivalent integral equation, allowing various fixed point theorems to be applied, establishing the solution's existence and uniqueness. However, further investigations have shown that similar features fail to apply for variable order fractional operators, indicating a different scenario than its constant order counterpart

$$\begin{aligned} I_{0^+}^{\psi_1(t)} I_{0^+}^{\psi_2(t)} &\neq I_{0^+}^{\psi_2(t)} I_{0^+}^{\psi_1(t)} \\ &\neq I_{0^+}^{\psi_1(t)+\psi_2(t)}, \end{aligned}$$

where $\psi_1(t)$ and $\psi_2(t)$ are general non negative functions. We shall give an example to prove these claimed arguments.

Example 2.4. *Let $\psi_1(t) = t + 1$, $\psi_2(t) = 1 - t$, $x(t) = t$, $0 \le t \le 1$.*

$$
\begin{aligned}
I_{0^+}^{\psi_1(t)} I_{0^+}^{\psi_2(t)} x(t) &= \int_0^t \frac{(t-s)^{t+1-1}}{\Gamma(t+1)} \left(\int_0^s \frac{(s-h)^{1-s-1}}{\Gamma(1-s)} x(h) dh \right) ds \\
&= \int_0^t \frac{(t-s)^t s^{2-s}}{\Gamma(t+1)\Gamma(3-s)} ds \\
I_{0^+}^{\psi_1(t)} I_{0^+}^{\psi_2(t)} x(t) \Big|_{t=\frac{1}{2}} &= \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{\frac{1}{2}} s^{2-s}}{\Gamma\left(\frac{3}{2}\right)\Gamma(3-s)} ds \\
&\approx 0.01436. \\
I_{0^+}^{\psi_2(t)} I_{0^+}^{\psi_1(t)} x(t) &= \int_0^t \frac{(t-s)^{1-t-1}}{\Gamma(1-t)} \left(\int_0^s \frac{(s-h)^{s+1-1}}{\Gamma(s+1)} x(h)\, dh \right) ds \\
&= \int_0^t \frac{(t-s)^{-t} s^{2+s}}{\Gamma(1-t)\Gamma(3+s)} ds \\
I_{0^+}^{\psi_2(t)} I_{0^+}^{\psi_1(t)} x(t) \Big|_{t=\frac{1}{2}} &= \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{-\frac{1}{2}} s^{2+s}}{\Gamma\left(\frac{1}{2}\right)\Gamma(3+s)} ds \\
&\approx 0.02437. \\
I_{0^+}^{\psi_1(t)+\psi_2(t)} x(t) \Big|_{t=\frac{1}{2}} &= \frac{1}{\Gamma(2)} \int_0^{\frac{1}{2}} (\frac{1}{2} - s) s\, ds \\
&\approx 0.02083.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_{0^+}^{\psi_1(t)} I_{0^+}^{\psi_2(t)} x(t) \Big|_{t=\frac{1}{2}} &\neq I_{0^+}^{\psi_2(t)} I_{0^+}^{\psi_1(t)} x(t) \Big|_{t=\frac{1}{2}} \\
&\neq I_{0^+}^{\psi_1(t)+\psi_2(t)} x(t) \Big|_{t=\frac{1}{2}}.
\end{aligned}
$$

Definition 2.5. *[8] The De Blasi measure of weak noncompactness $\omega : \mathcal{M}_E \longrightarrow \mathbb{R}^+$ is defined in the following way*

$$\omega(\Omega) = \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}_E \text{ such that } \Omega \subset W + B_r\}. \tag{3}$$

Proposition 2.6. *[4] Let I be a compact subset of $\mathbb{R}$, and let Ω be a bounded subset of $L^1(I, E)$ where E is a finite dimensional Banach space. Then, ω possesses the following form*

$$\omega(\Omega) = \lim_{\varepsilon \longrightarrow 0} \left\{ \sup_{\varphi \in \Omega} \left\{ \int_B \|\varphi(t)\| dt : \text{ meas}(B) \le \varepsilon \right\} \right\},$$

for any nonempty subset $B \subset \Omega$, where meas$(\cdot)$ denotes the Lebesgue measure.

We recall some basic properties of the measure of weak noncompactness.

Lemma 2.7. *[13] Let Ω_1, Ω_2 be two elements of $\mathcal{M}_E$. Then, the following properties are valid*

1) $\Omega_1 \subseteq \Omega_2$ *implies* $\omega(\Omega_1) \le \omega(\Omega_2)$.
2) $\omega(\Omega_1) = 0$ *if and only if,* $\overline{\Omega_1}^{\omega} \in \mathcal{W}_E$.
3) $\omega(\overline{\Omega_1}^{\omega}) = \omega(\Omega_1)$.
4) $\omega(\Omega_1 \cup \Omega_2) = \max\{\omega(\Omega_1), \omega(\Omega_2)\}$.

5) $\omega(\lambda\Omega_1) = |\lambda|\omega(\Omega_1)$, *for all* $\lambda \in \mathbb{R}$.
6) $\omega(conv(\Omega_1)) = \omega(\Omega_1)$.
7) $\omega(\Omega_1 + \Omega_2) \leq \omega(\Omega_1) + \omega(\Omega_2)$.
8) *if* $(\Omega_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of E with* $\lim_{n\to+\infty} \omega(\Omega_n) = 0$, *then* $\Omega_\infty := \bigcap_{n=1}^{+\infty} \Omega_n$ *is nonempty and belongs to* $\mathcal{W}_E$.

