Filomat 39:15 (2025), 5241–5251
https://doi.org/10.2298/FIL2515241P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Uniform structures on *E*-compact semilattice of topological groups

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Abstract. In this paper, we construct uniform structures on a E-compact semilattice of topological groups and study the structure of the uniform completion of a Hausdorff E-compact semilattice of topological groups.

1. Introduction

In 2001, Kunzi, Marin, and Romaguera[5] introduced the concept of quasi-uniformity on a topological semigroup. Also, in 2015, Mastellos[6] studied the quasi-uniform character of a topological semigroup. Due to the presence of the identity element and the homeomorphism property of the translation (left and right) maps in a topological group, so many topological structures exist in a topological group. In particular, a topological group has a compatible uniform structure. Furthermore, the uniform completion of a topological group can be characterized easily. However, due to the absence of an identity element and the homeomorphism property of the translation (left and right) maps in a topological semigroup, we cannot deal with uniform structures on topological semigroups. Recently, S.K. Maity and Monika Paul[7] studied a special type of topological semigroup, i.e., the semilattice of topological groups. In this paper, we construct compatible uniform structures in a particular type of semilattice of topological groups, viz., E-compact semilattice of topological groups. Moreover, analogous to the uniform completion of topological groups, here we study the structure of the uniform completion of a E-compact semilattice of topological groups. Also, we demonstrate the two-sided uniform completion of a E-compact semilattice of topological groups.

2. Preliminaries

In this section, we assemble some known information pertaining to topological semigroups and topological spaces.

2020 *Mathematics Subject Classification*. Primary 22A15; Secondary 22A05, 54E15.
Keywords. Topological semigroup, topological group, semilattice of topological groups, uniform structure, E-compact semilattice of topological groups.
Received: 11 May 2024; Revised: 02 April 2025; Accepted: 15 April 2025
Communicated by Ljubiša D. R. Kočinac
Amitabha Bhattacharyya's research is supported by UGC JRF, NTA Ref. No.: 201610149470, New Delhi, India.
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An element e in a semigroup S is said to be an idempotent element if $e^2 = e$. The set of all idempotent elements in a semigroup S is denoted by $E(S)$. A semigroup S is said to be a semilattice if S is commutative and $S = E(S)$. A congruence ρ on a semigroup S is said to be a semilattice congruence if S/ρ is a semilattice. In a semigroup S, an element $a \in S$ is said to be regular if $a = axa$, for some $x \in S$, and in this case, if we let $y = xax$, then $a = aya$ and $y = yay$. This element y is said to be an inverse of a and the set of inverse elements of a regular element $a \in S$ is denoted by $V(a)$. Naturally, a semigroup S is said to be regular if each of its elements is regular and clearly, in a regular semigroup S, $V(a) \neq \emptyset$, for each $a \in S$. If S is a regular semigroup, then the Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$ and $\mathscr{H}$ on S are defined by : for $a, b \in S$,

$$a \mathscr{L} b \text{ if and only if } Sa = Sb,$$
$$a \mathscr{R} b \text{ if and only if } aS = bS,$$
$$a \mathscr{J} b \text{ if and only if } SaS = SbS,$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}, \quad \mathscr{D} = \mathscr{L} \circ \mathscr{R}.$$

A regular semigroup S is said to be a Clifford semigroup if all its elements are central. In a Clifford semigroup S, for each element $a \in S$, there exists a unique element $x \in V(a)$ such that $ax = xa$. The unique element $x \in V(a)$ satisfying $ax = xa$ is denoted by a^{-1}. In a Clifford semigroup S, the Green's relation $\mathscr{J}(= \mathscr{H})$ is a semilattice congruence on S, and each $\mathscr{J}$-class is a group. For each element a in a Clifford semigroup S, the identity element of the group J_a is denoted by a^0, where $J_a(= H_a)$ is the $\mathscr{J}(= \mathscr{H})$-class containing the element a.

A non-empty subset T of a Clifford semigroup S is known as a full Clifford subsemigroup of S if $E(S) \subseteq T$ and for any $x, y \in T$, $x^{-1}y \in T$.

A semigroup S is said to be a semilattice Y of groups G_α $(\alpha \in Y)$ if S admits a semilattice congruence ρ on S such that $Y = S/\rho$ and each G_α is a ρ-class mapped onto α by the natural semigroup epimorphism $\rho^\# : S \longrightarrow Y$. It is also well known that a semigroup S is a Clifford semigroup if and only if it is a semilattice of groups.

Let X be a non-empty set. A filter on X is a non-empty family $\mathcal{F}$ of subsets of X such that

(1) $\Phi \notin \mathcal{F}$,

(2) $\mathcal{F}$ is closed under finite intersection,

(3) if $B \in \mathcal{F}$ and $B \subset A$ then $A \in \mathcal{F}$ for all $A, B \subset X$.

A uniformity on a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that the following conditions (1) - (5) are satisfied:

(1) each member of $\mathcal{U}$ contains the diagonal $\triangle = \{(x, x) : x \in X\}$,

(2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,

(3) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some V in $\mathcal{U}$,

(4) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$,

(5) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$, where $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ and $U \circ V = \{(x, z) \in X \times X : (x, y) \in U \text{ and } (y, z) \in V \text{ for some } y \in X\}$.

If $\mathcal{U}$ is uniformity for a set X, then we sometime write $X = (X, \mathcal{U})$, and each element of $\mathcal{U}$ is called an entourage of X.

A filter $\mathcal{F}$ on a set X with uniformity $\mathcal{U}$ is said to be a Cauchy filter if for any entourage U of X, there exists $A \in \mathcal{F}$ such that $A \times A \subset U$. The minimal elements (with respect to inclusion) of the set of Cauchy filters on $(X, \mathcal{U})$ are called minimal Cauchy filters on X. In addition, $(X, \mathcal{U})$ is said to have a uniform completion if each Cauchy filter on X is convergent in X. It is well known that any space $(X, \mathcal{U})$ may not be complete. [1]For the space $(X, \mathcal{U})$, there exists a uniformly complete Hausdorff space $\hat{X}$ and a uniformly continuous mapping $i : X \longrightarrow \hat{X}$ satisfying the following conditions:

(P) for any uniformly continuous mapping g from X into a uniformly complete Hausdorff space Y, there is a unique uniformly continuous mapping $h : \hat{X} \longrightarrow Y$ such that $g = h \circ i$.

If (i_1, X_1) is another pair consisting of a uniformly complete Hausdorff space X_1 and a uniformly continuous mapping $i_1 : X \longrightarrow X_1$ having the condition (P), then there is a unique isomorphism $\Phi : \hat{X} \longrightarrow X_1$ such that $i_1 = \Phi \circ i$. In this case, $\hat{X}$ is called the uniform completion of $(X, \mathcal{U})$.

In $(X, \mathcal{U})$, for each Cauchy filter $\mathcal{F}$ on X, there is a unique minimal Cauchy filter $\mathcal{F}_0$ coarser than $\mathcal{F}$. Additionally, in $(X, \mathcal{U})$, every Cauchy filter X, which is coarser than a filter converging to a point $x \in X$, also converges to x. Moreover, for any two topological spaces X and Y, let $f : X \longrightarrow Y$ be a mapping that is continuous at a point $a \in X$; then for every filter base $\mathscr{B}$ on X which converges to a, the filter base $f(\mathscr{B})$ converges to $f(a)$. For further study in semigroup theory, we refer to [2], and in topological space, we refer to [1], [3], etc.

It is well known that [1] for any two continuous functions f and g on a Hausdorff space $(X, \mathcal{U})$, if $f(x) = g(x)$ at all points of a dense subspace A of X, then $f = g$. Also, for any dense subset A of $(X, \mathcal{U})$ such that every Cauchy filter base on A converges in X, X has a uniform completion.

