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Nonlinear mixed $*$-Jordan type higher derivations on $*$-algebras

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Abstract. In this paper, we show that any nonlinear mixed $*$-Jordan-type higher derivation on unital $*$-algebras is an additive higher $*$-derivation. As applications, nonlinear mixed $*$-Jordan-type higher derivations on some classical unital $*$-algebras, such as prime $*$-algebras, von Neumann algebras of type I_1, factor von Neumann algebras and standard operator algebras, are characterized, and some conclusions are extended.

1. Introduction

Let $\mathfrak{B}$ be an unital $*$-algebra over the complex field $\mathfrak{C}$, where involution $*$ satisfies the relation $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$ and $((x)^*)^* = x$ for all $x, y \in \mathfrak{B}$. Now, given $A, B \in \mathfrak{B}$, the product symbols $A \circ B = AB + BA$, $A * B = AB + B^*A$ and $A \bullet B = A^*B + B^*A$ are called Jordan product, skew-Jordan product and bi-skew-Jordan product respectively. In addition, skew-Lie product $[A, B]_* = AB - B^*A$ and bi-skew Lie product $[A, B]_\bullet = A^*B - B^*A$ can be defined. Such kind of product plays a more and more important role in some research topics, and its study has attracted many authors' attention (see [1, 2, 5, 6, 14–21]). A mapping $\delta_1 : \mathfrak{B} \to \mathfrak{B}$ (without the additivity assumption) is called a nonlinear $*$-derivation if $\delta_1(y_1y_2) = \delta_1(y_1)y_2 + y_1\delta_1(y_2)$ and $\delta_1(x^*) = \delta_1(x)^*$. A mapping $\delta_1 : \mathfrak{B} \to \mathfrak{B}$ (without the additivity assumption) is called a nonlinear (resp. bi-skew) Jordan derivation if

$$\delta_1(y_1 \circ y_2) = \delta_1(y_1) \circ y_2 + y_1 \circ \delta_1(y_2)$$
$$(\text{resp.}\, \delta_1(y_1 \bullet y_2) = \delta_1(y_1) \bullet y_2 + y_1 \bullet \delta_1(y_2))$$

for all $y_1, y_2 \in \mathfrak{B}$. Many authors paid more attentions on the problem related to the Jordan $*$-derivations, bi-skew-Jordan $*$-derivations (see [14–21]).

In recent years, many scholars have paid attention to the mixed product operation of Jordan product, skew-Jordan product, bi-skew-Jordan product, and have obtained a lot of results (see [14–21]). With this picture in mind, authors of [16] studied the nonlinear mixed $*$-Jordan-type derivations $\Psi_1 : \mathfrak{B} \to \mathfrak{B}$ on $*$-algebras. They showed that the map Ψ_1 which satisfies

$$\Psi_1(x_1 \circ \cdots \circ x_{n-1} \bullet x_n) = \sum_{i=1}^{n} x_1 \circ \cdots \circ \Psi_1(x_i) \circ \cdots \circ x_{n-1} \bullet x_n$$

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is an additive $*$-derivation for $n \geq 3$, where the element $U_n(x_1, \cdots, x_n) = x_1 \circ \cdots \circ x_{n-1} \bullet x_n$ is a monomial of degree n and is calculated as follows: $x_1 \circ \cdots \circ x_{n-1} \bullet x_n = ((\cdots(x_1 \circ x_2) \circ \cdots) \circ x_{n-1}) \bullet x_n$.

Inspired by Ferreira and Wei[16], we introduce the concept of a nonlinear mixed $*$-Jordan-type higher derivations $\{\delta_m\}_{m\in\mathbb{N}}$ on $*$-algebras, which contains nonlinear mixed $*$-Jordan triple derivations, nonlinear mixed $*$-Jordan-type derivations, etc., as its special form. Let $\mathcal{N}$ be the set of all non-negative integers and $\Delta = \{\Psi_m\}_{m\in\mathcal{N}}$ be a family of mapping $\Psi_m : \mathfrak{B} \to \mathfrak{B}$ (without the additivity assumption) such that $\Psi_0 = id_{\mathfrak{B}}$. Δ is called:

(a) an additive higher $*$-derivation if

$$\Psi_m(xy) = \sum_{i+j=m} \Psi_i(x)\Psi_j(y), \ \ \Psi_m(x+y) = \Psi_m(x) + \Psi_m(y) \ \text{ and } \ \Psi_m(y^*) = \Psi_m(y)^* \tag{1.2}$$

for all $x, y \in \mathfrak{B}$ and for each $m \in \mathcal{N}$;

(b) a nonlinear mixed $*$-Jordan higher n-derivation if

$$\Psi_m(x_1 \circ \cdots \circ x_{n-1} \bullet x_n) = \sum_{i_1+\cdots+i_n=m} \Psi_{i_1}(x_1) \circ \cdots \circ \Psi_{i_{n-1}}(x_{n-1}) \bullet \Psi_{i_n}(x_n) \tag{1.3}$$

for all $x_1, \cdots, x_n \in \mathfrak{B}$ and for each $n, m \in \mathcal{N}$ such that $n \geq 3$.

This notion makes the best use of the definition of nonlinear mixed $*$-Jordan-type higher derivation. The main statement is as follows: when $m = 1$ in (1.2) and (1.3), the map $\Psi_1 : \mathfrak{B} \to \mathfrak{B}$ is an additive $*$-derivation and a nonlinear mixed $*$-Jordan-type derivation, respectively. Many mappings associated with nonlinear mixed $*$-Jordan-type derivations have been studied by scholars, see [3, 4, 6, 7, 10, 16].

