Filomat 39:15 (2025), 5177–5184
https://doi.org/10.2298/FIL2515177Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Further matrix inequalities with nonnegative functions

Feng Zhang[a], Hefang Jing[a,b,*]

[a] School of Mathematics and Information Science, North Minzu University, Yinchuan, 750021, China
[b] School of Civil Engineering, North Minzu University, Yinchuan 750021, China

Abstract. In this paper, we give some nonnegative function inequalities for positive semidefinite matrices.

1. Introduction

Let M_n be the algebra of $n \times n$ complex matrices with identity matrix I. For $A \in M_n$, let $\lambda_j(A)$ $(s_j(A))$, $j = 1, 2, \cdots, n$, be the eigenvalues (singular values) of A with $|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|$. The modulus of $A \in M_n$ is $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the conjugate transpose of A. We write $\lambda(A) = (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A))$, $s(A) = (s_1(A), s_2(A), \cdots, s_n(A))$ with $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$, $A \geq (>) B$ if $A - B$ is positive semidefinite (positive definite) and $A, B \in M_n$ are Hermitian matrices, i.e., the eigenvalues of $A - B$ are nonnegative. The notation $A \oplus B$ is used to mean the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where 0 is the zero matrix. A norm $\|\cdot\|$ on M_n is unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in M_n$ and any $n \times n$ unitary matrices U, V. For $A \in M_n$, $A = U|A|$, where U are some unitaty matrices, is the polar decomposition of A (see [12] for more details).

Let A, B be positive definite matrices, the weighted geometric mean of A and B is

$$A \#_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}, \quad 0 \leq t \leq 1.$$

$A \#_{\frac{1}{2}} B$ can be written as $A \# B$. The geometric mean has received a renowned attention in [4] and [20].

Let $A > 0$, $B > 0 \in M_n$ and $0 \leq t \leq 1$. Bhatia, Lim, and Yamazaki in [19] showed that

$$tr\left((A + B + 2(A \# B)\right) \leq tr\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2 \tag{1}$$

and

$$tr\left((A + B + 2(A \# B))^2\right) \leq tr\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^4, \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 47A30; Secondary 15A42, 15A60.
Keywords. Positive semidefinite matrices; Inequality; Block matrices
Received: 16 September 2024; Revised: 15 March 2025; Accepted: 10 June 2025
Communicated by Dijana Mosić
* Corresponding author: Hefang Jing
Email addresses: fzhang1024@163.com (Feng Zhang), Jinghef@163.com (Hefang Jing)
ORCID iDs: https://orcid.org/0000-0002-4349-3592 (Feng Zhang), https://orcid.org/0000-0001-6363-7584 (Hefang Jing)

where tr is the trace functional. Besides, the authors also proved in [19] that

$$tr\,(A\#_tB + B\#_tA) \le tr\left(A^{1-t}B^t + A^tB^{1-t}\right) \tag{3}$$

and

$$tr\left((A\#_tB + B\#_tA)^2\right) \le tr\left(\left|\left(A^{1-t}B^t + A^tB^{1-t}\right)\right|^2\right) \tag{4}$$

for $0 \le t \le 1$. Zou and Peng obtained that [18]

$$tr\left(\log\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2\right) \le tr(\log(A + B + 2(A\#B))). \tag{5}$$

One of the motivations for this paper is to give a new proof of (5). For $1 \le r \le 2$, they also proved that

$$tr\,((A + B + 2(A\#B))^r) \le (2 - r)tr\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2 + (r - 1)tr\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^4 \tag{6}$$

and

$$tr\,((A\#_tB + B\#_tA)^r) \le (2 - r)tr\left(A^{1-t}B^t + A^tB^{1-t}\right) + (r - 1)tr\left(\left|\left(A^{1-t}B^t + A^tB^{1-t}\right)\right|^2\right) \tag{7}$$

for $0 \le t \le 1$.

They pointed that inequalities (6) and (7) contain inequalities (1)-(4).