Definition 2.8. *Let* $I \subset \mathbb{R}^n$ *and let* E, F *be two Banach spaces. A function* $F : I \times E \longrightarrow F$ *is said to be Carathéodory if*

i) *the map* $t \mapsto F(t, x(t))$ *is measurable from I to F for any* $x \in E$, *and;*
ii) *the map* $x \mapsto F(t, x(t))$ *is continuous from E to F for all* $t \in I$.

Let $M(I, E)$ be the set of all measurable functions $x : I \times E \longrightarrow E$. If F is a Carathéodory function, then F defines a mapping $\mathcal{N}_F : M(I, E) \longrightarrow M(I, E)$ by $\mathcal{N}_F x(t) := F(t, x(t))$, for all $t \in I$. This mapping is called the Nemytskii's operator associated to F.

Lemma 2.9. *[17] Let* $I \in \mathbb{R}^n$ *and a Carathéodory function F, and* $p, q \geq 1$. *Nemytskii's operator defined on* $L^p(I, E)$ *with values in* $L^q(I, E)$ *is bounded and continuous. Moreover,* $\mathcal{N}$ *maps all of* $L^p(I, E)$ *into* $L^q(I, E)$ *if and only if the function F satisfies the following condition*

$$\begin{cases} \|F(t, x(t))\| \leq H(t) + C\|x\|^{p/q} \quad \text{with } H \in L^q,\ C > 0, & q < +\infty, \\ \|F(t, x(t))\| \leq C, & q = +\infty. \end{cases}$$

Obviously, we have

$$\|\mathcal{N}_F x\|_{L^1} \leq \|H\|_{L^1} + C\|x\|_{L^1}, \quad \forall x \in L^1(I, E).$$

Lemma 2.10. *[15] Let* $\psi > 0,\ 0 < a < b,\ x \in L^1(a, b),\ {}^{RL}\mathcal{D}^{\psi}_{a^+} x \in L^1(a, b)$. *Then,*

$$I^{\psi}_{a^+}\ {}^{RL}\mathcal{D}^{\psi}_{a^+} x(t) = x(t) + \varrho_0 + \varrho_1(t-a)^{\psi-1} + \varrho_2(t-a)^{\psi-2} + \cdots + \varrho_n(t-a)^{\psi-n},$$

with $n = [\psi] + 1,\ \varrho_k \in \mathbb{R},\ k \in \{0, 1, \ldots, n\}$.

We recall the following conditions $(C1)$, $(C2)$ that were considered in [12], and for some applications satisfying these conditions, see the monograph [13].

$$(C1) \quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{I}) \text{ is a weakly convergent sequence in } E, \\ \text{then } (\mathcal{I}x_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } E. \end{cases}$$

$$(C2) \quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(\mathcal{J}) \text{ is a weakly convergent sequence in } E, \\ \text{then } (\mathcal{J}x_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } E. \end{cases}$$

The following variant of fixed point theorem will play a fundamental role in our results.

Theorem 2.11. *[16] Let* $\Omega \subset E$ *be a nonempty bounded closed convex subset. Suppose that* $\mathcal{I} : \Omega \longrightarrow E$, *and* $\mathcal{J} : \Omega \longrightarrow E$ *are two operators such that*

i) $\mathcal{I}\Omega + \mathcal{J}\Omega \subseteq \Omega$,
ii) $\mathcal{I}$ *is continuous and satisfies* $(C1)$,
iii) $\mathcal{J}$ *is a strict contraction and sastisfies* $(C2)$,
iv) *there exists* $\gamma \in [0, 1)$ *such that* $\omega(\mathcal{I}S + \mathcal{J}S) \leq \gamma\omega(S)$ *for all* $S \subseteq \Omega$.

Then, there exists $u \in \Omega$ *such that* $\mathcal{I}u + \mathcal{J}u = u$.

3. Existence of solution

Based on the prior discussions, we give our main results in this section.

We begin by examining the problem (IVP) using the information provided above. Let

$$\psi_1 = \psi(0). \tag{4}$$

Hence, we consider the initial value problem defined in the interval $[0, T]$ as follows:

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\psi_1} x(t) = F(t, x(t)), \qquad 0 \le t \le T, \\ I_{0^+}^{1-\psi_1} x(t)\Big|_{t=0} = x_0. \end{cases} \tag{5}$$

Let $x_1 \in L^1([0, T], E)$ be a solution of the initial value problem (5). (By convention, we know that the initial value problem (5) has a measurable solution under some assumptions on the nonlinear term).

Since x_1 is measurable, then for an arbitrary ε_1, there exists $\delta_1^x > 0$ such that

$$\|x_1(t) - I_{0^+}^{1-\psi_1} x_1(0)\| = \|x_1(t) - x_0\| < \varepsilon_1, \quad \text{for } 0 < t \le \delta_1^x. \tag{6}$$

And because $\psi(t)$ is measurable at 0, then together with (4) and (6), for the above ε_1, there exists $\delta_1^{\psi} > 0$ such that

$$|\psi(t) - \psi(0)| = |\psi(t) - \psi_1| < \varepsilon_1, \quad \text{for } 0 < t \le \delta_1^{\psi}.$$

If $\min(\delta_1^x, \delta_1^{\psi}) < T$, then we take $T_1 = \min(\delta_1^x, \delta_1^{\psi})$ and repeat the same procedure. Otherwise, we take $T_1 = T$, and we stop here.

