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Generalized semidirect sums of Lie superalgebras and their modules: The osp(1,2) case

Rui Lu[a], **Youjun Tan**[b,*]

[a]*College of Mathematics, Chengdu Normal University, Chengdu 611130, China*
[b]*College of Mathematics, Sichuan University, Chengdu 610064, China*

Abstract. We introduce the notion of generalized semidirect sums of Lie superalgebras and their modules, which is applicable to construction of Lie superalgebras. Then we classify up to isomorphisms of Lie superalgebras all generalized semidirect sums of osp(1,2) and its finite-dimensional irreducible modules.

1. Introduction

Structures and representations of Lie superalgebras form important parts of Lie theory, which have been extensively studied since 1970s. See, for example, [1] [4] [9] [10] [11] [12] [16]. For more references one may consult seminal monographs [2] [13] [15]. Lie superalgebras and Lie algebras are related in the following basic way. Given a Lie superalgebra, the even part a is a Lie algebra and the odd part is a module of the even part via the adjoint action. Conversely, given a Lie algebra and a module, under some conditions one can construct a Lie superalgebra (see (1.1.3) in [9, §1.1.2]).

This paper is motivated by constructions or realizations of Lie superalgebras from some prescribed Lie superalgebras (not necessarily Lie algebras) and their modules. Recall that, given Lie superalgebras $\mathfrak{g}$ and $\mathfrak{h}$ with an action of $\mathfrak{g}$ on $\mathfrak{h}$, we have the semidirect sum $\mathfrak{g} \ltimes \mathfrak{h}$, which contains $\mathfrak{h}$ as an ideal. More generally, it would be interesting to classify all non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$ via some suitable cohomological groups, which has been studied recently for Lie algebras in [5] [7]. On the other hand, some Lie superalgebras, for example, semisimple Lie superalgebras in the module-theoretic sense (see the semsimplicity theorem [4, Theorem 2.1] of Djokovic and Hochschild), can be reconstructed from semisimple Lie superalgebras and their modules. See Corollary 2.10 and Example 3.3 below. This type of constructions can not be put into the framework of non-abelian extensions, especially for the case of classical simple Lie superalgebras. Therefore, we consider here some generalization of semidirect sums of Lie superalgebras and their modules.

Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. Let $\varphi\colon M \otimes M \to \mathfrak{g}$ and $\psi\colon M \otimes M \to M$ be super skew-symmetric bilinear maps preserving $\mathbb{Z}_2$-gradings. We define a superbracket $[-,-]_{(\varphi,\psi)}$ on the superspace $\mathfrak{g} \oplus M$ with $(\mathfrak{g} \oplus M)_\alpha := \mathfrak{g}_\alpha \oplus M_\alpha$ ($\alpha \in \mathbb{Z}_2$) as follows.

$$[(x_1, m_1), (x_2, m_2)]_{(\varphi,\psi)} := ([x_1, x_2] + \varphi(m_1 \otimes m_2),\ x_1.m_2 - (-1)^{|x_2||m_1|} x_2.m_1 + \psi(m_1 \otimes m_2)), \tag{1.1}$$

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* Corresponding author: Youjun Tan
Email addresses: 061064@cdnu.edu.cn (Rui Lu), ytan@scu.edu.cn (Youjun Tan)
ORCID iDs: https://orcid.org/0009-0009-2203-8943 (Rui Lu), https://orcid.org/0000-0001-8777-1788 (Youjun Tan)

where $x_i \in \mathfrak{g}$, $m_i \in M$ are homogeneous elements. One can show that $(\mathfrak{g}\oplus M, [-,-]_{(\varphi,\psi)})$ is a Lie superalgebra if and only if the following three conditions are satisfied (see Proposition 2.3 below).

(i) Both φ and ψ are $\mathfrak{g}$-module homomorphisms.
(ii) $(-1)^{|m_1||m_3|}\varphi(\psi(m_1 \otimes m_2) \otimes m_3) + c.p. = 0$ holds for any homogeneous elements $m_i \in M$.
(iii) $(-1)^{|m_1||m_3|}(\varphi(m_1 \otimes m_2).m_3 + \psi(\psi(m_1 \otimes m_2) \otimes m_3)) + c.p. = 0$ holds for any homogeneous elements $m_i \in M$.

In this case, we call the Lie superalgebra $(\mathfrak{g} \oplus M, [-,-]_{(\varphi,\psi)})$ the generalized semidirect sum of $\mathfrak{g}$ and M. Clearly, the semidirect sum $\mathfrak{g} \ltimes M$ is a generalized semidirect sum of $\mathfrak{g}$ and M with respect to $(\varphi,\psi) = (0,0)$. Moreover, the direct sum of two Lie superalgebras is also a generalized semidirect sum (see Example 2.6). For nontrivial generalized semidirect sums see Corollary 2.10, which is applicable to all semisimple Lie superalgebras due to the semisimplicty theorm of Djokovic and Hochschild [4, Theorem 2.1].

To classify all generalized semidirect sums of $\mathfrak{g}$ and M up to isomorphism one may compute the set $\mathscr{L}(\mathfrak{g}, M)$ of all pairs (φ,ψ) satisfying above conditions, which is involved in general (for examples see Examples 2.7, 2.8 and 2.9). In this paper we classify all generalized semidirect sums of $\mathfrak{g} = \mathrm{osp}(1,2)$ and its finite-dimensional irreducible modules.

Let $S^+(n)$ be the $2n+1$-dimensional irreducible module of osp(1,2). For details see (3.7) and (3.8) below. First, we obtain complete descriptions of all even osp(1,2)-module homomorphisms which are super skew-symmetric from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ and from $S^+(n) \otimes S^+(n)$ to osp(1,2) (the adjoint module). See Corollary 3.8 and Corollary 3.11 respectively. Then we compute $\mathscr{L}(\mathrm{osp}(1,2), S^+(n))$ to give the classification of generalized semidirect sums of osp(1, 2) and $S^+(n)$ in Proposition 4.1, which states that, if $n \neq 1,3$ then there are no nontrivial generalized semidirect sums; if $n = 1,3$ then there are only two isoclasses of generalized semidirect sums. Our computation depends heavily on semisimplicity of osp(1,2) (especially decompositions of tensor products of irreducible osp(1,2)-modules) and the complete classification of finite-dimensional irreducible osp(1,2)-modules. It is difficult for us to consider other semisimple Lie superalgebras at present.

The paper is organized as follows. In Section 2 we give the definition of generalized semidirect sums of Lie superalgebras and their modules with some examples, and prove Corollary 2.10. In Section 3 we review osp(1,2) and its finite-dimensional irreducible modules to determine all super skew-symmetric even osp(1,2)-module homomorphisms. In Section 4 we prove the classification result on all generalized semidirect sums of osp(1,2) and its finite-dimensional irreducible modules. Throughout the field is the complex number field $\mathbb{C}$ for brevity.

2. Preliminaries

2.1. Basic notations

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with superbracket $[-,-]$. By definition [9], $[-,-]$ satisfies that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ and

$$\begin{aligned}&\text{(super skew-symmetry)}\ \ [x,y] = -(-1)^{|x||y|}[y,x],\ \ x,y \in \mathfrak{g},\\ &\text{(super Jacobi identity)}\ \ [x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]],\ \ x,y,z \in \mathfrak{g},\end{aligned} \tag{2.1}$$

where $|x||y|$ is the multiplication in $\mathbb{Z}_2$. As usual, $|v|$ always means the degree of a homogeneous element v in a graded space. Since $[-,-]$ is super skew-symmetric, one may rewrite the super Jacobi identity in the following form [15]:

$$(-1)^{|x_1||x_3|}[[x_1,x_2],x_3] + c.p. := (-1)^{|x_1||x_3|}[[x_1,x_2],x_3] + (-1)^{|x_2||x_1|}[[x_2,x_3],x_1] + (-1)^{|x_3||x_2|}[[x_3,x_1],x_2] = 0. \tag{2.2}$$

Hereafter $c.p.$ denotes the permutation sum over the indices $1,2,3$.

Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be a $\mathfrak{g}$-module. Denote by $x.m$ the action of $x \in \mathfrak{g}$ on $m \in M$. By definition [9], we have

$$\mathfrak{g}_\alpha.M_\beta \subseteq M_{\alpha+\beta},\ \ [x,y].m = x.(y.m) - (-1)^{|x||y|}y.(x.m),\ \ x,y \in \mathfrak{g}, m \in M. \tag{2.3}$$

Let M be a $\mathfrak{g}$-module. Then $M \otimes M$ becomes a $\mathfrak{g}$-module [15, p. 38], with the $\mathbb{Z}_2$-gradation and $\mathfrak{g}$-module structure given by

$$(M \otimes M)_{\bar{0}} = (M_{\bar{0}} \otimes M_{\bar{0}}) \oplus (M_{\bar{1}} \otimes M_{\bar{1}}), \quad (M \otimes M)_{\bar{1}} = (M_{\bar{0}} \otimes M_{\bar{1}}) \oplus (M_{\bar{1}} \otimes M_{\bar{0}}), \tag{2.4}$$

and

$$x.(m_1 \otimes m_2) = x.m_1 \otimes m_2 + (-1)^{|x||m_1|} m_1 \otimes x.m_2, \quad x \in \mathfrak{g}, m_i \in M, \tag{2.5}$$

respectively.

2.2. A super skew-symmetric superbracket

Let φ: $M \otimes M \to \mathfrak{g}$ and ψ: $M \otimes M \to M$ be super skew-symmetric bilinear maps preserving $\mathbb{Z}_2$-gradations, i.e., for any homogeneous $m_i \in M$,

$$\varphi(m_1, m_2) = -(-1)^{|m_1||m_2|} \varphi(m_2, m_1) \in \mathfrak{g}_{|m_1|+|m_2|}, \quad \psi(m_1, m_2) = -(-1)^{|m_1||m_2|} \psi(m_2, m_1) \in M_{|m_1|+|m_2|}. \tag{2.6}$$

Hereafter we use the following notation for brevity.

Notation 2.1. *For any map γ: $M \otimes M \to N$ denote $\gamma(m_1 \otimes m_2)$ by $\gamma(m_1, m_2)$, $m_1, m_2 \in M$.*

Consider the bilinear operation $[-,-]_{(\varphi,\psi)}$ on $\mathfrak{g} \oplus M$ defined by

$$\begin{aligned} & [(x_1, m_1), (x_2, m_2)]_{(\varphi,\psi)} \\ := \;& ([x_1, x_2] + \varphi(m_1, m_2),\; x_1.m_2 - (-1)^{|x_2||m_1|} x_2.m_1 + \psi(m_1, m_2)), \end{aligned} \tag{2.7}$$

where $x_i \in \mathfrak{g}$, $m_i \in M$ are homogeneous. Consider the $\mathbb{Z}_2$-gradation on $\mathfrak{g} \oplus M$ given by

$$(\mathfrak{g} \oplus M)_{\bar{0}} := \mathfrak{g}_{\bar{0}} \oplus M_{\bar{0}}, \quad (\mathfrak{g} \oplus M)_{\bar{1}} := \mathfrak{g}_{\bar{1}} \oplus M_{\bar{1}}. \tag{2.8}$$

Since φ and ψ preserve $\mathbb{Z}_2$-gradations, by (2.4), (2.8) and (2.7) it follows that

$$[(\mathfrak{g} \oplus M)_\alpha, (\mathfrak{g} \oplus M)_\beta]_{(\varphi,\psi)} \subseteq (\mathfrak{g} \oplus M)_{\alpha+\beta}, \; \alpha, \beta \in \mathbb{Z}_2.$$

So, $[-.-]_{(\varphi,\psi)}$ is a superbracket on the superspace $\mathfrak{g} \oplus M$. To give a sufficient and necessary condition for $(\mathfrak{g} \oplus M, [-,-]_{(\varphi,\psi)})$ to be a Lie superalgebra, we need the following technical lemma on the bracket $[-,-]_{(\varphi,\psi)}$.