For any unitarily invariant norm $\|\cdot\|$ and $n \times n$ matrices A, B, it is known that [12]

$$\|A\| \le \|B\| \Leftrightarrow \sum_{i=1}^{k} s_i\,(A) \le \sum_{i=1}^{k} s_i\,(B)\,, \quad k = 1, 2, \cdots, n.$$

The quantity $\|A^pB^q + B^pA^q\|$ has been an essential research topic in the past 15 years due to a question raised by Bourin [13], which attracted more attentions from mathematicians. For this topic, we refer to [1]-[11]. Motivated by Bourin's work, Alakhrass and Sababheh in [8] studied $\|A^pB^q + B^pA^q\|$ in a more general setting, i.e., $\|f(A)g(B) \pm f(B)g(A)\|$. Let f and g be nonnegative functions on $[0, +\infty)$ and f^2, g^2 be convex with $f(0) = g(0) = 0$. It is proved in that

$$\|f(A)g(B) \pm f(B)g(A)\|^2 \le \left\|f^2(A + B)\right\| \|\|g^2(A + B)\| \tag{8}$$

for $A \ge 0, B \ge 0 \in M_n$.

We observe that $(f(A)g(B) \pm f(B)g(A)) \oplus 0$ can be rewritten as XY^*, where $X = \begin{bmatrix} f(A) & f(B) \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} g(B) & \pm g(A) \\ 0 & 0 \end{bmatrix}$. In [12], pages 263-295, a large number of theorems and inequalities indicate that it is meaningful to continue studying interpolation inequalities related to XKY^* based on the inequalities of XY^*, where $K \in M_n$.

In this paper, we prove inequality (5) by another method and give some sharpen results of inequalities (6) and (7). We also give some generalizations of the main results of section 2.2 in [8].

2. Main results

Before we start our discussion, we list some lemmas that are used in our proofs. The first lemma could be found on page 109 of [17].

Lemma 2.1. *Let $A > 0, B > 0 \in M_n$ and let U be a unitary matrix with $A^{\frac{1}{2}}UB^{\frac{1}{2}}$ is positive. Then $A^{\frac{1}{2}}UB^{\frac{1}{2}} = A\#B$.*

Lemma 2.2. *[14] Let $A > 0$, $B > 0 \in M_n$ with*

$$\prod_{j=1}^{k} \lambda_j(A) \leq \prod_{j=1}^{k} \lambda_j(B), \quad 1 \leq k \leq n.$$

Then

$$\det(I + A) \leq \det(I + B), \tag{9}$$

where $\det(X)$ *is the determinant of* $X \in M_n$.

Lemma 2.3. *[2] Let $A, X, B \in M_n$ and $r \geq 0$. Then*

$$\left\| |A^*XB|^r \right\|^2 \leq \left\| |AA^*X|^r \right\| \left\| |XBB^*|^r \right\|.$$

The following lemma extends the superadditivity behavior of convex functions to matrices, and has played an interesting role in matrix analysis, as one can see in [22] and [23].

Lemma 2.4. *[3] Let $A \geq 0$, $B \geq 0 \in M_n$. Then*

$$\left\| f(A) + f(B) \right\| \leq \left\| f(A + B) \right\|$$

for convex function f: $[0, +\infty) \to [0, +\infty)$ with $f(0) = 0$.

Now we present a new proof of inequality (5).

Theorem 2.5. *Let $A > 0$, $B > 0 \in M_n$. Then*

$$tr\left(\log\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2\right) \leq tr(\log(A + B + 2(A\#B))). \tag{10}$$

Proof. It follows from Lemma 2.1 that

$$A\#B = A^{\frac{1}{2}} U B^{\frac{1}{2}}$$

for a unitary matrix U. Using Horn's Theorem, i.e.,

$$\prod_{j=1}^{k} \left|\lambda_j(X)\right| \leq \prod_{j=1}^{k} s_j(X)$$

for $1 \leq k \leq n$ and $X \in M_n$ (see [21]) to get

$$\begin{aligned}
\prod_{j=1}^{k} \lambda_j(B^{\frac{1}{4}} A^{-\frac{1}{2}} B^{\frac{1}{4}}) &= \prod_{j=1}^{k} \lambda_j(A^{-\frac{1}{2}} B^{\frac{1}{2}}) \\
&= \prod_{j=1}^{k} \lambda_j(B^{\frac{1}{2}} A^{-\frac{1}{2}}) \\
&\leq \prod_{j=1}^{k} s_j(B^{\frac{1}{2}} A^{-\frac{1}{2}}) \\
&= \prod_{j=1}^{k} s_j(U B^{\frac{1}{2}} A^{-\frac{1}{2}}) \\
&= \prod_{j=1}^{k} \lambda_j(U B^{\frac{1}{2}} A^{-\frac{1}{2}}),
\end{aligned}$$