A semigroup S endowed with a topology τ is said to be a topological semigroup if the binary operation $\mu : \substack{S\times S \longrightarrow S \\ (x,y) \mapsto x\cdot y}$ is continuous, where $S \times S$ is considered as the product topological space. A topological semigroup (S, τ) is said to be a semilattice Y of topological groups $(G_\alpha, \tau_\alpha)\,(\alpha \in Y)$[7] if the semigroup S admits a semilattice congruence ρ such that $S/\rho = Y$, each G_α is a ρ-class mapped onto α by the natural semigroup epimorphism $\rho^\# : S \longrightarrow Y$ and $\bigcup_{\alpha \in Y} \tau_\alpha$ forms a base for the topology τ, i.e., $\bigcup_{\alpha \in Y} \tau_\alpha$ generates the topology τ. A semilattice of topological groups (S, τ) is said to be a E-compact semilattice of topological groups if $E(S)$ is compact. A collection of open sets $\mathscr{U}$ containing $E(S)$ is said to be a fundamental system of open sets of $E(S)$ or a base of $E(S)$ in S if for each open set G containing $E(S)$, there exists an open set $U \in \mathscr{U}$ such that $U \subseteq G$. Let S be a topological Clifford semigroup and V be a non-empty subset of S, define a set $(xV)^* = \{y \in S : x^{-1}y \in V \text{ and } x^0 = y^0\}$. It can be easily verified that $(xV)^* \subseteq xV$.

Let S and T be two E-compact semilattices of topological groups. Let $\mathscr{U}$ and $\mathscr{U}'$ be the bases of $E(S)$ and $E(T)$, respectively. A mapping $f : S \longrightarrow T$ is said to be

- left uniformly continuous if for each $U \in \mathscr{U}'$, there exists $V \in \mathscr{U}$ such that for all $x, y \in S$ with $x^{-1}y \in V$ and $x^0 = y^0$ imply that $(f(x))^{-1}f(y) \in U$ and $(f(x))^0 = (f(y))^0$.

- right uniformly continuous if for each $U \in \mathscr{U}'$, there exists $V \in \mathscr{U}$ such that for all $x, y \in S$ with $xy^{-1} \in V$ and $x^0 = y^0$ imply that $f(x)(f(y))^{-1} \in U$ and $(f(x))^0 = (f(y))^0$.

- left right uniformly continuous if for each $U \in \mathscr{U}'$, there exists $V \in \mathscr{U}$ such that for all $x, y \in S$ with $x^{-1}y \in V$ and $x^0 = y^0$ implies $f(x)(f(y))^{-1} \in U$ and $(f(x))^0 = (f(y))^0$.

- right left uniformly continuous if for each $U \in \mathscr{U}'$, there exists $V \in \mathscr{U}$ such that for all $x, y \in S$ with $xy^{-1} \in V$ and $x^0 = y^0$ imply that $(f(x))^{-1}f(y) \in U$ and $(f(x))^0 = (f(y))^0$.

A bijective mapping $f : S \longrightarrow T$ between two E-compact semilattices of topological groups is said to be left (resp. right, right-left, left-right) uniformly isomorphism if f and f^{-1} are both left (resp. right, right-left, left-right) uniformly continuous.

3. Uniform structures on a E-compact semilattice of topological groups

It is well known that any topological group has a uniform structure. In this section, we generalize this result for any E-compact semilattice of topological groups S by constructing a uniform structure on S for which the topology induced by that uniformity coincides with the topology defined on S. Also, we investigate the relation between two-sided uniformity, left uniformity, and right uniformity.

Theorem 3.1. *Let (S, τ) be a E-compact semilattice of topological groups and $\mathscr{U}$ be the collection of all open sets containing $E(S)$ in S. For each $U \in \mathscr{U}$, define $L(U) = \{(x, y) \in S \times S : x^{-1}y \in U, x^0 = y^0\}$. Let $\mathcal{L} = \{L(U) : U \in \mathscr{U}\}$. Then*

(1) $\mathcal{L}$ is a base for some uniformity $\mathscr{F}_{\mathcal{L}}$ on S.

(2) the topology induced by the uniformity $\mathscr{F}_{\mathcal{L}}$ coincides with the topology of S.

Proof. (1) For each $U \in \mathscr{U}$, $\triangle S = \{(x, x) : x \in S\} \subseteq L(U)$. Let $N \in \mathcal{L}$. Then $N = L(U)$, for some $U \in \mathscr{U}$. Now $(x, y) \in (L(U))^{-1}$ if and only if $(y, x) \in L(U)$ if and only if $y^{-1}x \in U$ and $y^0 = x^0$ if and only if $x^{-1}y \in U^{-1}$ and $x^0 = y^0$ if and only if $(x, y) \in L(U^{-1})$. Thus, we have $(L(U))^{-1} = L(U^{-1})$. Due to the continuity of the mapping $x \mapsto x^{-1}$, $U^{-1} \in \mathscr{U}$. Therefore, $N^{-1} \in \mathcal{L}$. Let $M_1, M_2 \in \mathcal{L}$. Then for $i = 1, 2$; $M_i = L(U_i)$, for some $U_i \in \mathscr{U}$. Now, $(x, y) \in M_1 \cap M_2$ if and only if $x^{-1}y \in U_1 \cap U_2$ and $x^0 = y^0$ if and only if $(x, y) \in L(U_1 \cap U_2)$. Therefore, $M_1 \cap M_2 = L(U_1 \cap U_2)$, where $U_1 \cap U_2 \in \mathscr{U}$ and thus $M_1 \cap M_2 \in \mathcal{L}$. Let $M \in \mathcal{L}$. Then $M = L(W)$, for some $W \in \mathscr{U}$. Since $(E(S))^2 = E(S)E(S) \subseteq E(S) \subseteq W$, and $E(S)$ is compact, so by [4, Theorem 1.1], there exist open sets V_1, V_2 containing $E(S)$ in S such that $V_1V_2 \subseteq W$. Take $V = V_1 \cap V_2$. Then $V^2 \subseteq W$. Let $P = L(V)$. Then $P \in \mathcal{L}$. Let $(x, z) \in P \circ P$. Then there exists $y \in S$ such that $(x, y), (y, z) \in P$. This implies that $x^{-1}y, y^{-1}z \in V$ and $x^0 = y^0 = z^0$. The condition $x^0 = z^0$ together with $x^{-1}z = (x^{-1}y)(y^{-1}z) \in V^2 \subseteq W$ implies $P \circ P \subseteq M$. Hence, $\mathcal{L}$ is a base for some uniformity on S.

(2) Let $\mathscr{F}_{\mathcal{L}}$ be the uniformity on S generated by $\mathcal{L}$ and $\tau_{\mathcal{L}}$ be the topology induced by the uniformity $\mathscr{F}_{\mathcal{L}}$ on S. Now, it is enough to show that $\tau_{\mathcal{L}} = \tau$. Clearly, for each $x \in S$ and $M \in \mathscr{F}_{\mathcal{L}}$, $M[x] = \{y \in S : (x, y) \in M\}$ is an open set in $(S, \tau_{\mathcal{L}})$. Moreover, for each $x \in S$, $\{M[x] : M \in \mathcal{L}\}$ is a base at x in $(S, \tau_{\mathcal{L}})$ and if $M = L(U) \in \mathcal{L}$ for some $U \in \mathscr{U}$, then $M[x] = (xU)^*$. Since S is a semilattice of topological groups, for each $x \in S$ and $U \in \mathscr{U}$, $M[x] = (xU)^*$ is open in (S, τ), where $M = L(U)$. Therefore, $\tau_{\mathcal{L}} \subseteq \tau$. For the reverse inclusion, let G be an open set in (S, τ) and $x \in G$. Then $x^0 \in (x^{-1}G)^*$. Let $V = (x^{-1}G)^* \cup \{J_y : y \notin J_x\}$. Then $V \in \mathscr{U}$. Let $M' = L(V)$. Then $M' \in \mathcal{L}$. We show that $x \in M'[x] \subseteq G$. For this, let $z \in M'[x]$. Then $x^{-1}z \in V$ and $x^0 = z^0$. We claim $x^{-1}z \notin \cup\{J_y : y \notin J_x\}$. Otherwise, $x^{-1}z \in J_p$, for some $p \notin J_x$ implies $x^0 = p^0$ and which in turn again imply that $p \in J_x$, a contradiction. Hence $x^{-1}z \in (x^{-1}G)^*$ and this implies $z \in G$. Therefore, $x \in M'[x] \subseteq G$ and hence $G \in \tau_{\mathcal{L}}$. Thus $\tau \subseteq \tau_{\mathcal{L}}$ and hence $\tau_{\mathcal{L}} = \tau$. □

Remark 3.2. The uniformity $\mathscr{F}_{\mathcal{L}}$ on a E-compact semilattice of topological groups S generated by $\mathcal{L}$, in Theorem 3.1, is called the left uniformity on S.