In order to consider the existence of a solution to (IVP) in $[T_1, T]$, we take

$$\psi_2 = \psi(T_1). \tag{7}$$

And since fractional operators are nonlocal, we have

$$\begin{aligned} {}^{RL}\mathcal{D}_{0^+}^{\psi_2} x(t) &= \left(\frac{d}{dt}\right) \int_0^t \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x(s)\, ds \\ &= \left(\frac{d}{dt}\right) \left[\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x(s)\, ds + \int_{T_1}^t \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x(s)\, ds \right] \\ &= \left(\frac{d}{dt}\right) \int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x_1(s)\, ds + \left(\frac{d}{dt}\right) \int_{T_1}^t \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x(s)\, ds \\ &= {}^{RL}\mathcal{D}_{T_1^+}^{\psi_2} x(t) + \Lambda_{\psi_2}(t, x_1(t)). \end{aligned}$$

Therefore, we consider the following initial value problem defined in the interval $[T_1, T]$:

$$\begin{cases} {}^{RL}\mathcal{D}_{T_1^+}^{\psi_2} x(t) = F(t, x(t)) - \Lambda_{\psi_2}(t, x_1(t)), \qquad T_1 \le t \le T, \\ I_{T_1^+}^{1-\psi_2} x(t)\Big|_{t=T_1} = x_1(T_1). \end{cases} \tag{8}$$

Let $x_2 \in L^1([T_1, T], E)$ be a solution of the initial value problem (8). (By convention, we know that the initial value problem (8) exists measurable solution under some assumptions on nonlinear term).

Since x_2 is measurabe, then for an arbitrary ε_2, there exists $\delta_2^x > 0$ such that

$$\|x_2(t) - I_{T_1^+}^{1-\psi_2} x_2(T_1)\| = \|x_2(t) - x_1(T_1)\| < \varepsilon_2, \quad \text{for } T_1 < t \le \delta_2^x. \tag{9}$$

And because $\psi(t)$ is measurable at T_1, then together with (7) and (9), for the above ε_2, there exists $\delta_2^{\psi} > 0$ such that

$$|\psi(t) - \psi(T_1)| < \varepsilon_2, \quad \text{for } T_1 < t \leq \delta_2^{\psi}.$$

If $\min(\delta_2^x, \delta_2^{\psi}) + T_1 < T$, then we take $T_2 = \min(\delta_2^x, \delta_2^{\psi}) + T_1$ and repeat the same procedure, otherwise we take $T_2 = T$, and we stop here.

In order to consider the existence of solution to (IVP) in $[T_2, T]$, we take

$$\psi_3 = \psi(T_2). \tag{10}$$

And since fractional operators are nonlocal, we have

$$\begin{aligned}
{}^{RL}\mathcal{D}_{0^+}^{\psi_3} x(t) &= \left(\frac{d}{dt}\right) \int_0^t \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds \\
&= \left(\frac{d}{dt}\right) \left[\int_0^{T_1} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds + \int_{T_1}^{T_2} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds + \int_{T_2}^{t} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds \right] \\
&= \left(\frac{d}{dt}\right) \int_0^{T_1} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds + \left(\frac{d}{dt}\right) \int_{T_1}^{T_2} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds + \left(\frac{d}{dt}\right) \int_{T_2}^{t} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds \\
&= \left(\frac{d}{dt}\right) \int_0^{T_1} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x_1(s) ds + \left(\frac{d}{dt}\right) \int_{T_1}^{T_2} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x_2(s) ds + \left(\frac{d}{dt}\right) \int_{T_2}^{t} \frac{(t-s)^{-\psi_3}}{\Gamma(1-\psi_3)} x(s) ds \\
&= {}^{RL}\mathcal{D}_{T_2^+}^{\psi_3} x(t) + \Lambda_{\psi_3}(t, x_1(t)) + \Lambda_{\psi_3}(t, x_2(t)).
\end{aligned}$$

Therefore, we consider the following initial value problem defined in the interval $[T_2, T]$:

$$\begin{cases} {}^{RL}\mathcal{D}_{T_2^+}^{\psi_3} x(t) = F(t, x(t)) - \Lambda_{\psi_3}(t, x_1(t)) - \Lambda_{\psi_3}(t, x_2(t)), & T_2 \leq t \leq T, \\ I_{T_2^+}^{1-\psi_3} x(t)\Big|_{t=T_2} = x_2(T_2). \end{cases} \tag{11}$$

Let $x_3 \in L^1([T_2, T], E)$ be a solution of the initial value problem (11). (By convention, we know that the initial value problem (11) exists measurable solution under some assumptions on nonlinear term).

Since x_3 is measurabe, then for an arbitrary ε_3, there exists $\delta_3^x > 0$ such that

$$\|x_3(t) - I_{T_2^+}^{1-\psi_3} x_3(T_2)\| = \|x_3(t) - x_2(T_2)\| < \varepsilon_3, \quad \text{for } T_2 < t \leq \delta_3^x. \tag{12}$$

And because $\psi(t)$ is measurable at T_2, then together with (10) and (12), for the above ε_3, there exists $\delta_3^{\psi} > 0$ such that

$$|\psi(t) - \psi(T_2)| < \varepsilon_3, \quad \text{for } T_2 < t \leq \delta_3^{\psi}.$$

If $\min(\delta_3^x, \delta_3^{\psi}) + T_2 < T$, then we take $T_3 = \min(\delta_3^x, \delta_3^{\psi}) + T_2$ and repeat the same procedure, otherwise we take $T_3 = T$, and we stop here.