Lemma 2.2. *Keep notations as above. Then the following results hold.*

(1) *$(\mathfrak{g} \oplus M, [-,-]_{(\varphi,\psi)})$ is a skew-symmetric superalgebra.*

(2) *For any $x_i \in \mathfrak{g}$, $m_i \in M$ with $|x_i| = |m_i| \in \mathbb{Z}_2$, $i = 1, 2, 3$, it holds that*

$$\begin{aligned} & (-1)^{|m_1||m_3|}[[(x_1, 0), (x_2, 0)]_{(\varphi,\psi)}, (0, m_3)]_{(\varphi,\psi)} + c.p. \\ & +(-1)^{|m_1||m_3|}[[(x_1, 0), (0, m_2)]_{(\varphi,\psi)}, (x_3, 0)]_{(\varphi,\psi)} + c.p. \\ & +(-1)^{|m_1||m_3|}[[(0, m_1), (x_2, 0)]_{(\varphi,\psi)}, (x_3, 0)]_{(\varphi,\psi)} + c.p. \;=\; 0. \end{aligned} \tag{2.9}$$

(3) *Both φ and ψ are $\mathfrak{g}$-module homomorphisms if and only if*

$$(-1)^{|x||m_2|}[[(x, 0), (0, m_1)]_{(\varphi,\psi)}, (0, m_2)]_{(\varphi,\psi)} + c.p. = 0$$

holds for any homogeneous elements $x \in \mathfrak{g}$, $m_1, m_2 \in M$.

(4) *For any homogeneous elements $m_i \in M$,*

$$(-1)^{|m_1||m_3|}[[(0, m_1), (0, m_2)]_{(\varphi,\psi)}, (0, m_3)]_{(\varphi,\psi)} + c.p. = 0$$

holds if and only if φ and ψ satisfy

$$(-1)^{|m_1||m_3|}\varphi(\psi(m_1,m_2),m_3) + c.p. = 0, \tag{2.10}$$

$$(-1)^{|m_1||m_3|}(\varphi(m_1,m_2).m_3 + \psi(\psi(m_1,m_2),m_3)) + c.p. = 0. \tag{2.11}$$

Proof. (1) It suffices to check that $[-,-]_{(\varphi,\psi)}$ is super skew-symmetric. We may assume that $x_i \in \mathfrak{g}$ and $m_i \in M$ satisfy that $|x_i| = |m_i|$ due to the $\mathbb{Z}_2$-gradation of $\mathfrak{g} \oplus M$ given by (2.8). Then, by (2.7) and (2.6) it follows that

$$\begin{aligned}
& [(x_1,m_1),(x_2,m_2)]_{(\varphi,\psi)} \\
= \ & ([x_1,x_2] + \varphi(m_1,m_2),\ x_1.m_2 - (-1)^{|x_2||m_1|}x_2.m_1 + \psi(m_1,m_2)) \\
= \ & (-(-1)^{|x_1||x_2|}[x_2,x_1] - (-1)^{|m_1||m_2|}\varphi(m_2,m_1),\ x_1.m_2 - (-1)^{|x_2||m_1|}x_2.m_1 - (-1)^{|m_1||m_2|}\psi(m_2,m_1)) \\
= \ & -(-1)^{|x_1||x_2|}([x_2,x_1] + \varphi(m_2,m_1),\ x_2.m_1 - (-1)^{|x_1||m_2|}x_1.m_2 + \psi(m_2,m_1)) \\
= \ & -(-1)^{|x_1||x_2|}[(x_2,m_2),(x_1,m_1)]_{(\varphi,\psi)}
\end{aligned}$$

as required.

(2) By (2.7) we have

$$\begin{aligned}
& [[(x_1,0),(x_2,0)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} = (0,\ [x_1,x_2].m_3), \\
& [[(x_1,0),(0,m_2)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} = (0,\ -(-1)^{|x_3|(|x_1|+|m_2|)}x_3.(x_1.m_2)), \\
& [[(0,m_1),(x_2,0)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} = (0,\ (-1)^{|x_2||m_1|+|x_3|(|x_2|+|m_1|)}x_3.(x_2.m_1)).
\end{aligned}$$

Then, L.H.S. of (2.9) = $(0,S_1) + (0,S_2) + (0,S_3)$, where

$$\begin{aligned}
S_1 &= (-1)^{|m_1||m_3|}[x_1,x_2].m_3 - (-1)^{|m_2||m_1|+|x_1|(|x_2|+|m_3|)}x_1.(x_2.m_3) + (-1)^{|m_2||m_3|+|x_1||m_3|+|x_2|(|x_1|+|m_3|)}x_2.(x_1.m_3) \\
&= (-1)^{|m_1||m_3|}([x_1,x_2].m_3 - x_1.(x_2.m_3) + (-1)^{|x_1||x_2|}x_2.(x_1.m_3)) \\
&\overset{(2.3)}{=} 0, \\
S_2 &= (-1)^{|m_3||m_2|}[x_3,x_1].m_2 - (-1)^{|m_1||m_3|+|x_3|(|x_1|+|m_2|)}x_3.(x_1.m_2) + (-1)^{|m_2||m_1|+|x_3||m_2|+|x_1|(|x_3|+|m_2|)}x_1.(x_3.m_2) \\
&= (-1)^{|m_3||m_2|}([x_3,x_1].m_2 - x_3.(x_1.m_2) + (-1)^{|x_3||x_1|}x_1.(x_3.m_2)) \\
&\overset{(2.3)}{=} 0, \\
S_3 &= (-1)^{|m_2||m_1|}[x_2,x_3].m_1 - (-1)^{|m_3||m_2|+|x_2|(|x_3|+|m_1|)}x_2.(x_3.m_1) + (-1)^{|m_1||m_3|+|x_2||m_1|+|x_3|(|x_2|+|m_1|)}x_3.(x_2.m_1) \\
&= (-1)^{|m_2||m_1|}([x_2,x_3].m_1 - x_2.(x_3.m_1) + (-1)^{|x_3||x_2|}x_3.(x_2.m_1)) \\
&\overset{(2.3)}{=} 0,
\end{aligned}$$

and hence (2.9) is proved.

(3) By (2.7) it follows that

$$\begin{aligned}
& (-1)^{|x||m_2|}[[(x,0),(0,m_1)]_{(\varphi,\psi)},(0,m_2)]_{(\varphi,\psi)} + c.p. \\
= \ & (-1)^{|x||m_2|}(\varphi(x.m_1,m_2),\ \ \psi(x.m_1,m_2)) + (-1)^{|m_1||x|}([\varphi(m_1,m_2),x],\ \ -(-1)^{(|m_1|+|m_2|)|x|}x.\psi(m_1,m_2)) \\
& -(-1)^{|m_1||m_2|+|x||m_2|}(\varphi(x.m_2,m_1),\ \ \psi(x.m_2,m_1)).
\end{aligned}$$

So, by (2.5) and comparing components we get that

$$(-1)^{|x||m_2|}[[(x,0),(0,m_1)]_{(\varphi,\psi)},(0,m_2)]_{(\varphi,\psi)} + c.p. = 0$$

holds if and only if

$$x.\varphi(m_1,m_2) = [x,\varphi(m_1,m_2)] = \varphi(x.m_1,m_2) + (-1)^{|x||m_1|}\varphi(m_1,x.m_2) = \varphi(x.(m_1 \otimes m_2)) \tag{2.12}$$

and

$$x.\psi(m_1,m_2) = \psi(x.m_1,m_2) + (-1)^{|x||m_1|}\psi(m_1,x.m_2) = \psi(x.(m_1 \otimes m_2)) \tag{2.13}$$

are satisfied. Now the statement follows by the fact that (2.12) (resp. (2.13)) is equivalent to that φ (resp. ψ) is a $\mathfrak{g}$-module homomorphism.

(4) By (2.7) we have

$$\begin{aligned} & (-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ = \ & (-1)^{|m_1||m_3|}(\varphi(\psi(m_1,m_2),m_3),\ \ \varphi(m_1,m_2).m_3 + \psi(\psi(m_1,m_2),m_3)) + c.p., \end{aligned}$$

and hence the statement follows. □

2.3. Generalized semidirect sums

We begin with the following result.

Proposition 2.3. *Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. Let φ: $M \otimes M \to \mathfrak{g}$ and ψ: $M \otimes M \to M$ be super skew-symmetric bilinear maps preserving $\mathbb{Z}_2$-gradations. Then, with respect to (2.7) and (2.8), $(\mathfrak{g} \oplus M, [-,-]_{(\varphi,\psi)})$ is a Lie superalgebra if and only if φ and ψ are $\mathfrak{g}$-module homomorphisms satisfying (2.10) and (2.11).*

Proof. By Lemma 2.2 (1) it suffices to check that $[-,-]_{(\varphi,\psi)}$ satisfies the super Jacobi identity if and only if φ and ψ are $\mathfrak{g}$-module homomorphisms satisfying (2.10) and (2.11).

Assume that $[-,-]_{(\varphi,\psi)}$ satisfies the super Jacobi identity. By (3) and (4) of Lemma 2.2, φ and ψ are $\mathfrak{g}$-module homomorphisms satisfying (2.10) and (2.11).

Conversely, assume that φ and ψ are $\mathfrak{g}$-module homomorphisms satisfying (2.10) and (2.11). It remains to verify that

$$(-1)^{|m_1||m_3|}[[(x_1,m_1),(x_2,m_2)]_{(\varphi,\psi)},(x_3,m_3)]_{(\varphi,\psi)} + c.p. = 0.$$

Without loss of generality we assume that $x_i \in \mathfrak{g}$, $m_i \in M$ with $|x_i| = |m_i|$. Then we have

$$\begin{aligned} & (-1)^{|m_1||m_3|}[[(x_1,m_1),(x_2,m_2)]_{(\varphi,\psi)},(x_3,m_3)]_{(\varphi,\psi)} + c.p. \\ = \ & \underbrace{(-1)^{|m_1||m_3|}[[(x_1,0),(x_2,0)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} + c.p.}_{=0 \text{ (by (2.7))}} + \Big\{(-1)^{|m_1||m_3|}[[(x_1,0),(x_2,0)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} \\ & +(-1)^{|m_1||m_3|}[[(x_1,0),(0,m_2)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} +(-1)^{|m_1||m_3|}[[(0,m_1),(x_2,0)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} + c.p.\Big\} \\ & +(-1)^{|m_1||m_3|}[[(x_1,0),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + (-1)^{|m_1||m_3|}[[(0,m_1),(x_2,0)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ & +(-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} + (-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p.. \end{aligned}$$

So, by (2.9) it follows that

$$\begin{aligned} & (-1)^{|m_1||m_3|}[[(x_1,m_1),(x_2,m_2)]_{(\varphi,\psi)},(x_3,m_3)]_{(\varphi,\psi)} + c.p. \\ = \ & \Big\{(-1)^{|m_1||m_3|}[[(x_1,0),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ & +(-1)^{|m_1||m_3|}[[(0,m_1),(x_2,0)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ & +(-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(x_3,0)]_{(\varphi,\psi)} + c.p.\Big\} \\ & +(-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ \overset{\text{Lemma 2.2 (3)}}{=} \ & (-1)^{|m_1||m_3|}[[(0,m_1),(0,m_2)]_{(\varphi,\psi)},(0,m_3)]_{(\varphi,\psi)} + c.p. \\ \overset{\text{Lemma 2.2 (4)}}{=} \ & 0, \end{aligned}$$

and the proof is completed. □

Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. Set

$$\begin{aligned} \mathscr{L}(\mathfrak{g},M) \ := \ & \{(\varphi,\psi) \in \mathrm{Hom}(M \otimes M,\mathfrak{g}) \times \mathrm{Hom}(M \otimes M, M) \mid \\ & \varphi \text{ and } \psi \text{ satisfy conditions in Proposition 2.3.}\} \end{aligned} \tag{2.14}$$

We make the following definition.

Definition 2.4. *Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. If $(\varphi,\psi) \in \mathscr{L}(\mathfrak{g}, M)$ then the Lie superalgebra $(\mathfrak{g} \oplus M, [-,-]_{(\varphi,\psi)})$ is called a generalized semidirect sum of $\mathfrak{g}$ and M.*

We have the following examples.

