Filomat 39:15 (2025), 5225–5240
https://doi.org/10.2298/FIL2515225G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Dominating sets in graph theory and algebraic hyperstructures

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Abstract. The study of hyperstructures and their connections with other branches of mathematics has been explored by various researchers. In this context, several studies have investigated the relationship between hyperstructures and Graph Theory. This paper aims to establish new connections between Hyperstructure Theory and Graph Theory by focusing on the concept of dominating sets and minimal dominating sets of a graph. Specifically, we define different semihypergroups on the set of all dominating sets and the set of all minimal dominating sets of a given graph. We also examine the conditions under which some of these semihypergroups can be hypergroups, and provide examples to illustrate them. Finally, we present some theorems that introduce and construct numerous graphs in which some of these semihypergroups are hypergroups. Through this research, we contribute to the understanding of how hyperstructures can enhance the study of graph properties and optimization problems, paving the way for future research in this interdisciplinary area.

1. Introduction

The relationship between graph theory and algebraic hyperstructures has gained considerable interest in recent years, revealing many connections that extend the reach of both fields [1, 14]. Notably, Corsini and Leoreanu laid foundational work by exploring the connections between hyperstructures and various mathematical domains, emphasizing their relevance in graph theory [4]. A key concept in algebraic hyperstructures is the hypergroup, which generalizes the notion of a group by allowing operations that do not necessarily yield a unique result for each pair of elements [5]. This structure has proven particularly useful in algebraic systems arising in hypergraph theory, coding theory, and network analysis [6, 15]. In parallel, graphs serve as mathematical representations of relationships among objects [2], offering a versatile platform to investigate hypergroup properties, especially in contexts where traditional group operations are insufficient, such as when considering all paths between two vertices [13, 17]. The exploration of

2020 *Mathematics Subject Classification*. Primary 03D50; Secondary 05C07.
Keywords. Dominating set in graphs, minimal dominating set in graphs, semihypergroup, hypergroup
Received: 05 May 2024; Revised: 06 March 2025; Accepted: 27 March 2025
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hypergroups within graph theory has yielded valuable insights into applications in network analysis, including efficient solutions to problems like finding specific dominating sets and graph coloring [9].

Recent studies have revealed significant connections between key concepts of algebraic hyperstructures, such as hypergroups, regular relations, and join spaces [16], and fundamental concepts in graph theory, including directed graphs and hypergraphs [20]. One investigation focuses on regular relations in two types of hypergroups: one derived from the vertices of a hypergraph and the other from its edges, which are crucial for understanding their algebraic properties [21]. In 2023, Kalampakas and Spartalis made significant contributions to this field by establishing key results on path hyperoperations in directed graphs, highlighting their algebraic properties and their relevance to applications in image processing and complex network dynamics [10]. However, several concepts in graph theory remain underexplored and require further study to fully understand their implications through algebraic hyperstructures, such as dominating sets, bipartite graphs, graph homomorphisms, and spectral properties [22]. The purpose of the present paper is to establish new connections between hypergroups and dominating sets and minimal dominating sets in graphs. This exploration aims to deepen our understanding of how these concepts interact and contribute to advancements in both fields, potentially leading to new methodologies for solving complex problems in graph theory and related applications.

In this context, let $G = (V, E)$ be a graph and H be the set of all dominating sets of G. Then, by some properties of dominating sets of a graph, we could define two hyperoperations $*,\ o : H \times H \to \mathcal{P}^*(H)$ and show that $(H, *)$ and (H, o) are semihypergroups. These semihypergroups help us to compare dominating sets in different ways. We also find three strongly regular relations on semihypergroups $(H, *)$ and (H, o), including the smallest strongly regular relation β^* on (H, o). There is a point in [7] which says that $(H/\beta^*, \otimes)$ is a hypergroup. In this note, wepresent some graphs to give a counterexample to clarify this point.

Moreover, let S be the set of all minimal dominating sets of G and S' be the set of all complement minimal dominating sets. Then, by considering the number of isolated vertices in a minimal dominating set, we could define a hyperoperation $*_i : S \times S \to \mathcal{P}^*(S)$ on S. We also define a hyperoperation $*' : S' \times S' \to \mathcal{P}^*(S')$ on S', by considering the number of vertices in a complement minimal dominating set which are dominated by just one vertex. We illustrate $(S, *_i)$ and $(S', *')$ are semihypergroups. We also investigate some situations in which these semihypergroups are hypergroups and give some examples to clarify them. Eventually, we prove some theorems to construct indefinite graphs in which $(S, *_i)$ or $(S', *')$ are hypergroups.

2. Preliminaries on hyperstructures and graph theory

Let H be a non- empty set and $o : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple (H, o) is called a *hypergroupoid*. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A\ o\ B = \bigcup_{a\in A\ \&\ b\in B} a\ o\ b, \quad A\ o\ x = A\ o\ \{x\} \quad and \quad x\ o\ B = \{x\}\ o\ B.$$

Definition 2.1. ([7]) A hypergroupoid (H, o) is called a *semihypergroup* if for all a, b, c of H we have $(aob)\,oc = ao(boc)$, which means that

$$\bigcup_{u\in aob} uoc = \bigcup_{v\in boc} aov.$$

A semihypergroup (H, o) is called a *hypergroup* if for all a of H we have $a\ o\ H = H\ o\ a = H$. Furthermore, a hypergroup (H, o) is called *commutative* if for all $a, b \in H$, it holds $aob = boa$.

Definition 2.2. ([7]) Let (H, o) be a semihypergroup and R be an equivalence relation on H. If A and B are non- empty subsets of H, then

$A\ \overline{R}\ B$ means that $\forall a \in A, \exists b \in B$ such that aRb and $\forall b' \in B, \exists a' \in A$ such that $a'Rb'$.

$A\ \overline{\overline{R}}\ B$ means that $\forall a \in A,\ \forall b \in B$ we have $a\ R\ b$.