We continue this procedure and we obtain the following general case. That is, there exists $n \in \mathbb{N}$, such that for each $i \in \{1, 2, \dots, n\}$, we have $T_{i-2} + \min(\delta_{i-1}^x, \delta_{i-1}^{\psi}) = T_{i-1} < T$. Thus, we obtain intervals $[0, T_1], [T_1, T_2], \dots, [T_{i-1}, T]$, and solutions $x_i \in L^1([T_{i-1}, T], E)$ of the following constant order fractional initial value problem (For convenience, let $T_0 = 0,\ x_0(T_0) = x_0,\ \tilde{\Lambda}(t, x_0(t)) = 0$).

$$\begin{cases} {}^{RL}\mathcal{D}_{T_{i-1}^+}^{\psi_i} x(t) = F(t, x(t)) + \tilde{\Lambda}(t, x_{i-1}(t)), & T_{i-1} \leq t \leq T, \\ I_{T_{i-1}^+}^{1-\psi_i} x(t)\Big|_{t=T_{i-1}} = x_{i-1}(T_{i-1}). \end{cases} \tag{13}$$

Where $\psi_i = \psi(T_{i-1})$ satisfying

$$|\psi(t) - \psi(T_{i-1})| < \varepsilon, \quad \text{for } T_{i-1} < t \leq T, \tag{14}$$

and

$$\tilde{\Lambda}(t, x_{i-1}(t)) = -\sum_{k=2}^{i} \Lambda_{\psi_i}(t, x_{k-1}(t) = -\sum_{k=2}^{i} \left(\frac{d}{dt}\right) \int_{T_{k-2}}^{T_{k-1}} \frac{(t-s)^{-\psi_i}}{\Gamma(1-\psi_i)} x_{k-1}(s) ds, \quad \text{for all } k \in \{2, 3, \ldots, i\}.$$

Definition 3.1. *We say that the problem* (IVP) *has a solution $x \in L^1([0,T], E)$, if there exists functions x_i, $i \in \{1, \ldots, n\}$, such that $x_1 \in L^1([0,T], E)$ satisfying the problem* (5)*, and $x_1(0) = x_0$; $x_2 \in L^1([T_1, T], E)$ satisfying the problem* (8)*, and $x_2(T_1) = x_1(T_1)$; $x_i \in L^1([T_{i-1}, T], E)$ satisfying the problem* (13)*, and $x_i(T_{i-1}) = x_{i-1}(T_{i-1})$ for all $i \in \{3, 4, \ldots, n\}$, and given by*

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq T_1, \\ x_2(t), & T_1 \leq t \leq T_2, \\ \vdots & \\ x_n(t), & T_{n-1} \leq t \leq T. \end{cases}$$

Lemma 3.2. *Let $i \in \{1, \ldots, n\}$. Then the functions x_i are solutions of* (13) *if and only if x_i are solutions of the integral equations for $t \in [T_{i-1}, T]$.*

$$x_i(t) = \frac{x_{i-1}(T_{i-1})}{\Gamma(\psi_i)} (t - T_{i-1})^{\psi_i - 1} + \frac{1}{\Gamma(\psi_i)} \int_{T_{i-1}}^{t} (t-s)^{\psi_i - 1} F(s, x_i(s))\, ds$$
$$+ \frac{1}{\Gamma(\psi_i)} \int_{T_{i-1}}^{t} (t-s)^{\psi_i - 1} \tilde{\Lambda}(s, x_{i-1}(s))\, ds. \tag{15}$$

Proof. Assume x_i satisfies (13); then we transforme 13 into an equivalent integral equation as follows. For each $i \in \{1, 2, \ldots, n\}$, let $T_{i-1} \leq t \leq T$, then Lemma 2.10 implies

$$I_{T_{i-1}^+}^{\psi_i} \, {}^{RL}\mathcal{D}_{T_{i-1}^+}^{\psi_i} x_i(t) = I_{T_{i-1}^+}^{\psi_i} \left[F(t, x_i(t)) + \tilde{\Lambda}(t, x_{i-1}(t)) \right]$$

so

$$x_i(t) = \varrho\, (t - T_{i-1})^{\psi_i - 1} + I_{T_{i-1}^+}^{\psi_i} \left[F(t, x_i(t)) + \tilde{\Lambda}(t, x_{i-1}(t)) \right].$$

Using the boundary conditions $I_{T_{i-1}^+}^{1-\psi_i} x_i(t) \Big|_{t=T_{i-1}} = x_{i-1}(T_{i-1})$, we obtain

$$x_i(t) = \frac{x_{i-1}(T_{i-1})}{\Gamma(\psi_i)} (t - T_{i-1})^{\psi_i - 1} + \frac{1}{\Gamma(\psi_i)} \int_{T_{i-1}}^{t} (t-s)^{\psi_i - 1} F(s, x_i(s))\, ds$$
$$+ \frac{1}{\Gamma(\psi_i)} \int_{T_{i-1}}^{t} (t-s)^{\psi_i - 1} \tilde{\Lambda}(s, x_{i-1}(s))\, ds.$$

□

Some assumptions are required to complete the main results:

(H1) F is a Carathéodory function and there exists a function $H \in L^1_+([0,T], E)$ and a nonnegative constant $C \geq 0$ such that

$$\|F(t, u(t))\| \leq H(t) + C\|u(t)\|,$$

for all $(t, u) \in [0, T] \times L^1([0,T], E)$.