In the scope of the author's research, many researchers have paid attention to the additivity of maps (without assuming additivity) associated with various products on $*$-algebra $\mathfrak{B}$, and studied the relationship between maps and $*$-derivations. Rehman and co-authors[18] studied the structure of the first nonlinear mixed Jordan triple derivation associated with the mixed product $A \star B \circ C$ and show that it is additive $*$-derivation. Rehman and co-authors[19] proved that the first nonlinear mixed Jordan triple derivation defined via the mixed product $A \circ B \star C$ on the $*$-algebra is additive $*$-derivation. It should be noted that Peng and Ma[20], independent of [19], studied the nonlinear mixed Jordan triple derivation defined via the mixed product $A \circ B \star C$ on the factor von Neumann algebras, which is also called the first nonlinear mixed Jordan triple derivation, and it is shown to be an additive $*$-derivation. Ashraf and co-authors[1] proved that every nonlinear bi-skew Jordan-type derivation on factor von Neumann algebra is an additive $*$-derivation. Meanwhile, Zhao and co-authors[12] generalized the results of [1, 5] to $*$-algebra $\mathfrak{B}$, that is, every nonlinear bi-skew Jordan-type derivation on unital $*$-algebra $\mathfrak{B}$ is an additive $*$-derivation. At the same time, some scholars have studied the structural properties of some higher derivations on algebras along the framework of Herstein Lie type mapping[8]. Wani and his collaborators[11] have studied the structure of multiplicative $*$-Jordan-type higher derivations on von Neumann algebras without nonzero central abelian projections, and proved that every multiplicative $*$-Jordan-type higher derivations on von Neumann algebras is an additive higher $*$-derivation. After that, in 2024, Liang and co-authors[9] extended the results of [1, 5, 12] to nonlinear bi-skew Jordan-type higher derivations and proved that every nonlinear bi-skew Jordan-type higher derivation is an additive higher $*$-derivation. After that, it was surprising to find that Ferreira and Wei[16] studied the structure of the nonlinear mixed $*$-Jordan-type derivation associated with mixed product $U_n(x_n, \cdots, x_1)$, and proved that every nonlinear mixed $*$-Jordan-type derivation is an additive $*$-derivation. Inspired by [9, 16], an interesting question is raised:

Qestion 1.1. *Is a nonlinear mixed $*$-Jordan-type higher derivation on an unital $*$-algebra an additive higher $*$-derivation?*

The subject of this article is to give positive responses to Question 1.1, that is, it proves that every nonlinear mixed $*$-Jordan-type higher derivation on unital $*$-algebra is an additive higher $*$-derivation. The affirmative solution would allow one to obtain structure of nonlinear mixed $*$-Jordan-type higher derivations on some operator algebras, such as standard operator algebras, prime $*$-algebras, factor von Neumann algebras and von Neumann algebras of type I_1, but also generalizes many meaningful conclusions, such as [16, Theorem 1].

2. Nonlinear mixed $*$-Jordan-type higher derivation

In this part, we will study the structure of nonlinear mixed $*$-Jordan-type higher derivations on unital $*$-algebras. For this purpose, we first introduce the concept of unital $*$-algebras and some important symbols.

Suppose that the symbol $\mathfrak{B}$ represents unital $*$-algebra with identity I and a nontrivial projection e_1 (that is, $e_1 \neq 0$ and $e_1^2 = e_1 = e_1^*$) and write $e_2 = I - e_1$. With the help of Peirce decomposition, we have the decomposition form of the algebra $\mathfrak{B}$ as follows

$$\mathfrak{B} = e_1\mathfrak{B}e_1 + e_1\mathfrak{B}e_2 + e_2\mathfrak{B}e_1 + e_2\mathfrak{B}e_2.$$

Below we introduce the symbols $\mathfrak{B}_{11}, \mathfrak{B}_{12}, \mathfrak{B}_{21}$ and $\mathfrak{B}_{22}$ for $e_1\mathfrak{B}e_1, e_1\mathfrak{B}e_2, e_2\mathfrak{B}e_1$ and $e_2\mathfrak{B}e_2$ respectively, which satisfy the multiplicative relations which also satisfy the relation $\mathfrak{B}_{ij}\mathfrak{B}_{lk} = \{0\}$ if $j \neq l$ and $\mathfrak{B}_{ij}\mathfrak{B}_{jk} = \mathfrak{B}_{ik}$ for $i, j, k \in \{1, 2\}$.

Throughout the paper, we assume that unital $*$-algebra $\mathfrak{B}$ is consistent with the following condition:

$$\mathfrak{C} = \begin{cases} Y\mathfrak{B}e_1 = 0 \text{ implies } Y = 0, \\ Y\mathfrak{B}e_2 = 0 \text{ implies } Y = 0. \end{cases}$$

By means of $\mathfrak{C}$, standard operator algebras, factor von Neumann algebras, von Neumann algebras of type I_1 and prime $*$-algebras satisfy condition $\mathfrak{C}$, and thus they become typical examples of unital $*$-algebras.

Theorem 2.1. *Let $\mathfrak{B}$ be an unital $*$-algebra with identity element I that satisfies condition $\mathfrak{C}$. Then every nonlinear mixed $*$-Jordan-type higher derivation satisfies equation* (1.3) *on $\mathfrak{B}$ is an additive higher $*$-derivation.*

To obtain the theorem, we use mathematical induction for m, which appears in equation (1.3). When $m = 1$ in Eq (1.3), Ψ_1 is a nonlinear mixed $*$-Jordan n-derivation on $\mathfrak{B}$, which provides the results underlying the use of mathematical induction in this paper. With aid of [16, Theorem 1], every nonlinear mixed $*$-Jordan n-derivation $\Psi_1 : \mathfrak{B} \to \mathfrak{B}$ is an additive $*$-derivation on $\mathfrak{B}$ and satisfies the following conditions:

$$\mathfrak{H}_1 = \begin{cases} \Psi_1(0) = 0;\ \Psi_1(I) = \Psi_1(iI) = 0; \\ \Psi_1(a_{11} + a_{12} + a_{21} + a_{22}) = \Psi_1(a_{11}) + \Psi_1(a_{12}) + \Psi_1(a_{21}) + \Psi_1(a_{22}) \text{ for all } a_{ij} \in \mathfrak{B}_{ij}(i, j \in \{1, 2\}); \\ \Psi_1(a_{ij} + b_{ij}) = \Psi_1(a_{ij}) + \Psi_1(b_{ij}) \text{ for all } a_{ij}, b_{ij} \in \mathfrak{B}_{ij} \text{ for } i \neq j \in \{1, 2\}; \\ \Psi_1(a_{ii} + b_{ii}) = \Psi_1(a_{ii}) + \Psi_1(b_{ii}) \text{ for all } a_{ii}, b_{ii} \in \mathfrak{B}_{ii}, i \in \{1, 2\}; \\ \Psi_1(A)^* = \Psi_1(A^*);\ \Psi_1(\mathrm{i}A) = \mathrm{i}\Psi_1(A) \text{ for all } A \in \mathfrak{B}. \end{cases}$$