Definition 2.3. ([7]) The equivalence relation R is called

1. *Regular* on the right (on the left) if for all x of H, from $a\,R\,b$, it follows that $(a\ o\ x)\ \overline{R}\ (b\ o\ x)$ $((x\ o\ a)\ \overline{R}\ (x\ ob)$ respectively);
2. *Strongly Regular* on the right (on the left) if for all x of H, from $a\ R\ b$, it follows that $(a\ o\ x)\ \overline{\overline{R}}(b\ o\ x)$ $((x\ o\ a)\ \overline{\overline{R}}(x\ ob)$ respectively);
3. R is called *regular* (*strongly regular)* if it is regular (strongly regular) on the right and on the left.

Theorem 2.4. ([7]) *Let (H, o) be a semihypergroup and R be an equivalence relation on H.*

1. *If R is strongly regular, then H/R is a semigroup, with respect to the following operation: $\overline{x} \otimes \overline{y} = \overline{z}$ for all $z \in x\ o\ y$, where $\overline{x}$ presents the equivalence class of an element x in the semihypergroup H under the equivalence relation R.*
2. *If the hyperoperation $\otimes$ is well defined on H/R, then R is strongly regular.*

Definition 2.5. ([7]) For all $n > 1$, we define the relation β_n on a semihypergroup H, as follows:

$$a\ \beta_n\ b \Longleftrightarrow \exists\ (x_1, \ldots,\ x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i.$$

and $\beta = \bigcup_{n\geq 1} \beta_n$ where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H.

Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Theorem 2.6. ([7] *β^* is the smallest strongly regular relation on H.*

Proposition 2.7. ([7]) *The quotient $(H/\beta^*,\ \otimes)$ is a group.*

Definition 2.8. ([7]) β^* is called the *fundamental equivalence relation* on H and H/β^* is called the *fundamental group.*

A vertex v in a graph G is said to *dominate* itself and each of its neighbors, that is, v dominates the vertices in its *closed neighborhood* $N[v]$ which includes the vertex v along with all its neighbours. A set S of vertices of G is a *dominating set* of G if every vertex of G is dominated by at least one vertex of S. Equivalently, a set S of vertices of G is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the *dominating number* of G and is denoted by $\Upsilon(G)$. A dominating set of cardinality $\Upsilon(G)$ is then referred to as a *minimum dominating set*. Moreover, a set of vertices S is *independent* if no two vertices in S are adjacent; that is, there are no edges connecting any pair of vertices within the set.

Definition 2.9. *[3] A set S of vertices in a graph G is called an independent dominating set of G if S is both an independent and dominating set of G.*

Theorem 2.10. ([3]) *A set S of vertices in a graph is an independent dominating set if and only if S is maximal independent.*

Corollary 2.11. ([3]) *Every maximal independent set of vertices in a graph is a minimal dominating set.*

Theorem 2.12. ([3]) *A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v in S satisfies at least one of the following two properties:*

1. *There exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$, $N(w)$ encompasses only the neighbors of w;*
2. *v is adjacent to no vertex of S.*

Theorem 2.13. ([3]) *If G is a graph without isolated vertices and S is a minimal dominating set of G, then $V(G) - S$ is a dominating set.*

3. Dominating sets and hyperstructures

In what follows, we expand the research on making relationships between graphs and hyperstructures. First, by some new definitions, we introduce two semihypergroups on the set of all minimal dominating sets of a graph. Then, by defining three strongly regular relations on these semihypergroups, we construct three semigroups. This new connection between graphs and hyperstructures provides the opportunity for us to give a counterexample for Proposition 2.7.

Let $G = (V, E)$ be a graph. We denote the set of all dominating sets of G by " H ". In order to compare dominating sets of graph G, we define the following hyperoperations on H.

Definition 3.1. Let $G = (V, E)$ be a graph and $H_i \in H$ be a dominating set. Then $\theta(H_i)$ is the maximum number of vertices of H_i, that we can omit from H_i to convert it to a minimal dominating set.

Definition 3.2. Let $G = (V, E)$ represent a graph. We define the hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ in the following way:

For every $H_i, H_j \in H$, we have

$$H_i * H_j = \begin{cases} H_i & \text{if } \theta(H_i) < \theta\left(H_j\right) \\ H_j & \text{if } \theta\left(H_j\right) < \theta(H_i) \\ H_i & \text{if } \theta(H_i) = \theta\left(H_j\right) \textit{ and } |H_i| < \left|H_j\right|, \\ H_j & \text{if } \theta(H_i) = \theta\left(H_j\right) \textit{ and } \left|H_j\right| < |H_i| \\ \left\{H_i, H_j\right\} & \text{if } \theta(H_i) = \theta\left(H_j\right) \textit{ and } |H_i| = \left|H_j\right| \end{cases}$$

Theorem 3.3. *Let $G = (V, E)$ be a graph. Then the hypergroupoid $(H, *)$ is a commutative semihypergroup.*

Proof. By definition of the hyperoperation "$*$", it is easy to see that the hypergroupoid $(H, *)$ is commutative. Now, it is enough to check the associativity of " $*$ ", i.e. $\left(H_i * H_j\right) * H_k = H_i * (H_j * H_k)$ for all $H_i, H_j, H_k \in H$. To achieve this aim we check the following situations for every $H_i, H_j, H_k \in H$:

1. $\theta(H_i) \neq \theta(H_j) \neq \theta(H_k) \neq \theta(H_i)$, $\min\{\theta(H_i), \theta(H_j), \theta(H_k)\} = \theta(H_l)$ and $l \in \{i, j, k\}$

   $$\left(H_i * H_j\right) * H_k = H_l = H_i * (H_j * H_k).$$

2. Let $\{i_1, i_2, i_3\} \in \{i, j, k\}$ *and* $\theta(H_{i_1}) = \theta(H_{i_2}) < \theta(H_{i_3})$. Then

   $$\left(H_i * H_j\right) * H_k = H_{i_1} * H_{i_2} = H_i * (H_j * H_k).$$

3. Let $\{i_1, i_2, i_3\} \in \{i, j, k\}$ *and* $\theta(H_{i_1}) = \theta(H_{i_2}) > \theta(H_{i_3})$. Then

   $$\left(H_i * H_j\right) * H_k = H_{i_3} = H_i * \left(H_j * H_k\right).$$

From now on, we assume that $\theta(H_i) = \theta\left(H_j\right) = \theta(H_k)$. For completing the proof, it is enough to consider following casas:

