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Notes on the strong Pytkeev property in paratopological groups

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Abstract. In this paper, we discuss certain strong Pytkeev property in topological spaces, and prove the following results:

(1) a Hausdorff topological space with the strong Pytkeev property is sequential if it is a k-space, which gives an affirmative answer to the question proposed by F.C. Lin, A. Ravsky and J. Zhang [25, Question 4.3];

(2) a regular κ-Fréchet-Urysohn paratopological group with countable sp^*-character is submetrizable, and if it still has the property (⊛), then it is first-countable, which gives a partial answer to the question proposed by Z.Y. Cai, P.Q. Ye, S. Lin and B. Zhao [7, Question 2.8].

1. Introduction

Spaces with a $\mathfrak{G}$-base are known in Functional Analysis in 2003 when B. Cascales, J. Kąkol, and S. Saxon [8] characterized quasi-barreled locally convex spaces with a $\mathfrak{G}$-base. In 2006, the concept of a $\mathfrak{G}$-base first appeared in [12] as a tool for studying locally convex spaces that belong to the class of $\mathfrak{G}$-bases introduced by B. Cascales and J. Qrihuela [9]. A systematic study of locally convex spaces and topological groups with $\mathfrak{G}$-bases has been started in [5, 16, 18, 19] and continued in [14, 15, 17, 25]. Nowadays, topological groups with $\mathfrak{G}$-bases play an important role, and they have many beautiful results in topological algebra. For example, the k-property for topological groups with $\mathfrak{G}$-bases has the following characterization.

Theorem 1.1. ([19]) *Let G be a Hausdorff topological group with a $\mathfrak{G}$-base. The following are equivalent:*
(1) *G is a k-space.*
(2) *G is a sequential space.*
(3) *G is metrizable or contains a submetrizable open k_ω-subgroup.*

S.S. Gabriyelyan, J. Kąkol and A. Leiderman [19] proved that every topological group with a $\mathfrak{G}$-base has countable cs^*-character, and said that "it would be interesting to know whether the k-property and sequentiality are equivalent for the class of all topological groups having countable cs^*-character". F.C. Lin, A. Ravsky and J. Zhang gave a negative answer to this question.

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Example 1.2. ([25, Example 4.2]) There exists a Hausdorff topological group G such that it is a k-space with countable cs^*-character. However, G is not sequential.

T. Banakh discussed Pytkeev networks and strict Pytkeev networks [3], and proved that each countable (strict) Pytkeev network is a k-network (resp. cs^*-network) in topological spaces, and the converse is also true for a k-space (resp. Fréchet-Urysohn space) [6]. Subsequently, various types of topological spaces with certain Pytkeev networks have been defined and studied, which played an important role in generalized metric spaces, cardinal functions, function spaces, topological groups and topological vector spaces [3, 6, 15, 16, 18, 19, 29].

We know that the strong Pytkeev property for topological groups is closely related to the notion of a $\mathfrak{G}$-base. For instance, each topological group that is a k-space with a $\mathfrak{G}$-base has the strong Pytkeev property [18]. But the topological group G in Example 1.2 does not have the strong Pytkeev property, hence it is natural to pose the following problem.

Problem 1.3. ([25, Question 4.3]) *Let G be a Hausdorff topological group. If G is a k-space having the strong Pytkeev property, is it sequential?*

Spaces with a $\mathfrak{G}$-base are also called spaces with an ω^ω-base [24]. Gabriyelyan, Kąkol and Leiderman proved the following result.

Theorem 1.4. ([19]) *A Hausdorff topological group is first-countable if and only if it is a Fréchet-Urysohn space having an ω^ω-base.*

A natural question is whether it also holds in paratopological groups? A lot of works have been done in this area [7, 11, 22, 33]. Z.Y. Cai, P.Q. Ye, S. Lin and B. Zhao [7] proved the following result using the property (∗∗) introduced by C. Liu [27].

Lemma 1.5. ([7]) *Every Fréchet-Urysohn Hausdorff paratopological group having the property (∗∗) with an ω^ω-base is first-countable, hence submetrizable.*

Subsequently, they posed the following problem.