We assume that the mappings Ψ_s holds for all $1 < s < m$ on an unital $*$-algebra $\mathfrak{B}$ satisfies the following:

$$\mathfrak{H}_s = \begin{cases} \Psi_s(0) = 0;\ \Psi_s(I) = \Psi_s(iI) = 0; \\ \Psi_s(a_{11} + a_{12} + a_{21} + a_{22}) = \Psi_s(a_{11}) + \Psi_s(a_{12}) + \Psi_s(a_{21}) + \Psi_s(a_{22}) \text{ for all } a_{ij} \in \mathfrak{B}_{ij}(i, j \in \{1, 2\}); \\ \Psi_s(a_{ij} + b_{ij}) = \Psi_s(a_{ij}) + \Psi_s(b_{ij}) \text{ for all } a_{ij}, b_{ij} \in \mathfrak{B}_{ij} \text{ for } i \neq j \in \{1, 2\}; \\ \Psi_s(a_{ii} + b_{ii}) = \Psi_s(a_{ii}) + \Psi_s(b_{ii}) \text{ for all } a_{ii}, b_{ii} \in \mathfrak{B}_{ii}, i \in \{1, 2\}; \\ \Psi_s(A)^* = \Psi_s(A^*);\ \Psi_s(\mathrm{i}A) = \mathrm{i}\Psi_s(A) \text{ for all } A \in \mathfrak{B}. \end{cases}$$

The remainder of this section will devoted to show that the nonlinear mixed $*$-Jordan higher n-derivations Ψ_m still satisfies condition $\mathfrak{H}_s$ for $s = m$, and then prove that nonlinear mixed Jordan higher n-derivations $\Delta = \{\Psi_m\}_{m \in \mathcal{N}}$ are additive higher $*$-derivations.

In order to simplify the proof process, we will use the symbol $\xi_a(x_{n-1}, x_n)$ to denote $U_n(a, \cdots, a, x_{n-1}, x_n) = a \circ a \circ \cdots \circ a \circ x_{n-1} \bullet x_n$.

The induction process can be realized through a series of lemmas.

Lemma 2.2. $\Psi_m(0) = 0$.

Proof. By the hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, i.e., $\Psi_s(0) = 0$, we have

$$\begin{aligned}\Psi_m(0) &= \Psi_m(U_n(0, 0, \cdots, 0))\\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(0), \Psi_{i_2}(0), \cdots, \Psi_{i_n}(0))\\ &= U_n(\Psi_m(0), 0, \cdots, 0) + U_n(0, \Psi_m(0), 0, \cdots, 0) + \cdots + U_n(0, \cdots, 0, \Psi_m(0))\\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n<m}} U_n(\Psi_{i_1}(0), \Psi_{i_2}(0), \cdots, \Psi_{i_n}(0))\\ &= 0.\end{aligned}$$

□

Lemma 2.3. $\Psi_m(a_{ii} + a_{ij}) = \Psi_m(a_{ii}) + \Psi_m(a_{ij})$ *for every* $a_{ii} \in \mathfrak{B}_{ii}, a_{ij} \in \mathfrak{B}_{ij}$ $i \neq j \in \{1, 2\}$.

Proof. In the following we consider the case where $(i, j) = (1, 2)$.

For every $a_{11} \in \mathfrak{B}_{11}, a_{12} \in \mathfrak{B}_{12}$, consider $t = \Psi_m(a_{11} + a_{12}) - \Psi_m(a_{11}) - \Psi_m(a_{12})$, for any $x_{21} \in \mathfrak{B}_{21}$, by the fact inductive hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$ and equation $\xi_I(a_{12}, \frac{x_{21}}{2^{n-2}}) = 0$, we have

$$\begin{aligned}&\Psi_m(\xi_I(a_{11} + a_{12}, \frac{x_{21}}{2^{n-2}}))\\ =&\Psi_m(\xi_I(a_{11}, \frac{x_{21}}{2^{n-2}})) + \Psi_m(\xi_I(a_{12}, \frac{x_{21}}{2^{n-2}}))\\ =&\sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}), \Psi_{i_n}(\frac{x_{21}}{2^{n-2}}))\\ &+ \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{12}), \Psi_{i_n}(\frac{x_{21}}{2^{n-2}}))\\ =&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}) + \Psi_{i_{n-1}}(a_{12}), \Psi_{i_n}(\frac{x_{21}}{2^{n-2}}))\\ &+ \xi_I(\Psi_m(a_{ii}) + \Psi_m(a_{12}), \frac{x_{21}}{2^{n-2}}).\end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned}&\Psi_m(\xi_I(a_{11} + a_{12}, \frac{x_{21}}{2^{n-2}}))\\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11} + a_{12}, \Psi_{i_n}(\frac{x_{21}}{2^{n-2}}))\\ &= \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11} + a_{12}, \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}}))\\ &\quad + \xi_I(\Psi_m(a_{ii} + a_{ij}), \frac{x_{lk}}{2^{n-2}}).\end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that $\xi_I(t, \frac{x_{21}}{2^{n-2}}) = 0$. It follows from condition $\mathfrak{C}$ that $t_{22} = 0$.

By replacing ix_{21} in the above two equations (2.1) and (2.2) with x_{21} and following a similar calculation process, the results can be obtained $t_{21} = 0$.

In the following we show that $t_{11} = t_{12} = 0$ holds.

To arrive at the conclusion, we will use two different expansions of the equation $\Psi_m(\xi_I(\xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}}))$. On the one hand for $l \neq k \in \{1, 2\}$, we have

$$
\begin{aligned}
&\Psi_m(\xi_I(\xi_I(a_{11} + a_{12}, \frac{x_{21}}{2^{n-2}}), \frac{e_1}{2^{n-2}}))\\
=&\Psi_m(\xi_I(\xi_I(a_{11}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}})) + \Psi_m(\xi_I(\xi_I(a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_l}{2^{n-2}}))\\
=&\Psi_m(U_n(I, \cdots, I, \xi_I(a_{11}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}}) + \Psi_m(U_n(I, \cdots, I, \xi_I(a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}})\\
=&\sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{11}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+\sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{12}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
=&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{11}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+\xi_I(\Psi_m(\xi_I(a_{11}, \frac{x_{12}}{2^{n-2}})), \frac{e_1}{2^{n-2}})\\
&+\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{12}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+\xi_I(\Psi_m(\xi_I(a_{ij}, \frac{x_{12}}{2^{n-2}})), \frac{e_l}{2^{n-2}})\\
=&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{11}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+U_n(I, \cdots, I, \sum_{\substack{j_1+\cdots+j_n=m,\\ j_{n-1}\neq m}} U_n(\Psi_{j_1}(I), \cdots, \Psi_{j_{n-2}}(I), \Psi_{j_{n-1}}(\xi_I(a_{11}, \Psi_{j_n}(\frac{x_{12}}{2^{n-2}})), \frac{e_1}{2^{n-2}})\\
&+\xi_I(\xi_I(\Psi_m(a_{11}), \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}})\\
&+\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{12}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+U_n(I, \cdots, I, \sum_{\substack{j_1+\cdots+j_n=m,\\ j_{n-1}\neq m}} U_n(\Psi_{j_1}(I), \cdots, \Psi_{j_{n-2}}(I), \Psi_{j_{n-1}}(\xi_I(a_{12}, \Psi_{j_n}(\frac{x_{12}}{2^{n-2}})), \frac{e_1}{2^{n-2}})\\
&+\xi_I(\xi_I(\Psi_m(a_{12}), \frac{x_{21}}{2^{n-2}}), \frac{e_1}{2^{n-2}}).
\end{aligned}
\tag{2.3}
$$