Remark 3.3. If (S, τ) is a E-compact semilattice of topological groups, then $\mathcal{R} = \{R(U) : U \in \mathscr{U}\}$, where $\mathscr{U}$ is the collection of all open sets containing $E(S)$ in S and for $U \in \mathscr{U}$, $R(U) = \{(x, y) \in S \times S : xy^{-1} \in U, x^0 = y^0\}$ is a base for some uniformity $\mathscr{F}_{\mathcal{R}}$ on S. Moreover, $\tau_{\mathcal{R}} = \tau$, where $\tau_{\mathcal{R}}$ is the topology induced by the uniformity $\mathscr{F}_{\mathcal{R}}$ on S. The uniformity $\mathscr{F}_{\mathcal{R}}$ generated by $\mathcal{R} = \{R(U) : U \in \mathscr{U}\}$ is called the right uniformity on S. The E-compact semilattice of topological groups S with respect to the left (respectively, the right) uniformity is denoted by S_s (respectively, S_d).

Remark 3.4. If (S, τ) is a E-compact semilattice of topological groups, then $\mathcal{O} = \{O(U) = R(U) \cap L(U) : U \in \mathscr{U}\}$, where $\mathscr{U}$ is the collection of all open sets containing $E(S)$ in S, is a base for some uniformity $\mathscr{O}$ on S. Moreover, $\tau_{\mathscr{O}} = \tau$, where $\tau_{\mathscr{O}}$ is the topology induced by the uniformity $\mathscr{O}$. The uniformity $\mathscr{O}$ generated by $\mathcal{O} = \{O(U) = R(U) \cap L(U) : U \in \mathscr{U}\}$ is called the two-sided uniformity on S. The E-compact semilattice of topological groups S with respect to the two-sided uniformity $\mathscr{O}$ is denoted by S_o. In addition, if S is either commutative or compact, then the left uniformity, the right uniformity, and the two-sided uniformity coincide.

Theorem 3.5. *For any E-compact semilattice of topological groups S, the two-sided uniformity $\mathscr{O}$ is the coarsest uniformity on S, which is finer than left uniformity as well as right uniformity on S.*

Proof. We first prove that the two-sided uniformity $\mathscr{O}$ is finer than the left uniformity as well as the right uniformity on S. Let $\mathscr{U}$ be the collection of all open sets containing $E(S)$ in S. Since $O(V) = L(V) \cap R(V)$ for any $V \in \mathscr{U}$, it follows that $\mathscr{O}$ is finer than both left uniformity $\mathscr{F}_{\mathcal{L}}$ and right uniformity $\mathscr{F}_{\mathcal{R}}$. Moreover, to show $\mathscr{O}$ is the coarsest uniformity on S which is finer than left uniformity as well as right uniformity, let $\mathscr{W}$ be any other uniformity on S that is finer than both $\mathscr{F}_{\mathcal{L}}$ and $\mathscr{F}_{\mathcal{R}}$. Let $P \in \mathscr{O}$. Then there exists $V \in \mathscr{U}$ such that $O(V) \subset P$. Since $\mathscr{W}$ is finer than both $\mathscr{F}_{\mathcal{L}}$ and $\mathscr{F}_{\mathcal{R}}$, there exist $U_1, U_2 \in \mathscr{W}$ such that $U_1 \subset L(V)$ and $U_2 \subset R(V)$. Then $U = U_1 \cap U_2 \in \mathscr{W}$ and $U \subset L(V) \cap R(V) = O(V) \subset P$. Consequently, $\mathscr{W}$ is finer than $\mathscr{O}$. □

Remark 3.6. Throughout the rest of the paper, we only consider the left uniformity, if not specified, on a E-compact semilattice of topological groups.

For further study, we first state the following useful result from [1].

Theorem 3.7. ([1]) *If X has a uniform structure and $\mathcal{B}$ is the base for the corresponding uniformity on X, then X is Hausdorff if and only if $\bigcap_{U \in \mathcal{B}} U = \triangle X$.*

Proposition 3.8. *A E-compact semilattice of topological groups S is Hausdorff if and only if $\bigcap_{U \in \mathscr{U}} U = E(S)$, where $\mathscr{U}$ is a base of $E(S)$ in S.*

Proof. Since S is a E-compact semilattice of topological groups, by Theorem 3.1, it follows that S has a uniform structure. Let $\mathcal{L} = \{L(U) : U \in \mathscr{U}\}$ be a basis for the left uniformity $\mathscr{F}_{\mathcal{L}}$ on S, where $\mathscr{U}$ is the collection of all open sets in S containing $E(S)$. We first prove that $\bigcap_{M \in \mathcal{L}} M = \triangle S$ if and only if $\bigcap_{U \in \mathscr{U}} U = E(S)$. For this purpose, let $\bigcap_{M \in \mathcal{L}} M = \triangle S$. Clearly, $E(S) \subseteq \bigcap_{U \in \mathscr{U}} U$. For the reverse inclusion, let $x \in \bigcap_{U \in \mathscr{U}} U$. Then $x \in U$, for all $U \in \mathscr{U}$. This implies that $(x^0, x) \in L(U)$, for all $U \in \mathscr{U}$. Since $\bigcap_{M \in \mathcal{L}} M = \triangle S$, it follows that $x = x^0$ and so $\bigcap_{U \in \mathscr{U}} U \subseteq E(S)$. Therefore, $\bigcap_{U \in \mathscr{U}} U = E(S)$. On the other hand, let $\bigcap_{U \in \mathscr{U}} U = E(S)$. Clearly, $\triangle S \subseteq \bigcap_{M \in \mathcal{L}} M$. For the reverse inclusion, let $(a, b) \in \bigcap_{M \in \mathcal{L}} M$. Then $a^{-1}b \in U$ and $a^0 = b^0$, for all $U \in \mathscr{U}$. Since $\bigcap_{U \in \mathscr{U}} U = E(S)$, we must have $a^{-1}b \in E(S)$. Now, $a^0 = b^0$ implies $a \mathscr{J} b$. This implies $a^{-1}b \in J_a = J_b$. Since J_a is a group and $a^{-1}b$ is an idempotent element, we have $a^{-1}b = a^0 = b^0$. Therefore, $a = aa^0 = a(a^{-1}b) = (aa^{-1})b = a^0 b = b^0 b = b$ and thus $(a, b) \in \triangle S$. Hence $\bigcap_{M \in \mathcal{L}} M = \triangle S$. Consequently, $\bigcap_{M \in \mathcal{L}} M = \triangle S$ if and only if $\bigcap_{U \in \mathscr{U}} U = E(S)$ and applying Theorem 3.7, the result follows. □

Proposition 3.9. *Let S be a E-compact semilattice of topological groups. Then the inversion mapping $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ is a right-left as well as a left-right uniformly isomorphism.*