1. $|H_i| \neq \left|H_j\right| \neq |H_k| \neq |H_i|$.
2. Let $\{i_1, i_2, i_3\} \in \{i, j, k\}$ *and* $\left|H_{i_1}\right| = \left|H_{i_2}\right| >$ *or* $< \left|H_{i_3}\right|$. The proof is similar to the previous section.
3. $|H_i| = \left|H_j\right| = |H_k|$.

$$\left(H_i * H_j\right) * H_k = \{H_i, H_j, H_k\} = H_i * (H_j * H_k).$$

So, the hypergroupoid $(H, *)$ is a commutative semihypergroup. □

Definition 3.4. Let $G = (V, E)$ be a graph. Now, we define the relation R on H as follows:

$$H_i \, R \, H_j \Longleftrightarrow \theta\,(H_i) = \theta\left(H_j\right).$$

It is easy to see that R is reflexive, symmetric and transitive. Hence, it is an equivalence relation on H.

Theorem 3.5. *The equivalence relation R on the semihypergroup $(H, *)$ is strongly regular.*

Proof. Let $H_i,\ H_j \in H$, $H_i \, R \, H_j$ and H_k be an arbitrary member of H. Then by definition of R, we know that $\theta\,(H_i) = \theta(H_j)$. Now, to prove this theorem we consider following situations:

1. $\theta\,(H_k) < \theta\,(H_i) = \theta(H_j)$. We know that "$*$" is a commutative hyperoperation. So, it is enough to prove " R " is strongly regular on the right. By definition of hyperoperation " $*$ " , we have:

$$\begin{cases} H_i * H_k = H_k \\ H_j * H_k = H_k \\ H_k R H_k \end{cases} \Longrightarrow \ (H_i * H_k)\,\overline{\overline{R}}\,(H_j * H_k).$$

2. $\theta\,(H_k) > \theta\,(H_i) = \theta(H_j)$.
   By definition of hyperoperation " $*$ " , we get that:

$$\begin{cases} H_i * H_k = H_i \\ H_j * H_k = H_j \\ H_i \, R \, H_j \end{cases} \Longrightarrow \ (H_i * H_k)\,\overline{\overline{R}}\,(H_j * H_k).$$

3. $\theta\,(H_k) = \theta\,(H_i) = \theta(H_j)$. By definition of relation "R " and hyperoperation " $*$ ", we have:

$$\begin{cases} H_i \, R \, H_j \\ H_i \, R \, H_k \\ H_j \, R \, H_k \end{cases} \Longrightarrow (H_i * H_k)\,\overline{\overline{R}}\,(H_j * H_k).$$

   Therefore, R is strongly regular relation on the semihypergroup $(H, *)$.

□

Theorem 3.6. $(H/R, \otimes)$ *is a semigroup.*

Proof. Since R is a strongly regular relation on the semihypergroup $(H, *)$, by Theorem , $(H/R, \otimes)$ is a semigroup.

□

Theorem 3.7. *The smallest strongly regular relation β^* on the semihypergroup $(H, *)$ is defined as follows:*

$$H_i \, \beta^* H_j \Longleftrightarrow \begin{cases} 1) & \theta\,(H_i) = \theta(H_j) \\ 2) & |H_i| = \left|H_j\right| \end{cases},$$

Proof. By definition of the hyperoperation " $*$ " and the relation " β ", we have:

$$H_i \, \beta^* H_j \Leftrightarrow \exists\, n \in \mathbb{N} \,\left|H_i \, \beta_n \, H_j \right. \Longleftrightarrow \exists\ (H_1, \ldots,\ H_n) \in H^n : \{H_i, H_j\} \subseteq \prod_{m=1}^{n} H_m$$

$$\Longleftrightarrow \begin{cases} 1) & \theta\,(H_i) = \theta(H_j) \\ 2) & |H_i| = \left|H_j\right| \end{cases}.$$

□

Now we want to study two examples with regards to $\theta(H_i)$ and $|H_i|$ for every $H_i \in H$. We know that all of isolated vertices in a given graph should be placed in all of dominating sets of the graph. So, the number of isolated vertices in graphs below is not important.

First, consider the empty graph, which does not have any edges, see Figure 1. In this case, there is only one dominating set, as follows:

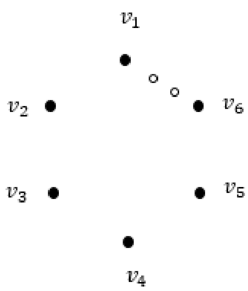

Figure 1:

$$\begin{cases} H_1 = V(G) \\ \theta(H_1) = 0 \\ |H_1| = |V(G)| \end{cases}.$$

In the Figure 1, the small white circles mean we can have different numbers of vertices.

Now, assume a graph with one edge as Figure 2. Then, we have the following dominating sets:

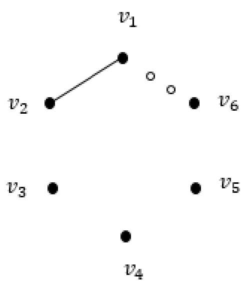

Figure 2:

$$\begin{cases} H_1 = V(G) \\ \theta(H_1) = 1 \\ |H_1| = |V(G)| \\ \\ \\ for\ i \in \{1,2\} \\ H_2 = H_3 = V(G) - \{v_i\} \\ \theta(H_2) = \theta(H_3) = 0 \\ |H_2| = |H_3| = |V(G)| - 1 \end{cases}.$$

In this example, by considering β^*, we have two equivalence classes. We denote these classes by $\overline{H_1}$ and $\overline{H_2}$ such that $H_1 \in \overline{H_1}$ and $H_2, H_3 \in \overline{H_2}$. Then by definition of $'' \otimes ''$ in Theorem 3, we have:

$\otimes$	$\overline{H_1}$	$\overline{H_2}$
$\overline{H_1}$	$\overline{H_1}$	$\overline{H_2}$
$\overline{H_2}$	$\overline{H_2}$	$\overline{H_2}$

The above table shows that $(H/\beta^*, \otimes)$ is not a group. Because by this table, we find that $\overline{H_1}$ is an identity member. On the other hand, by definition of a group with two members we should have:

$$\overline{H_2} \otimes \overline{H_2} = \overline{H_1}.$$

But above table shows that:

$$\overline{H_2} \otimes \overline{H_2} = \overline{H_2}.$$

This example is a counterexample for Proposition 2.7 which is mentioned in [7].