Problem 1.6. ([7, Question 2.8]) *Is every Fréchet-Urysohn Hausdorff paratopological group with an ω^ω-base submetrizable?*

This paper is organized as follows. In Section 3, we mainly discuss the strong Pytkeev property for topological spaces, and prove that a topological space with the strong Pytkeev property is sequential if it is a k-space, which gives an affirmative answer to Problem 1.3. In Section 4, we mainly discuss paratopological groups with countable sp^*-character, and prove that a κ-Fréchet-Urysohn topological space with countable sp^*-character has countable π-character, hence is submetrizable if it is a paratopological group, which gives a partial answer to Problem 1.6. We also introduce the concept of the property (⊛), and prove that every κ-Fréchet-Urysohn paratopological group having the property (⊛) with countable sp^*-character is first-countable.

2. The strong Pytkeev property and weak first-countability

In this section, we introduce the necessary notation, terminology, and describe some relationships among spaces defined by these notions. Throughout this paper, all topological spaces are assumed to be Hausdorff.

Definition 2.1. Let $\mathscr{P}$ be a family of subsets of a topological space X and $x \in X$.

(1) $\mathscr{P}$ is called a *network* at x for X [10, p. 127], if for any neighborhood U of x in X, there exists $P \in \mathscr{P}$ such that $x \in P \subset U$.

(2) $\mathscr{P}$ is called a *cs^*-network* at x for X [20, Definition 3], if for every sequence $\{x_n\}_{n\in\omega}$ converging to the point $x \in U$ with U open in X, there is $P \in \mathscr{P}$ such that some subsequence $\{x_{n_i}\}_{i\in\omega}$ of $\{x_n\}_{n\in\omega}$ is contained in P and $x \in P \subset U$.

(3) $\mathscr{P}$ is called a *Pytkeev network* [3, Definition 1.1] (resp., *Pytkeev* network* [4, Definition 3.1]) at x, if $\mathscr{P}$ is a network at x, and for each neighborhood U of x in X and each subset (resp., countable subset) A of X accumulating at x, there exists $P \in \mathscr{P}$ such that $P \cap A$ is infinite and $P \subset U$.

(4) $\mathscr{P}$ is called an *sp-network* [30] (resp., *sp^*-network*) at x, if for each $x \in U \cap \overline{A}$ with U open and A subset (resp., countable subset) in X, there is $P \in \mathscr{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$.

The space X has *the strong Pytkeev property* [3] (resp., *the strong Pytkeev* property* [4, Definition 4.2]), if it has a countable Pytkeev network (resp., Pytkeev* network) network at each point in X. The space X has countable *sp-character* [30](resp., *sp^*-character*, or *cs^*-character*), if it has a countable *sp*-network (resp., sp^*-network, or cs^*-network) at each point in X.

Definition 2.2. Let X be a topological space.

(1) X is a *k-space* [10, p. 152], if whenever a subset A of X satisfying that $A \cap K$ is closed in K for each compact subset K of X, A is closed in X.

(2) A subset A of X is *sequentially closed* in X if S is a sequence in A converging to a point $x \in X$, then $x \in A$. The space X is a *sequential space* [10, p. 53] if every sequentially closed subset of X is closed.

(3) X is a *Fréchet-Urysohn space* [10, p. 53] (resp., *κ-Fréchet-Urysohn space* [28, p. 391]) if, for each subset (resp., open subset) $A \subset X$ and $x \in \overline{A}$, there is a sequence in A converging to x in X.

(4) X is of *countable tightness* [31, Proposition 8.5] if, for each $A \subset X$ and $x \in \overline{A}$, there is a countable subset C of A such that $x \in \overline{C}$.

Lemma 2.3. ([4, Corollary 3.7]) *A topological space X has the strong Pytkeev property if and only if X has the strong Pytkeev* property and countable tightness.*

Since each space with countable *sp*-character has the strong Pytkeev property, the following corollary is obvious.

Corollary 2.4. *A topological space X has countable sp-character if and only if X has countable sp^*-character and countable tightness.*

We consider the Cartesian product ω^ω with the natural partial order, i.e., $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for each $i \in \omega$, where $\alpha = (\alpha_i)_{i\in\omega}$ and $\beta = (\beta_i)_{i\in\omega}$.