Proof. Clearly, γ is a bijective mapping. To complete the proof, we only show that γ is a right-left uniformly isomorphism. For this purpose, let $\mathscr{U}$ be a base of $E(S)$ in S. Let $U \in \mathscr{U}$. Set $V = U$. Let $x, y \in S$ with $xy^{-1} \in V$ and $x^0 = y^0$. Then $(\gamma(x))^{-1}\gamma(y) = (x^{-1})^{-1}y^{-1} = xy^{-1} \in U$ and $(\gamma(x))^0 = (\gamma(y))^0$. Therefore, γ is a right-left uniformly continuous. Since $\gamma^{-1} = \gamma$, it follows that γ^{-1} is also a right-left uniformly continuous and therefore, γ is a right-left uniformly isomorphism. Similarly, one can easily show that γ is also a left-right uniformly isomorphism. □

Proposition 3.10. *Let S and T be two E-compact semilattices of topological groups. Then every continuous homomorphism $f : S \longrightarrow T$ is left as well as right uniformly continuous.*

Proof. Let $f : S \longrightarrow T$ be a continuous homomorphism. We show that f is a left uniformly continuous mapping. Let $\mathscr{U}$ be a base of $E(S)$ in S and $\mathscr{U}'$ be a base of $E(T)$ in T. Let $U' \in \mathscr{U}'$. Then $f^{-1}(U')$ is an open set containing $E(S)$ in S. Then there exists $U \in \mathscr{U}$ such that $E(S) \subseteq U \subseteq f^{-1}(U')$. Let $x, y \in S$ with $x^{-1}y \in U$ and $x^0 = y^0$. Since f is a homomorphism, it follows that $(f(x))^0 = f(x^0) = f(y^0) = (f(y))^0$ and $(f(x))^{-1}f(y) = f(x^{-1})f(y) = f(x^{-1}y) \in U'$. Therefore, f is a left uniformly continuous mapping. Similarly, one can easily prove that f is also a right uniformly continuous mapping. □

Lemma 3.11. *Any left (respectively, right) translation on a E-compact semilattice of topological groups is left (respectively, right) uniformly continuous.*

Proof. Let $a \in S$ and $\lambda_a : \underset{x \mapsto ax}{S \longrightarrow S}$ be a left translation on S. Let U be an open set containing $E(S)$ in S. Since $E(S)$ is compact, there exists an open set V containing $E(S)$ in S such that $V^2 \subseteq U$. Let $x, y \in S$ be with $(x, y) \in L(V)$. Then $x^{-1}y \in V$ and $x^0 = y^0$. Now, $(\lambda_a(x))^{-1}\lambda_a(y) = (ax)^{-1}(ay) = x^{-1}ya^0 \in V^2 \subseteq U$ and $(\lambda_a(x))^0 = (ax)^0 = a^0x^0 = a^0y^0 = (ay)^0 = (\lambda_a(y))^0$. This implies that λ_a is left uniformly continuous. Similarly, one can show that any right translation on a E-compact semilattice of topological groups is right uniformly continuous. □

Theorem 3.12. *Let S and T be two E-compact semilattices of topological groups. Then the left (resp. right, two-sided) uniformities of $S \times T$ coincide with the product of the left (resp. right, two-sided) uniformities of S and T.*

Proof. Clearly, $Z = S \times T$ is a E-compact semilattice of topological groups. Let $\mathscr{U}$ be a family of open neighborhoods of $E(S)$ in S and $\mathscr{V}$ be a family of open neighborhoods of $E(T)$ in T. Then $\{U \times V : U \in \mathscr{U}, V \in \mathscr{V}\}$ constitutes a base of the open neighborhoods of $E(S \times T) = E(S) \times E(T)$. Now, the E-compact semilattice of topological groups Z has a left uniformity $\mathscr{F}^Z_{\mathcal{L}}$ generated by $\mathscr{G} = \{L(U \times V) : U \in \mathscr{U}, V \in \mathscr{V}\}$. Let $\mathscr{F}^S_{\mathcal{L}}$ and $\mathscr{F}^T_{\mathcal{L}}$ be the uniformities of S and T generated by $\{L(U) : U \in \mathscr{U}\}$ and $\{L(V) : V \in \mathscr{V}\}$ respectively. Now $(S, \mathscr{F}^S_{\mathcal{L}})$ and $(T, \mathscr{F}^T_{\mathcal{L}})$ have uniform structures, $\mathscr{F}^S_{\mathcal{L}} \times \mathscr{F}^T_{\mathcal{L}}$ is a uniformity on $S \times T$ generated by the sets $\mathscr{G}_{U,V} = \{((x, y), (x_1, y_1)) \in Z \times Z : x^{-1}x_1 \in U,\ y^{-1}y_1 \in V \text{ with } x^0 = x_1^0,\ y^0 = y_1^0\}$, where $U \in \mathscr{U}$ and $V \in \mathscr{V}$. For any $U \in \mathscr{U}$ and $V \in \mathscr{V}$, $((x, y), (x_1, y_1)) \in \mathscr{G}_{U,V}$ if and only if $x^{-1}x_1 \in U$, $y^{-1}y_1 \in V$ with $x^0 = x_1^0$, $y^0 = y_1^0$ if and only if $(x^{-1}x_1, y^{-1}y_1) \in U \times V$ with $x^0 = x_1^0$ and $y^0 = y_1^0$ if and only if $(x, y)^{-1}(x_1, y_1) \in U \times V$ with $(x, y)^0 = (x_1, y_1)^0$ if and only if $((x, y), (x_1, y_1)) \in L(U \times V)$. So, for any $U \in \mathscr{U}$ and $V \in \mathscr{V}$, $\mathscr{G}_{U,V} = L(U \times V)$ and it follows that $\mathscr{F}^Z_{\mathcal{L}} = \mathscr{F}^S_{\mathcal{L}} \times \mathscr{F}^T_{\mathcal{L}}$. Analogously, one can easily verify the results for right uniformity and two-sided uniformity. □

4. Structure of the uniform completion of a Hausdorff *E*-compact semilattice of topological groups

We know that the uniform completion of a Hausdorff topological group is also a Hausdorff topological group. In this section, we analyze the structure of the uniform completion of a Hausdorff E-compact semilattice of topological groups. For this purpose, let us first establish some basic results.

Definition 4.1. A E-compact semilattice of topological groups is said to be complete if it is complete with respect to its left and right uniformities.

Proposition 4.2. *Let V be a subset of a E-compact semilattice of topological groups S such that V is complete with respect to the left (or right) uniformity. Then for any $x \in S$, $(xV)^*$ and $(Vx)^*$ are complete with respect to the left (or right) uniformity.*