(H2) The function $\tilde{\Lambda} : [0, T] \times E \longrightarrow E$ is measurable, and Lipschitzian with respect to the second variable, i.e., there exists a positive constant $\lambda > 0$ such that

$$\|\tilde{\Lambda}(t, u(t)) - \tilde{\Lambda}(t, v(t))\| \leq \lambda \|u(t) - v(t)\|,$$

for all $t \in [0, T]$ and $u, v \in L^1([0, T], E)$.

Lemma 3.3. *[16] Let E be a finite dimensional Banach space. Assume that (H2) holds. Then, the Nemytskii operator $\mathcal{N}_F$ satisfies condition (C2).*

Theorem 3.4. *Assume that (H1) and (H2) hold. If*

$$(C + \lambda)\frac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} < 1,$$

then the problem (13) has at least one solution for each $\in \{1, 2, \ldots, n\}$.

Proof. We can write the equation (15) as the following for each $i \in \{1, 2, \ldots, n\}$

$$x_i = \mathcal{I} x_i + \mathcal{J} x_i,$$

where

$$L^1([T_{i-1}, T], E) \ni u \mapsto (\mathcal{I}u)(t) := I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u(t) \in L^1([T_{i-1}, T], E),$$
$$L^1([T_{i-1}, T], E) \ni u \mapsto (\mathcal{J}u)(t) := I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u(t) + \frac{x_{i-1}(T_{i-1})}{\Gamma(\psi_i)}(t - T_{i-1})^{\psi_i - 1} \in L^1([T_{i-1}, T], E).$$

Clearly, $\mathcal{I}$ and $\mathcal{J}$ are well defined for each $i \in \{1, 2 \ldots, n\}$, and let $B_{R_i} = \left\{u \in L^1([T_{i-1}, T], E) : \|u\|_{L^1} \leq R_i\right\}$ be a non-empty, closed, bounded, convex subset of $L^1([T_{i-1}, T], E)$, where

$$R_i \geq \frac{\dfrac{\|x_{i-1}(T_{i-1})\|}{\Gamma(\psi_i)}(T - T_{i-1})^{\psi_i - 1} + \|H\|_{L^1}\dfrac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)}}{1 - (C + \lambda)\dfrac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)}}.$$

We will show that the conditions of Theorem 2.11 are satisfied.

Step 1: $\mathcal{I}(B_{R_i}) + \mathcal{J}(B_{R_i}) \subseteq B_{R_i}$. For each $i \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned}
\|\mathcal{I}u + \mathcal{J}u\|_{L^1} &\leq \left\| I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u + I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u \right\|_{L^1} + \frac{\|x_{i-1}(T_{i-1})\|}{\Gamma(\psi_i)} (t - T_{i-1})^{\psi_i - 1} \\
&\leq \|I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u\|_{L^1} + \|I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u\|_{L^1} + \frac{\|x_{i-1}(T_{i-1})\|}{\Gamma(\psi_i)} (T - T_{i-1})^{\psi_i - 1} \\
&\leq \frac{\|x_{i-1}(T_{i-1})\|}{\Gamma(\psi_i)} (T - T_{i-1})^{\psi_i - 1} + \frac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} (\|H\|_{L^1} + C\|u\|_{L^1}) \\
&\quad + \frac{\lambda (T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} \|u\|_{L^1} \\
&\leq R_i,
\end{aligned}$$

which is what we wanted to show.

Step 2: $\mathcal{I}$ is a continuous and satisfies $(C1)$.

Based on Lemma 2.9 and using (H1), we can see that $\mathcal{I}$ is continuous maps from $L^1([T_{i-1}, T], E)$ into itself, and we show now that $\mathcal{I}$ satisfies $(C1)$. To this end, let $(u_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence of $L^1([T_{i-1}, T], E)$, then by Lemma 3.3, it follows that $(\mathcal{N}_F u_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence $(\mathcal{N}_F u_{n_k})_{k\in\mathbb{N}}$. From the boundedness of the Riemann-Liouville fractional integral operator, it follows that the sequence $(I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u_{n_k})_{k\in\mathbb{N}}$ converges pointwise for almost all $t \in [T_{i-1}, T]$ for each $i \in \{1, 2, \ldots, n\}$. Now, when applying Vitali convergence [10, page 150], we deduce that the sequence $(\mathcal{I} u_{n_k})_{k\in\mathbb{N}}$ converges strongly in $L^1([T_{i-1}, T], E)$. Therefore, $\mathcal{I}$ satisfies $(C1)$ for each $i \in \{1, 2, \ldots, n\}$, which is what we intended to illustrate.

Step 3: $\mathcal{J}$ is a contraction, and satifies $(C2)$.

Let $u, v \in L^1([T_{i-1}, T], E)$ and by assumption (H2), it follows for all $t \in [T_{i-1}, T]$, and for each $i \in \{1, 2, \ldots, n\}$ that

$$\begin{aligned} \|\mathcal{J}u - \mathcal{J}v\|_{L^1} &= \left\| I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u - I^{\psi_i}_{T_{i-1}} \mathcal{N}_{\tilde{\Lambda}} v \right\|_{L^1} \\ &\le \frac{\lambda (T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} \|u - v\|_{L^1}. \end{aligned}$$

Hence, $\mathcal{J}$ is a contraction on $L^1([T_{i-1}, T], E)$ by Theorem 3.4 for each $i \in \{1, 2, \ldots, n\}$.