On the other hand, with the aid of another decomposition of the element $\Psi_m(\xi_I(\xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}}))$, we have

$$
\begin{aligned}
&\Psi_m(\xi_I(\xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}}))\\
=&\Psi_m(\xi_n(I, \cdots, I, \xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}}), \frac{e_1}{2^{n-2}})\\
=&\sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
=&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\xi_I(a_{11} + a_{12}, \frac{x_{12}}{2^{n-2}})), \Psi_{i_n}(\frac{e_1}{2^{n-2}}))
\end{aligned}
$$

$$
\begin{aligned}
&+\xi_I(\Psi_m(\xi_I(a_{11}+a_{12},\frac{x_{12}}{2^{n-2}}),\frac{e_1}{2^{n-2}}))\\
=&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I),\cdots,\Psi_{i_{n-2}}(I),\Psi_{i_{n-1}}(\xi_I(a_{11}+a_{12},\frac{x_{12}}{2^{n-2}})),\Psi_{i_n}(\frac{e_1}{2^{n-2}}))\\
&+\xi_I(\sum_{\substack{j_1+\cdots+j_n=m,\\ j_{n-1}\neq m}} U_n(\Psi_{j_1}(I),\cdots,\Psi_{j_{n-2}}(I),\Psi_{j_{n-1}}(a_{11}+a_{12}),\Psi_{j_n}(\frac{x_{12}}{2^{n-2}})),\frac{e_1}{2^{n-2}})\\
&+\xi_I(\xi_I(\Psi_m(a_{11}+a_{12}),\frac{x_{12}}{2^{n-2}}),\frac{e_1}{2^{n-2}})
\end{aligned} \tag{2.4}
$$

With respect to (2.3) and (2.4), we have

$$\xi_I(\xi_I(t,\frac{x_{12}}{2^{n-2}}),\frac{e_1}{2^{n-2}})=0.$$

It can be obtained by the above equation

$$0=x_{12}^*t_{11}+t_{11}^*x_{12}$$

We have $x_{12}^*t_{11}=0$ or $t_{11}^*xe2=0$ for all $x\in\mathfrak{B}$. In accordance with condition $\mathfrak{C}$, we get $t_{11}=0$. Similarly, we can show that $t_{12}=0$ by applying e_2 instead of e_1 in above.

By similar computational tricks and methods we can show that the case $(i,j)=(2,1)$ also holds. □

Lemma 2.4. *With notations as above, we obtain*

$$\Psi_m(a_{11}+a_{12}+a_{21})=\Psi_m(a_{11})+\Psi_m(a_{12})+\Psi_m(a_{21})$$

and

$$\Psi_m(a_{22}+a_{12}+a_{21})=\Psi_m(a_{22})+\Psi_m(a_{12})+\Psi_m(a_{21})$$

for all $a_{ij}\in\mathfrak{B}_{ij}$ $(i,j\in\{1,2\})$.

Proof. To prove this lemma, we introduce symbol $V^m=\Psi_m(a_{11}+a_{12}+a_{21})-\Psi_m(a_{11})-\Psi_m(a_{12})-\Psi_m(a_{21})$. In agreement with Lemma 2.3 and inductive hypothesis $\mathfrak{H}_s$ $(1\leq s\leq m-1)$, we get

$$
\begin{aligned}
&\xi_I(\Psi_m(a_{11}+a_{12}+a_{21}),\frac{x_{lk}}{2^{n-2}})\\
&+\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I),\cdots,\Psi_{i_{n-2}}(I),\Psi_{i_{n-1}}(a_{11}+a_{12}+a_{21}),\Psi_{i_n}(\frac{x_{lk}}{2^{n-2}}))\\
&=\Psi_m(U_n(I,\cdots,I,a_{11}+a_{12}+a_{21},\frac{x_{lk}}{2^{n-2}}))\\
&=\Psi_m(\xi_I(a_{11}+a_{12}+a_{21},\frac{x_{lk}}{2^{n-2}}))\\
&=\Psi_m(\xi_I(a_{11}+a_{12},\frac{x_{lk}}{2^{n-2}}))+\Psi_m(\xi_I(a_{21},\frac{x_{lk}}{2^{n-2}}))
\end{aligned}
$$

$$
\begin{aligned}
&= \xi_I(\Psi_m(a_{11}+a_{12}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}+a_{12}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}})) \\
&+ \xi_I(\Psi_m(a_{21}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{21}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}})) \\
&= \xi_I(\Psi_m(a_{11}) + \Psi_m(a_{12}) + \Psi_m(a_{21}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}+a_{12}+a_{21}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}}))
\end{aligned} \tag{2.5}
$$

for any x_{lk} such that $(l,k) = (1,2)$. It follows from the above two equations that $\xi_I(V^m, x_{lk}) = 0$, which implies that

$$V_{11}^m = 0 \text{ for } (l,k) = (1,2).$$

In the above operation, if $(l,k) = (2,1)$ in (2.5), I use a similar operation method for the relationship using Eq.

$$\Psi_m(\xi_I(a_{11}+a_{12}+a_{21}, \frac{x_{lk}}{2^{n-2}})) = \Psi_m(\xi_I(a_{11}+a_{21}, \frac{x_{lk}}{2^{n-2}})) + \Psi_m(\xi_I(a_{12}, \frac{x_{lk}}{2^{n-2}})) \tag{2.6}$$

to obtain that $V_{22}^m = 0$ holds.