Definition 2.5. A topological space X is said to have an *ω^ω-base* or a local *$\mathfrak{G}$-base* [12] at a point $x \in X$, if there exists a local base $\{U_\alpha : \alpha \in \omega^\omega\}$ at the point x such that $U_\beta \subset U_\alpha$ for all elements $\alpha \leq \beta$ in ω^ω. The space X is said to have an ω^ω-base if it has an ω^ω-base at every point $x \in X$.

Lemma 2.6. ([5, Theorem 6.4.1]) *Every topological space with an ω^ω-base has the strong Pytkeev* property.*

A topological space with an ω^ω-base and countable sp^*-character does not have countable tightness, see Example 3.3. A natural question is whether a space with an ω^ω-base has countable sp^*-character? In fact, the answer is negative.

Example 2.7. There is a topological group with an ω^ω-base which does not have countable sp^*-character.

Proof. Denote by $e = \{e_n\}_{n\in\omega}$ the sequence in the direct sum $\mathbb{Z}^\omega$ with $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, ...)$,...Then the sequence $\{e_n\}_{n\in\omega}$ converges to zero in the topology induced by the product topology on $\mathbb{Z}^\omega$. Denote by τ_e the finest group topology on $\mathbb{Z}^\omega$ in which $e_n \to 0$. Then the free Abelian group $A(e) = (\mathbb{Z}^\omega, \tau_e)$ [13], and $A(e)$ is a countable sequential group with an ω^ω-base which is not Fréchet-Urysohn [17, Corollary 4.20].

Assume that $A(e)$ has countable sp^*-character. It follows from $A(e)$ being a sequential space that $A(e)$ has countable tightness. By Corollary 2.4, $A(e)$ has countable *sp*-character. A topological group with countable *sp*-character is metrizable [30, Corollary 6.5]. Thus $A(e)$ does not have countable sp^*-character. □

Example 2.8. The Arens space S_2: a regular sequential space with the strong Pytkeev property which does not have countable sp^*-character.

Proof. Let $X = \{x\} \cup \{x_n : n \in \omega\} \cup \{x_{n,m} : n, m \in \omega\}$, where every x_n, $x_{n,m}$ and x are different from each other. The set X endowed with the following topology is called the *Arens space* [10, Example 1.6.19] and denoted briefly as S_2: each $x_{n,m}$ is isolated; a basic neighborhood of each x_n has the form $\{x_n\} \cup \{x_{n,m} : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of x has the form $\{x\} \cup \bigcup_{n \geq k} V_n$ for some $k \in \omega$, where each V_n is a neighborhood of x_n.

It is easy to see that the space X is a non-Fréchet-Urysohn, regular sequential space with countable cs^*-character [26, Example 1.8.6]. Thus X has the strong Pytkeev property [3, Proposition 1.8]. But X dose not have countable sp-character [30, Example 3.7]. It follows from Corollary 2.4 that X dose not have countable sp^*-character. □

It is obvious that every first-countable space is a Fréchet-Urysohn space, and every Fréchet-Urysohn space is a κ-Fréchet-Urysohn space. But both sequential spaces and k-spaces are independent of κ-Fréchet-Urysohn spaces.

Example 2.9. ([28, Examples 2.2 and 2.3]) There is a κ-Fréchet-Urysohn space where it is neither a k-space nor a space of countable tightness.

Some relationships among the weakly first-countable spaces in Definitions 2.1, 2.2 and 2.5 are illustrated in Figure 2.1.

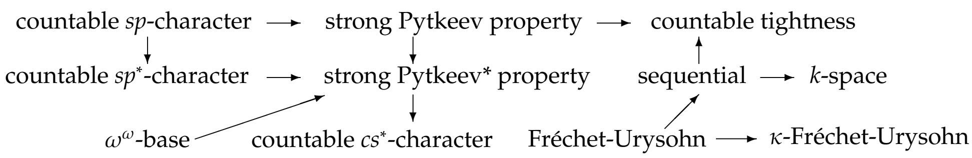

Figure 2.1 Weak first-countability

3. The strong Pytkeev property in topological spaces

In this section, we give a positive answer to Problem 1.3.