where the final step follows from the fact that $UB^{\frac{1}{2}}A^{-\frac{1}{2}}$ is positive definite. Thereof, by Lemma 2.2, we obtain

$$\det(I + A^{-\frac{1}{2}}B^{\frac{1}{2}}) = \det(I + B^{\frac{1}{4}}A^{-\frac{1}{2}}B^{\frac{1}{4}}) \leq \det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}}).$$

With the same method, we conclude that

$$\det(I + B^{\frac{1}{2}}A^{-\frac{1}{2}}) \leq \det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}}).$$

Therefore,

$$\det(I + A^{-\frac{1}{2}}B^{\frac{1}{2}})\det(I + B^{\frac{1}{2}}A^{-\frac{1}{2}}) \leq \det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}})\det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}}).$$

A calculation shows that

$$\begin{aligned} & \det(I + A^{-\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{-\frac{1}{2}} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\ = \; & \det(I + A^{-\frac{1}{2}}B^{\frac{1}{2}})\det(I + B^{\frac{1}{2}}A^{-\frac{1}{2}}) \\ \leq \; & \det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}})\det(I + UB^{\frac{1}{2}}A^{-\frac{1}{2}}) \\ = \; & \det(I + 2UB^{\frac{1}{2}}A^{-\frac{1}{2}} + UB^{\frac{1}{2}}A^{-\frac{1}{2}}UB^{\frac{1}{2}}A^{-\frac{1}{2}}). \end{aligned} \tag{11}$$

Hence, by multiplying both sides of (11) by $\det(A)$,

$$\det(A + A^{\frac{1}{2}}B^{\frac{1}{2}} + B^{\frac{1}{2}}A^{\frac{1}{2}} + B) \leq det(A + 2A^{\frac{1}{2}}UB^{\frac{1}{2}} + A^{\frac{1}{2}}UB^{\frac{1}{2}}A^{-\frac{1}{2}}UB^{\frac{1}{2}}). \tag{12}$$

It is obvious that

$$A^{\frac{1}{2}}UB^{\frac{1}{2}}A^{-\frac{1}{2}}UB^{\frac{1}{2}} = B^{\frac{1}{2}}U^{*}A^{\frac{1}{2}}A^{-\frac{1}{2}}UB^{\frac{1}{2}} = B. \tag{13}$$

According to (13), (12) is equivalent to

$$\det\left(A^{\frac{1}{2}} + B^{\frac{1}{2}}\right)^2 \leq \det(A + B + 2A\#B).$$

The desired result follows from the fact [18]

$$\log(\det X) = tr(\log X).$$

□

The following lemma is related to Young's inequality, which played a vital role in advancing matrix inequalities, as one can see in [10]-[9].

Lemma 2.6. *[16] For nonnegative numbers a, b,*

$$a^v b^{1-v} \leq \left(va^{\frac{1}{2}} + (1-v)\,b^{\frac{1}{2}}\right)^2,$$

where $0 \leq v \leq 1$

Lemma 2.7 is a quick consequence of Corollary IV.2.6 on page 95 of [12].

Lemma 2.7. *Let* $p > 0$, $q > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$tr\,(|AB|) \leq tr^{\frac{1}{p}}(|A|^p)tr^{\frac{1}{q}}(|B|^q)$$

for $A, B \in M_n$.

The following theorem is important for potential applications in [7].