Proof. Let $x \in S$. Then H_x is a topological group and so has a left uniformity, say $\mathcal{L}_x$. Now, we show that $\{U \cap H_x : U \in \mathscr{U}\}$ is a fundamental set of neighborhoods of x^0, where $\mathscr{U}$ is a base of $E(S)$ in S. Let W be an open set containing x^0 in H_x. Since S is a semilattice of topological groups, by [7, Theorem 2.15] we have H_x is open in S. This concludes that W is open in S. Moreover, because $S \setminus H_x = \bigcup_{y \notin H_x} H_y$, we have $W \bigcup (S \setminus H_x)$ is open in S. Therefore, $E(S) = \{x^0\} \bigcup (E(S) \setminus \{x^0\}) \subseteq W \cup (S \setminus H_x)$. As $\mathscr{U}$ is a base of $E(S)$ in S, there exists a basic open set $U \in \mathscr{U}$ such that $U \subset W \cup (S \setminus H_x)$. This implies $U \cap H_x \subseteq W$, and thus it follows that $\{U \cap H_x : U \in \mathscr{U}\}$ is a fundamental set of neighborhoods of x^0. Therefore, $\{L(U \cap H_x) : U \in \mathscr{U}\}$ is a base for the left uniformity $\mathcal{L}_x$, where $L(U \cap H_x) = \{(a, b) \in H_x \times H_x : a^{-1}b \in U \cap H_x\}$. Now, since S has a uniform structure and H_x is a subset of S, let $\mathcal{L}'_x$ be the left uniformity induced on H_x by the left uniformity of S. Then $\{L(U) \cap (H_x \times H_x) : U \in \mathscr{U}\}$ is a base for the left uniformity $\mathcal{L}'_x$ on H_x. Now, we show that $\{L(U) \cap (H_x \times H_x) : U \in \mathscr{U}\} = \{L(U \cap H_x) : U \in \mathscr{U}\}$. Clearly, for any $U \in \mathscr{U}$, $L(U) \cap (H_x \times H_x) = L(U \cap H_x)$. This implies that $\{L(U) \cap (H_x \times H_x) : U \in \mathscr{U}\} = \{L(U \cap H_x) : U \in \mathscr{U}\}$ and thus $\mathcal{L}_x = \mathcal{L}'_x$. Also, the restriction of the left translation $\lambda_x : S \longrightarrow S$ on H_x is a left translation on H_x and therefore, $\lambda_x|_{H_x} : H_x \longrightarrow H_x$ is an isomorphism in the sense of uniformity but not in the sense of homomorphism. Since H_x is a closed subset of S, it follows that $V \cap H_x$ is a closed subset of the complete set V, and so $V \cap H_x$ is complete. Now, $\lambda_x|_{H_x}(V \cap H_x) = x(V \cap H_x)$ and $(xV)^* = x(V \cap H_x)$ imply that $(xV)^*$ is complete. Similarly, one can prove that $(Vx)^*$ is complete. □

From Proposition 3.9, it is easy to show that a E-compact semilattice of topological groups is complete if it is complete with respect to one of its uniformities, either left or right. Therefore, for the completeness of the E-compact semilattice of topological groups, we consider the left uniform structure.

Proposition 4.3. *If in a E-compact semilattice of topological groups S, there is a neighborhood V containing $E(S)$ in S which is complete with respect to either the right or the left uniformity, then S is complete.*

Proof. Let V be complete with respect to the left uniformity, and let $\mathscr{F}$ be a Cauchy filter on S_s. Then there exists an element A in $\mathscr{F}$ such that $A \times A \subseteq L(V)$ and this implies that for any $a, x \in A$, $x^{-1}a \in V$, $x^0 = a^0$. So, for any $x \in A$, $a \in (xV)^*$, for all $a \in A$ and therefore, the trace of $\mathscr{F}$ on the complete subspace $(xV)^*$ of S_s is a Cauchy filter which converges to a point x'. Since x' is a cluster point of $\mathscr{F}$, by [1, Chapter II, §3, No. 2, Corollary 2], it follows that $\mathscr{F}$ converges to x'. Hence S is complete. □

Using Proposition 4.3, we have the following result.

Theorem 4.4. *Any locally compact E-compact semilattice of topological groups is complete.*

Proof. Since S is a semilattice of topological groups, by [7, Theorem 2.15], we have H_x is open in S for all $x \in S$. Moreover, $S = \bigcup_{e \in E(S)} H_e$. This leads to $\mathscr{U} = \{H_e : e \in E(S)\}$ is an open cover of $E(S)$. Due to the compactness property of $E(S)$, we have $E(S) \subseteq \bigcup_{i=1}^{n} H_{e_i}$ for some $e_1, e_2, \ldots, e_n$ in $E(S)$. Since for each $i = 1, \ldots, n$, H_{e_i} is a group, this concludes that $E(S)$ is a finite set. Let $E(S) = \{e_1, e_2, ..., e_n\}$ be the set of idempotents of S and let U be an open set containing $E(S)$. For each $i = 1, 2, 3, ..., n$, since S locally compact, there exists an open set V_i containing e_i in S such that $e_i \in \overline{V_i} \subset U$, where $\overline{V_i}$ is compact. Therefore, $E(S) \subset \bigcup_{i=1}^{n} \overline{V_i} \subset U$ and thus $E(S) \subset \overline{\bigcup_{i=1}^{n} V_i} \subset U$. Set $V = \bigcup_{i=1}^{n} V_i$. Then V is an open neighborhood of $E(S)$ in S and $\overline{V}$ is compact with $\overline{V} \subseteq U$. Since every compact space is complete with respect to its unique uniformity, by Proposition 4.3, it follows that S is complete. □

Since any E-compact semilattice of topological groups is complete if it is complete with respect to one of its uniformities, either left or right, for the uniform completion of a Hausdorff E-compact semilattice of topological groups, we consider the left uniform structure.

Proposition 4.5. *Let S and S' be two E-compact semilattices of topological groups in which S' is Hausdorff, and let N and N' be dense full Clifford subsemigroups of S and S', respectively. Then, every continuous homomorphism f from N into N' can be uniquely extended to a continuous homomorphism $\hat{f}$ of S into S'. Furthermore, if S is Hausdorff and complete, and if f is an isomorphism from N onto N', then $\hat{f}$ is an isomorphism from S onto S'.*

Proof. Since f is a continuous homomorphism from N into N', by Proposition 3.10, f is a left uniformly continuous function from N into N'. So, by [1, Chapter II, §3, no. 6, Theorem 2], f can be extended uniquely to a mapping $\hat{f}$ of S into S' such that $\hat{f}$ is left uniformly continuous. Now, we show that $\hat{f} : S \longrightarrow S'$ is a homomorphism. For this, let $x, y \in S$. If possible, let $\hat{f}(xy) \neq \hat{f}(x)\hat{f}(y)$. Due to the Hausdorff property of $\hat{S}$, there exist two disjoint open sets U and V that contain $\hat{f}(xy)$ and $\hat{f}(x)\hat{f}(y)$, respectively, in S'. Since $\hat{f}$ is left uniformly continuous, it is also continuous. This implies that $\hat{f}^{-1}(U)$ is an open set containing xy in S. As S is a topological semigroup, there exist open neighborhoods U_1 of x and U_2 of y in S such that $U_1U_2 \subseteq \hat{f}^{-1}(U)$.

Moreover, since S' is a topological semigroup, there exist open neighborhoods V_1 of $\hat{f}(x)$ and V_2 of $\hat{f}(y)$ in S' such that $V_1V_2 \subseteq V$. Then $\hat{f}^{-1}(V_1)$ and $\hat{f}^{-1}(V_2)$ are open sets containing x and y respectively in S. Set $W_1 = U_1 \cap \hat{f}^{-1}(V_1)$ and $W_2 = U_2 \cap \hat{f}^{-1}(V_2)$. Then W_1 and W_2 are open sets containing x and y respectively in S. Since $\overline{N} = S$, $W_1 \cap N \neq \emptyset$ and $W_2 \cap N \neq \emptyset$. Let $n_1 \in W_1 \cap N$ and $n_2 \in W_2 \cap N$. Therefore, $n_1n_2 \in N$. Also, $\hat{f}|_N = f$ and f is a homomorphism implies that $\hat{f}(n_1n_2) = f(n_1n_2) = f(n_1)f(n_2) = \hat{f}(n_1)\hat{f}(n_2)$. Thus, $\hat{f}(n_1n_2) \in U$ and $\hat{f}(n_1n_2) = \hat{f}(n_1)\hat{f}(n_2) \in V_1V_2 \subseteq V$ imply that $U \cap V \neq \emptyset$, a contradiction. This contradiction ensures that $\hat{f}$ is a homomorphism.