Lemma 3.1. *Let X be a regular countably compact space. If X has the strong Pytkeev property, then X is first-countable.*

Proof. Assume that a regular countably compact space X has the strong Pytkeev property at a non-isolated point $x \in X$. Fix a countable Pytkeev network $\mathscr{P}$ at the point x and an open neighborhood U of the point x. Consider the subfamily $\mathscr{P}_U = \{P \in \mathscr{P} : P \subset U\}$ and write it as $\mathscr{P}_U = \{P_i\}_{i \in \omega}$ with $x \in P_0$. Suppose that for each $n \in \omega$, $x \notin (\bigcup_{i \leq n} P_i)^\circ$. Put $A_n = U \setminus \bigcup_{i \leq n} P_i$. We have that $x \in \overline{A_n \setminus \{x\}}$.

Put

$$S = \{y \in X : \text{there exists a sequence } \{y_n\}_{n\in\omega} \text{ in } X \text{ such that each } y_n \in A_n \text{ and } y \text{ is an accumulation point of the sequence } \{y_n\}_{n\in\omega}\}.$$

Case 1. $x \in \overline{S}$.

Assume the contrary. Then the set $V = X \setminus \overline{S}$ is an open neighborhood of the point x. By the regularity of X, there is a neighborhood W of x in X such that $\overline{W} \subset V$. Then for each $n \in \omega$, $W \cap A_n$ is an infinite set. Therefore, we can select a sequence $\xi = \{y_n\}_{n\in\omega}$ in X such that each $y_n \in A_n \cap W$ and $y_n \neq y_m$ if $n \neq m$. Because X is countably compact, there is an accumulation point y of the sequence ξ in X. Clearly, $y \in S \cap \overline{W} \subset V = X \setminus \overline{S}$, which is a contradiction. It shows that $x \in \overline{S}$.

By Lemma 2.3, X has countable tightness. Then there exists a countable subset M of S such that $x \in \overline{M}$. Put $M = \{z_n : n \in \omega\}$ and for each $n \in \omega$ select a sequence $\xi^n = \{z_i^n\}_{i\in\omega}$ accumulating at the point z_n. Define $K_n = \{z_n^0, z_n^1, ..., z_n^n\}$. We have that for each $i \le j$, $z_j^i \in K_j$.

Case 2. $x \in \overline{\bigcup_{n\in\omega} K_n}$.

For each open neighborhood H of the point x, $H \cap M \neq \emptyset$. Take a point $z_j \in H \cap M$, H is also an open neighborhood of z_j. Because the point z_j is an accumulation point of the sequence ξ^j, we have that $H \cap \xi^j$ is an infinite set. From the selection of the set K_n, it can be seen that $\xi^j \setminus \{z_0^j, z_1^j, ..., z_{j-1}^j\} \subset \bigcup_{n\in\omega} K_n$. Thus $H \cap \bigcup_{n\in\omega} K_n$ is an infinite set. Therefore, $x \in \overline{\bigcup_{n\in\omega} K_n}$.

Because $\mathscr{P}$ is a Pytkeev network at the point x in X, there exists $P_i \in \mathscr{P}_U$ such that $P_i \cap \bigcup_{n\in\omega} K_n$ is an infinite set. For each $m \ge i$, $K_m \subset A_m \subset U \setminus P_i$. Thus $P_i \cap \bigcup_{n\in\omega} K_n \subset \bigcup_{m<i} K_m$ is finite, which is a contradiction. Hence there is $n \in \omega$ such that $x \in (\bigcup_{i\le n} P_i)^\circ$. This implies the family $\{(\bigcup \mathscr{P}')^\circ : x \in (\bigcup \mathscr{P}')^\circ \text{ and } \mathscr{P}' \in [\mathscr{P}]^{<\omega}\}$ is a countable base at the point x in X. Therefore, X is first-countable. □

The following result gives an affirmative answer to Problem 1.3.

Theorem 3.2. *Every k-space having the strong Pytkeev property is a sequential space.*

Proof. Let X be a k-space having the strong Pytkeev property. Assume that a subset A of X is not closed. Then there is a compact subset K of X such that $A \cap K$ is not closed in K. It follows from Lemma 3.1 that the subspace K is first-countable. There exists a sequence $\{a_n\}_{n\in\omega}$ in $A \cap K$ converging to a point $y \in K \setminus A$. Then the limit point y of the sequence $\{a_n\}_{n\in\omega}$ is not in A. Thus X is sequential. □

The condition "the strong Pytkeev property" in Theorem 3.2 cannot be replaced by "ω^ω-base" or "countable sp^*-character", and the condition "topological group" in Theorem 1.1 cannot be omitted.