Step 4: Let Ω be a bounded subset of $L^1([T_{i-1}, T], E)$, then for all $u \in \Omega$, for all $\varepsilon > 0$ and any nonempty subset $I \subset [T_{i-1}, T]$, we have for each $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned} \int_I \left\| I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u(t) + I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u(t) \right\| &\le \int_I \left\| I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_F u(t) \right\| + \int_I \left\| I^{\psi_i}_{T^+_{i-1}} \mathcal{N}_{\tilde{\Lambda}} u(t) \right\| \\ &\le \frac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} \left[\int_I (\|H(t)\| + C\|u(t)\|)\, dt + \int_I \left(\|\tilde{\Lambda}(t, 0)\| + \lambda \|u(t)\| \right) dt \right] \\ &\quad + \int_I \left\| \frac{x_{i-1}(T_{i-1})}{\Gamma(\psi_i)} (t - T_{i-1})^{\psi_i - 1} \right\| dt. \end{aligned}$$

Taking into account the fact that all sets consisting of one element are weakly compact, then by Proposition 2.6 we obtain

$$\begin{aligned} &\lim_{\varepsilon \longrightarrow 0} \sup \left\{ \int_I \|H(t)\| dt : \; meas(I) \le \varepsilon \right\} = 0, \\ &\lim_{\varepsilon \longrightarrow 0} \sup \left\{ \int_I \|\tilde{\Lambda}(t, 0)\| dt : \; meas(I) \le \varepsilon \right\} = 0, \\ &\lim_{\varepsilon \longrightarrow 0} \sup \left\{ \int_I \left\| \frac{x_{i-1}(T_i - 1)}{\Gamma(\psi_i)} (t - T_{i-1})^{\psi_i - 1} \right\| dt : \; meas(I) \le \varepsilon \right\} = 0. \end{aligned}$$

Therefore, we obtain

$$\omega(\mathcal{I}\Omega + \mathcal{J}\Omega) \le (C + \lambda) \frac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} \omega(\Omega).$$

Then by Theorem 2.11, we conclude that the problem (13) have at least one fixed point in B_{R_i} for each $i \in \{1, 2, \ldots, n\}$. Hence, in view of Definition 3.1, the initial value problem (IVP) has at least one approximate integrable solution in $L^1([0, T], E)$. □

Example 1

In this section, we illustrate the usefulness of the results obtained in this paper. Consider the following initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}+\frac{t}{1000(1+t^2)}} x(t) = \frac{1}{2000}(t+\sin(x(t))), \qquad 0 \le t \le 1, \\ I_{0^+}^{\frac{1}{2}-\frac{t}{1000(1+t^2)}} x(t)\Big|_{t=0} = 0. \end{cases} \tag{16}$$

Notice that $F(t,x(t)) = \frac{1}{2000}(t+\sin x(t))$. Since

$$\begin{aligned} |F(t,x(t))| &= \left|\frac{1}{2000}(t+\sin x(t))\right| \\ &\le \frac{t}{2000} + \left|\frac{1}{2000}\sin(x(t))\right| \\ &\le \frac{t}{2000} + \frac{1}{2000}|x(t)|, \end{aligned}$$

so condition (H1) is satisfied with $H(t) = \frac{t}{2000}$, $C = \frac{1}{2000}$.

We start by setting $\varepsilon_1 = 55 \times 10^{-5}$.

Let $\psi_1 = \psi(0) = \frac{1}{2}$. Hence, we consider the first constant order initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{1}{2}} x(t) = \frac{t}{2000} + \frac{1}{2000}\sin(x(t)), \qquad 0 \le t \le 1, \\ I_{0^+}^{\frac{1}{2}} x(t)\Big|_{t=0} = 0. \end{cases} \tag{17}$$

Now, we check if the condition of Theorem 3.4 is satisfied for $i = 1$.

$$(C+\lambda)\frac{(T-T_{i-1})^{\psi_i}}{\Gamma(\psi_i+1)} = \frac{1}{2000\Gamma(\frac{3}{2})} \approx 56418 \times 10^{-8} < 1.$$

Therefore, the problem (17) have at least one solution x_1 given by

$$x_1(t) = \frac{1}{2000\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}(s+\sin(x_1(s)))\, ds, \quad 0 \le t \le 1.$$

Since x_1 is measurable at point 0, then for the above $\varepsilon_1 = 55 \times 10^{-5}$, we take $\delta_1^x = 5546 \times 10^{-4}$, such that

$$\left|x_1(t) - I_{0^+}^{\frac{1}{2}} x_1(t)\Big|_{t=0}\right| = \left|\frac{1}{2000\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}(s+\sin(x_1(s)))\, ds\right| < \varepsilon, \quad \text{for } 0 < t \le \delta_1^x. \tag{18}$$

Notice that $\psi(t)$ is measurable at point 0, then together with (18), for the above $\varepsilon_1 = 55 \times 10^{-5}$, we take $\delta_1^\psi = 0.5$, such that

$$|\psi(t) - \psi_1| = |\psi(t) - \psi(0)| = \left|\frac{t}{1000(1+t^2)}\right| < \varepsilon, \quad \text{for } 0 < t \le \delta_1^\psi. \tag{19}$$

Since $\min(\delta_1^x, \delta_1^\psi) = \delta_1^\psi < 1$, we take $T_1 = \delta_1^\psi$, and let $\psi_2 = \psi(T_1) \approx 50040 \times 10^{-5}$.