Similarly by applying ix_{lk} in the above equations (2.5) and (2.6) we obtain

$$V_{12}^m = 0 \text{ for } (l,k) = (1,2) \text{ and } V_{21}^m = 0 \text{ for } (l,k) = (2,1).$$

Using similar computational techniques, we can obtain $\Psi_m(a_{22}+a_{12}+a_{21}) = \Psi_m(a_{22}) + \Psi_m(a_{12}) + \Psi_m(a_{21})$ for all $a_{ij} \in \mathfrak{B}_{ij}$ $(i,j \in \{1,2\})$. □

Lemma 2.5. *With notations as above, we have*

$$\Psi_m(a_{11}+a_{12}+a_{21}+a_{22}) = \Psi_m(a_{11}) + \Psi_m(a_{12}) + \Psi_m(a_{21}) + \Psi_m(a_{22})$$

for all $a_{ij} \in \mathfrak{B}_{ij}$ $(i,j \in \{1,2\})$.

Proof. Let us prove this lemma by introducing the notation $V^m = \Psi_m(a_{11}+a_{12}+a_{21}+a_{22}) - \Psi_m(a_{11}) - \Psi_m(a_{12}) - \Psi_m(a_{21}) - \Psi_m(a_{22})$ for all $a_{ij} \in \mathfrak{B}_{ij}$ $(i,j \in \{1,2\})$. In accordance with Lemma 2.4, induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, we know

$$\begin{aligned}
&\xi_I(\Psi_m(a_{11}+a_{12}+a_{21}+a_{22}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}) + \Psi_{i_{n-1}}(a_{12}) + \Psi_{i_{n-1}}(a_{21}) + \Psi_{i_{n-1}}(a_{22}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}})) \\
&= \xi_I(\Psi_m(a_{11}+a_{12}+a_{21}+a_{22}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \Psi_{i_2}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}+a_{12}+a_{21}+a_{22}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}})) \\
&= \Psi_m(\xi_I(a_{11}+a_{12}+a_{21}+a_{22}, \frac{x_{lk}}{2^{n-2}})) \\
&= \Psi_m(\xi_I(a_{11}+a_{12}+a_{21}, \frac{x_{lk}}{2^{n-2}})) + \Psi_m(\xi_I(a_{22}, \frac{x_{lk}}{2^{n-2}})) \\
&= \xi_I(\Psi_m(a_{11}) + \Psi_m(a_{12}) + \Psi_m(a_{21}) + \Psi_m(a_{22}), \frac{x_{lk}}{2^{n-2}}) \\
&+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{11}) + \Psi_{i_{n-1}}(a_{12}) + \Psi_{i_{n-1}}(a_{21}) + \Psi_{i_{n-1}}(a_{22}), \Psi_{i_n}(\frac{x_{lk}}{2^{n-2}}))
\end{aligned} \tag{2.7}$$

for arbitrary $x_{lk} \in \mathfrak{B}_{lk}$ such that $(l,k) = (1,2)$. Then, we have $\xi_I(V^m, \frac{x_{lk}}{2^{n-2}}) = 0$, which implies that

$$V_{11}^m = 0 \text{ for } (l,k) = (1,2).$$

In the above operation, if $(l,k) = (2,1)$ in (2.7), I use a similar operation method for the relationship using Eq.

$$\Psi_m(\xi_I(a_{11}+a_{12}+a_{21}+a_{22}, \frac{x_{lk}}{2^{n-2}})) = \Psi_m(\xi_I(a_{12}+a_{21}+a_{22}, \frac{x_{lk}}{2^{n-2}})) + \Psi_m(\xi_I(a_{11}, \frac{x_{lk}}{2^{n-2}})) \tag{2.8}$$

to obtain that $V_{22}^m = 0$ holds.

Similarly by applying ix_{lk} in (2.7) and (2.8), where i is imaginary unity, in the above equation we get $V_{12}^m = 0$ for $(l,k) = (1,2)$ and $V_{21}^m = 0$ for $(l,k) = (2,1)$.
□

Lemma 2.6. *With notations as above, we have*

$$\Psi_m(a_{ij} + b_{ij}) = \Psi_m(a_{ij}) + \Psi_m(b_{ij})$$

for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ such that $i \neq j$.

Proof. To prove this, we introduce notation $V^m = \Psi_m(a_{ij}+b_{ij}) - \Psi_m(a_{ij}) - \Psi_m(b_{ij})$ for all $a_{ij}, b_{ij} \in \mathfrak{B}_{ij}$ such that $i \neq j \in \{1,2\}$. According to Lemma 2.5 and the induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, we know that

$$\begin{aligned}
&\Psi_m(a_{ij}+b_{ij}) + \Psi_m(a_{ij}^* + b_{ij}^*) \\
=&\Psi_m(\xi_I(e_i + a_{ij}^*, \frac{e_j + b_{ij}}{2^{n-2}})) \\
=& \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}, i_n\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(e_i + a_{ij}^*), \Psi_{i_n}(\frac{e_j + b_{ij}}{2^{n-2}})) \\
&+ \xi_I(\Psi_m(e_i) + \Psi_m(a_{ij}^*), \frac{e_j + b_{ij}}{2^{n-2}}) + \xi_I(e_i + a_{ij}^*, \Psi_m(\frac{e_j}{2^{n-2}}) + \Psi_m(\frac{b_{ij}}{2^{n-2}})) \\
=&\Psi_m(\xi_I(a_{ij}^*, \frac{e_j}{2^{n-2}})) + \Psi_m(\xi_I(e_i, \frac{b_{ij}}{2^{n-2}})) \\
=&\Psi_m(a_{ij}) + \Psi_m(b_{ij}) + \Psi_m(a_{ij}^*) + \Psi_m(b_{ij}^*).
\end{aligned} \tag{2.9}$$