Example 3.3. Let $X = [0, \omega_1]$ be endowed with the ordered topology. Then

(1) X is a k-space;

(2) X has an ω^ω-base under $\omega_1 = \mathfrak{b}$;

(3) X has countable sp^*-character;

(4) X dose not have countable tightness.

Proof. In [5, After Corollary 8.6.7], X is a compact space with an ω^ω-base under $\omega_1 = \mathfrak{b}$. For each $x \in X$, since each point of $[0, \omega_1)$ is first-countable in X, we can assume that $x = \omega_1$. If $x \in \overline{A}$ with A countable in X, then $x \in A$. Thus $\{\{x\}\}$ is an sp^*-network at x in X. Hence X has countable sp^*-character. Obviously, X dose not have countable tightness at the point ω_1. It shows that X is not a sequential space. □

4. Paratopological groups with countable *sp**-character

In this section, we mainly discuss paratopological groups with countable sp^*-character, and prove that a κ-Fréchet-Urysohn space with countable sp^*-character has countable π-character, hence submetrizable, which gives a partial answer to Problem 1.6. We also introduce the concept of the property (⊛), as a generalization of the property (∗∗), and prove that a κ-Fréchet-Urysohn paratopological group having the property (⊛) with countable sp^*-character is first-countable.

Let X be a topological space. A family $\mathscr{B}$ of open subsets of X is called a *π-base* at a point $x \in X$ [26], if for any neighborhood U of x in X, there exists $B \in \mathscr{B}$ such that $B \subset U$. The topological space X has countable *π-character*, if it has a countable π-base at each point in X.

Theorem 4.1. *Let X be a regular κ-Fréchet-Urysohn space with countable sp^*-character. Then X has countable π-character.*

Proof. Let $\mathscr{P}$ be a countable sp^*-network at a point x in X. Enlarging $\mathscr{P}$ by a larger family, we can assume that $\mathscr{P}$ is closed under finite unions and every set $P \in \mathscr{P}$ is closed. Put

$$\mathscr{B} = \{\bigcup \mathscr{P}' : \mathscr{P}' \in [\mathscr{P}]^{<\omega}\}.$$

Then $\mathscr{B}$ is countable. Next we will show that the family $\mathscr{B}^\circ = \{B^\circ : B \in \mathscr{B}\}$ is a π-base at x.

Assume that there exists an open neighborhood U of x such that U contains no element of $\mathscr{B}^\circ$. We can assume that $x \in \overline{U \setminus \{x\}}$. Because X is κ-Fréchet-Urysohn, there exists a countable subset $A = \{x_i\}_{i\in\omega}$ of $U \setminus \{x\}$ such that $x \in \overline{A}$. Write the family $\{P \in \mathscr{P} : x \in P \subset U\}$ as $\{P_i\}_{i\in\omega}$. For each $i \in \omega$, there is an open neighborhood U_i of the point x_i such that $U_i \subset U$ and $x \notin \overline{U}_i$. It follows from $(\bigcup_{j\le i} P_j)^\circ \subset U$ that $(\bigcup_{j\le i} P_j)^\circ = \varnothing$, thus $x_i \in U_i \cap \overline{X \setminus \bigcup_{j\le i} P_j} \subset \overline{U_i \setminus \bigcup_{j\le i} P_j}$. Because the set $U_i \setminus \bigcup_{j\le i} P_j$ is open and X is κ-Fréchet-Urysohn, there exists a countable subset A_i of $U_i \setminus \bigcup_{j\le i} P_j$ such that $x_i \in \overline{A_i}$; thus $x \notin \overline{A_i}$ and $P_j \cap A_i = \varnothing$ for each $i \ge j$. Clearly, $\overline{A} \subset \overline{\bigcup_{i\in\omega} A_i}$. Because $\mathscr{P}$ is a countable sp^*-network at the point x, there exists $P \in \mathscr{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap \bigcup_{i\in\omega} A_i}$. Thus we can choose $m \in \omega$ such that $P = P_m$. It shows that $x \in \overline{P_m \cap \bigcup_{i<m} A_i} \subset \bigcup_{i<m} \overline{A_i}$, which is a contradiction. Hence $\mathscr{B}^\circ$ is a countable π-base at the point x in X. □

Next, we provide several corollaries to Theorem 4.1.