To prove the last part, let S be Hausdorff and complete, and $f : N \longrightarrow N'$ is an isomorphism. Then there exists an isomorphism $g : N' \longrightarrow N$ such that $f^{-1} = g$. By the first part, g has a continuous extension $\hat{g} : S' \longrightarrow S$, which is also a homomorphism. Now, $g \circ f = id_{N}$ and $f \circ g = id_{N'}$. Since S is Hausdorff, it follows that $\hat{g} \circ \hat{f} = id_{S}$. Similarly, $\hat{f} \circ \hat{g} = id_{S'}$. This implies that $\hat{f}$ is bijective and, consequently, $\hat{f}$ is an isomorphism. □

Theorem 4.6. *Let S be a Hausdorff E-compact semilattice of topological groups.*

(1) *For any two Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ in S_s, the image of the filter $\mathcal{F} \times \mathcal{G}$ under the binary operation $\mu : S \times S \longrightarrow S$ is a Cauchy filter in S_s.*

(2) *S is a dense full Clifford subsemigroup of a complete E-compact semilattice of topological groups $\hat{S}$ if and only if the image under the mapping $x \mapsto x^{-1}$ of a Cauchy filter with respect to the left uniformity of S is a Cauchy filter with respect to this uniformity. Moreover, up to isomorphism, the complete E-compact semilattice of topological groups $\hat{S}$ is unique.*

Proof. (1) Let W be an open set containing $E(S)$ in S. Since $E(S)$ is compact, by the Wallace theorem, there exists an open set V containing $E(S)$ in S such that $V^3 \subseteq W$. As $\mathcal{G}$ is a Cauchy filter, there is an element $B \in \mathcal{G}$ such that $B \times B \subseteq L(V)$. Let $b \in B$. Then it is easy to verify that $U = \{x \in S : b^{-1}xb \in V\}$ is an open set containing $E(S)$ in S. Again, since $\mathcal{F}$ is a Cauchy filter, there is an element $A \in \mathcal{F}$ such that $A \times A \subseteq L(U)$. Now, $A \times B \in \mathcal{F} \times \mathcal{G}$ and $\mu(A \times B) = AB$. We show that $AB \times AB \subseteq L(W)$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Then $(b_1, b), (b, b_2) \in L(V)$. This implies that $b_1^{-1}b, b^{-1}b_2 \in V$ and $b_1^0 = b^0 = b_2^0$. Also, $(a_1, a_2) \in A \times A \subseteq L(U)$ implies that $a_1^{-1}a_2 \in U$ and $a_1^0 = a_2^0$. Then $b^{-1}a_1^{-1}a_2b \in V$. Now, $(a_1b_1)^{-1}(a_2b_2) = b_1^{-1}a_1^{-1}a_2b_2 = (b_1^{-1}b)(b^{-1}a_1^{-1}a_2b)(b^{-1}b_2) \in V^3 \subseteq W$ and $(a_1b_1)^0 = a_1^0b_1^0 = a_2^0b_2^0 = (a_2b_2)^0$ imply $AB \times AB \subseteq L(W)$. Consequently, the image of the filter $\mathcal{F} \times \mathcal{G}$ under the binary operation $\mu : S \times S \longrightarrow S$ is a Cauchy filter in S_s.

(2) Since S_s is a Hausdorff topological space, S_s has a uniform completion $\hat{S}$ with respect to the left uniformity such that S_s is a dense subspace of $\hat{S}$. By virtue of [1, Chapter II, §3, no. 6, Proposition II] and the above result, it is clear that the binary operation $\mu : \hat{S} \times \hat{S} \longrightarrow \hat{S}$ is continuous. Now, we show that $\hat{S}$ is a topological semigroup. Consider the functions $f, g : \hat{S} \times \hat{S} \times \hat{S} \longrightarrow \hat{S}$ defined by $f(x, y, z) = x(yz)$ and $g(x, y, z) = (xy)z$ for all $x, y, z \in \hat{S}$. Since $f = g$ on the dense subspace S, it follows that $f = g$ on $\hat{S}$, i.e. the law $(x, y) \longmapsto xy$ is associative on $\hat{S}$ and, therefore, $\hat{S}$ is a topological semigroup.

Now, we assume that the image under the mapping $x \longmapsto x^{-1}$ of a Cauchy filter with respect to the left uniformity of S is again a Cauchy filter with respect to this uniformity. It follows that the mapping $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ has a continuous extension on $\hat{S}$. Let $\hat{\gamma} : \hat{S} \longrightarrow \hat{S}$ be the continuous extension of $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$. Since S is dense in $\hat{S}$ and $\hat{S}$ has a Hausdorff uniform structure, it follows that $\hat{\gamma}$ is unique. Now, we show that $\hat{S}$ is a completely regular semigroup. For this, let $a \in \hat{S}$. Since S is dense in $\hat{S}$, there exists a net (a_α) in S such that $a_\alpha \longrightarrow a$. Since $\hat{\gamma}$ is continuous, we must have $\hat{\gamma}(a_\alpha) \longrightarrow \hat{\gamma}(a)$. Again, since $\hat{\gamma}$ is an extension of γ, it follows that $a_\alpha^{-1} \longrightarrow \hat{\gamma}(a)$. Now, for each α, $a_\alpha a_\alpha^{-1} a_\alpha = a_\alpha$ and $a_\alpha^{-1} a_\alpha = a_\alpha a_\alpha^{-1}$. This implies that $a \cdot \hat{\gamma}(a) \cdot a = a$ and $\hat{\gamma}(a) \cdot a = a \cdot \hat{\gamma}(a)$. This implies that a is completely regular and $a^{-1} = \hat{\gamma}(a)$. Hence, $\hat{S}$ is a completely regular semigroup. Now, we show that $E(\hat{S}) = E(S)$. Clearly, $E(S) \subseteq E(\hat{S})$. For the reverse inclusion, let $\hat{e} \in E(\hat{S})$. Clearly, $\hat{\gamma}(\hat{e}) = \hat{e}$. Since $\hat{e} \in \hat{S} = \overline{S}$, there is a net (e_α) in S such that $e_\alpha \longrightarrow \hat{e}$. This implies $e_\alpha e_\alpha^{-1} \longrightarrow \hat{e}\hat{\gamma}(\hat{e}) = \hat{e}$. Now, for each α, $e_\alpha e_\alpha^{-1} \in E(S)$ and $E(S)$ compact in S implies that $\hat{e} \in E(S)$. Therefore, $E(\hat{S}) = E(S)$. This implies that S is a dense full Clifford subsemigroup of $\hat{S}$. Let $e \in E(\hat{S})$ and $y \in \hat{S}$. Then $e \in E(S)$ and there is a net (y_α) in S such that $y_\alpha \longrightarrow y$. Now, since S is a Clifford semigroup, so for each α, $ey_\alpha = y_\alpha e$, and hence $ey = ye$. Therefore, idempotents of $\hat{S}$ are central and, consequently, $\hat{S}$ is a topological Clifford semigroup. Now, we show that each $\mathscr{J}$-class is open in $\hat{S}$. From [7, Lemma 2.7], it follows that $\varphi : \underset{x \mapsto x^0}{\hat{S} \longrightarrow \hat{S}}$ is a continuous function. We denote the $\mathscr{J}$-class containing an element a in $\hat{S}$ by $\hat{J}_a$. Now, for any $a \in \hat{S}$, $\varphi^{-1}(\{a^0\}) = \hat{J}_a$ and this implies that each $\mathscr{J}$-class is closed in $\hat{S}$. Since $\hat{S}$ is a Clifford semigroup and $E(\hat{S}) = E(S)$ is finite, we must have each $\mathscr{J}$-class open in $\hat{S}$. Therefore, by [7, Theorem 2.15], we conclude that $\hat{S}$ is a semilattice of topological groups. Since $E(\hat{S}) = E(S)$ is compact, it follows that $\hat{S}$ is a E-compact semilattice of topological groups.