Now it is easy to see that $\Psi_m(a_{ij}+b_{ij})-\Psi_m(a_{ij})-\Psi_m(b_{ij})\in\mathfrak{B}_{ii}\oplus\mathfrak{B}_{ij}$. Indeed, with the help of $\xi_I(a_{ij},\frac{e_j}{2^{n-2}})=0$, we have

$$\begin{aligned}&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}}U_n(\Psi_{i_1}(I),\cdots,\Psi_{i_{n-2}}(I),\Psi_{i_{n-1}}(a_{ij}+b_{ij}),\Psi_{i_n}(\frac{e_j}{2^{n-2}}))\\ &+\xi_I(\Psi_m(a_{ij}+b_{ij}),\frac{e_j}{2^{n-2}})\\ =&\Psi_m(\xi_I(a_{ij}+b_{ij},\frac{e_j}{2^{n-2}}))\\ =&\Psi_m(\xi_I(a_{ij},\frac{e_j}{2^{n-2}}))+\Psi_m(\xi_I(b_{ij},\frac{e_j}{2^{n-2}}))\\ =&\sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}}U_n(\Psi_{i_1}(I),\cdots,\Psi_{i_{n-2}}(I),\Psi_{i_{n-1}}(a_{ij})+\Psi_{i_{n-1}}(b_{ij}),\Psi_{i_n}(\frac{e_j}{2^{n-2}}))\\ &+\xi_I(\Psi_m(a_{ij})+\Psi_m(b_{ij}),\frac{e_j}{2^{n-2}})\end{aligned}$$

And then it yields that $\xi_I(V^m,e_j)=0$ and $\xi_I(V^m,\mathrm{i}e_j)=0$. And then

$$(V^m)^*e_j+e_j(V^m)=0 \text{ and } (V^m)^*e_j-e_j(V^m)=0.$$

Furthermore, we have $e_j(V^m)=0$. By multiplying e_j and e_i by equation $e_j(V^m)=0$, we get that equation $V^m_{jj}=e_j(V^m)e_j=0$ and $V^m_{ji}=e_j(V^m)e_i=0$ hold. So

$$V^m=\Psi_m(a_{ij}+b_{ij})-\Psi_m(a_{ij})-\Psi_m(b_{ij})=V^m_{ii}+V^m_{ij}\in\mathfrak{B}_{ii}\oplus\mathfrak{B}_{ij}.$$

In the same way available:

$$V^m=\Psi_m(a_{ji}+b_{ji}-\Psi_m(a_{ji})-\Psi_m(b_{ji})=V^m_{jj}+V^m_{ji}\in\mathfrak{B}_{ji}\oplus\mathfrak{B}_{jj}.$$

By virtue of (2.9), it follows that:

$$\Psi_m(a_{ij}+b_{ij})-\Psi_m(a_{ij})-\Psi_m(b_{ij})=\Psi_m(a^*_{ij})+\Psi_m(b^*_{ij})-\Psi_m(a^*_{ij}+b^*_{ij}).$$

With the help of $(\mathfrak{B}_{ii}\oplus\mathfrak{B}_{ij})\cap\mathfrak{B}_{ji}\oplus\mathfrak{B}_{jj}=\{0\}$ for $i\neq j\in\{1,2\}$. Therefore, $\Psi_m(a_{ij}+b_{ij})=\Psi_m(a_{ij})+\Psi_m(b_{ij})$.
□

Lemma 2.7. *With notations as above, we have*

$$\Psi_m(a_{ii}+b_{ii})=\Psi_m(a_{ii})+\Psi_m(b_{ii})$$

for all $a_{ii},b_{ii}\in\mathfrak{B}_{ii}, i\in\{1,2\}$.

Proof. The main purpose of this lemma is to prove that the map Ψ_m agrees with additivity on $\mathfrak{B}_{ii}$ $(i\in\{1,2\})$. Consider $V^m=\Psi_m(a_{ii}+b_{ii})-\Psi_m(a_{ii})-\Psi_m(b_{ii})$ for all $a_{ii},b_{ii}\in\mathfrak{B}_{ii}, i\in\{1,2\}$.

It follows from the induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$ and $\xi_I(b_{ii}, \frac{e_j}{2^{n-2}}) = 0$ for $i \neq j \in \{1, 2\}$ that

$$\begin{aligned}
&\xi_I(\Psi_m(a_{ii}+b_{ii}), \frac{e_j}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{ii}+b_{ii}), \Psi_{i_n}(\frac{e_j}{2^{n-2}}))\\
=&\Psi_m(\xi_I(a_{ii}+b_{ii}, \frac{e_j}{2^{n-2}}))\\
=&\Psi_m(\xi_I(a_{ii}, \frac{e_j}{2^{n-2}})) + \Psi_m(\xi_I(b_{ii}, \frac{e_j}{2^{n-2}}))\\
=&\xi_I(\Psi_m(a_{ii}), \frac{e_j}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{ii}), \Psi_{i_n}(\frac{e_j}{2^{n-2}}))\\
&+ \xi_I(\Psi_m(b_{ii}), \frac{e_j}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(b_{ii}), \Psi_{i_n}(\frac{e_j}{2^{n-2}}))\\
=&\xi_I(\Psi_m(a_{ii}) + \Psi_m(b_{ii}), \frac{e_j}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{ii}+b_{ii}), \Psi_{i_n}(\frac{e_j}{2^{n-2}}))
\end{aligned}$$

And then, we have $\xi_I(V^m, e_j) = 0$ and $\xi_I(V^m, \mathrm{i}e_j) = 0$, which implies that $V^m_{ji} = V^m_{jj} = 0$, where i is imaginary unity.

In agreement with Lemma 2.5 and Lemma 2.6, and the induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$, we have

$$\begin{aligned}
&\xi_I(\Psi_m(a_{ii}+b_{ii}), \frac{x_{ij}}{2^{n-2}})) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{ii}+b_{ii}), \Psi_{i_n}(\frac{x_{ij}}{2^{n-2}}))\\
=&\Psi_m(\xi_I(a_{ii}+b_{ii}, \frac{x_{ij}}{2^{n-2}}))\\
=&\Psi_m(a_{ii}^*x_{ij} + b_{ii}^*x_{ij}) + \Psi_m(x_{ij}^*a_{ii} + x_{ij}^*b_{ii})\\
=&\Psi_m(a_{ii}^*x_{ij}) + \Psi_m(b_{ii}^*x_{ij}) + \Psi_m(x_{ij}^*a_{ii}) + \Psi_m(x_{ij}^*b_{ii})\\
=&\Psi_m(a_{ii}^*x_{ij} + x_{ij}^*a_{ii}) + \Psi_m(b_{ii}^*x_{ij} + x_{ij}^*b_{ii})\\
=&\Psi_m(\xi_I(a_{ii}, \frac{x_{ij}}{2^{n-2}})) + \Psi_m(\xi_I(b_{ii}, \frac{x_{ij}}{2^{n-2}}))\\
=&\xi_I(\Psi_m(a_{ii}), \frac{x_{ij}}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(a_{ii}), \Psi_{i_n}(\frac{x_{ij}}{2^{n-2}}))\\
&+ \xi_I(\Psi_m(b_{ii}), \frac{x_{ij}}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(b_{ii}), \Psi_{i_n}(\frac{x_{ij}}{2^{n-2}}))\\
=&\xi_I(\Psi_m(a_{ii}) + \Psi_m(b_{ii}), \frac{x_{ij}}{2^{n-2}}) + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_{n-1}\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_n-1}(a_{ii}+b_{ii}), \Psi_{i_n}(\frac{x_{ij}}{2^{n-2}}))
\end{aligned}$$