Example 2.5. *Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. Then $(0,0) \in \mathscr{L}(\mathfrak{g}, M)$, and the corresponding generalized semidirect sum of $\mathfrak{g}$ and M is the semidirect sum $\mathfrak{g} \ltimes M$.*

Example 2.6. *Let $\mathfrak{g}$, $\mathfrak{h}$ be Lie superalgebras. Regard $\mathfrak{h}$ as a trivial $\mathfrak{g}$-module, that is, $x.h = 0$ for any $x \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Then the direct sum of $\mathfrak{g}$ and $\mathfrak{h}$ is a generalized semidirect sum of $\mathfrak{g}$ and $\mathfrak{h}$: $\mathfrak{g} \oplus \mathfrak{h} = \mathfrak{g} \oplus_{(0,\psi)} \mathfrak{h}$, where ψ is the Lie superbracket of $\mathfrak{h}$.*

Example 2.7. *Let $\mathfrak{g}$ be any Lie superalgebra with the superbracket $[-,-]$. Let $M = \mathfrak{g}$ be the adjoint $\mathfrak{g}$-module. Define $\varphi \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ by $\varphi(x,y) = [x,y]$ and $\psi = 0 \in \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$. Then, by the super Jacobi identity for $[-,-]$ it follows that $(\varphi, 0) \in \mathcal{L}(\mathfrak{g}, \mathfrak{g})$.*

Example 2.8. *Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. If there is no nonzero skew-symmetric $\mathfrak{g}$-module homomorphism from $M \otimes M$ to $\mathfrak{g}$, then by (2.14) it follows that*

$$\mathcal{L}(\mathfrak{g}, M) = \{(0,\psi) \mid \psi \in \mathrm{Hom}_{\mathfrak{g}}(M \otimes M, M) \text{ and } \psi \text{ is a Lie superbracket on } M\}.$$

Example 2.9. *Let $\mathfrak{g}$ be a Lie superalgebra and M a $\mathfrak{g}$-module. Then, for any $(\varphi,\psi) \in \mathcal{L}(\mathfrak{g}, M)$ it holds that $(c^2\varphi, c\psi) \in \mathcal{L}(\mathfrak{g}, M)$ for any $c \in \mathbb{C}$, which can be verified directly by (2.14). Moreover, in this case the Lie superalgebras $\mathfrak{g} \oplus_{(\varphi,\psi)} M$ and $\mathfrak{g} \oplus_{(c^2\varphi,c\psi)} M$ ($c \neq 0$) are isomorphic via $(x,m) \mapsto (x, c^{-1}m)$. In particular, if one of the following two conditions is satisfied:*
(1) $(\varphi,\psi) \in \mathcal{L}(\mathfrak{g}, M)$ implies that $\psi = 0$.
(2) There is a constant c such that $\mathcal{L}(\mathfrak{g}, M) = \{(c_1\varphi_0, c_2\psi_0) : c_1/c_2^2 = c\} \cup \{(0,0)\}$.
Then there are at most two isoclasses of generalized semidirect sums of $\mathfrak{g}$ and M.

The following corollary shows that, there are many Lie superalgebras which can be realized as generalized semidirect sums.

Corollary 2.10. *Let $\mathfrak{L}$ be a Lie superalgebra and $\mathfrak{g}$ is a proper subsuperalgebra of $\mathfrak{L}$. Assume that $\mathfrak{L} = \mathfrak{g} \oplus M$ as $\mathfrak{g}$-modules via the adjoint action such that $\mathfrak{L}_\alpha = \mathfrak{g}_\alpha \oplus M_\alpha$, $\alpha \in \mathbb{Z}_2$. Then $\mathfrak{L}$ is a generalized semidirect sum $\mathfrak{L} = \mathfrak{g} \oplus_{(\varphi,\psi)} M$ of $\mathfrak{g}$ and M for some $(\varphi,\psi) \in \mathcal{L}(\mathfrak{g}, M)$.*

Proof. Let $\mathrm{pr}_{\mathfrak{g}}\colon \mathfrak{L} \to \mathfrak{g}$ and $\mathrm{pr}_M\colon \mathfrak{L} \to M$ be the projections. Let $[-,-]|_M\colon M \times M \to \mathfrak{L}$ be the restriction of the Lie superbracket $[-,-]$ of $\mathfrak{L}$ to M. Then we get $\mathbb{C}$-linear maps

$$\varphi := \mathrm{pr}_{\mathfrak{g}} \circ [-,-]|_M \in \mathrm{Hom}(M \otimes M, \mathfrak{g}),\ \psi := \mathrm{pr}_M \circ [-,-]|_M \in \mathrm{Hom}(M \otimes M, M).$$

In particular,

$$[m_1, m_2] = (\varphi(m_1,m_2), \psi(m_1,m_2)) \in \mathfrak{L} = \mathfrak{g} \oplus M,\ m_i \in M.$$

Note that the Lie superbracket on $\mathfrak{L}$ is given by

$$\begin{aligned} & [(x_1,m_1),(x_2,m_2)] \\ = \ & ([x_1,x_2] + \varphi(m_1,m_2),\ [x_1,m_2] + [m_1,x_2] + \psi(m_1,m_2)) \\ = \ & ([x_1,x_2] + \varphi(m_1,m_2),\ x_1.m_2 - (-1)^{|x_2||m_1|}x_2.m_1 + \psi(m_1,m_2)), \\ & \quad x_i \in \mathfrak{g},\ m_i \in M, \end{aligned} \tag{2.15}$$

which is of the form given by (2.7). Since the superbracket $[-,-]$ given by (2.15) is a Lie superbracket, by Proposition 2.3 it follows that $(\varphi,\psi) \in \mathcal{L}(\mathfrak{g}, M)$ as required. □

For an application of Corollary 2.10 see Example 3.3 below.

3. The Lie superalgebra osp(1,2) and its finite-dimensional simple modules

3.1. The Lie superalgebra osp(1,2)

Recall that osp(1,2) is the 5-dimensional complex Lie superalgebra of type $\mathbf{B}(0,1)$, which is a subsuperalgebra of the general linear Lie superalgebra $\mathrm{gl}(1,2)$ [13, §2.3.1]. More precisely,

$$\mathrm{osp}(1,2)_{\bar 0}=\left\{\begin{pmatrix}0&0&0\\0&a&b\\0&c&-a\end{pmatrix},\quad a,b,c\in\mathbb{C}\right\}\cong \mathrm{sl}_2(\mathbb{C}),\ \ \mathrm{osp}(1,2)_{\bar 1}=\left\{\begin{pmatrix}0&x&y\\-y&0&0\\x&0&0\end{pmatrix},\quad x,y\in\mathbb{C}\right\}. \tag{3.1}$$

As in [13, A.4.4], if we set

$$\begin{aligned}&e:=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},\ h:=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix},\ f:=\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\in \mathrm{osp}(1,2)_{\bar 0},\\ &E:=\begin{pmatrix}0&0&-1\\1&0&0\\0&0&0\end{pmatrix},\ F:=\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\end{pmatrix}\in \mathrm{osp}(1,2)_{\bar 1},\end{aligned} \tag{3.2}$$

then e,h,f,E,F form a basis of osp(1,2) such that $\mathrm{osp}(1,2)_{\bar 0}=\mathbb{C}\{e,h,f\}$ and $\mathrm{osp}(1,2)_{\bar 1}=\mathbb{C}\{E,F\}$ and the multiplication table is given by

$$\begin{aligned}&[h,e]=2e,\ [h,f]=-2f,\ [e,f]=h,\quad [E,E]=-2e,\ [F,F]=2f,\ [E,F]=h,\\ &[h,E]=E,\ [h,F]=-F,\ [e,E]=0,\ [e,F]=E,\ \ [f,E]=F,\ [f,F]=0,\end{aligned} \tag{3.3}$$

where other obvious zero superbrackets are omitted.

Similar to $\mathrm{sl}_2(\mathbb{C})$, osp(1,2) can be embedded into some other classical Lie superalgebras. Here we shall use the following two embeddings. First, since the Lie superalgebra sl(1,2) is the subsuperlagebra of gl(1,2) consisting of matrices with supertrace being 0, osp(1,2) is itself a subsperalgebra of sl(1,2). Second, osp(1, 2) can be embedded into osp(3, 2) as follows, where osp(3, 2) is the subsuperalgebra of gl(3,2) given by ([13, §2.3.1])

$$\begin{aligned}&\mathrm{osp}(3,2)_{\bar 0}=\left\{\begin{pmatrix}0&-u&-v&0&0\\v&a&0&0&0\\u&0&-a&0&0\\0&0&0&d&e\\0&0&0&f&-d\end{pmatrix},\ a,d,e,f,u,v\in\mathbb{C}\right\},\\ &\mathrm{osp}(3,2)_{\bar 1}=\left\{\begin{pmatrix}0&0&0&x_1&x_2\\0&0&0&y_1&y_2\\0&0&0&z_1&z_2\\-x_2&-z_2&-y_2&0&0\\x_1&z_1&y_1&0&0\end{pmatrix},\ x_1,x_2,y_1,y_2,z_1,z_2\in\mathbb{C}\right\}.\end{aligned} \tag{3.4}$$

Then, by a direct check we have the embedding of osp(1, 2) into osp(3, 2) given by

$$h\mapsto\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&-1\end{pmatrix},\ e\mapsto\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix},\ f\mapsto\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix},$$

$$E \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ F \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

3.2. Finite-dimensional irreducible modules of osp(1,2)

Classification of finite-dimensional irreducible osp(1,2)-modules is given in [9, p. 86] via Verma module theory of Lie superalgebras, see also [2, p. 85] and [13, p. 457]. Since we shall use explicit osp(1,2)-actions on its finite-dimensional irreducible modules, here we follow the idea in [3] to construct such modules, using only representation theory of $\mathrm{sl}_2(\mathbb{C})$. Since 1-dimensional irreducible osp(1,2)-modules are trivial, for convenience we consider irreducible osp(1,2)-modules of dimension greater than 1. We recall the following known result, which has been generalized to the enveloping algebra of osp(1,2) in [14].

Lemma 3.1. *If $S = S_{\bar{0}} \oplus S_{\bar{1}}$ is a finite-dimensional irreducible $\mathrm{osp}(1,2)$-module, then $S_{\bar{0}}$ and $S_{\bar{1}}$ are irreducible $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-modules with $|\dim S_{\bar{0}} - \dim S_{\bar{1}}| = 1$.*

This can be verified as follows. Both $S_{\bar{0}}$ and $S_{\bar{1}}$ are $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-modules due to (2.3). If $S_{\bar{0}} = 0$ then, for any $0 \neq w \in S_{\bar{1}}$, we have $E.w = F.w = 0$ since $S_{\bar{0}} = 0$. Therefore, $e.w = f.w = 0 \in S_{\bar{1}}$ due to $2e = -[E,E]$ and $2f = [F,F]$, which means that $0 \oplus \langle w \rangle$ is a submodule of S, and hence S is a trivial osp(1,2)-module. So, we assume that $S_{\bar{0}} \neq 0$. Let $0 \neq S'_{\bar{0}}$ be an $\mathrm{osp}(1,2)_{\bar{0}}$-submodule of $S_{\bar{0}}$. By (3.3) it follows that $S'_{\bar{1}} := \mathbb{C}\{E.v, F.v, v \in S'_{\bar{0}}\} \subseteq S_{\bar{1}}$ is an $\mathrm{osp}(1,2)_{\bar{0}}$-submodule of $S_{\bar{1}}$. Therefore, $S'_{\bar{0}} \oplus S'_{\bar{1}}$ is a nonzero osp(1,2)-submodule of S. But S is an irreducible osp(1,2)-module, which means that $S'_{\bar{0}} \oplus S'_{\bar{1}} = S_{\bar{0}} \oplus S_{\bar{1}}$. So, $S'_{\bar{0}} = S_{\bar{0}}$ since $S'_{\bar{0}} \subseteq S_{\bar{0}}$ and $S'_{\bar{1}} \subseteq S_{\bar{1}}$. Therefore, $S_{\bar{0}}$ must be an irreducible $\mathrm{osp}(1,2)_{\bar{0}}$-module. Similarly, we can show that $S_{\bar{1}}$ is an irreducible $\mathrm{osp}(1,2)_{\bar{0}}$-module.

Assume that the irreducible $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-module $S_{\bar{0}}$ has highest weight $n-1$. Then $\dim S_{\bar{0}} = n$. Let v_0 be a highest weight vector of $S_{\bar{0}}$. Note that $e.E.v_0 = 0$ due to $[e,E] = 0$ and $e.v_0 = 0$. There are the following two exclusive cases.

Case a: $E.v_0 \neq 0$. By $e.E.v_0 = 0$ it follows that $E.v_0$ is a highest weight vector of $S_{\bar{1}}$ of weight n, and hence $\dim S_{\bar{1}} = n+1$ in this case.