Now, we want to define another hyperoperation on H. First, we need some definitions.

Definition 3.8. Let $G = (V, E)$ be a graph and H_i be a dominating set. Then $\lambda(H_i)$ is a set of all minimal dominating sets which have minimum cardinality among all minimal dominating sets that are obtained from H_i.

We know that all dominating sets in $\lambda(H_i)$ have the same cardinality. We denote this cardinality by $\delta(H_i)$.

Definition 3.9. Let $G = (V, E)$ be a graph. Then we define the hyperoperation $o : H \times H \to \mathcal{P}^*(H)$ as follows: For every $H_i, H_j \in H$, we have

$$H_i \, o \, H_j = \begin{cases} \lambda(H_i) & if\ \delta(H_i) < \delta(H_j) \\ \lambda(H_j) & if\ \delta(H_j) < \delta(H_i) \\ \lambda(H_i) \cup \lambda(H_j) & if\ \delta(H_i) = \delta(H_j) \end{cases}.$$

Theorem 3.10. *Let $G = (V, E)$ be a graph. Then the hypergroupoid (H, o) is a commutative semihypergroup.*

Proof. By definition of the hyperoperation "o", it is easy to see that the hypergroupoid (H, o) is commutative. It is enough to check the associativity of "o", i.e. $(H_i \, o \, H_j) \, o \, H_k = H_i o \, (H_j \, o \, H_k)$ For all $H_i, H_j, H_k \in H$. To achieve this aim we check the following situations for every $H_i, H_j, H_k \in H$:

1. $\delta(H_i) \neq \delta(H_j) \neq \delta(H_k) \neq \delta(H_i)$, $\min\{\delta(H_i), \delta(H_j), \}\delta(H_k) = \delta(H_l)$ and $l \in \{i, j, k\}$
$$(H_i \, o \, H_j) \, o \, H_k = \lambda(H_l) = H_i \, o \, (H_j \, o \, H_k).$$
2. Let $\{i_1, i_2, i_3\} \in \{i, j, k\}$ *and* $\delta(H_{i_1}) = \delta(H_{i_2}) < \delta(H_{i_3})$. Then
$$(H_i \, o \, H_j) \, o \, H_k = \lambda(H_{i_1}) \cup \lambda(H_{i_2}) = H_i \, o \, (H_j \, o \, H_k).$$
3. Let $\{i_1, i_2, i_3\} \in \{i, j, k\}$ *and* $\delta(H_{i_1}) = \delta(H_{i_2}) > \delta(H_{i_3})$. Then
$$(H_i \, o \, H_j) \, o \, H_k = \lambda(H_{i_3}) = H_i \, o \, (H_j \, o \, H_k).$$
4. $\delta(H_i) = \delta(H_k) = \delta(H_j)$,
$$(H_i \, o \, H_j) \, o \, H_k = \lambda(H_i) \cup \lambda(H_j) \cup \lambda(H_k) = H_i \, o \, (H_j \, o \, H_k).$$

□

Definition 3.11. Let $G = (V, E)$ be a graph. Now, we define the relation $\sim$ on H as follows:

$$H_i \sim H_j \Longleftrightarrow \delta(H_i) = \delta\left(H_j\right).$$

It is easy to see that $\sim$ is reflexive, symmetric and transitive. Hence, it is an equivalence relation on H.

Theorem 3.12. *The equivalence relation $\sim$ on the semihypergroup (H, o) is strongly regular.*

Proof. Let $H_i, H_j \in H, H_i \sim H_j$ and H_k be an arbitrary member of H. Then $\delta(H_i) = \delta(H_j)$. Now we consider following situations:

1. $\delta(H_k) < \delta(H_i) = \delta(H_j)$. We know that $''o''$ is a commutative hyperoperation. So, it is enough to prove that $'' \sim ''$ is strongly regular on the right. By definition of the hyperoperation $''o''$, we have:

$$\begin{cases} H_i \, o \, H_k = \lambda(H_k) \\ H_j \, o \, H_k = \lambda(H_k) \\ \delta(H_k) = \delta(H_k) \end{cases} \Longrightarrow (H_i o \, H_k) \overline{\overline{\sim}} (H_j \, o \, H_k).$$

2. $\delta(H_k) > \delta(H_i) = \delta\left(H_j\right)$. By definition of the hyperoperation $''o''$, we get that:

$$\begin{cases} H_i \, o \, H_k = \lambda(H_i) \\ H_j \, o \, H_k = \lambda(H_j) \\ H_i \sim H_j \end{cases} \Longrightarrow (H_i \, o \, H_k) \overline{\overline{\sim}} (H_j \, o \, H_k).$$

3. $\delta(H_k) = \delta(H_i) = \delta(H_j)$
   By definition of the relation $'' \sim ''$, it is obvious that:

$$\begin{cases} H_i \sim H_j \\ H_i \sim H_k \\ H_j \sim H_k \end{cases} \Longrightarrow (H_i \, o \, H_k) \overline{\overline{\sim}} (H_j \, o \, H_k).$$

□

Theorem 3.13. $(H/\sim, \otimes)$ *is a semigroup.*

Proof. Since $\sim$ is a strongly regular relation on the semihypergroup (H, o), by Theorem 3, $(H/\sim, \otimes)$ is a semigroup. □

4. Minimal dominating sets and hyperstructures

In this section, we want to make a connection between minimal dominating sets and hyperstructures. First, we define different semihypergroups on the set of all minimal dominating sets of a graph. Then, by properties of minimal dominating sets and proving some theorems, we could introduce lots of graphs in which these semihypergrous are hypergroups.