With the help of $\xi_I(V^m, x_{ij}) = 0$ and $\xi_I(V^m, ix_{ij}) = 0$, we have $(V^m)^*x_{ij} + x_{ij}^*V^m = 0$ and $(V^m)^*x_{ij} - x_{ij}^*V^m = 0$, and then $0 = (V^m)^*x_{ij} = (V^m_{ii})^*x_{ij} + (V^m_{ij})^*x_{ij}$. Since $\mathfrak{B}_{ij} \ni (V^m_{ii})^*x_{ij} = -(V^m_{ij})^*x_{ij} \in \mathfrak{B}_{jj}$, combined with $\mathfrak{B}_{ij} \cap \mathfrak{B}_{jj} = \{0\}$ for $i \neq j$, we get $(V^m_{ii})^*x_{ij} = 0 = -(V^m_{ij})^*x_{ij}$. With the help of condition $\mathbb{C}$, we have we know that $V^m_{ii} = V^m_{ij} = 0$ is true. To sum up, it can be seen that $V^m = 0$.

□

Lemma 2.8. *$\Psi_m(I) = \Psi_m(iI) = 0$.*

Proof. Using induction hypothesis $\mathfrak{H}_s$ $(1 \le s \le m-1)$, we obtain

$$\begin{aligned}\Psi_m(\xi_I(I, \frac{I}{2^{n-3}})) &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(I), \Psi_{i_n}(\frac{I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-1} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \frac{I}{2^{n-3}}) + U_n(I, \cdots, I, \Psi_m(\frac{I}{2^{n-3}})) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(I), \Psi_{i_n}(\frac{I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-1} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \frac{I}{2^{n-3}}) + U_n(I, \cdots, I, \Psi_m(\frac{I}{2^{n-3}}))\end{aligned}$$

Hence, $\Psi_m(I) = \frac{n}{2}(\Psi_m(I) + \Psi_m(I)^*)$. Therefore, $\Psi_m(I) = 0$.

Also, notice the equation

$$\begin{aligned}&\Psi_m(\xi_I(I, \frac{\mathrm{i}I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-1} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \frac{\mathrm{i}I}{2^{n-3}}) + U_n(I, \cdots, I, \Psi_m(\frac{\mathrm{i}I}{2^{n-3}})) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(I), \Psi_{i_n}(\frac{\mathrm{i}I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-1} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \frac{\mathrm{i}I}{2^{n-3}}) + U_n(I, \cdots, I, \Psi_m(\frac{\mathrm{i}I}{2^{n-3}})).\end{aligned}$$

It follows that

$$\Psi_m(\mathrm{i}I)^* = -\Psi_m(\mathrm{i}I). \tag{2.10}$$

Hence,

$$\begin{aligned}&\Psi_m(\xi_I(\mathrm{i}I, \frac{\mathrm{i}I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-2} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \mathrm{i}I, \frac{\mathrm{i}I}{2^{n-3}}) \\ &\quad + U_n(I, \cdots, I, \Psi_m(\mathrm{i}I), \frac{\mathrm{i}I}{2^{n-3}}) + U_n(I, \cdots, \mathrm{i}I, \Psi_m(\frac{\mathrm{i}I}{2^{n-3}})) \\ &\quad + \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(\mathrm{i}I), \Psi_{i_n}(\frac{\mathrm{i}I}{2^{n-3}})) \\ &= \sum_{k=1}^{n-2} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \mathrm{i}I, \frac{\mathrm{i}I}{2^{n-3}}) \\ &\quad + U_n(I, \cdots, I, \Psi_m(\mathrm{i}I), \frac{\mathrm{i}I}{2^{n-3}}) + U_n(I, \cdots, \mathrm{i}I, \Psi_m(\frac{\mathrm{i}I}{2^{n-3}})).\end{aligned}$$

It follows that

$$\Psi_m(\mathrm{i}I)^* - \Psi_m(\mathrm{i}I) = 0. \tag{2.11}$$

Finally, in accordance with (2.10) and (2.11), we get $\Psi_m(\mathrm{i}I) = 0$.
□

Lemma 2.9. *Following the notations above, we realize Ψ_m preserves involution, i.e., $\Psi_m(A^*) = \Psi_m(A)^*$ for arbitrary $A \in \mathfrak{B}$.*

Proof. In harmony with the induction hypothesis $\mathfrak{H}_s$ ($1 \le s \le m-1$) and Lemma 2.8, we have

$$\begin{aligned}\Psi_m(\xi_I(I, \frac{A}{2^{n-2}})) = &\sum_{k=1}^{n-1} U_n(I, I, \cdots, I, \underbrace{\Psi_m(I)}_{k-component}, I, \cdots, I, \frac{A}{2^{n-2}}) \\ &+ \xi_I(I, \Psi_m(\frac{A}{2^{n-2}})) \\ &+ \sum_{\substack{i_1+\cdots+i_n=m,\\ i_1,\cdots,i_n\neq m}} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(I), \Psi_{i_n}(\frac{A}{2^{n-2}})) \\ &= \xi_I(I, \Psi_m(\frac{A}{2^{n-2}})).\end{aligned}$$

So we have $\Psi_m(A + A^*) = \Psi_m(A) + \Psi_m(A)^*$. Then $\Psi_m(A)^* = \Psi_m(A^*)$.
□

Lemma 2.10. *According to the above registered symbols, we obtain*

$$\Psi_m(iA) = i\Psi_m(A)$$

for arbitrary $A \in \mathfrak{B}$.