It is easy to check that a Fréchet-Urysohn space with an ω^ω-base is a κ-Fréchet-Urysohn space with countable sp^*-character. The following corollary generalizes Theorem 1.4.

Corollary 4.2. *A topological group is first-countable if and only if it is a κ-Fréchet-Urysohn space with countable sp^*-character.*

A topological space (X, τ) is *submetrizable* [26] if there exists a topology τ' on X such that $\tau' \subset \tau$ and (X, τ') is metrizable. A subset B of a topological space X is said to be *bounded* [2] in X if every infinite family $\mathscr{U}$ of open subsets of X such that $U \cap B \ne \varnothing$, for every $U \in \mathscr{U}$, has an accumulation point in X. If X is bounded in itself, then we say that X is *pseudocompact*.

The following corollary gives a partial answer to Problem 1.6, and generalizes the following results: (1) every first-countable paratopological group has a regular G_δ-diagonal [27, Theorem 2.1]; (2) every bounded subspace of a regular first-countable paratopological group is metrizable [27, Corollary 2.2].

Corollary 4.3. *Let G be a regular κ-Fréchet-Urysohn paratopological group with countable sp^*-character. Then G is submetrizable, and every bounded subspace of G is metrizable.*

Proof. Let G be a regular κ-Fréchet-Urysohn paratopological group with countable sp^*-character. It follows from Theorem 4.1 that G has countable countable π-character. Since every paratopological group of countable π-character is submetrizable [32, Lemma 2.8], G is submetrizable. It is known that every bounded subset of a regular space with a regular G_δ-diagonal is metrizable [1, Theorem 1]. Since every submetrizable space has regular G_δ-diagonal, every bounded subspace of G is metrizable. □

Assume $\mathbf{V}[G_{\omega_2}]$, every separable Fréchet-Urysohn regular paratopological group has countable π-character [21]. Without the set-theoretic hypothesis, we have the following result.

Corollary 4.4. *If X is a κ-Fréchet-Urysohn space with countable sp^*-character, then X is separable if and only if it has a countable π-base.*

Proof. Sufficiency is obvious. We just need to prove the Necessity . Assume that X is separable and has countable sp^*-character. Take any $x \in X$, by Theorem 4.1, X has a countable π-base $\mathscr{P}_x$ at x. Because X is separable, we can choose a countable subset $D = \{d_n\}_{n\in\omega}$ of X such that $\overline{D} = X$. We claim that the countable family $\mathscr{P} = \bigcup_{n\in\omega} \mathscr{P}_{d_n}$ is a countable π-base for X. For each non-empty open subset U of X, there exists a point $d_n \in D \cap U$. Then we can choose a set $P \in \mathscr{P}_{d_n}$ such that $P \subset U$. It means that $\mathscr{P}$ is a countable π-base for X. □

Finally, we focus on special property (∗∗) and discuss a generalization of Lemma 1.5.

Definition 4.5. Let G be a paratopological group.

(1) G is said to have *the property* (∗∗) [27] if there exists a non-trivial sequence $\{x_n\}_{n\in\omega}$ in G such that both $\{x_n\}_{n\in\omega}$ and $\{x_n^{-1}\}_{n\in\omega}$ converge to the identity of G.

(2) G is said to have *the property* (⊛) if there exists a non-trivial sequence $\{x_n\}_{n\in\omega}$ of G such that for any neighborhood U of the identity e in G, there exists $n \in \omega$ such that $x_n \in U \cap U^{-1}$.

Remark 4.6. (1) It is clear that every paratopological group with the property (∗∗) has the property (⊛).

(2) It is easy to check that every topological group with countable tightness has the property (⊛).

(3) Not every first-countable paratopological group has the property (⊛). The Sorgenfrey line $\mathbb{S}$ does not have the property (⊛).