=Let $G = (V, E)$ be a graph. We denote the set of all minimal dominating sets of G by $''S''$. In the previous section, we defined the hyperoperation $'' * ''$ on H. Now, we consider this hyperoperation on $''S''$. We know that for every minimal dominating set $S_i \in S$, we have $\theta(S_i) = 0$. Hence, the definition of hyperoperation $'' * ''$ changes as follows on S:

Definition 4.1. Let $G = (V, E)$ be a graph. Then we define the hyperoperation $* : S \times S \to \mathcal{P}^*(S)$ as follows:
For every $S_i, S_j \in S$, we have

$$S_i * S_j = \begin{cases} S_i, & if \;\; |S_i| < \left|S_j\right| \\ S_j, & if \;\; \left|S_j\right| < |S_i| \\ \left\{S_i, S_j\right\}, & if \;\; |S_i| = \left|S_j\right| \end{cases} .$$

Theorem 4.2. *Let $G = (V, E)$ be a graph. Then the hypergroupoid $(S, *)$ is a commutative semihypergroup.*

Proof. By Theorem 3.3, we know that $(H, *)$ is a commutative semihypergroup. Since S is subset of H. We conclude that the hypergroupoid $(S, *)$ is a commutative semihypergroup. □

Theorem 4.3. *Let $G = (V, E)$ be a graph. Then all minimal dominating sets of G have the same cardinality if and only if the semihypergroup $(S, *)$ is a hypergroup.*

Proof. If all minimal dominating sets of G have the same cardinality, then by definition of hyperoperation "$*$", for every $S_i, S_j \in S$, we have:

$$S_i * S_j = \{S_i, S_j\}.$$

So, $S_i * S = S = S * S_i$ and $(S, *)$ is a hypergroup.

Now, let $(S, *)$ be a hypergroup and $\exists\, S_i,\ S_j \in S$ such that $|S_i| \neq \left|S_j\right|$. Then, we have

$$\textit{either } S_i \notin S_j * S_i \textit{ or } S_j \notin S_i * S_j \Rightarrow \textit{either } S_j * S \neq S \textit{ or } S_i * S \neq S.$$

This opposed to the fact that $(S, *)$ is a hypergroup. ■ □

Example 4.4. In each of the graphs presented as Figure 3, the cardinality of minimal dominating sets is 2. Thus, in all of them $(S, *)$ is a hypergroup. To clarify this further, we will illustrate one example of a minimal dominating set in graphs below by encircling the relevant vertices.

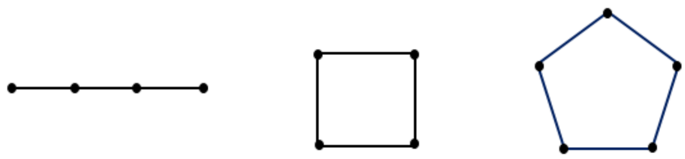

Figure 3:

Example 4.5. Consider a complete graph as Figure 4. In this graph, the cardinality of all minimal dominating set is 1. So $(S, *)$ is a hypergroup.

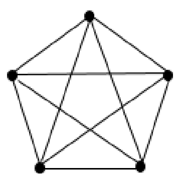

Figure 4:

Example 4.6. Let $G = (V, E)$ be the graph as Figure 5. If $|V(G)| = 2n$, then the cardinality of every minimal dominating sets of G is n. So $(S, *)$ is a hypergroup. We could say that if G is a perfect matching, then $(S, *)$ is a hypergroup.

Theorem 4.7. *Let $G = (V, E)$ be a separate union of subgraphs such that $(S, *)$ in all of them is a hypergroup. Then $(S, *)$ in G is a hypergroup.*

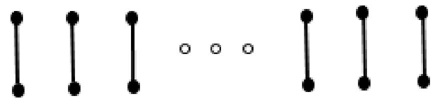

Figure 5:

Proof. Every minimal dominating set in G is the union of the minimal dominating sets in its subgraphs. According to the Theorem 4.3, we know that the cardinality of a minimal dominating set in each subgraph is unique. So, all minimal dominating sets in G have the same cardinality. Thus, $(S, *)$ is a hypergroup. □

Example 4.8. The given graph as Figure 6 consists of a disjoint union of three subgraphs. The cardinality of a minimal dominating set for the square subgraph is 2, for the line segment is 1, and for the triangle subgraph is also 1. Therefore, the total cardinality of a minimal dominating set for the entire graph is 2 + 1 + 1 = 4. Now, by considering Theorem 2.12, we want to define two hyperoperations and give many examples to clarify them.

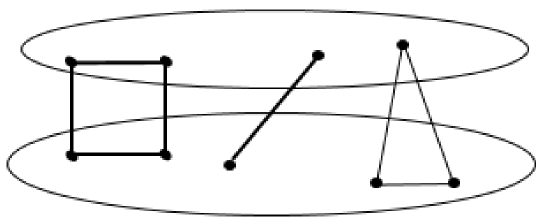

Figure 6:

Definition 4.9. Let S_i be a minimal dominating set. Then we define $\varphi(S_i)$ by

$$\varphi\left(S_i\right) = the\ number\ of\ isolated\ vertices\ of\ G\left[S_i\right] / \left|S_i\right|.$$

By Theorem 2.10 and Corollary 2.11, we find that every independent dominating set is a minimal dominating set. So, the set of all independent dominating sets is a subset of the set of all minimal dominating sets. We denote this set by "I".

Definition 4.10. Let $G = (V, E)$ be a graph. Then we define the hyperoperation $*_i : S \times S \to \mathcal{P}^*(S)$ as follows: For every $S_m, S_n \in S$, we have

$$S_m *_i S_n = \begin{cases} S_n & if\ \ \varphi(S_m) < \varphi(S_n) \\ S_m & if\ \ \varphi(S_m) > \varphi(S_n) \\ \{S_m, S_n\} & if\ \ \varphi\left(S_m\right) = \varphi(S_n) \end{cases} .$$

We call this hyperoperation, *independent hyperoperation*.