Now, suppose that $\varepsilon_2 = 11 \times 10^{-1}$. Hence, we consider the second constant order initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{T_1^+}^{\psi_2} x(t) = \frac{1}{2000}(t+\sin(x(t))) - \Lambda_{\psi_2}(t, x_1(t)), \qquad T_1 \le t \le 1, \\ I_{T_1^+}^{1-\psi_2} x(t)\Big|_{t=T_1} = x_1(T_1). \end{cases} \tag{20}$$

Notice that $\Lambda_{\psi_2}(t, x_1(t)) = \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x_1(s)ds$. Since

$$\|\Lambda_{\psi_2}(t,u) - \Lambda_{\psi_2}(t,v)\|_{L^1} = \int_0^1 \left| \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} u(s)\, ds - \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} v(s)\, ds \right| dt$$
$$\leq 86413 \times 10^{-5} \|u - v\|_{L^1},$$

so condition (H2) is satisfied with $\lambda = 86413 \times 10^{-5}$.

Now, we check if the condition of Theorem 3.4 is satisfied for $i = 2$.

$$(C + \lambda)\frac{(T - T_{i-1})^{\psi_i}}{\Gamma(\psi_i + 1)} = \left(\frac{1}{2000} + 86413 \times 10^{-5}\right)\frac{(1-0.5)^{\psi_2}}{\Gamma(1+\psi_2)} \approx 68967 \times 10^{-5} < 1.$$

Therefore, the problem (20) have at least one solution x_2 given by

$$x_2(t) = \frac{x_1(T_1)}{\Gamma(\psi_2)}(t - T_1)^{\psi_2 - 1} + \frac{1}{\Gamma(\psi_2)}\int_{T_1}^t (t-s)^{\psi_2-1}\left(s + \sin\left(\frac{1}{8}x_2(s)\right) - \Lambda_{\psi_2}(s, x_1(s))\right) ds, \quad T_1 \leq t \leq 1. \tag{21}$$

Since x_2 is measurable at point T_1, then for the above ε_2, we take $\delta_2^x = 1$, such that

$$\left| x_2(t) - I_{T_1^+}^{\psi_2} x_2(t)\Big|_{t=T_1} \right| = |x_2(t) - x_1(T_1)| < \varepsilon_2, \quad \text{for } T_1 < t \leq 1. \tag{22}$$

And since $\psi(t)$ is measurable at point T_1, then together with (22), for the above $\varepsilon_2 = 11 \times 10^{-1}$, we take $\delta_2^\psi = 1$, such that

$$|\psi(t) - \psi_2| = |\psi(t) - \psi(T_1)| = \left|\frac{1}{2} + \frac{t}{1000(1+t^2)} - 50040 \times 10^{-5}\right| < \varepsilon_2, \quad \text{for } T_1 < t \leq \delta_2^\psi. \tag{23}$$

So $T_1 + \min(\delta_2^x, \delta_2^\psi) = T_1 + 1 > 1$, then we stop here.

According to Definition 3.1, the initial value problem (16) has at least one approximate solution $x \in L^1([0,1], \mathbb{R})$ given by

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq T_1, \\ x_2(t), & T_1 \leq t \leq 1. \end{cases}$$

Example 2

Consider the following initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4} + \frac{t}{1000(1+t^2)}} x(t) = t, & 0 \leq t \leq 1, \\ I_{0^+}^{\frac{3}{4} - \frac{t}{1000(1+t^2)}} x(t)\Big|_{t=0} = 0. \end{cases} \tag{24}$$

Notice that the condition (H1) is satisfied with $H(t) = t$, $C = 0$.

We start by setting $\varepsilon_1 = 4 \times 10^{-4}$.

Let $\psi_1 = \psi(0) = \frac{1}{4}$. Hence, we consider the first constant order initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{1}{4}} x(t) = t, & 0 \leq t \leq 1, \\ I_{0^+}^{\frac{3}{4}} x(t)\Big|_{t=0} = 0. \end{cases} \tag{25}$$

Now, we check if the condition of Theorem 3.4 is satisfied for $i = 1$.

$$(C+\lambda)\frac{(T-T_{i-1})^{\psi_i}}{\Gamma(\psi_i+1)} = 0 < 1.$$

Therefore, the problem (17) have at least one solution x_1 given by

$$x_1(t) = \frac{1}{\Gamma(\frac{1}{4})}\int_0^t (t-s)^{-\frac{3}{4}} s\, ds = \frac{1}{\Gamma(\frac{9}{4})} t^{\frac{5}{4}}, \quad 0 \le t \le 1.$$

Since x_1 is measurable at point 0, then for the above $\varepsilon_1 = 4\times 10^{-4}$, we take $\delta_1^x = 2.1\times 10^{-4}$, such that

$$\left| x_1(t) - I_{0^+}^{\frac{3}{4}} x(t)\Big|_{t=0} \right| = |x_1(t)| = \left| \frac{1}{\Gamma(\frac{9}{4})} t^{\frac{5}{4}} \right| < \varepsilon_1, \quad \text{for } 0 < t \le \delta_1^x. \tag{26}$$

Notice that $\psi(t)$ is measurable at point 0, then together with (26), for the above $\varepsilon_1 = 4\times 10^{-4}$, we take $\delta_1^{\psi} = 4995\times 10^{-4}$, such that

$$|\psi(t) - \psi_1| = |\psi(t) - \psi(0)| = \left| \frac{t}{1000(1+t^2)} \right| < \varepsilon_1, \quad \text{for } 0 < t \le \delta_1^{\psi}.$$

Since $\min(\delta_1^x, \delta_1^{\psi}) = \delta_1^x < 1$, we take $T_1 = \delta_1^x$, and let $\psi_2 = \psi(T_1) \approx 25\times 10^{-3}$.