Proof. By an easy calculation we can see

$$\Psi_m(\xi_I(\mathrm{i}A, \frac{I}{2^{n-3}})) = \Psi_m(\xi_I(-A, \mathrm{i}\frac{I}{2^{n-3}})),$$

then

$$\xi_I(\Psi_m(\mathrm{i}A), \frac{I}{2^{n-3}})) = \xi_I(\Psi_m(-A), \mathrm{i}\frac{I}{2^{n-3}})),$$

thus

$$\Psi_m(\mathrm{i}A)^* + \Psi_m(\mathrm{i}A) = \mathrm{i}(\Psi_m(-A)^* - \Psi_m(-A)). \tag{2.7}$$

Also,

$$\Psi_m(\xi_I(\mathrm{i}A, \frac{\mathrm{i}I}{2^{n-3}})) = \Psi_m(\xi_I(-A, \frac{-I}{2^{n-3}})),$$

then

$$\xi_I(\Psi_m(\mathrm{i}A), \frac{\mathrm{i}I}{2^{n-3}})) = \xi_I(\Psi_m(-A), \frac{-I}{2^{n-3}})),$$

thus

$$-\Psi_m(\mathrm{i}A)^* + \Psi_m(\mathrm{i}A) = \mathrm{i}(\Psi_m(A)^* + \Psi_m(A)). \tag{2.8}$$

Therefore, from (2.7) and (2.8), we have $\Psi_m(\mathrm{i}A) = \mathrm{i}\Psi_m(A)$ for all $A \in \mathfrak{B}$.
□

Lemma 2.11. *According to the above marked symbols, we obtain that the mapping Ψ_m is an additive higher $*$-derivation on $\mathfrak{B}$.*

Proof. For arbitrary $A, B \in \mathfrak{B}$, in concert with induction hypothesis $\mathfrak{H}_s$ $(1 \leq s \leq m-1)$ and Lemma 2.9, we have

$$\begin{aligned}\Psi_m(AB + B^*A^*) &= \Psi_m(\xi_I(A^*, \frac{B}{2^{n-2}}))\\ &= \sum_{i_1+\cdots+i_n=m} U_n(\Psi_{i_1}(I), \cdots, \Psi_{i_{n-2}}(I), \Psi_{i_{n-1}}(A^*), \Psi_{i_n}(\frac{B}{2^{n-2}}))\\ &= \sum_{i_{n-1}+i_n=m} \xi_I(\Psi_{i_{n-1}}(A^*), \Psi_{i_n}(\frac{B}{2^{n-2}}))\\ &= \sum_{i_{n-1}+i_n=m} (\Psi_{i_{n-1}}(A^*)^*\Psi_{i_n}(B) + \Psi_{i_n}(B)^*\Psi_{i_{n-1}}(A^*)),\end{aligned}$$

which implies that

$$\Psi_m(AB + B^*A^*) = \sum_{i_{n-1}+i_n=m} (\Psi_{i_{n-1}}(A)\Psi_{i_n}(B) + \Psi_{i_n}(B)^*\Psi_{i_{n-1}}(A^*)). \tag{2.9}$$

On the other hand, since Ψ_m preserves involution, we obtain

$$\begin{aligned}\Psi_m(\mathrm{i}(AB - B^*A^*)) &= \Psi_m(A(\mathrm{i}B) + (\mathrm{i}B)^*A^*)\\ &= \sum_{i_{n-1}+i_n=m} (\Psi_{i_n}(\mathrm{i}B)^*\Psi_{i_{n-1}}(A)^* + \Psi_{i_{n-1}}(A)\Psi_{i_n}(\mathrm{i}B)).\end{aligned}$$

Therefore, from Lemma 2.10, it follows that

$$\Psi_m(AB - B^*A^*) = \sum_{i_{n-1}+i_n=m} (-\Psi_{i_n}(B)^*\Psi_{i_{n-1}}(A)^* + \Psi_{i_{n-1}}(A)\Psi_{i_n}(B)). \tag{2.10}$$

Thus, from equations (2.9) and (2.10), we get

$$\Psi_m(AB) = \sum_{i_{n-1}+i_n=m} \Psi_{i_{n-1}}(A)\Psi_{i_n}(B).$$

In summary, it can be concluded that mapping Ψ_m is an additive higher $*$-derivation on $\mathfrak{B}$.
□

It immediately follows from Theorem 2.1 and [16] that the following corollary holds.

Corollary 2.12. *[16, Theorem 1] Let $\mathfrak{B}$ be an unital $*$-algebra with identity element I that satisfies condition $\mathfrak{C}$. Then every nonlinear mixed $*$-Jordan-type derivation is an additive $*$-derivation.*

By Theorem 2.1, we obtain the following corollaries on typical examples prime $*$-algebras, factor von Neumann algebra, von Neumann algebra of type I_1 and standard operator algebra, etc.

Corollary 2.13. *Let $\mathfrak{B}$ be a prime $*$-algebra. Then every nonlinear mixed $*$-Jordan-type higher derivation on $\mathfrak{B}$ is an additive higher $*$-derivation.*

Corollary 2.14. *[16, Corollary 3] Let $\mathfrak{B}$ be a prime $*$-algebra. Then every nonlinear mixed $*$-Jordan-type derivation is an additive $*$-derivation.*

Corollary 2.15. *Let $\mathbb{A}$ be a factor von Neumann algebra acting on complex Hilbert space with $dim(\mathbb{A}) \geq 2$. Then every nonlinear mixed $*$-Jordan-type higher derivation on $\mathbb{A}$ is an additive higher $*$-derivation.*

Corollary 2.16. *[16, Corollary 5] Let $\mathbb{A}$ be a factor von Neumann algebra acting on complex Hilbert space with $dim(\mathbb{A}) \geq 2$. Then every nonlinear mixed $*$-Jordan-type derivation on $\mathbb{A}$ is an additive $*$-derivation.*

Corollary 2.17. *Let $\mathfrak{S}$ be an infinite dimensional complex Hilbert space and $\mathfrak{R}$ be a standard operator algebra on $\mathfrak{S}$ containing the identity operator I. Suppose that $\mathfrak{R}$ is closed under the adjoint operation. Then every nonlinear mixed $*$-Jordan-type higher derivation on $\mathfrak{R}$ is an additive higher $*$-derivation.*

Corollary 2.18. *[9, Theorem 2.18] Let $\mathfrak{S}$ be an infinite dimensional complex Hilbert space and $\mathfrak{R}$ be a standard operator algebra on $\mathfrak{S}$ containing the identity operator I. Suppose that $\mathfrak{R}$ is closed under the adjoint operation. Then every nonlinear mixed $*$-Jordan-type derivation on $\mathfrak{R}$ is an additive $*$-derivation.*

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