Theorem 4.11. *Let $G = (V, E)$ be a graph. Then the hypergroupoid $(S, *_i)$ is a commutative semihypergroup.*

Proof. By definition of hyperoperation "$*_i$", it is easy to see that the hypergroupoid $(S, *_i)$ is commutative. Now, it is enough to check the associativity of "$*_i$", i.e. $(S_m\ *_i\ S_n) *_i\ S_l = S_m *_i (S_n\ *_i\ S_l)$ for all $S_m, S_n, S_l \in S$. To achieve this aim we check the following situations for every $S_m, S_n, S_l \in S$:

1. $\varphi\left(S_m\right) \neq \varphi\left(S_n\right) \neq \varphi\left(S_l\right) \neq \varphi\left(S_m\right)$, $\min\left\{\ \varphi\left(S_m\right), \varphi(S_n), \varphi\left(S_l\right)\right\} = \varphi(S_k)$ and $k \in \{m, n, l\}$

$$\left(S_m\ *_i\ S_n\right)\ *_i S_l = S_k = S_m\ *_i \left(S_n\ *_i S_l\right).$$

2. Let $\{i_1, i_2, i_3\} \in \{m.n.l\}$ *and* $\varphi(S_{i_1}) = \varphi(S_{i_2}) < \varphi(S_{i_3})$. Then

$$(S_m *_i S_n) *_i S_l = \{S_{i_1}, S_{i_2}\} = S_m *_i (S_n *_i S_l).$$

3. Let $\{i_1, i_2, i_3\} \in \{m.n.l\}$ *and* $\varphi(S_{i_1}) = \varphi(S_{i_2}) > \varphi(S_{i_3})$. Then

$$(S_m *_i S_n) *_i S_l = S_{i_3} = S_m *_i (S_n *_i S_l).$$

4. $\varphi(S_m) = \varphi(S_n) = \varphi(S_l)$,

$$(S_m *_i S_n) *_i S_l = \{S_m, S_n, S_l\} = S_m *_i (S_n *_i S_l).$$

So, the hypergroupoid $(S, *_i)$ is a commutative semihypergroup. ■
□

Theorem 4.12. *Let $G = (V, E)$ be a graph. Then if every minimal dominating set of G is independent, then the semihypergroup $(S, *_i)$ is a hypergroup.*

Proof. By definition of hyperoperation " $*_i$ ", for every $S_m, S_n \in S$, we have

$$S_m, S_n \in I,\ \varphi(S_m) = \varphi(S_n) = 1 \text{ and } S_m *_i S_n = \{S_m, S_n\}.$$

So, $S_m *_i S = S = S *_i S_m$ and $(S, *_i)$ is a hypergroup. ■ □

Example 4.13. In all graphs presented in Figurs 7, and 8, every minimal dominating set is independent. So, by Theorem 4.12, $(S, *_i)$ in all of them is a hypergroup. In the following, we bring all states of minimal dominating sets of each graph. We do not consider some states which are equivalent to other states. To clarify this further, we also illustrate a minimal dominating set in all graphs by encircling the relevant vertices.

Theorem 4.14. *Let for every minimal dominating set S_l of graph G we have $\varphi(S_l) = k$ and k be a constant number. Then the semihypergroup $(S, *_i)$ is a hypergroup.*

Proof. By definition of hyperoperation " $*_i$ ", for every $S_m, S_n \in S$, we have

$$\varphi(S_m) = \varphi(S_n) = k \text{ and } S_m *_i S_n = \{S_m, S_n\}.$$

So, $S_m *_i S = S = S *_i S_m$ and $(S, *_i)$ is a hypergroup. □

The following theorem makes it possible to find lots of graphs in which $(S, *_i)$ is a hypergroup.

Theorem 4.15. *Let in graph G the semihypergroup $(S, *_i)$ be a hypergroup and for every minimal dominating set S_n of S we have $\varphi(S_n) = 1$. Then for every $n \in N$, the semihypergroup $(S, *_i)$ in graph $G \times K_n$ is a hypergroup.*

Proof. A minimal dominating set in $G \times K_n$ is a vertex in K_n or a minimal dominating set in G. Since $\varphi(S_n) = 1$ for every $S_n \in S$ in graph G, we find that all minimal dominating sets in G are independent. So, all minimal dominating sets in $G \times K_n$ are independent and by Theorem 4.12, it is easy to see that$(S, *_i)$ in $G \times K_n$ is a hypergroup. □

Example 4.16. By Example 4.13, we know that in P_5, all minimal dominating sets are independent. So, by previous theorem, $(S, *_i)$ in $P_5 \times K_2$ is a hypergroup.

By Example 4.13 and Theorem 4.15 we could build numerous graphs in which $(S, *_i)$ is a hypergroup.

Now, by considering Theorem 2.12, we try to define a semihypergroup which it seems to be the dual of the semihypergroup $(S, *_i)$.

Let G be a graph with no isolated vertices. Then by Theorem 2.13, we know that for every minimal dominating set S_i of G, $V(G) - S_i$ is a dominating set. We define S' by the following way:

$$S' = \{V(G) - S_i \mid S_i \in S\},$$

and we denote $V(G) - S_i$ by $S_i{}'$. In the following, we try to define a hyperoperation on S'.