Theorem 2.8. *Let $A > 0$, $B > 0 \in M_n$ and $1 \le r \le 2$. Then*

$$tr\left((A+B+2(A\#B))^r\right) \le \left((2-r)\left(tr\left(A^{\frac{1}{2}}+B^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} + (r-1)\left(tr\left(A^{\frac{1}{2}}+B^{\frac{1}{2}}\right)^4\right)^{\frac{1}{2}}\right)^2.$$

Proof. When $r = 1$ or $r = 2$, it is clear that Theorem 2.8 holds. For $1 < r < 2$, we assume that $p = \frac{1}{2-r}$, $q = \frac{1}{r-1}$. By Lemma 2.7, we get

$$\begin{aligned} & tr\left((A+B+2(A\#B))^r\right) \\ = \; & tr\left((A+B+2(A\#B))^{2-r}(A+B+2(A\#B))^{2(r-1)}\right) \\ \le \; & tr^{\frac{1}{p}}\left((A+B+2(A\#B))^{p(2-r)}\right) tr^{\frac{1}{q}}\left((A+B+2(A\#B))^{2q(r-1)}\right) \\ = \; & [tr(A+B+2(A\#B))]^{2-r}\left[tr(A+B+2(A\#B))^2\right]^{r-1}. \end{aligned}$$

Using inequalities (1), (2) and Lemma 2.6, we obtain

$$tr\left((A+B+2(A\#B))^r\right) \le \left((2-r)\left(tr\left(A^{\frac{1}{2}}+B^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} + (r-1)\left(tr\left(A^{\frac{1}{2}}+B^{\frac{1}{2}}\right)^4\right)^{\frac{1}{2}}\right)^2.$$

□

Remark 2.9. *Using the convexity of the function $f(x) = x^2$, we get the right side of Theorem 2.8 is smaller than inequality (6).*

Theorem 2.10. *Let $A > 0$, $B > 0 \in M_n$ and $1 \le r \le 2$. Then*

$$\begin{aligned} & tr\left((A\#_tB + B\#_tA)^r\right) \\ \le \; & \left((2-r)\left(tr\left(A^{1-t}B^t + A^tB^{1-t}\right)\right)^{\frac{1}{2}} + (r-1)\left(tr\left(\left|\left(A^{1-t}B^t + A^tB^{1-t}\right)\right|^2\right)\right)^{\frac{1}{2}}\right)^2. \end{aligned} \tag{14}$$

Proof. For $r = 1$ or $r = 2$, we see that Theorem 2.10 holds. Let $p = \frac{1}{2-r}$, $q = \frac{1}{r-1}$ for $1 < r < 2$. It follows Lemma 2.7 that

$$\begin{aligned} & tr\left((A\#_tB + B\#_tA)^r\right) \\ = \; & tr\left((A\#_tB + B\#_tA)^{2-r}(A\#_tB + B\#_tA)^{2(r-1)}\right) \\ \le \; & tr^{\frac{1}{p}}\left((A\#_tB + B\#_tA)^{p(2-r)}\right) tr^{\frac{1}{q}}\left((A\#_tB + B\#_tA)^{2q(r-1)}\right) \\ = \; & [tr(A\#_tB + B\#_tA)]^{2-r}\left[tr(A\#_tB + B\#_tA)^2\right]^{r-1}. \end{aligned}$$

By inequalities (3), (4) and Lemma 2.6, we obtain

$$tr\left((A\#_tB + B\#_tA)^r\right) \le \left((2-r)\left(tr\left(A^{1-t}B^t + A^tB^{1-t}\right)\right)^{\frac{1}{2}} + (r-1)\left(tr\left(\left|\left(A^{1-t}B^t + A^tB^{1-t}\right)\right|^2\right)\right)^{\frac{1}{2}}\right)^2.$$

□

Remark 2.11. *Since $f(x) = x^2$ is convex, it follows that the right side of Theorem 2.10 is sharper than the right side of inequality (7).*

Theorem 2.12. *Let A, B in M_n be positive semidefinite matrices. Then*

$$\left\|\left|f(A)Ug(B)+f(B)Vg(A)\right|^{\alpha}\right\|^2 \le \left\|\left(f^2(A)+f^2(B)\right)^{\alpha}\right\| \left\|\left(g^2(A)+g^2(B)\right)^{\alpha}\right\|$$

for $\alpha > 0$, nonnegative functions f, g and all unitary matrices U, V.