Case b: $E.v_0 = 0$. Then $F.v_0 \neq 0$. (Otherwise, by $2f = [F,F]$ it follows that $f.v_0 = 0$, and hence S is a trivial osp(1,2)-module.) Also, we have $e.F.v_0 = 0$ due to $[e,F] = E$ and $e.v_0 = 0$. So, by $F.v_0 \neq 0$ and $e.F.v_0 = 0$ it follows that $F.v_0$ is a highest weight vector of $S_{\bar{1}}$ of weight $n-2$, and hence $\dim S_{\bar{1}} = n-1$ in this case.

Summing up we get Lemma 3.1, which implies that, to obtain all finite-dimensional irreducible osp(1,2)-modules, it suffices to define E, F-actions on superspaces of the form $S^{\pm}(n) = S^{\pm}(n)_{\bar{0}} \oplus S^{\pm}(n)_{\bar{1}}$ with

$$S^+(n)_{\bar{0}} = V(n-1), \ S^+(n)_{\bar{1}} = V(n); \quad S^-(n)_{\bar{0}} = V(n), \ S^-(n)_{\bar{1}} = V(n-1), \tag{3.6}$$

where $V(-1) = 0$ and $V(k)$ is a finite-dimensional irreducible $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-module of highest weight k. Note that $\dim S^{\pm}(n) = 2n+1$, and $S^-(n)$ is obtained from $S^+(n)$ by changing even and odd parts. Moreover, $S^{\pm}(0)$ becomes an irreducible osp(1,2)-module with trivial actions of h, e, f, E, F.

To fix notation we choose a basis $\{v_{k,i}\}_{i=0}^{k}$ of $V(k)$ such that

$$h.v_{k,i} = (k-2i)v_{k,i}, \ e.v_{k,i} = (k-i+1)v_{k,i-1}, \ f.v_{k,i} = (i+1)v_{k,i+1}, \tag{3.7}$$

where $v_{k,-1} = v_{k,k+1} = 0$. See, for example, [6, §7.2]. In particular, $v_{k,0}$ is a highest weight vector of $V(k)$. Then we have the following lemma.

Lemma 3.2. *Keep notations as above. Assume that $\dim S^{\pm}(n) > 1$.*

(1) *If $S^+(n)$ is an irreducible $\mathrm{osp}(1,2)$-module then $E.v_{n-1,0} \neq 0$, and actions of E, F are uniquely determined by $E.v_{n-1,0}$. Conversely, if there are actions of E, F given by*

$$\begin{aligned} &E.v_{n-1,i} = (n-i)v_{n,i}, \ F.v_{n-1,i} = (i+1)v_{n,i+1}, \ 0 \le i \le n-1, \\ &E.v_{n,j} = -v_{n-1,j-1}, \ F.v_{n,j} = v_{n-1,j}, \ 0 \le j \le n, \end{aligned} \tag{3.8}$$

then $S^+(n)$ *is an irreducible* $\mathrm{osp}(1,2)$*-module.*

(2) *If* $S^-(n)$ *is an irreducible* $\mathrm{osp}(1,2)$*-module then* $E.v_{n,0} = 0$*,* $F.v_{n,0} \neq 0$*, and actions of* E*,* F *are uniquely determined by* $F.v_{n,0}$*. Conversely, if there are actions of* E, F *given by (3.8), then* $S^-(n)$ *is an irreducible* $\mathrm{osp}(1,2)$*-module.*

Proof. (1) Assume that $S^+(n)$ is an irreducible osp(1,2)-module. Since $[h, E] = E$, $E.S^+(n)_{\bar{0}} \subseteq S^+(n)_{\bar{1}}$ and the weights of $v_{n-1,i}$ and $v_{n,i}$ are $n-1-2i$ and $n-2i$ respectively, we must have $E.v_{n-1,i} = a_i v_{n,i}$ for some $a_i \in \mathbb{C}$, $0 \leq i \leq n-1$. Then, by (3.7) and $[e, E].v_{n-1,i} = e.E.v_{n-1,i} - E.e.v_{n-1,i} = 0$ it follows that $a_i(n+1-i) - a_{i-1}(n-i) = 0$. If $a_0 = 0$ then all $a_i = 0$, and hence, by

$$F.v_{n-1,i} = [f, E].v_{n-1,i} = f.E.v_{n-1,i} - E.f.v_{n-1,i} = (i+1)(a_i - a_{i+1})v_{n,i+1}$$

it follows that $F.v_{n-1,i} = 0$ for all i, a contradiction to irreducibility of $S^+(n)$ since $\dim S^+(n) > 1$. So, $a_0 \neq 0$, and hence $E.v_{n-1,i}$ and $F.v_{n-1,i}$ are uniquely determined by a_0. Moreover, by a similar weight argument we must have $E.v_{n,j} = b_j v_{n-1,j-1}$ for some $b_j \in \mathbb{C}$. Then, by

$$-2e.v_{n,j} = [E, E].v_{n,j} = 2E.(E.v_{n,j}) = 2E.(b_j v_{n-1,j-1}), \quad E.v_{n-1,j-1} = a_{j-1}v_{n,j-1}$$

it follows that $E.v_{n,j}$ is uniquely determined by a_0. Finally, by $F.v_{n,j} = [f, E].v_{n,j} = f.E.v_{n,j} - E.f.v_{n,j}$ it follows that $F.v_{n,j}$ is uniquely determined by a_0 as well. In particular, if $a_0 = n$ then $a_i = n - i$, $b_j = -1$, then we obtain (3.8).

Conversely, if (3.8) is satisfied, then by the former part $S^+(n)$ becomes an osp(1,2)-module, since other identities in (3.3) can be verified directly. $S^+(n)$ is irreducible since the even (resp. odd) part of any nonzero submodule must be $V(n-1)$ (resp. $V(n)$).

(2) If $S^-(n)$ is an irreducible $\mathrm{osp}(1,2)$-module then $E.v_{n,0} = 0$ since the highest weight of $V(n-1)$ is $n-1$ and the weight of $v_{n,0}$ in $V(n)$ is n. The remaining argument is similar to that of (1). In particular, in this case actions given in (3.8) are determined uniquely by $F.v_{n,0} = v_{n-1,0}$. □

From now on we always assume that $S^+(n)$ and $S^-(n)$ are irreducible osp(1,2)-modules satisfying (3.8). Then $S^+(n) \cong S^-(n)$ as osp(1,2)-modules and any finite-dimensional irreducible osp(1,2)-module S is isomorphic to $S^+(k)$ for some k.

Example 3.3. (1) *The Lie superalgebra* $\mathrm{sl}(1,2)$ *is a generalized semidirect sum of* $\mathrm{osp}(1,2)$ *and* $S^+(1)$*.*
(2) *The Lie superalgebra* $\mathrm{osp}(3,2)$ *is a generalized semidirect sum of* $\mathrm{osp}(1,2)$ *and* $S^+(3)$*.*

Indeed, by the canonical embedding of osp(1, 2) into sl(1,2) and semisimplicity of osp(1,2) [4, Theorem 4.1], we have $\mathrm{sl}(1,2) = \mathrm{osp}(1,2) \oplus M$ as $\mathrm{osp}(1,2)$-modules under the adjoint action. Note that $\dim M = 3$ since $\dim \mathrm{sl}(1,2) = 8$ and $\dim \mathrm{osp}(1,2) = 5$. Therefore, by semisimplicity of osp(1,2) again it follows that, either $M \cong S^+(1)$ or $M \cong S^+(0) \oplus S^+(0) \oplus S^+(0)$. The latter can not happen since, otherwise, sl(1, 2) has 1-dimensional ideals, which is impossible. So, (1) follows by Corollary 2.10. Similarly, due to the embedding of osp(1, 2) into osp(3,2) given by (3.5) and semisimplicity of osp(1,2), we have $\mathrm{osp}(3,2) = \mathrm{osp}(1,2) \oplus N$ as osp(1, 2)-modules under the adjoint action, where N is irreducible or a sum of irreducible osp(1,2)-modules and $\dim N = 7$. However, if $N \ncong S^+(3)$ then N has a 1-dimensional direct summand since all irreducible osp(1,2)-modules have odd dimension, and hence osp(3,2) has 1-dimensional ideals, which is impossible. So, $N \cong S^+(3)$ and (2) follows by Corollary 2.10.

3.3. *Some osp(1,2)-module homomorphisms*

By Proposition 2.3 we consider only even homomorphisms of modules of Lie superalgebras.

Let $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^+(n))_{\bar{0}}$ be the space of *even* osp(1,2)-module homomorphisms. By (2.4),

$$\begin{aligned}(S^+(n) \otimes S^+(n))_{\bar{0}} &= (V(n-1) \otimes V(n-1)) \oplus (V(n) \otimes V(n)),\\ (S^+(n) \otimes S^+(n))_{\bar{1}} &= (V(n-1) \otimes V(n)) \oplus (V(n) \otimes V(n-1)),\end{aligned} \tag{3.9}$$

which can be decomposed further as sums of irreducible $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-modules. Note that, as an irreducible $\mathrm{sl}_2(\mathbb{C})$-module, $V(n-1) \otimes V(n-1)$ (resp. $V(n) \otimes V(n)$) has weights of the form $2(n-1) - 2i - 2j$

(resp. $2n-2i-2j$), and weights of $V(n-1)$ are of the form $n-1-2k$. Any $\psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}}$ is an $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2(\mathbb{C})$-module homomorphism from $S^+(n)\otimes S^+(n)$ to $S^+(n)$ satisfying that

$$\psi((S^+(n)\otimes S^+(n))_{\bar{0}}) \subseteq V(n-1),\ \ \psi((S^+(n)\otimes S^+(n))_{\bar{1}}) \subseteq V(n). \tag{3.10}$$

Moreover, if $\psi \neq 0$ then both inclusions in (3.10) must be equalities since $S^+(n)$ is an irreducible osp(1, 2)-module. At first we have the following observation.

Lemma 3.4. *If n is even then* $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}} = 0$.

Proof. If n is even, then $2(n-1)-2i-2j = n-1$ and $2n-2i-2j = n-1$ can not happen, which means that, for any $\psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}}$ we must have $\psi = 0$ since the highest weight of $V(n-1) = S^+(n)_{\bar{0}}$ is $n-1$. □

Now we assume that n is odd. For any $\psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}}$, by (3.10) and weight argument we have (recall Notation 2.1)

$$\begin{cases} \psi(v_{n-1,i},\ v_{n-1,j}) = a_{ij}v_{n-1,k} \in S^+(n)_{\bar{0}}, & a_{ij}\in\mathbb{C},\ k=(i+j)-(n-1)/2, \\ \psi(v_{n,i},\ v_{n,j}) = x_{ij}v_{n-1,\ell} \in S^+(n)_{\bar{0}}, & x_{ij}\in\mathbb{C},\ \ell=(i+j)-(n+1)/2, \\ \psi(v_{n-1,i},\ v_{n,j}) = y_{ij}v_{n,m} \in S^+(n)_{\bar{1}}, & y_{ij}\in\mathbb{C},\ m=(i+j)-(n-1)/2, \\ \psi(v_{n,i},\ v_{n-1,j}) = z_{ij}v_{n,p} \in S^+(n)_{\bar{1}}, & z_{ij}\in\mathbb{C},\ p=(i+j)-(n-1)/2. \end{cases} \tag{3.11}$$

By (3.8) and applying the action of E to both sides of $\psi(v_{n-1,i}, v_{n,j}) = y_{ij}v_{n,m}$ we get

$$-y_{ij} = (n-i)x_{ij} - a_{i,j-1}. \tag{3.12}$$

Similarly, by applying the action of E to both sides of $\psi(v_{n,i}, v_{n-1,j}) = z_{ij}v_{n,p}$ we get

$$-z_{ij} = -(n-j)x_{ij} - a_{i-1,j}, \tag{3.13}$$

and, by applying the action of F to both sides of $\psi(v_{n,i}, v_{n,j}) = x_{ij}v_{n-1,\ell}$ we get

$$(\ell+1)x_{ij} = y_{ij} - z_{ij}. \tag{3.14}$$

Since $\ell+1+2n-i-j = 2n-(n-1)/2 \neq 0$, by (3.12)-(3.14) it follows that x_{ij} is uniquely determined by $a_{i-1,j}$ and $a_{i,j-1}$ via

$$x_{ij} = \frac{a_{i,j-1} - a_{i-1,j}}{2n-(n-1)/2}, \tag{3.15}$$

and hence y_{ij} and z_{ij} are also uniquely determined by $a_{i-1,j}$ and $a_{i,j-1}$. Based on this we shall prove the following

Lemma 3.5. $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}} \neq 0$ *if and only if n is odd. In this case it holds that*

$$\dim \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}} = 1.$$

Proof. By Lemma 3.4 it suffices to check the "if" part. Assume that n is odd. First we show that $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}} \neq 0$. Choose any $0 \neq a_{0,(n-1)/2}$. We obtain a nonzero $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2$-module homomorphism ψ from $V(n-1)\otimes V(n-1)$ to $V(n-1)$. Then, extend ψ to a map from $S^+(n)\otimes S^+(n)$ to $S^+(n)$ by using (3.11), where x_{ij}, y_{ij}, z_{ij} are given by (3.12)-(3.14). Finally, it is direct to check that such a ψ commutes with the actions of h, e, f, E and F.