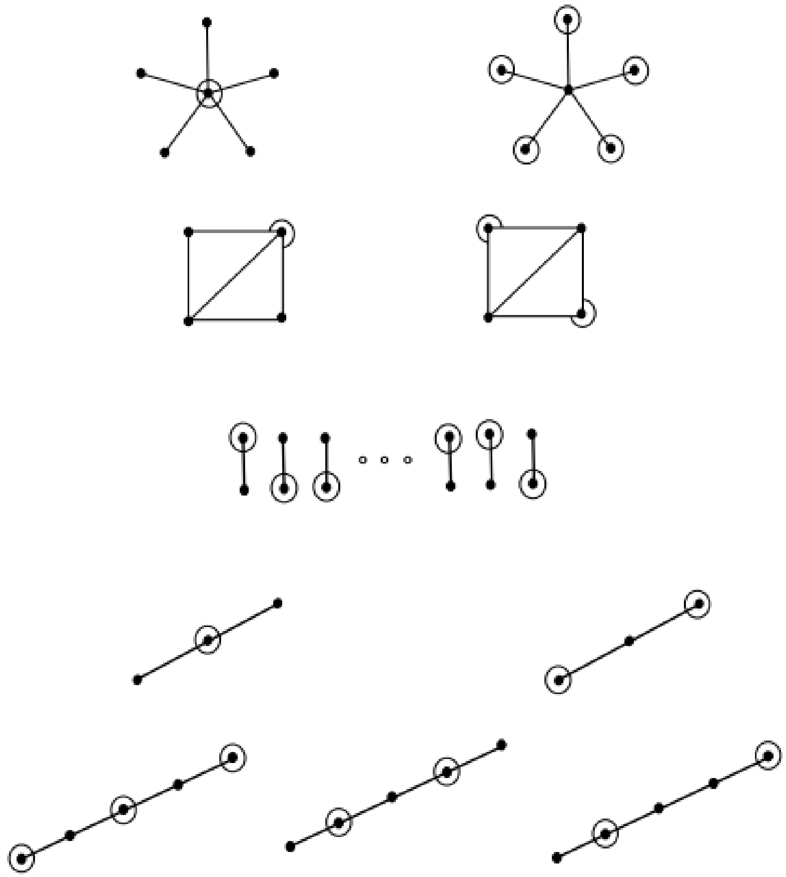

Figure 7:

Definition 4.17. Let G be a graph with no isolated vertices and $S_i' \in S'$. Then we define $\mu(S_i')$ as follows:

$$\mu(S_i') = \begin{cases} \left|\left\{w \in S_i' \mid N(w) \cap S_i = \{v\}\right\}\right| / \left|S_i'\right| & for\ |S_i| \neq 1 \\ \mu(S_i') = 0 & for\ |S_i| = 1 \end{cases}.$$

Definition 4.18. Let $G = (V, E)$ be a graph with no isolated vertices. Then we define the hyperoperation $*' : S' \times S' \to \mathcal{P}*(S')$ as follows:
For every $S_m', S_n' \in S'$, we have

$$S_m' *' S_n' = \begin{cases} S_n' & if\ \mu(S_m') < \mu(S_n') \\ S_m' & if\ \mu(S_m') > \mu(S_n') \\ \{S_m', S_n'\} & if\ \mu(S_m') = \mu(S_n') \end{cases}$$

Theorem 4.19. *Let $G = (V, E)$ be a graph with no isolated vertices. Then the hypergroupoid $(S', *')$ is a commutative semihypergroup.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.11. □

Theorem 4.20. *Let for every minimal dominating set S_n' of graph G we have $\mu(S_n') = k$ and k be a constant number. Then, the semihypergroup $(S', *')$ is a hypergroup.*

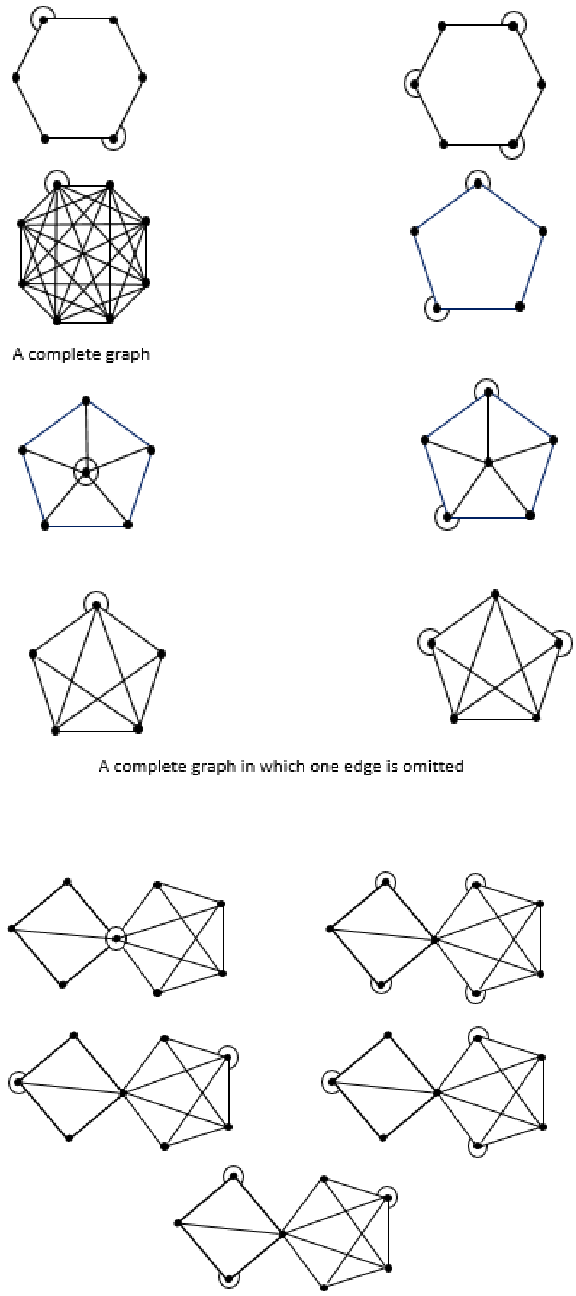

Figure 8:

Proof. The proof of this theorem is similar to the proof of Theorem 4.14. □

Theorem 4.21. *Let $G = (V, E)$ be a graph with no isolated vertices, $(S', *')$ be a hypergroup and for every $S_n{}' \in S'$, $\mu(S_n{}') = 0$. Then the semihypergroup $(S, *_i)$ is a hypergroup.*

Proof. Since for every $S_n{}' \in S'$, $\mu(S_n{}') = 0$ and S_n is a dominating set, we get that for every vertex $w \in S_n{}'$, if $|S_n| \neq 1$, then we have

$$|N(w) \cap S_n| \geq 2.$$

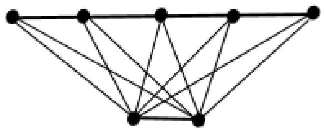

Figure 9: $P_5 \times K_2$.