Proof. Let

$$\begin{aligned} X &= \begin{bmatrix} f(A) & f(B) \\ 0 & 0 \end{bmatrix}, \\ M &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, \\ Y &= \begin{bmatrix} g(B) & g(A) \\ 0 & 0 \end{bmatrix} \end{aligned}$$

for any unitary matrices U, V. Then

$$\begin{aligned} & \left\|\left|f(A)Ug(B)+f(B)Vg(A)\right|^{\alpha} \oplus 0\right\|^2 \\ = & \left\||XMY^*|^{\alpha}\right\|^2 \\ \le & \left\||X^*XM|^{\alpha}\right\| \left\||MY^*Y|^{\alpha}\right\| \\ \le & \left\||XX^*|^{\alpha}\right\| \left\||YY^*|^{\alpha}\right\| \\ \le & \left\|\left(f^2(A)+f^2(B)\right)^{\alpha} \oplus 0\right\| \left\|\left(g^2(A)+g^2(B)\right)^{\alpha} \oplus 0\right\|, \end{aligned}$$

where the second inequality follows from M is a unitary matrix. □

Remark 2.13. *Let A, B in M_n and let $A = V|A|$, $B = U|B|$ be the polar decomposition of A, B, we have*

$$\left\|\left|f(|A|)Ug(|B|)+f(|B|)Vg(|A|)\right|^{\alpha}\right\|^2 \le \left\|\left(f^2(|A|)+f^2(|B|)\right)^{\alpha}\right\| \left\|\left(g^2(|A|)+g^2(|B|)\right)^{\alpha}\right\|$$

from Theorem 2.12 for $\alpha > 0$, nonnegative functions f, g. In fact,

$$Ug(|B|) = g(|B^*|)U$$

and

$$Vg(|A|) = g(|A^*|)V$$

for $|A^*| = V|A|V^*$ and $|B^*| = U|B|U^*$. Thus, we get

$$\left\|\left|f(|A|)g(|B^*|)U+f(|B|)g(|A^*|)V\right|^{\alpha}\right\|^2 \le \left\|\left(f^2(|A|)+f^2(|B|)\right)^{\alpha}\right\| \left\|\left(g^2(|A|)+g^2(|B|)\right)^{\alpha}\right\|. \quad (15)$$

Theorem 2.14. *Let $A \ge 0$, $B \ge 0$ in M_n. Then*

$$\left\|\left|f(A)Ug(B)-f(B)Vg(A)\right|^{\alpha}\right\|^2 \le \left\|\left(f^2(A)+f^2(B)\right)^{\alpha}\right\| \left\|\left(g^2(A)+g^2(B)\right)^{\alpha}\right\|$$

for $\alpha > 0$, nonnegative functions f, g and all unitatily matrices U, V.

Proof. Since f, g are nonnegative, we have

$$\begin{aligned} & \begin{bmatrix} f(A) & 0 \\ 0 & g(B) \end{bmatrix} \begin{bmatrix} I & U \\ U^* & I \end{bmatrix} \begin{bmatrix} f(A) & 0 \\ 0 & g(B) \end{bmatrix} \\ = & \begin{bmatrix} f^2(A) & f(A)Ug(B) \\ g(B)U^*f(A) & g^2(B) \end{bmatrix} \ge 0. \end{aligned}$$

Similarly, we also have

$$\begin{aligned}
& \begin{bmatrix} f(B) & 0 \\ 0 & g(A) \end{bmatrix} \begin{bmatrix} I & -V \\ -V^* & I \end{bmatrix} \begin{bmatrix} f(B) & 0 \\ 0 & g(A) \end{bmatrix} \\
= & \begin{bmatrix} f^2(B) & -f(B)Vg(A) \\ -g(A)V^*f(B) & g^2(A) \end{bmatrix} \geq 0.
\end{aligned}$$

Using the fact that the sum of positive semidefinite matrices is positive semidifinite, we get

$$H = \begin{bmatrix} f^2(A) + f^2(B) & f(A)Ug(B) - f(B)Vg(A) \\ g(B)U^*f(A) - g(A)V^*f(B) & g^2(A) + g^2(B) \end{bmatrix} \geq 0.$$