It remains to show that, any $\psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n)\otimes S^+(n), S^+(n))_{\bar{0}}$ is uniquely determined by

$$\psi(v_{n-1,0},\ v_{n-1,(n-1)/2}) = a_0 v_{n-1,0},\ \ a_0\in\mathbb{C}.$$

By argument as above, it suffices to verify that all a_{ij}'s in (3.11) are uniquely determined by a_0, which is $a_{0,(n-1)/2}$. But this is clear due to the Clebsch-Gordon formula for the $\mathrm{osp}(1,2)_{\bar{0}} \cong \mathrm{sl}_2$-module $V(n-1)\otimes V(n-1)$, which implies that $\dim \mathrm{Hom}_{\mathrm{sl}_2}(V(n-1)\otimes V(n-1), V(n-1)) = 1$. □

Remark 3.6. *One may apply the Clebsch-Gordon decomposition of the osp(1,2)-module $S^+(n) \otimes S^+(n)$, which is unique up to isomorphism, to derive directly Lemma 3.4 and Lemma 3.5. However, the approach as above gives explicit constructions which will be important to our further computations.*

Now we describe super skew-symmetric and even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$. Recall the following decompositions of sl_2-modules

$$\wedge^2 V(2t+1) \cong \oplus_{i=0}^{t} V(4(t-i)), \quad \wedge^2 V(2t) \cong \oplus_{j=0}^{t-1} V(4(t-j)-2), \tag{3.16}$$

where $t \geq 0$ is an integer and $\wedge^2 M = M \wedge M$ is the exterior product of M. So, by (3.16) and Schur Lemma we have

$$\begin{cases} \mathrm{Hom}_{\mathrm{sl}_2}(\wedge^2 V(n), V(n-1)) \neq 0 \iff n = 4t+1, \\ \mathrm{Hom}_{\mathrm{sl}_2}(\wedge^2 V(n-1), V(n-1)) \neq 0 \iff n = 4t+3. \end{cases} \tag{3.17}$$

Lemma 3.7. *There are nonzero super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ if and only if $n = 4t+3$ ($t \geq 0$). In this case, such super skew-symmetric even homomorphisms are uniquely determined up to scalar.*

Proof. Assume that there are nonzero super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$. In particular, $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^+(n))_{\bar{0}} \neq 0$. Then n is odd due to Lemma 3.5. Fix any $0 \neq \psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^+(n))_{\bar{0}}$. Then ψ is surjective and $\psi|_{V(n-1)\otimes V(n-1)} \neq 0$. (Otherwise, it follows that $\psi|_{V(n)\otimes V(n)} = 0$ due to (3.11) and (3.15), and hence $\psi(S^+(n) \otimes S^+(n)) \subseteq S^+(n)_{\bar{1}} = V(n)$, a contradiction to the fact that ψ is surjective.) Note that $\psi|_{V(n-1)\otimes V(n-1)} \in \mathrm{Hom}_{\mathrm{sl}_2}(\wedge^2 V(n-1), V(n-1))$ since ψ is even and $V(n-1) \subseteq S^+(n)_{\bar{0}}$. Therefore, by (3.17) we must have $n = 4t+3$ for some integer $t \geq 0$.

Conversely, assume that $n = 4t+3$. Then, by (3.17) we may choose

$$0 \neq \psi_0 \in \mathrm{Hom}_{\mathrm{sl}_2}(\wedge^2 V(n-1), V(n-1))$$

and extend it uniquely to $\psi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^+(n))_{\bar{0}}$ via (3.11), where coefficients a_{ij}, x_{ij}, y_{ij} and z_{ij} are related by (3.12)-(3.14). It's routine to check that ψ is super skew-symmetric. For example, by (3.15) and $\psi(v_{n-1,i}, v_{n-1,j-1}) + \psi(v_{n-1,j-1}, v_{n-1,i}) = (a_{i,j-1} + a_{j-1,i})v_{n-1,k} = 0$ it follows that

$$x_{ij} - x_{ji} = \frac{(a_{i,j-1} - a_{i-1,j}) - (a_{j,i-1} - a_{j-1,i})}{2n - (n-1)/2} = 0,$$

which implies that $\psi(v_{n,i}, v_{n,j}) - \psi(v_{n,j}, v_{n,i}) = 0$, and hence

$$\psi(v_{n-1,i}, v_{n,j}) + \psi(v_{n,j}, v_{n-1,i}) = (y_{ij} + z_{ji})v_{n,m} = (-(n-i)(x_{ij} - x_{ji}) + (a_{i,j-1} + a_{j-1,i}))v_{n,m} = 0$$

as desired. □

Based on Lemma 3.7 all skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ can be described further as follows, which will be used in Section 4.

Corollary 3.8. *Assume that $n = 4t+3$. Let $\widehat{\psi}$ be the super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ which is uniquely determined by*

$$\widehat{\psi}(v_{n-1,0}, v_{n,(n-1)/2}) = 2nv_{n,0}. \tag{3.18}$$

Then any skew-symmetric even homomorphism from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ is a multiple of $\widehat{\psi}$. Moreover, the following identities hold.

$$\widehat{\psi}(v_{n,0}, v_{n-1,(n-1)/2}) = -\widehat{\psi}(v_{n-1,(n-1)/2}, v_{n,0}) = 2nv_{n,0}, \tag{3.19}$$

$$\widehat{\psi}(v_{n,0}, v_{n,(n+1)/2}) = 2v_{n-1,0}, \tag{3.20}$$

$$\widehat{\psi}(v_{n-1,0}, v_{n,n-1}) = -\widehat{\psi}(v_{n,n-1}, v_{n-1,0}) = \frac{2\,(((n+1)/2)!)^2}{(n-1)!} v_{n,(n-1)/2}, \tag{3.21}$$

$$\widehat{\psi}(v_{n,0}, v_{n,n}) = \widehat{\psi}(v_{n,n}, v_{n,0}) = \frac{2\,(((n-1)/2)!)^2}{(n-1)!} v_{n-1,(n-1)/2}, \tag{3.22}$$

$$\widehat{\psi}(v_{n,(n-1)/2}, v_{n,n-1}) = \widehat{\psi}(v_{n,n-1}, v_{n,(n-1)/2}) = -\frac{1}{2}(n-1)v_{n-1,n-2}. \tag{3.23}$$

Proof. It remains to verify (3.19)-(3.23) by using (3.18), (3.7), (3.8). At first we consider (3.19). Since $\widehat{\psi}$ is weight-preserving and maps $V(n-1) \otimes V(n-1)$ to $S^+(n)_{\bar{0}} = V(n-1)$, we have that, if $\widehat{\psi}(v_{n-1,0}, v_{n-1,\,(n-1)/2-1})$ were nonzero then it has weight

$$n - 1 + n - 1 - 2((n-1)/2 - 1) = n - 1 + 2 = n + 1.$$

But $n+1$ is not a weight of $V(n-1)$. So, $\widehat{\psi}(v_{n-1,0}, v_{n-1,\,(n-1)/2-1}) = 0$. Applying the action of F to $\widehat{\psi}(v_{n-1,0}, v_{n-1,\,(n-1)/2-1}) = 0$ we get that

$$\begin{aligned} 0 &= \widehat{\psi}(F.v_{n-1,0}, v_{n-1,(n-1)/2-1}) + (-1)^{|F||v_{n-1,0}|}\widehat{\psi}(v_{n-1,0}, F.v_{n-1,(n-1)/2-1}) \\ &\overset{(3.8)}{=} \widehat{\psi}(v_{n,1}, v_{n-1,(n-1)/2-1}) + \frac{1}{2}(n-1)\widehat{\psi}(v_{n-1,0}, v_{n,(n-1)/2}) \\ &\overset{(3.18)}{=} \widehat{\psi}(v_{n,1}, v_{n-1,(n-1)/2-1}) + n(n-1)v_{n,0}, \end{aligned}$$

which implies $\widehat{\psi}(v_{n,1}, v_{n-1,\frac{1}{2}(n-1)-1}) = -n(n-1)v_{n,0}$. By applying this identity and the action of f to

$$\widehat{\psi}(v_{n,0}, v_{n-1,\frac{1}{2}(n-1)-1}) = 0$$

we get that

$$\begin{aligned} 0 &= \widehat{\psi}(f.v_{n,0}, v_{n-1,(n-1)/2-1}) + (-1)^{|f||v_{n,0}|}\widehat{\psi}(v_{n,0}, f.v_{n-1,(n-1)/2-1}) \\ &= \widehat{\psi}(v_{n,1}, v_{n-1,(n-1)/2-1}) + \frac{1}{2}(n-1)\widehat{\psi}(v_{n,0}, v_{n-1,(n-1)/2}) \\ &= -n(n-1)v_{n,0} + \frac{1}{2}(n-1)\widehat{\psi}(v_{n,0}, v_{n-1,(n-1)/2}), \end{aligned}$$

from which we get (3.19) by super skew-symmetry of $\widehat{\psi}$.

By weight argument we have $\widehat{\psi}(v_{n,0}, v_{n,(n+1)/2}) = xv_{n-1,0}$ for some $x \in \mathbb{C}$. Applying the action of E we get

$$\begin{aligned} & xE.v_{n-1,0} = xnv_{n,0} = \widehat{\psi}(E.v_{n,0}, v_{n,(n+1)/2}) + (-1)^{|E||v_{n,0}|}\widehat{\psi}(v_{n,0}, E.v_{n,(n+1)/2}) \\ &\overset{(3.8)}{=} 0 + \widehat{\psi}(v_{n,0}, v_{n-1,(n-1)/2}) \overset{(3.19)}{=} 2nv_{n,0}. \end{aligned}$$

So $x = 2$, and (3.20) follows.

By weight argument we have $\widehat{\psi}(v_{n-1,0}, v_{n,n-1}) = yv_{n,(n-1)/2}$ for some $y \in \mathbb{C}$. Applying the action of $e^{(n-1)/2}$ and using $\widehat{\psi}(e.v_{n-1,0}, -) = 0$ we get that

$$\begin{aligned} & ye^{(n-1)/2}.v_{n,(n-1)/2} = y.\frac{n!}{((n+1)/2)!}v_{n,0} = e^{(n-1)/2}.\widehat{\psi}(v_{n-1,0}, v_{n,n-1}) \\ &= \widehat{\psi}(v_{n-1,0}, e^{(n-1)/2}.v_{n,n-1}) \\ &\overset{(3.7)}{=} ((n+1)/2)!\widehat{\psi}(v_{n-1,0}, v_{n,(n-1)/2}) \overset{(3.18)}{=} 2n\,((n+1)/2)!v_{n,0}. \end{aligned}$$

So $y = \frac{2(((n+1)/2)!)^2}{(n-1)!}$, and (3.21) follows by using the super skew-symmetry of $\widehat{\psi}$. In a complete similar way, by using the action of $e^{(n-1)/2}$ to $\widehat{\psi}(v_{n,0}, v_{n,n}) = zv_{n-1,\frac{1}{2}(n-1)}$ ($z \in \mathbb{C}$) and using $\widehat{\psi}(e.v_{n,0}, -) = 0$ we get (3.22).