Now, suppose that $\varepsilon_2 = 99\times 10^{-2}$. Hence, we consider the second constant order initial value problem

$$\begin{cases} {}^{RL}\mathcal{D}_{T_1^+}^{\psi_2} x(t) = t - \Lambda_{\psi_2}(t, x_1(t)), & T_1 \le t \le 1, \\ I_{T_1^+}^{1-\psi_2} x(t)\Big|_{t=T_1} = x_1(T_1). \end{cases} \tag{27}$$

Where, $\Lambda_{\psi_2}(t, x_1(t)) = \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} x_1(s) ds$, and we have

$$\begin{aligned} \|\Lambda_{\psi_2}(t,u) - \Lambda_{\psi_2}(t,v)\|_{L^1} &= \int_0^1 \left| \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} u(s)\, ds - \left(\frac{d}{dt}\right)\int_0^{T_1} \frac{(t-s)^{-\psi_2}}{\Gamma(1-\psi_2)} v(s)\, ds \right| dt \\ &\le 1781\times 10^{-6} \|u-v\|_{L^1}, \end{aligned}$$

so condition (H2) is satisfied with $\lambda = 1781\times 10^{-6}$.

Now, we check if the condition of Theorem 3.4 is satisfied for $i = 2$.

$$(C+\lambda)\frac{(T-T_{i-1})^{\psi_i}}{\Gamma(\psi_i+1)} = 1964\times 10^{-6} < 1.$$

Therefore, the problem (27) have at least one solution x_2 given by

$$x_2(t) = \frac{x_1(T_1)}{\Gamma(\psi_2)}(t-T_1)^{\psi_2-1} + \frac{1}{\Gamma(\psi_2)}\int_{T_1}^t (t-s)^{\psi_2-1}\left(s - \Lambda_{\psi_2}(s, x_1(s))\right) ds, \quad T_1 \le t \le 1. \tag{28}$$

Since x_2 is measurable at point T_1, then for the above $\varepsilon_2 = 99\times 10^{-2}$, we take $\delta_2^x = 1$, such that

$$\begin{aligned} \left| x_2(t) - I_{T_1^+}^{\psi_2} x_2(t)\Big|_{t=T_1} \right| = \Bigg| &\frac{x_1(T_1)}{\Gamma(\psi_2)}(t-T_1)^{\psi_2-1} - x_1(T_1) \\ &+ \frac{1}{\Gamma(\psi_2)}\int_{T_1}^t (t-s)^{\psi_2-1}\left(s - \Lambda_{\psi_2}(s, x_1(s))\right) ds \Bigg| < \varepsilon_2, \quad \text{for } T_1 < t \le \delta_2^x. \end{aligned} \tag{29}$$

And since $\psi(t)$ is measurable at point T_1, then together with (29), for the above $\varepsilon_2 = 99 \times 10^{-2}$, we take $\delta_2^{\psi} = 1$, such that

$$|\psi(t) - \psi_2| = |\psi(t) - \psi(T_1)| = \left|\frac{1}{4} + \frac{t}{1000(1+t^2)} - 25 \times 10^{-3}\right| < \varepsilon_2, \quad \text{for } T_1 < t \le \delta_2^{\psi}.$$

So $T_1 + \min(\delta_2^x, \delta_2^{\psi}) = T_1 + 1 > 1$, then we stop here.

According to Definition 3.1, the initial value problem (16) has at least one approximate solution $x \in L^1([0,1], \mathbb{R})$ given by

$$x(t) = \begin{cases} x_1(t), & 0 \le t \le T_1, \\ x_2(t), & T_1 \le t \le 1. \end{cases}$$

4. Results and Discussion

Fractional calculus under weak topology has significant applications in modeling complex systems with memory, hereditary properties, and irregular dynamics. For instance, in viscoelastic materials, fractional derivatives effectively capture the stress-strain behavior of polymers and biological tissues, as demonstrated by Meral, et al. [19]. In control theory, it is used to design controllers for systems with long-range dependencies, as highlighted by Monje, et al. [20]. Additionally, in biomedical engineering, fractional models describe phenomena like the links between stress and strain in load-bearing tissues, such as cartilage, the electrical impedance of implanted cardiac pacemaker electrodes, or in predicting changes in the shear modulus of tumors developing in breast tissue, as reviewed by Magin [18]. Weak topology provides a robust mathematical framework to handle solutions in these systems, as seen in the work of Jeribi, et al. [13, 14]. These applications underscore the practicality of combining fractional calculus with weak topology to address real-world problems in engineering, physics, and biology.

Conclusion

In addition to its contributions, the paper opens several avenues for future research. One direction could involve extending the proposed framework to more complex fractional operators or exploring other types of fractional derivatives, such as Caputo or Hadamard, within weak topology settings. Another potential area is the investigation of nonlinear or multi-term fractional differential equations under similar conditions using numerical methods and computational algorithms to approximate weak solutions, providing practical tools for applications in physics, engineering, and biology. Lastly, studying the stability, uniqueness, and regularity of weak solutions in variable order fractional systems could further enhance the theoretical and practical impact of this research.

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