(4) A paratopological group having the property (⊛) need not be a topological group. Let $(\mathbb{R}, +)$ be the real line with the usual topology. Then $\mathbb{S} \times \mathbb{R}$ is a paratopological group having the property (⊛) but not a topological group [27, P. 637].

The following result generalizes Lemma 1.5 and gives a partial answer to Problem 1.6.

Theorem 4.7. *Let X be a regular κ-Fréchet-Urysohn paratopological group with countable sp^*-character. If X has the property* (⊛)*, then X is first-countable.*

Proof. Fix a countable sp^*-network $\mathscr{P}$ at the unit e of X. Enlarging $\mathscr{P}$ by a larger family, we can assume that $\mathscr{P}$ is closed under finite unions and every set $P \in \mathscr{P}$ is closed. Because X has the property (⊛), we can find a sequence $\{x_n\}_{n\in\omega}$ in $X \setminus \{e\}$ such that for any neighborhood U of the unit e in X, there exists $n \in \omega$ such that $x_n \in U \cap U^{-1}$. Consider the countable family $\mathscr{N} = \{x_n^{-1}P : n \in \omega, P \in \mathscr{P}\}$ and let $\mathscr{N}^\circ = \{N^\circ : N \in \mathscr{N}\}$. We claim that some subfamily of $\mathscr{N}^\circ$ is a local base at e; consequently, X is first-countable.

Given any neighborhood O of e in X. By the continuity of the multiplication in the paratopological group X, there exists a neighborhood U of e such that $UU \subset O$. Let $\mathscr{P}' = \{P \in \mathscr{P} : P \subset U\}$ and $\Omega = \{n \in \omega : x_n \in U \cap U^{-1}\}$. We claim that there exist $P \in \mathscr{P}'$ and $n \in \Omega$ such that $x \in (x_n^{-1}P)^\circ$. To derive a contradiction, we assume that $e \notin (x_n^{-1}P)^\circ$, for each $P \in \mathscr{P}'$ and $n \in \Omega$. Then $x_n \notin P^\circ$ for all $P \in \mathscr{P}'$ and $n \in \Omega$.

Write the countable family $\mathscr{P}'$ as $\{P_k\}_{k\in\omega}$. For each $n \in \omega$, there is an open neighborhood W_n of the point x_n such that $e \notin \overline{W_n}$. For each $n \in \omega$, because the family $\mathscr{P}$ is closed under finite unions, the set $\bigcup_{k\le n} P_k$ belongs to $\mathscr{P}'$. It shows that for each $n \in \Omega$, $x_n \notin (\bigcup_{k\le n} P_k)^\circ$. Thus $x_n \in \overline{W_n \setminus \bigcup_{k\le n} P_k}$. Because X is κ-Fréchet-Urysohn, there exists a countable subset

$$A_n \subset W_n \setminus \bigcup_{k\le n} P_k$$

such that $x_n \in \overline{A_n}$. Put $A = \bigcup_{n\in\Omega} A_n$, then $e \in \overline{A}$. Since $\mathscr{P}$ is an sp^*-network at e, there exists a set $P \in \mathscr{P}$ such that $P \subset U$ and $e \in \overline{A \cap P}$. Then $P \in \mathscr{P}'$ and hence $P = P_k$ for some $k \in \omega$. Now observe that

$$A \cap P \subset P_k \cap \bigcup_{n\in\Omega}(W_n \setminus \bigcup_{i\le n} P_i) \subset P_k \cap \bigcup_{n<k}(W_n \setminus \bigcup_{i\le n} P_i) \subset \bigcup_{n<k} W_n,$$

thus $e \notin \overline{A \cap P}$, which is a contradiction. □

It is easy to check that every Fréchet-Urysohn space with countable cs^*-character has countable sp^*-character. We have the following corollary, which is also a generalization of Lemma 1.5.

Corollary 4.8. *Let G be a regular Fréchet-Urysohn paratopological group with countable cs^*-character. If G has the property* (⊛)*, then G is first-countable.*

The following problem is natural.

Problem 4.9. *Is every regular Fréchet-Urysohn paratopological group with countable cs^*-character first-countable?*

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