To check (3.23) we shall need the following identity

$$\widehat{\psi}(v_{n,(n-1)/2+j}, v_{n,n-j}) = \widehat{\psi}(v_{n,n-j}, v_{n,(n-1)/2+j}) = 2(-1)^j \binom{(n+1)/2}{j} v_{n-1,n-1} \tag{3.24}$$

for $0 \le j \le (n-1)/2$, where $\binom{p}{q} = \frac{p!}{q!(p-q)!}$ denotes the binomial coefficient. Indeed, by weight argument we have $\widehat{\psi}(v_{n,(n-1)/2+j}, v_{n,n-j}) = u_j v_{n-1,n-1}$ for some $u_j \in \mathbb{C}$. Applying the action of e to $\widehat{\psi}(v_{n,(n+1)/2+j}, v_{n,n-j}) = 0$ we get

$$\begin{aligned} 0 &= \widehat{\psi}(e.v_{n,(n+1)/2+j}, v_{n,n-j}) + (-1)^{|e||v_{n,(n+1)/2+j}|}\widehat{\psi}(v_{n,(n+1)/2+j}, e.v_{n,n-j}) \\ &\overset{(3.7)}{=} ((n+1)/2 - j)\,\widehat{\psi}(v_{n,(n-1)/2+j}, v_{n,n-j}) + (j+1)\widehat{\psi}(v_{n,(n+1)/2+j}, v_{n,n-j-1}) \\ &= \Big(((n+1)/2 - j)\,u_j + (j+1)u_{j+1}\Big)\, v_{n-1,n-1}, \end{aligned}$$

from which we deduce that $\dfrac{u_{j+1}}{u_j} = -\dfrac{(n+1)/2 - j}{j+1}$. Then

$$\frac{u_j}{u_0} = \frac{u_j}{u_{j-1}}\frac{u_{j-1}}{u_{j-2}}\cdots\frac{u_1}{u_0} = \prod_{k=1}^{j}\left(-\frac{(n+1)/2 - k + 1}{k}\right) = (-1)^j\binom{(n+1)/2}{j},$$

that is $u_j = u_0(-1)^j\binom{(n+1)/2}{j}$, where u_0 can be calculated as follows. Applying the action of $f^{(n-1)/2}$ to (3.22) and using $\widehat{\psi}(-, f.v_{n,n}) = 0$ we get that

$$\begin{aligned} & \frac{2\,(((n-1)/2)!)^2}{(n-1)!} f^{(n-1)/2}.v_{n-1,(n-1)/2} \overset{(3.7)}{=} 2\,((n-1)/2)!v_{n-1,n-1} \\ =\ & f^{(n-1)/2}.\widehat{\psi}(v_{n,0}, v_{n,n}) = \widehat{\psi}(f^{(n-1)/2}.v_{n,0}, v_{n,n}) \\ =\ & ((n-1)/2)!\widehat{\psi}(v_{n,(n-1)/2}, v_{n,n}) = u_0\,((n-1)/2)!v_{n-1,n-1}. \end{aligned}$$

So, $u_0 = 2$ and (3.24) follows by super skew-symmetry of $\widehat{\psi}$.

By weight argument we have $\widehat{\psi}(v_{n,\frac{1}{2}(n-1)}, v_{n,n-1}) = wv_{n-1,n-2}$ for some $w \in \mathbb{C}$. By applying the action of f we get that

$$\begin{aligned} & wf.v_{n-1,n-2} \overset{(3.7)}{=} w(n-1)v_{n-1,n-1} \\ =\ & \widehat{\psi}(f.v_{n,(n-1)/2}, v_{n,n-1}) + (-1)^{|f||v_{n,(n-1)/2}|}\widehat{\psi}(v_{n,(n-1)/2}, f.v_{n,n-1}) \\ =\ & \frac{1}{2}(n+1)\widehat{\psi}(v_{n,(n-1)/2+1}, v_{n,n-1}) + n\widehat{\psi}(v_{n,(n-1)/2}, v_{n,n}) \\ \overset{(3.24)}{=}\ & -\frac{1}{2}(n+1)^2 v_{n-1,n-1} + 2nv_{n-1,n-1} = -\frac{1}{2}(n-1)^2 v_{n-1,n-1}. \end{aligned}$$

So, $w = -\frac{1}{2}(n-1)$, and (3.23) follows by super skew-symmetry of $\widehat{\psi}$. □

Motivated by Proposition 2.3 we consider super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to the adjoint module osp(1,2), which can be identified with $S^-(2)$ via

$$\begin{cases} \mathrm{osp}(1,2)_{\bar 0} = S^-(2)_{\bar 0} = V(2): & e \mapsto -v_{2,0},\ h \mapsto v_{2,1},\ f \mapsto v_{2,2}, \\ \mathrm{osp}(1,2)_{\bar 1} = S^-(2)_{\bar 1} = V(1): & E \mapsto v_{1,0},\ F \mapsto v_{1,1}. \end{cases} \tag{3.25}$$

(see (3.2), (3.6), (3.7) and Lemma 3.2.) Let $\mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^-(2))_{\bar 0}$ be the space of *even* homomorphisms of osp(1,2)-modules. At first we have the following lemma (recall Notation 2.1).

Lemma 3.9. *Let $\varphi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n) \otimes S^+(n), S^-(2))_{\bar{0}}$. Then φ is uniquely determined by $\varphi(v_{n,0}, v_{n,n-1})$.*

Proof. By (3.9) and weight argument we may set

$$\begin{cases} \varphi(v_{n-1,i},\ v_{n-1,j}) = a'_{ij} v_{2,k'} \in V(2), & a'_{ij} \in \mathbb{C},\ k' = (i+j) - (n-2), \\ \varphi(v_{n,i},\ v_{n,j}) = x'_{ij} v_{2,\ell'} \in V(2), & x'_{ij} \in \mathbb{C},\ \ell' = (i+j) - (n-1), \\ \varphi(v_{n-1,i},\ v_{n,j}) = y'_{ij} v_{1,m'} \in V(1), & y'_{ij} \in \mathbb{C},\ m' = (i+j) - (n-1), \\ \varphi(v_{n,i},\ v_{n-1,j}) = z'_{ij} v_{1,p'} \in V(1), & z'_{ij} \in \mathbb{C},\ p' = (i+j) - (n-1). \end{cases} \tag{3.26}$$

By (3.8) and applying the action of F to both sides of $\varphi(v_{n-1,i},\ v_{n,j}) = y'_{ij} v_{1,m'}$ we get

$$(i + j - (n-2)) y'_{ij} = a'_{ij} + (i+1) x'_{i+1,j}. \tag{3.27}$$

Similarly, by applying the action of F to both sides of $\varphi(v_{n,i},\ v_{n-1,j}) = z'_{ij} v_{1,p'}$ we get

$$(i + j - (n-2)) z'_{ij} = a'_{ij} - (j+1) x'_{i,j+1}, \tag{3.28}$$

and, by applying the action of E to both sides of $\varphi(v_{n-1,i},\ v_{n-1,j}) = a'_{ij} v_{2,k'}$ we get

$$-a'_{ij} = (n-i) z'_{ij} + (n-j) y'_{ij}. \tag{3.29}$$

By (3.27)-(3.29) it follows that a'_{ij} is uniquely determined by $x'_{i,j+1}$ and $x'_{i+1,j}$ via

$$a'_{ij} = \frac{(n-i)(j+1)x'_{i,j+1} - (n-j)(i+1)x'_{i+1,j}}{n+2}, \tag{3.30}$$

and hence y'_{ij} and z'_{ij} are also uniquely determined by $x'_{i,j+1}$ and $x'_{i,j+1}$. It remains to check that all x'_{ij} are uniquely determined by $\varphi(v_{n,0},\ v_{n,n-1})$. Since the restriction of φ to $V(n) \otimes V(n)$ is an sl_2-module homomorphism from $V(n) \otimes V(n)$ to $V(2)$, the result follows by Schur Lemma and the Clebsch-Gordon decomposition of $V(n) \otimes V(n)$, which implies that $\dim \mathrm{Hom}_{\mathrm{sl}_2}(V(n) \otimes V(n), V(2)) = 1$. □

Now we prove the following result.

Lemma 3.10. *There are nonzero super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $\mathrm{osp}(1,2)$ (i.e., the module $S^-(2)$) if and only if n is odd. In this case, such super skew-symmetric even homomorphisms are uniquely determined up to scalar.*

Proof. ⇒: Let φ be a nonzero super skew-symmetric even homomorphism from $S^+(n) \otimes S^+(n)$ to $\mathrm{osp}(1,2) = S^-(2)$. Set $\varphi_0 := \varphi|_{V(n)\otimes V(n)}$. Then $\varphi_0 \neq 0$ by Lemma 3.9. Since $S^+(n)_{\bar{1}} = V(n)$, it follows that φ_0 is a *symmetric* sl_2-module homomorphism from $V(n) \otimes V(n)$ to $V(2)$. Thus, $\mathrm{Hom}_{\mathrm{sl}_2}(S^2V(n), V(2)) \neq 0$, where $S^2V(n)$ is the submodule of symmetric tensors in $V(n) \otimes V(n)$. By (3.16) and $V(n) \otimes V(n) \cong S^2V(n) \oplus \wedge^2 V(n)$ we get

$$S^2V(2t) \cong \oplus_{i=0}^{t} V(4(t-i)), \quad S^2V(2t+1) \cong \oplus_{j=0}^{t} V(4(t-j)+2), \tag{3.31}$$

from which we deduce that n is odd by Schur Lemma.

⇐: Assume that n is odd. By (3.31) we may choose $0 \neq \varphi_0 \in \mathrm{Hom}_{\mathrm{sl}_2}(S^2V(n), V(2))$ and extend it uniquely via (3.26) to

$$\varphi \in \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n),\ S^+(n), S^-(2))_{\bar{0}} = \mathrm{Hom}_{\mathrm{osp}(1,2)}(S^+(n),\ S^+(n), \mathrm{osp}(1,2))_{\bar{0}},$$

where coefficients $a'_{ij}, x'_{ij}, y'_{ij}$ and z'_{ij} are related by (3.27)-(3.29). It's direct to check that φ is super skew-symmetric. For example, by (3.26) and $\varphi(v_{n,i},\ v_{n,j}) = \varphi(v_{n,j},\ v_{n,i})$ we have $x'_{ij} = x'_{ji}$, and hence by (3.30) we have $a'_{ij} + a'_{ji} = 0$, which implies that $\varphi(v_{n-1,i},\ v_{n-1,j}) + \varphi(v_{n-1,j},\ v_{n-1,i}) = 0$. Moreover, to see that $\varphi(v_{n-1,i},\ v_{n,j}) + \varphi(v_{n,j},\ v_{n-1,i}) = 0$, we may assume that $(i+j)-(n-2) \neq 0$ without loss of generality (otherwise $\varphi(v_{n-1,i},\ v_{n,j}) = \varphi(v_{n,j},\ v_{n-1,i}) = 0$ by (3.26)). Then, by (3.26), (3.27) and (3.28) we have

$$\begin{aligned} & \varphi(v_{n-1,i},\ v_{n,j}) + \varphi(v_{n,j},\ v_{n-1,i}) \ = \ (y'_{ij} + z'_{ji}) v_{1,m'} \\ = \ & \frac{1}{(i+j)-(n-2)}((a'_{ij} + a'_{ji}) + (i+1)(x'_{i+1,j} - x'_{j,i+1})) v_{1,m'} = 0 \end{aligned}$$

as desired. □

Similar to Corollary 3.8, by Lemma 3.10 all skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $\mathrm{osp}(1,2) = S^-(2)$ can be described further as follows, which will be used in Section 4.