Now, since $\hat{S}$ is a E-compact semilattice of topological groups, there is a left uniformity on $\hat{S}$. Let $\hat{\mathcal{U}}$ be the left uniformity on the E-compact semilattice of topological groups $\hat{S}$. We show that $(\hat{S}, \hat{\mathcal{U}})$ has a uniform completion. Since $\hat{S}$ is itself complete, there is a uniformity on $\hat{S}$ and let $\mathcal{U}$ be a complete uniformity on $\hat{S}$. Then $\hat{\mathcal{U}}$ and $\mathcal{U}$ induce the same uniformity on S and this implies that every Cauchy filter base on S with respect to $\hat{\mathcal{U}}$ is also a Cauchy filter base on S with respect to $\mathcal{U}$. Let $\mathcal{B}$ be a Cauchy filter base on S with respect to the uniformity $\hat{\mathcal{U}}$. Due to the completeness of uniform structure on $\mathcal{U}$, we conclude $\mathcal{B}$ converges in $\hat{S}$. Now, $\hat{\mathcal{U}}$ and $\mathcal{U}$ induce the same topology on $\hat{S}$. So, $\hat{\mathcal{U}}$ is complete uniformity. Hence, $\hat{S}$ has a uniform completion. In particular, $\mathcal{U}$ and $\hat{\mathcal{U}}$ coincide. The uniqueness of $\hat{\mathcal{U}}$ follows from Proposition 4.5.

Conversely, suppose that S is isomorphic to a dense full Clifford subsemigroup of a complete E-compact semilattice of topological groups $\hat{S}$. Let $\mathcal{F}$ be a Cauchy filter in S with respect to the left uniformity. Then $\mathcal{F}$ converges in $\hat{S}$. Since the mapping $\gamma : \underset{x \mapsto x^{-1}}{\hat{S} \longrightarrow \hat{S}}$ is continuous, $\mathcal{F}^{-1}$ converges in $\hat{S}$, so it is a Cauchy filter. Now, for any $A \in \mathcal{F}^{-1}$, $A \subseteq S$ implies that $\gamma(\mathcal{F}) = \mathcal{F}^{-1}$ is a Cauchy filter in S with respect to the left uniformity. □

Definition 4.7. Let S and T be two E-compact semilattices of topological groups. A mapping $f : S \longrightarrow T$ is said to be uniformly continuous with respect to two-sided uniformity if it is both left as well as right uniformly continuous, i.e., for any open set V containing $E(T)$ in T, there exists an open set U containing $E(S)$ in S such that for any $x, y \in S$ with $(x, y) \in O(U)$, $(f(x), f(y)) \in O(V)$.

Theorem 4.8. *A Hausdorff E-compact semilattice of topological groups S is isomorphic to a dense full Clifford subsemigroup of a complete E-compact semilattice of topological groups $\hat{S}$ with respect to the two-sided uniformity. Moreover, up to isomorphism, the complete E-compact semilattice of topological groups $\hat{S}$ is unique.*

Proof. We first show that the binary operation $\mu : \hat{S} \times \hat{S} \longrightarrow \hat{S}$ is continuous. For this, it is enough to show that for any two Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ in $S_\circ$, the image of the filter $\mathcal{F} \times \mathcal{G}$ under binary operation $\mu : \underset{(x,y) \mapsto xy}{S \longrightarrow S}$ is a Cauchy filter in $S_\circ$. Let W be an open set containing $E(S)$ in S. Due to the compactness of $E(S)$, there exists an open set U containing $E(S)$ in S such that $U^3 \subseteq W$. Since $\mathcal{F}$ is a Cauchy filter, there exists an element $A_1 \in \mathcal{F}$ such that $A_1 \times A_1 \subseteq O(U)$. Similarly, $B_1 \times B_1 \subseteq O(U)$, for some $B_1 \in \mathcal{G}$. Let $a \in A_1$ and $b \in B_1$. Set $V_1 = \{x \in S : axa^{-1} \in U\}$ and $V_2 = \{x \in S : b^{-1}xb \in U\}$. Then V_1, V_2 are open sets containing $E(S)$ in S. As $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters, $A_2 \times A_2 \subseteq O(V_2)$ and $B_2 \times B_2 \subseteq O(V_1)$ for some $A_2 \in \mathcal{F}$ and $B_2 \in \mathcal{G}$. Set $A = A_1 \cap A_2$ and $B = B_1 \cap B_2$. Then $A \in \mathcal{F}$ and $B \in \mathcal{G}$. We show that $AB \times AB \subseteq O(W)$. Let $(x, y), (x', y') \in A \times B$. Then $x, x' \in A$ and $y, y' \in B$. Here $x^0 = x'^0 = a^0$ and $y^0 = y'^0 = b^0$. Now $A \times A \subseteq A_2 \times A_2 \subseteq O(V_2)$. This implies that $x^{-1}x' \in V_2$ and so $b^{-1}x^{-1}x'b \in U$. Similarly, $ayy'^{-1}a^{-1} \in U$. Also, $y^{-1}b, b^{-1}y', xa^{-1}, ax'^{-1} \in U$. Now $(xy)^{-1}(x'y') = (y^{-1}b)(b^{-1}x^{-1}x'b)(b^{-1}y') \in U^3 \subseteq W$ and $(xy)(x'y')^{-1} = (xa^{-1})(ayy'^{-1}a^{-1})(ax'^{-1}) \in U^3 \subseteq W$. Also, $(xy)^0 = (x'y')^0$. Therefore, $AB \times AB \subseteq O(W)$. Hence, the image of the filter $\mathcal{F} \times \mathcal{G}$ is a Cauchy filter in $S_\circ$. So, the binary operation $\mu : \hat{S} \times \hat{S} \longrightarrow \hat{S}$ is continuous. Now we show that $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ is uniformly continuous with respect to two-sided uniformity. Let V be an open set containing $E(S)$ in S. Set $U = V$. Let $x, y \in S$ with $(x, y) \in O(U)$. Then $x^{-1}y, xy^{-1} \in U$ and $x^0 = y^0$. Now $(\gamma(x))^{-1}\gamma(y) = xy^{-1} \in V$ and $\gamma(x)(\gamma(y))^{-1} = x^{-1}y \in V$ with $(\gamma(x))^0 = (\gamma(y))^0$. This implies that γ is uniformly continuous. Then by [1, Chapter 2, §3, no. 6, Theorem 2], $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ has a continuous extension $\hat{\gamma} : \hat{S} \longrightarrow \hat{S}$. The rest of the proof follows in a similar fashion as in Theorem 4.6. □

Now we prove that the two-sided uniform completion is universal.

Theorem 4.9. *Let S and T be two Hausdorff E-compact semilattices of topological groups, and let $f : S \longrightarrow T$ be a continuous homomorphism. If T is complete relative to its two-sided uniformity, then f can be extended uniquely to a continuous homomorphism from the two-sided uniform completion of S into T.*

Proof. Since f is a continuous homomorphism, f is both left and right uniformly continuous. Consequently, f is uniformly continuous relative to its two-sided uniformity. So by [1, Chapter 2, §3, no. 6, Theorem 2], f has a continuous extension $\hat{f}$ from the two sided uniform completion $\hat{S}$ of S into T. To complete the proof, it is now enough to verify that $\hat{f}$ is a homomorphism from $\hat{S}$ to T. Suppose $\hat{f}$ is not a homomorphism. Then there exist elements $x, y \in \hat{S}$ such that $\hat{f}(xy) \neq \hat{f}(x)\hat{f}(y)$. Due to the Hausdorff property of T, there exist two disjoint open sets P and Q containing $\hat{f}(xy)$ and $\hat{f}(x)\hat{f}(y)$, respectively, in T. Then there exist open sets U and V containing $\hat{f}(x)$ and $\hat{f}(y)$ respectively in T such that $UV \subset Q$. By the continuity of $\hat{f}$, there are open sets U_1, V_1 and W_1 of x, y and xy respectively in $\hat{S}$ such that $\hat{f}(U_1) \subset U$, $\hat{f}(V_1) \subset V$ and $\hat{f}(W_1) \subset P$. Since $\hat{S}$ is a topological semigroup, there exist open sets U_2 and V_2 respectively in $\hat{S}$ such that $U_2V_2 \subseteq W_1$. Set $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Since S is dense in $\hat{S}$, there exist $x_1 \in U_3 \cap S$ and $y_1 \in V_3 \cap S$. Since f is a homomorphism and $\hat{f}|_S = f$, we must have $\hat{f}(x_1y_1) = f(x_1y_1) = f(x_1)f(y_1) = \hat{f}(x_1)\hat{f}(y_1) \in UV \subset Q$. But we have $\hat{f}(x_1y_1) \in \hat{f}(U_2V_2) \subset \hat{f}(W_1) \subset P$, a contradiction. This contradiction ensures that $\hat{f}$ is a homomorphism. □

It can be easily verified that in a commutative E-compact semilattice of topological groups, left and right uniform structures coincide. So, in this section, we use only the term uniformity instead of right or left uniformities.