Since $H \geq 0$, we write

$$H = \begin{bmatrix} P & Q \end{bmatrix}^* \begin{bmatrix} P & Q \end{bmatrix}$$

and $P_0 = \begin{bmatrix} P & 0 \end{bmatrix}$, $Q_0 = \begin{bmatrix} Q & 0 \end{bmatrix}$ with $P_0, Q_0 \in M_{2n}$ and P, Q are two matrices with $2n$ rows, n columns. Then

$$\begin{aligned}
& \left\| \left| f(A)Ug(B) - f(B)Vg(A) \right|^\alpha \oplus 0 \right\| \\
= & \left\| \left| P_0^* Q_0 \right|^\alpha \right\| \\
\leq & \left\| \left(P_0 P_0^* \right)^\alpha \right\|^{\frac{1}{2}} \times \left\| \left(Q_0 Q_0^* \right)^\alpha \right\|^{\frac{1}{2}} \\
= & \left\| (P^*P)^\alpha \oplus 0 \right\|^{\frac{1}{2}} \times \left\| (Q^*Q)^\alpha \oplus 0 \right\|^{\frac{1}{2}}
\end{aligned}$$

for Lemma 2.3. It is clear that

$$P^*P = f^2(A) + f^2(B)$$

and

$$Q^*Q = g^2(A) + g^2(B).$$

Therefore,

$$\left\| \left| f(A)Ug(B) - f(B)Vg(A) \right|^\alpha \right\| \leq \left\| \left(f^2(A) + f^2(B) \right)^\alpha \right\|^{\frac{1}{2}} \times \left\| \left(g^2(A) + g^2(B) \right)^\alpha \right\|^{\frac{1}{2}}.$$

□

Remark 2.15. *Although Theorem 2.14 is a consequence of Theorem 2.12, in order to illustrate the diversity of the proofs in matrix norm inequalities, we choose a method which is distinct from that in Theorem 2.12 to obtain Theorem 2.14.*

Remark 2.16. *We assume that $B = U|B|$ be the polar decomposition of $B \in M_n$ and get $-B = -U|B|$,*

$$\left\| \left| f(|A|)\, g(|-B^*|)\, (-U) + f(|B|)\, g(|A^*|)\, V \right|^\alpha \right\|^2 \leq \left\| \left(f^2(|A|) + f^2(|-B|) \right)^\alpha \right\| \left\| \left(g^2(|A|) + g^2(|-B|) \right)^\alpha \right\|$$

from inequality (15) for A, B in M_n, $A = V|A|$, $\alpha > 0$, nonnegative functions f, g, which is equivalent to

$$\left\| \left| f(|A|)\, g(|B^*|)\, U - f(|B|)\, g(|A^*|)\, V \right|^\alpha \right\|^2 \leq \left\| \left(f^2(|A|) + f^2(|B|) \right)^\alpha \right\| \left\| \left(g^2(|A|) + g^2(|B|) \right)^\alpha \right\|.$$

Remark 2.17. *Combing Theorem 2.12 and Theorem 2.14, we get*

$$\left\|\left|f(A)Ug(B) \pm f(B)Vg(A)\right|^{\alpha}\right\|^{2} \leq \left\|\left(f^{2}(A)+f^{2}(B)\right)^{\alpha}\right\| \left\|\left(\left(g^{2}(A)+g^{2}(B)\right)\right)^{\alpha}\right\| \quad (16)$$

for $\alpha > 0$, nonnegative functions f, g and all unitatily matrices U, V.

Remark 2.18. *Putting $U = V = I$ in (16), we obtain Lemma 2.6 in [8].*

Theorem 2.19. *Let $A \geq 0, B \geq 0 \in M_n$. Then*

$$\left\|f(A)Ug(B) \pm f(B)Vg(A)\right\|^{2} \leq \left\|f^{2}(A+B)\right\| \left\|g^{2}(A+B)\right\|,$$

where f, g are nonnegative functions on $[0, +\infty)$, f^2, g^2 are convex with $f(0) = g(0) = 0$ and U, V are unitarily matrices.