Corollary 3.11. *Assume that n is odd. Let $\widehat{\varphi}$ be the super skew-symmetric homomorphisms from $S^+(n) \otimes S^+(n)$ to $\mathrm{osp}(1,2) = S^-(2)$ which is uniquely determined by*

$$\widehat{\varphi}(v_{n,0}, v_{n,n-1}) = 2(-1)^{n+1}e. \tag{3.32}$$

Then any skew-symmetric even homomorphism from $S^+(n) \otimes S^+(n)$ to $\mathrm{osp}(1,2) = S^-(2)$ is a multiple of $\widehat{\varphi}$. Moreover, the following identities hold.

$$\widehat{\varphi}(v_{n-1,0}, v_{n-1,n-2}) = 2(n-1)e, \tag{3.33}$$
$$\widehat{\varphi}(v_{n-1,0}, v_{n,n-1}) = -\widehat{\varphi}(v_{n,n-1}, v_{n-1,0}) = -E, \tag{3.34}$$
$$\widehat{\varphi}(v_{n,0}, v_{n,n}) = \widehat{\varphi}(v_{n,n}, v_{n,0}) = -h. \tag{3.35}$$

Proof. It is similar to and simpler than the proof of Corollary 3.8 by using (3.32), (3.7), (3.8). As an example we consider only (3.33), since (3.34) and (3.35) can be deduced similarly. Since $\widehat{\varphi}$ is weight-preserving and maps $V(n) \otimes V(n-1)$ to $\mathrm{osp}(1,2)_{\bar{1}} = \mathbb{C}\langle E, F\rangle$, we have that, if $\widehat{\varphi}(v_{n,0}, v_{n-1,n-2})$ is nonzero then it has weight $n + n - 1 - 2(n-2) = 3$. But 3 is not a weight of the adjoint module osp(1,2), a contradiction. So, we must have $\widehat{\varphi}(v_{n,0}, v_{n-1,n-2}) = 0$. Applying the action of F we get that

$$\begin{aligned} 0 &= \widehat{\varphi}(F.v_{n,0}, v_{n-1,n-2}) + (-1)^{|F||v_{n,0}|}\widehat{\varphi}(v_{n,0}, F.v_{n-1,n-2}) \\ &\overset{(3.8)}{=} \widehat{\varphi}(v_{n-1,0}, v_{n-1,n-2}) - (n-1)\widehat{\varphi}(v_{n,0}, v_{n,n-1}) \overset{(3.32)}{=} \widehat{\varphi}(v_{n-1,0}, v_{n-1,n-2}) - 2(n-1)e. \end{aligned}$$

Hence $\widehat{\varphi}(v_{n-1,0}, v_{n-1,n-2}) = 2(n-1)e$ as required. □

4. The classification of generalized semidirect sums of osp(1,2) and $S^+(n)$

Keep notations as above. For all generalized semidirect sums of osp(1,2) and $S^+(n)$ (see Lemma 3.2) we have the following result. Recall that, $(0,0) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(n))$ and the generalized semidirect sum $\mathrm{osp}(1,2) \oplus_{(0,0)} S^+(n)$ is just the semidirect sum $\mathrm{osp}(1,2) \ltimes S^+(n)$ (see Example 2.5).

Proposition 4.1.

(1) *If $n \neq 1, 3$ then $\mathrm{osp}(1,2) \ltimes S^+(n)$ is the unique generalized semidirect sum of $\mathrm{osp}(1,2)$ and $S^+(n)$.*
(2) *If $n = 1$ then any generalized semidirect sum of $\mathrm{osp}(1,2)$ and $S^+(n)$ is isomorphic to either $\mathrm{osp}(1,2) \ltimes S^+(1)$ or $\mathrm{sl}(1,2)$.*
(3) *If $n = 3$ then any generalized semidirect sum of $\mathrm{osp}(1,2)$ and $S^+(n)$ is isomorphic to either $\mathrm{osp}(1,2) \ltimes S^+(3)$ or $\mathrm{osp}(3,2)$.*

To give a proof we need to compute the set $\mathscr{L}(\mathrm{osp}(1,2), S^+(n))$ (see (2.14) and Proposition 2.3), which involves super skew-symmetric even homomorphisms from $S^+(n) \otimes S^+(n)$ to $S^+(n)$ and $\mathrm{osp}(1,2)$, respectively. We need in the following lemmas.

Lemma 4.2. *Any element of $\mathcal{L}(\mathrm{osp}(1,2), S^+(n))$ has the form $(a\widehat{\varphi}, b\widehat{\psi})$, $a, b \in \mathbb{C}$, where $\widehat{\varphi}$ is given by Corollary 3.11 and $\widehat{\psi}$ is given by Corollary 3.8. Moreover, if $n > 1$ and $(a\widehat{\varphi}, 0) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(n))$ then $a = 0$.*

Proof. The first statement follows by Corollary 3.11, Corollary 3.8. Assume that $n > 1$ and $(a\widehat{\varphi}, 0) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(n))$. Recall that $\widehat{\varphi}$ maps $V(n) \otimes V(n)$ into $\mathrm{osp}(1,2)_{\bar{0}} = S^-(2)_{\bar{0}} = V(2)$ (see (3.25)). So, if $\widehat{\varphi}(v_{n,0}, v_{n,0}) \neq 0$ then it has weight $n + n = 2n$, which is impossible since $2n$ ($n > 1$) is not a weight of $V(2)$.

Hence we must have $\widehat{\varphi}(v_{n,0}, v_{n,0}) = 0$. (This can not be deduced by (2.6), though $\widehat{\varphi}$ is super skew-symmetric.) Therefore, by (2.14) and Proposition 2.2 we have

$$\begin{aligned} 0 &= (-1)^{|v_{n,0}||v_{n,n}|} a\widehat{\varphi}(v_{n,0}, v_{n,0}).v_{n,n} + c.p. = 0 - a\widehat{\varphi}(v_{n,n}, v_{n,0}).v_{n,0} - a\widehat{\varphi}(v_{n,0}, v_{n,n}).v_{n,0} \\ &\overset{(3.35)}{=} -a(-h.v_{n,0} - h.v_{n,0}) \overset{(3.7)}{=} 2anv_{n,0}, \end{aligned}$$

which implies that $a = 0$. The second statement is proved. □

Lemma 4.3. *If $n \neq 1, 3$ then $\mathscr{L}(\mathrm{osp}(1,2), S^+(n)) = \{(0,0)\}$.*

Proof. If n is even then the result follows by Lemma 3.7 and Lemma 3.10. So, we assume that $n > 3$ is odd. By Lemma 4.2, it suffices to check that, if $(a\widehat{\varphi}, b\widehat{\psi}) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(n))$ then $a = b = 0$. Note that, if $b = 0$ then $a = 0$ due to Lemma 4.2. So, it remains to verify that $b = 0$.

Assume contrarily that $b \neq 0$. (In this case we have $n = 4t + 3$ for $t > 0$ by Lemma 3.7.) By Example 2.9 we have $(\frac{a}{b^2}\widehat{\varphi}, \widehat{\psi}) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(n))$. To get a contradiction we compute $\frac{a}{b^2}$ in two different ways by using (2.11).

By $\widehat{\varphi}(v_{n,0}, v_{n,0}) = 0$ (see the proof of Lemma 4.2) we get that

$$\begin{aligned} & (-1)^{|v_{n,0}||v_{n,n}|} \frac{a}{b^2}\widehat{\varphi}(v_{n,0}, v_{n,0}).v_{n,n} + c.p. \\ = \; & 0 - \frac{a}{b^2}\widehat{\varphi}(v_{n,n}, v_{n,0}).v_{n,0} - \frac{a}{b^2}\widehat{\varphi}(v_{n,0}, v_{n,n}).v_{n,0} \\ \overset{(3.35)}{=} \; & -\frac{a}{b^2}(-h.v_{n,0} - h.v_{n,0}) \overset{(3.7)}{=} \frac{2a}{b^2} n v_{n,0}. \end{aligned}$$

Note that $\widehat{\psi}(v_{n,0}, v_{n,0}) = 0$ since $2n$ is not a weight of $V(n-1)$. So we have that

$$\begin{aligned} & (-1)^{|v_{n,0}||v_{n,n}|} \widehat{\psi}(\widehat{\psi}(v_{n,0}, v_{n,0}), v_{n,n}) + c.p. \\ = \; & 0 - \widehat{\psi}(\widehat{\psi}(v_{n,n}, v_{n,0}), v_{n,0}) - \widehat{\psi}(\widehat{\psi}(v_{n,0}, v_{n,n}), v_{n,0}) \\ \overset{(3.22)}{=} \; & -\frac{4\,(((n-1)/2)!)^2}{(n-1)!}\widehat{\psi}(v_{n-1,(n-1)/2}, v_{n,0}) \overset{(3.19)}{=} \frac{8n\,(((n-1)/2)!)^2}{(n-1)!} v_{n,0}. \end{aligned}$$

Therefore, by (2.11) we get that

$$\frac{a}{b^2} = -\frac{4\,(((n-1)/2)!)^2}{(n-1)!} < 0. \tag{4.1}$$

Similarly, by $\widehat{\varphi}(v_{n,n-1}, v_{n,n-1}) = 0$ we have

$$\begin{aligned} & (-1)^{|v_{n-1,0}||v_{n,n-1}|} \frac{a}{b^2}\widehat{\varphi}(v_{n-1,0}, v_{n,n-1}).v_{n,n-1} + c.p. \\ = \; & \frac{a}{b^2}\widehat{\varphi}(v_{n-1,0}, v_{n,n-1}).v_{n,n-1} - \frac{a}{b^2}\widehat{\varphi}(v_{n,n-1}, v_{n-1,0}).v_{n,n-1} + 0 \\ \overset{(3.34)}{=} \; & -\frac{2a}{b^2}E.v_{n,n-1} \overset{(3.8)}{=} \frac{2a}{b^2} v_{n-1,n-2}, \end{aligned}$$

and by $\widehat{\psi}(v_{n,n-1}, v_{n,n-1}) = 0$ we have

$$\begin{aligned} & (-1)^{|v_{n-1,0}||v_{n,n-1}|} \widehat{\psi}(\widehat{\psi}(v_{n-1,0}, v_{n,n-1}), v_{n,n-1}) + c.p. \\ = \; & \widehat{\psi}(\widehat{\psi}(v_{n-1,0}, v_{n,n-1}), v_{n,n-1}) - \widehat{\psi}(\widehat{\psi}(v_{n,n-1}, v_{n-1,0}), v_{n,n-1}) + 0 \\ \overset{(3.21)}{=} \; & \frac{4\,(((n+1)/2)!)^2}{(n-1)!}\widehat{\psi}(v_{n,(n-1)/2}, v_{n,n-1}) \overset{(3.23)}{=} -\frac{2\,(((n+1)/2)!)^2}{(n-2)!} v_{n-1,n-2}. \end{aligned}$$

Therefore, by (2.11) we get that $\frac{a}{b^2} = \frac{(((n+1)/2)!)^2}{(n-2)!} > 0$, which contradicts to (4.1). □

Now we consider the case $n = 1$. In this case we have Table 1 due to Corollary 3.11.

Table 1 The values of $\widehat{\varphi}$ on $S^+(1) \otimes S^+(1)$

$\widehat{\varphi}(-,-)$	$v_{0,0}$	$v_{1,0}$	$v_{1,1}$
$v_{0,0}$	0	$-E$	$-F$
$v_{1,0}$	E	$2e$	$-h$
$v_{1,1}$	F	$-h$	$-2f$

Lemma 4.4. *It holds that* $\mathscr{L}(\mathrm{osp}(1,2), S^+(1)) = \{(a\widehat{\varphi}, 0) : a \in \mathbb{C}\}$.

Proof. By Lemma 3.7 it remains to check that $(a\widehat{\varphi}, 0) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(1))$ for any $a \in \mathbb{C}$. The case $a = 0$ is clear. So we assume that $a \neq 0$. By Example 2.9 it suffices to check $(\widehat{\varphi}, 0) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(1))$. By (2.14) and Proposition 2.3 it suffices to check that $\widehat{\varphi}$ satisfies (2.11), which reduces to

$$(-1)^{|m_1||m_3|}\widehat{\varphi}(m_1, m_2).m_3 + c.p. = 0 \tag{4.2}$$

for any homogeneous elements $m_i \in S^+(1) = V(0) \oplus V(1)$ $(i = 1, 2, 3)$. Clearly we may assume that m_i is one of $v_{0,0}, v_{1,0}, v_{1,1}$ (see (3.7)). We check (4.2) case by case, using (3.8), (3.7) and Table 1. The verification is long but straightforward. We give only the following two cases as examples.

(i) For $m_1 = v_{0,0}, m_2 = v_{1,0}, m_3 = v_{1,1} \in S^+(1)$, (4.2) holds since

$$\begin{aligned} & (-1)^{|v_{0,0}||v_{1,1}|}\widehat{\varphi}(v_{0,0}, v_{1,0}).v_{1,1} + c.p. \\ = \ & \widehat{\varphi}(v_{0,0}, v_{1,0}).v_{1,1} - \widehat{\varphi}(v_{1,1}, v_{0,0}).v_{1,0} + \widehat{\varphi}(v_{1,0}, v_{1,1}).v_{0,0} \ = \ -E.v_{1,1} - F.v_{1,0} - h.v_{0,0} = v_{0,0} - v_{0,0} - 0 = 0. \end{aligned}$$

(ii) For $m_1 = v_{1,0}, m_2 = v_{1,1}, m_3 = v_{1,1} \in S^+(1)$, (4.2) holds since

$$\begin{aligned} & (-1)^{|v_{1,0}||v_{1,1}|}\widehat{\varphi}(v_{1,0}, v_{1,1}).v_{1,1} + c.p. \\ = \ & -\widehat{\varphi}(v_{1,0}, v_{1,1}).v_{1,1} - \widehat{\varphi}(v_{1,1}, v_{1,0}).v_{1,1} - \widehat{\varphi}(v_{1,1}, v_{1,1}).v_{1,0} \\ = \ & h.v_{1,1} + h.v_{1,1} + 2f.v_{1,0} = -v_{1,1} - v_{1,1} + 2v_{1,1} = 0. \end{aligned}$$

Other cases are similar and omitted. □

Finally we consider the case $n = 3$. In this case we have Table 2 and Table 3 due to Corollary 3.11 and Corollary 3.8, respectively.