Theorem 4.10. *Let S be a commutative Hausdorff E-compact semilattice of topological groups. Then the mappings $\mu : \underset{(x,y) \mapsto xy}{S \times S \longrightarrow S}$ and $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ are uniformly continuous on $S \times S$ and S, respectively. Moreover, S admits a Hausdorff uniform completion $\hat{S}$, and $\hat{S}$ is a commutative E-compact semilattice of topological groups.*

Proof. Since the mappings $\mu : \underset{(x,y) \mapsto xy}{S \times S \longrightarrow S}$ and $\gamma : \underset{x \mapsto x^{-1}}{S \longrightarrow S}$ are continuous homomorphisms, by Theorem 3.10, it follows that these mappings are uniformly continuous. Since S is Hausdorff and its left and right uniformities coincide, by Theorem 4.6, S admits a Hausdorff uniform completion $\hat{S}$, which is a E-compact semilattice of topological groups. Now, consider the functions $f, g : \hat{S} \times \hat{S} \longrightarrow \hat{S}$ by $f(x, y) = xy$ and $g(x, y) = yx$. Since $f = g$ on the dense subspace S, it follows that $\hat{S}$ is commutative. □

Theorem 4.11. *Let S be a commutative regular semigroup. Let (S, τ_1) and (S, τ_2) be two Hausdorff E-compact semilattices of topological groups. Suppose that τ_1 is finer than τ_2 and there is a fundamental system of open sets of $E(S)$ in (S, τ_1) that are closed in (S, τ_2). Let S_1 and S_2 be the uniform completions of (S, τ_1) and (S, τ_2), respectively. Then*

(1) *the continuous homomorphism $f : S_1 \longrightarrow S_2$ extending the identity mapping of S is injective.*

(2) *any complete subspace A of S with respect to the uniformity $\mathcal{U}_2$ corresponding to the topology τ_2 is also complete with respect to the uniformity $\mathcal{U}_1$ corresponding to the topology τ_1.*

Proof. (1) Let $\mathcal{U}_1$ be the uniformity of S corresponding to the topology τ_1. Since the mapping $id : (S, \tau_1) \longrightarrow (S, \tau_2)$ is uniformly continuous and $\tau_2 \subseteq \tau_1$, $id : (S, \tau_1) \longrightarrow (S, \tau_2)$ is uniformly continuous. Hence, the identity mapping can be extended uniquely to a continuous homomorphism f. Let $a_1, a_2 \in S_1$ and $f(a_1) = f(a_2) = a$ (say). Then there exist two filters $\mathcal{F}'_1$ and $\mathcal{F}'_2$ in (S, τ_1) such that $\mathcal{F}'_1 \longrightarrow a_1$ and $\mathcal{F}'_2 \longrightarrow a_2$. This implies that $\mathcal{F}'_1$ and $\mathcal{F}'_2$ are Cauchy filters on (S, τ_1). Since (S, τ_1) has a uniform structure and $\mathcal{F}'_1$ and $\mathcal{F}'_2$ are Cauchy filters on (S, τ_1), there exist minimal Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on (S, τ_1) such that $\mathcal{F}_1 \subseteq \mathcal{F}_1'$ and $\mathcal{F}_2 \subseteq \mathcal{F}_2'$. Clearly, $\mathcal{F}_1 \longrightarrow a_1$ and $\mathcal{F}_2 \longrightarrow a_2$. Since $f : S_1 \longrightarrow S_2$ is continuous, it follows that $f(\mathcal{F}_1) \longrightarrow f(a_1)$ and $f(\mathcal{F}_2) \longrightarrow f(a_2)$ on S_2, i.e., $\mathcal{F}_1 \longrightarrow a$ and $\mathcal{F}_2 \longrightarrow a$ on S_2. Let V be an open set containing $E(S)$ in (S, τ_1) such that V is closed in (S, τ_2). Then there exists a symmetric open set U containing $E(S)$ in (S, τ_1) such that $U \cdot U \subseteq V$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are Cauchy filters on (S, τ_1), there exist $M_1 \in \mathcal{F}_1$ and $M_2 \in \mathcal{F}_2$ such that $M_1 \times M_1 \subseteq L(U)$ and $M_2 \times M_2 \subseteq L(U)$. Then for any $x, y \in M_1$, $x^{-1}y \in U$ and $x^0 = y^0$. This implies $M_1 \subseteq (xU)^* = \{z \in S : x^{-1}z \in U,\ x^0 = z^0\}$, for each $x \in M_1$. Let $\overline{U}$ and $\overline{V}$ be the closures of U and V, respectively, in S_2. Now, $\mathcal{F}_1 \longrightarrow a$ implies $a \in \overline{M_1}$, the closure of M_1 in S_2. Thus, we have $a \in \overline{M_1} \subseteq \overline{(xU)^*} \subseteq (x\overline{U})^*$. This implies that $x^{-1}a \in \overline{U}$ with $a^0 = x^0$, for each $x \in M_1$. Similarly, $x'^{-1}a \in \overline{U}$ with $x'^0 = a^0$, for each $x' \in M_2$. Let $M = M_1 \cap M_2$. Then for any $p, q \in M$, $(p^{-1}a)(q^{-1}a)^{-1} \in \overline{U} \cdot \overline{U}^{-1} = \overline{U} \cdot \overline{U} \subseteq \overline{V} = V$, because V is closed in S_2. Also, $p^0 = a^0 = q^0$. Therefore, $M \times M \subseteq L(V)$ with $M \in \mathcal{F}_1 \cap \mathcal{F}_2$. This implies that $\mathcal{F}_1 \cap \mathcal{F}_2$ is a Cauchy filter in (S, τ_1). Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are minimal Cauchy filters in (S, τ_1), it follows that $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_1 \cap \mathcal{F}_2$. Due to the Hausdorff property of (S, τ_1) and the convergence of $\mathcal{F}_1 \longrightarrow a_1$ and $\mathcal{F}_2 \longrightarrow a_2$ with $\mathcal{F}_1 = \mathcal{F}_2$ implies that $a_1 = a_2$. Hence, f is injective.

(2) Let A be a complete subspace of S with respect to the uniformity $\mathcal{U}_2$ corresponding to the topology τ_2. Let A_1 be the closure of A in (S, τ_1). Since f is continuous, we must have $f(A_1)$ is contained in the closure of A in (S, τ_2). Moreover, as A is a complete subspace of (S, τ_2), this concludes that A is closed in (S, τ_2). This implies $f(A_1) \subseteq A$. Again, $f(A) = A$ and $f(A) \subseteq f(A_1)$. Therefore, $f(A_1) = f(A)$. Since f is injective, we have $A_1 = A$. The proof is complete. □

Acknowledgments

We thank the Editor and anonymous referees for their insightful comments that greatly improved this work, and Dr. Sunil Kumar Maity for his valuable suggestions that enhanced the presentation.

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