By considering Theorem 2.12, we conclude that every vertex v in $G[S_n]$, is isolated. So, for every $S_n \in S$ and $|S_n| \neq 1$, $\varphi(S_n) = 1$. We also know that if $|S_n| = 1$, then $\varphi(S_n) = 1$. Therefore, by Theorem 4.12, the semihypergroup $(S, *_i)$ is a hypergroup. □

In the following, there are some examples for above theorem. In all graphs presented in Figure 10, we try to bring all members of S' which are not equivalente to other members.

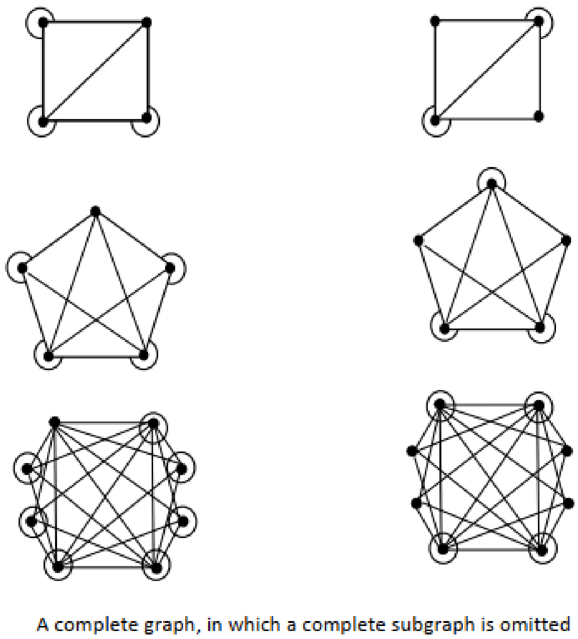

Figure 10:

Theorem 4.22. *Let $G = (V, E)$ be a complete graph in which the edges of one complete subgraph are omitted. Then the semihypergroup $(S, *_i)$ and the semihypergroup $(S', *')$ are hypergroups.*

Proof. Every minimal dominating set in this graph has just two forms:

1. A vertex which is not appear in omitted complete subgraph.
2. A set of all vertices of omitted subgraph.

So, every minimal dominating set in graph G is independent and for every $S_n \in S$, we have $\varphi(S_n) = 1$. Thus, by Theorem 4.12, $(S, *_i)$ is a hypergroup.

On the other hand, by definition of "μ" for every $S_n{}' \in S'$, in states 1 and 2, we have $\mu(S_n{}') = 0$. Hence, the semiypergroup $(S', *')$ is a hypergroup. □

Theorem 4.23. *Let $G = (V, E)$ be a separate union of complete graphs. Then the semihypergroup $(S, *_i)$ and the semihypergroup $(S', *')$ are hypergroups.*

Proof. We know that in complete graphs, every minimal dominating set has just one vertex. So every vertex in minimal dominating set in graph G is isolated and for every $S_n \in S$, we have $\varphi(S_n) = 1$. So by Theorem 4.12, $(S, *_i)$ is a hypergroup.

On the other hand, it is easy to see that in complete graphs, each vertex is dominated by just one vertex. So, for every $S_n{}' \in S'$, $\mu(S_n{}') = 1$. Then the semiypergroup $(S', *')$ is a hypergroup. □

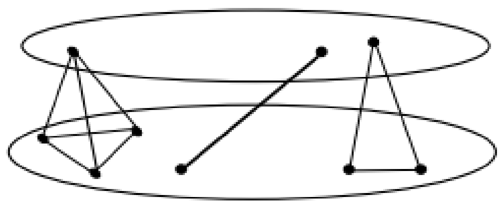

Figure 11:

Theorem 4.24. *Let the semihypergroup $(S', *')$ in graph G be a hypergroup and for every minimal dominating set S_n of S we have $\mu(S_n{}') = 0$. Then for every $n \in N$, the semihypergroup $(S', *')$ in graph $G \times K_n$ is a hypergroup.*

Proof. A minimal dominating set in $G \times K_n$is a vertex in K_n or a minimal dominating set in G. Now, we have two situations:

1. Let a minimal dominating set S_n in $G \times K_n$ is a vertex in K_n. Then all vertices in $G \times K_n$ are dominated by just one vertex. By definition of "μ", we get that $\mu(S_n{}') = 0$.
2. Let a minimal dominating set S_n in $G \times K_n$ is a minimal dominating set in G. We know that in graph G for all minimal dominating set S_n of S we have $\mu(S_n{}') = 0$.Thus, since all vertices in K_n are dominated by all vertices of a minimal dominating set of G, we get that $\mu(S_n{}') = 0$.

Therefore, for every $n \in N$, the semihypergroup $(S', *')$ in graph $G \times K_n$ is a hypergroup. In the following, we bring an example for this theorem.

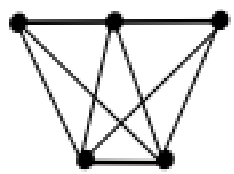

Figure 12: $P_3 \times K_2$

Above theorem and Theorem 4.15, provide the opportunity for us to construct numerous graphs in which $(S, *_i)$ or $(S', *')$ are hypergroups. □

5. Conclusion

This paper makes significant contributions to the intersection of graph theory and algebraic hyperstructure theory by establishing novel connections between dominating sets in graphs and hypergroups. The study introduces innovative approaches for defining semihypergroups on the sets of dominating sets and minimal dominating sets of graphs, effectively leveraging the algebraic properties of hyperstructures to deepen the understanding of graph properties. Furthermore, the study examines the conditions under which these semihypergroups transform into hypergroups. These results enhance the current theoretical framework of hypergroups, particularly in their application to graph theory, and provide a richer algebraic perspective for analyzing graph properties.

The findings have both theoretical and practical implications, particularly in graph optimization and network analysis. Future research could explore hypergroup structures for other graph properties, such as bipartite graphs, spectral graph theory, or graph homomorphisms, and examine their applications in dynamic networks and combinatorial optimization. The methodologies and results here lay a foundation for extending hyperstructure theory's role in addressing complex problems in graph-based systems and beyond.

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