Proof. By inequality (16) and Lemma 2.4, we get

$$\begin{aligned} & \left\|f(A)Ug(B) \pm f(B)Vg(A)\right\|^{2} \\ \leq & \left\|f^{2}(A)+f^{2}(B)\right\| \left\|g^{2}(A)+g^{2}(B)\right\| \\ \leq & \left\|f^{2}(A+B)\right\| \left\|g^{2}(A+B)\right\|. \end{aligned}$$

□

Remark 2.20. *Theorem 2.7 in [8] is a special case of Theorem 2.19 by letting $U = V = I$.*

Funding

This work was Supported by the National Natural Science Foundation of China (12461084) and the Natural Science Foundation of Ningxia, China (2023AAC02049).

Declarations

Conflict of interest: No conflicts of interest.

References

[1] M. Ghaemi, N. Gharakhanlu, T. Rassias, R. Saadati, Advances in Matrix Inequalities, Springer (2021).
[2] R. Bhatia, C. Davis, A Cauchy–Schwarz inequality for operators with applications, Linear Algebra Appl., 223 (1995) 119–129.
[3] J. Bourin, M. Uchiyama, A matrix subadditivity inequality for $f(A + B)$ and $f(A) + f(B)$. Linear Algebra Appl., 423 (2007) 512-518.
[4] M. Sababheh, H. Moradi, I. Gümüş, S. Furuichi, A note on Kantorovich and Ando inequalities, Filomat, 37 (2023) 4171-4183.
[5] M. Sababheh, D. Choi, A complete refinement of Young's inequality, J. Math. Anal. Appl., 440 (2016) 379-393.
[6] M. Sababheh, M. Moslehian, Advanced refinements of Young and Heinz inequalities, J. Number Theory, 172 (2017) 178-199.
[7] C. Yang, Y. Li, A new young type inequality involving Heinz mean, Filomat, 34 (2020) 3639-3654.
[8] M. Alakhrass, M. Sababheh, Functional inequalities for positive matrices, Linear Multilinear A., 68 (2020) 2462-2472.
[9] M. Sababheh, Graph indices via the AG-GM inequality, Discrete Appl. Math., 230 (2017) 100-111.
[10] M. Sababheh, A. Yousef, R. Khalil, Interpolated young and Heinz inequalities, Linear and Multilinear A., 63 (2015) 2232-2244.
[11] M. Alakhrass, Inequalities related to Heinz mean, Linear and Multilinear A., 64 (2016) 1562–1569.
[12] R. Bhatia, Matrix Analysis, Springer (1997).
[13] J. Bourin, Matrix subadditivity inequalities and block-matrices. Int. J. Math., 20 (2009) 679-691.
[14] M. Lin, On a determinantal inequality arising from diffusion tensor imaging, Commun. Contemp. Math., 19 (2017) 1650044-1650049.
[15] A. Yousef, M. Sababheh, R. Khalil, Operator-norm inequalities and interpolated trace inequalities, Linear and Multilinear A., 63 (2015) 455-465.
[16] M.Lin, On Drury's solution of Bhatia and Kittaneh's question, Linear Algebra Appl., 528 (2017) 33-39.
[17] R. Bhatia, Positive Definite Matrices, Princeton University Press, Princeton (2007).
[18] L. Zou, Y. Peng, Some trace inequalities for matrix means, J. Inequal. Appl., 2016 (2016) 283-287.
[19] R. Bhatia, Y. Lim, T. Yamazaki, Some norm inequalities for matrix means. Linear Algebra Appl., 501 (2016) 112-122.
[20] S. Furuichi, H. Moradi, M. Sababheh, New sharp inequalities for operator means, Linear and Multilinear A., 67 (2019) 1567-1578.
[21] A. Horn, On the eigenvalues of a matrix with prescribed singular values, Proc. Am. Math. Soc., 5 (1954) 4-7.
[22] T. Ando, X. Zhan, Norm inequalities related to operator monotone functions, Math. Ann., 315 (1999) 771-780.
[23] H. Moradi, Z. Heydarbeygi, M. Sababheh, Subadditive inequalities for operators, Math. Inequal. Appl., 23 (2020) 317-327.