Table 2 The values of $\widehat{\varphi}$ on $S^+(3) \otimes S^+(3)$

$\widehat{\varphi}(-,-)$	$v_{2,0}$	$v_{2,1}$	$v_{2,2}$	$v_{3,0}$	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$
$v_{2,0}$	0	$4e$	$-2h$	0	0	$-E$	$-F$
$v_{2,1}$	$-4e$	0	$-4f$	0	$2E$	$2F$	0
$v_{2,2}$	$2h$	$4f$	0	$-E$	$-F$	0	0
$v_{3,0}$	0	0	E	0	0	$2e$	$-h$
$v_{3,1}$	0	$-2E$	F	0	$-4e$	h	$-2f$
$v_{3,2}$	E	$-2F$	0	$2e$	h	$4f$	0
$v_{3,3}$	F	0	0	$-h$	$-2f$	0	0

Table 3 The values of $\widehat{\psi}$ on $S^+(3) \otimes S^+(3)$

$\widehat{\psi}(-,-)$	$v_{2,0}$	$v_{2,1}$	$v_{2,2}$	$v_{3,0}$	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$
$v_{2,0}$	0	$10v_{2,0}$	$5v_{2,1}$	0	$6v_{3,0}$	$4v_{3,1}$	$2v_{3,2}$
$v_{2,1}$	$-10v_{2,0}$	0	$10v_{2,2}$	$-6v_{3,0}$	$-2v_{3,1}$	$2v_{3,2}$	$6v_{3,3}$
$v_{2,2}$	$-5v_{2,1}$	$-10v_{2,2}$	0	$-2v_{3,1}$	$-4v_{3,2}$	$-6v_{3,3}$	0
$v_{3,0}$	0	$6v_{3,0}$	$2v_{3,1}$	0	0	$2v_{2,0}$	$v_{2,1}$
$v_{3,1}$	$-6v_{3,0}$	$2v_{3,1}$	$4v_{3,2}$	0	$-4v_{2,0}$	$-v_{2,1}$	$2v_{2,2}$
$v_{3,2}$	$-4v_{3,1}$	$-2v_{3,2}$	$6v_{3,3}$	$2v_{2,0}$	$-v_{2,1}$	$-4v_{2,2}$	0
$v_{3,3}$	$-2v_{3,2}$	$-6v_{3,3}$	0	$v_{2,1}$	$2v_{2,2}$	0	0

Lemma 4.5. *It holds that* $\mathscr{L}(\mathrm{osp}(1,2), S^+(3)) = \{(a\widehat{\varphi}, b\widehat{\psi}) : a, b \in \mathbb{C}, \frac{a}{b^2} = -2\} \cup \{(0,0)\}$*, where* $\widehat{\varphi}$ *is given by Corollary 3.11 and* $\widehat{\psi}$ *is given by Corollary 3.8.*

Proof. By Lemma 2.5, $(0,0) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(3))$. If $(a\widehat{\varphi}, 0) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(3))$ then $a = 0$ by Lemma 4.2. Assume that $(0, b\widehat{\psi}) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(3))$. Due to (2.11),

$$b^2(-1)^{|m_1||m_3|}\widehat{\psi}(\widehat{\psi}(m_1, m_2), m_3)) + c.p. = 0 \tag{4.3}$$

holds for any homogeneous elements $m_i \in S^+(3)$. For $m_1 = v_{2,0}, m_2 = m_3 = v_{3,2}$, due to (4.3) and Table 2, we have

$$0 = (-1)^{|v_{2,0}||v_{3,2}|} b^2 \widehat{\psi}(\widehat{\psi}(v_{2,0}, v_{3,2}), v_{3,2}) + c.p. = 12b^2 v_{2,1},$$

which implies that $b = 0$.

Assume that $(a\widehat{\varphi}, b\widehat{\psi}) \in \mathscr{L}(\mathrm{osp}(1,2), S^+(3))$ with $a, b \neq 0$. Then, by Example 2.9 it follows that $(\frac{a}{b^2}\widehat{\varphi}, \widehat{\psi}) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(3))$. Applying (2.11) to $m_1 = v_{2,0}, m_2 = m_3 = v_{3,2} \in S^+(3)$ and using Table 2, Table 3 we get that

$$0 = \left((-1)^{|v_{2,0}||v_{3,2}|}\frac{a}{b^2}\widehat{\varphi}(v_{2,0}, v_{3,2}).v_{3,2} + (-1)^{|v_{2,0}||v_{3,2}|}\widehat{\psi}(\widehat{\psi}(v_{2,0}, v_{3,2}), v_{3,2})\right) + c.p. = \frac{6a}{b^2}v_{2,1} + 12v_{2,1},$$

which implies that $\frac{a}{b^2} = -2$. Therefore, it remains to verify that

$$\{(a\widehat{\varphi}, b\widehat{\psi}) : a, b \in \mathbb{C}, \frac{a}{b^2} = -2\} \subset \mathscr{L}(\mathrm{osp}(1,2), S^+(3)).$$

Furthermore, by Example 2.9 it suffices to check that $(-2\widehat{\varphi}, \widehat{\psi}) \in \mathcal{L}(\mathrm{osp}(1,2), S^+(3))$. By (2.14) and Proposition 2.3 it suffices to check that $\widehat{\varphi}, \widehat{\psi}$ satisfies (2.10) and (2.11), which are equivalent to the following two identities.

$$(-1)^{|m_1||m_3|}\widehat{\varphi}(\widehat{\psi}(m_1, m_2), m_3) + c.p. = 0, \tag{4.4}$$

$$2((-1)^{|m_1||m_3|}\widehat{\varphi}(m_1, w_2).m_3 + c.p.) = (-1)^{|m_1||m_3|}\widehat{\psi}(\widehat{\psi}(m_1, m_2), m_3) + c.p., \tag{4.5}$$

where $m_i \in S^+(3)$ is homogeneous. In fact, it suffices to check that both (4.4) and (4.5) hold for m_i being $v_{k,j} \in S^+(3) = V(2) \oplus V(3)$ (see (3.6) and (3.7)). Similar to the proof of Lemma 4.4, we check (4.4) and (4.5) for m_i being $v_{k,j} \in S^+(3) = V(2) \oplus V(3)$ case by case, using (3.8), (3.7), Table 2 and Table 3. The verification is long but straightforward. We give only the following two cases as examples.

(i) For $m_1 = v_{2,0}, m_2 = v_{2,1}, m_3 = v_{2,2} \in S^+(3)$, both (4.4) and (4.5) hold since

$$\begin{aligned}
& (-1)^{|v_{2,0}||v_{2,2}|}\widehat{\varphi}(\widehat{\psi}(v_{2,0}, v_{2,1}), v_{2,2}) + c.p. \\
= \;& \widehat{\varphi}(\widehat{\psi}(v_{2,0}, v_{2,1}), v_{2,2}) + \widehat{\varphi}(\widehat{\psi}(v_{2,2}, v_{2,0}), v_{2,1}) + \widehat{\varphi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{2,0}) \\
= \;& 10\widehat{\varphi}(v_{2,0}, v_{2,2}) - 5\widehat{\varphi}(v_{2,1}, v_{2,1}) + 10\widehat{\varphi}(v_{2,2}, v_{2,0}) \\
= \;& -20h - 0 + 20h = 0, \\
& (-1)^{|v_{2,0}||v_{2,2}|}\widehat{\varphi}(v_{2,0}, v_{2,1}).v_{2,2} + c.p. \\
= \;& \widehat{\varphi}(v_{2,0}, v_{2,1}).v_{2,2} + \widehat{\varphi}(v_{2,2}, v_{2,0}).v_{2,1} + \widehat{\varphi}(v_{2,1}, v_{2,2}).v_{2,0} \\
= \;& 4e.v_{2,2} + 2h.v_{2,1} - 4f.v_{2,0} = 4v_{2,1} + 0 - 4v_{2,1} = 0, \\
& (-1)^{|v_{2,0}||v_{2,2}|}\widehat{\psi}(\widehat{\psi}(v_{2,0}, v_{2,1}), v_{2,2}) + c.p. \\
= \;& \widehat{\psi}(\widehat{\psi}(v_{2,0}, v_{2,1}), v_{2,2}) + \widehat{\psi}(\widehat{\psi}(v_{2,2}, v_{2,0}), v_{2,1}) + \widehat{\psi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{2,0}) \\
= \;& 10\widehat{\psi}(v_{2,0}, v_{2,2}) - 5\widehat{\psi}(v_{2,1}, v_{2,1}) + 10\widehat{\psi}(v_{2,2}, v_{2,0}) \\
= \;& 50v_{2,1} + 0 - 50v_{2,1} = 0.
\end{aligned}$$

(ii) For $m_1 = v_{2,1}, m_2 = v_{2,2}, m_3 = v_{3,1} \in S^+(3)$, both (4.4) and (4.5) hold since

$$\begin{aligned}
& (-1)^{|v_{2,1}||v_{3,1}|}\widehat{\varphi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{3,1}) + c.p. \\
= \;& \widehat{\varphi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{3,1}) + \widehat{\varphi}(\widehat{\psi}(v_{3,1}, v_{2,1}), v_{2,2}) + \widehat{\varphi}(\widehat{\psi}(v_{2,2}, v_{3,1}), v_{2,1}) \\
= \;& 10\widehat{\varphi}(v_{2,2}, v_{3,1}) + 2\widehat{\varphi}(v_{3,1}, v_{2,2}) - 4\widehat{\varphi}(v_{3,2}, v_{2,1}) \;=\; -10F + 2F + 8F = 0, \\
& (-1)^{|v_{2,1}||v_{3,1}|}\widehat{\varphi}(v_{2,1}, v_{2,2}).v_{3,1} + c.p. \\
= \;& \widehat{\varphi}(v_{2,1}, v_{2,2}).v_{3,1} + \widehat{\varphi}(v_{3,1}, v_{2,1}).v_{2,2} + \widehat{\varphi}(v_{2,2}, v_{3,1}).v_{2,1} \\
= \;& -4f.v_{3,1} - 2E.v_{2,2} - F.v_{2,1} = -8v_{3,2} - 2v_{3,2} - 2v_{3,2} = -12v_{3,2}, \\
& (-1)^{|v_{2,1}||v_{3,1}|}\widehat{\psi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{3,1}) + c.p. \\
= \;& \widehat{\psi}(\widehat{\psi}(v_{2,1}, v_{2,2}), v_{3,1}) + \widehat{\psi}(\widehat{\psi}(v_{3,1}, v_{2,1}), v_{2,2}) + \widehat{\psi}(\widehat{\psi}(v_{2,2}, v_{3,1}), v_{2,1}) \\
= \;& 10\widehat{\psi}(v_{2,2}, v_{3,1}) + 2\widehat{\psi}(v_{3,1}, v_{2,2}) - 4\widehat{\psi}(v_{3,2}, v_{2,1}) \\
= \;& -40v_{3,2} + 8v_{3,2} + 8v_{3,2} = -24v_{3,2}.
\end{aligned}$$

Other cases are similar and omitted. □

To close this section we give the proof of Proposition 4.1 as follows.

Proof. (1) Since $n \neq 1, 3$, the result follows by Lemma 4.3 and Example 2.5.

(2) Since $n = 1$, by Lemma 4.4 and Example 2.9 there are exactly two isoclasses of generalized semidirect sums of osp(1,2) by $S^+(1)$. So, the result follows by Example 2.5 and Example 3.3.

(3) Since $n = 3$, by Lemma 4.5 and Example 2.9 there are exactly two isoclasses of generalized semidirect sums of osp(1,2) by $S^+(3)$. So, the result follows by Example 2.5 and Example 3.3. □

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