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Quadratic Fock space calculus (III): Inner product of the quadratic exponential vectors

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Abstract. This paper falls within the first phase of the Fock quadratic quantization program, where we define the appropriate framework for the program, namely the quadratic Fock space. During this phase, our goal is to synthesize foundational results and standardize notations in alignment with those employed in the series of papers focusing on the one–mode case (J Math Anal App 439(1): 135–153, 2016). The primary focus of this paper is on the inner product of quadratic exponential vectors. We begin by revisiting the expression for the inner product of two one–mode quadratic exponential vectors as established in a previous work (J Math Anal App 439(1): 135–153, 2016). We then extend this result to a more general scenario, specifically the continuous form of quadratic exponential vectors. Although the result is known, our exposition deviates from the one outlined in (J Math Phys 51:2, 2010). Specifically, we offer several modifications. Our contribution entails introducing a distinct approach customized for the one–mode scenario, along with providing a reference that consolidates the fundamental findings related to this topic.

1. Introduction

The exploration of quadratic quantization has been a significant endeavor in the realm of mathematical physics. As an integral part of the quadratic quantization program [1], our focus lies in establishing some tools and results on the quadratic Fock space as the appropriate mathematical framework for this program. This paper marks the third installment in the series entitled: Quadratic Fock Space Calculus. It focus on the inner product between two quadratic exponential vectors.

In the initial phase, our objective was to study the quadratic Fock space which is the suitable space for the Fock quadratic quantization program. Among our objective is to bring about a harmonization of symbols while scrutinizing previously obtained results. For example in the definition of the re–normalized

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square of white noise algebra (RSWN–algebra, later) over a test function space $\mathcal{K}$ (see [2] and [3]), one of the commutation relation driving by their generators is the following:

$$[\mathbf{b}_f^-, \mathbf{b}_g^+] = 2c\langle f, g\rangle \mathbf{1} + 4\mathbf{n}_{\bar{f}g}\ , \quad c > 0 \tag{1}$$

It is wiser, for some reasonable consideration explained in [4], to make the following change

$$B_f^+ := \mathbf{b}^+_{\frac{f}{2}} \quad ; \quad B_f^- := \mathbf{b}^-_{\frac{f}{2}} \quad ; \quad N_f := \mathbf{n}_f,\ f \in \mathcal{K} := L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

From this point, we shall adapt the following definition of the RSWN–algebra.

Definition 1.1. *The re–normalized square of white noise algebra with test functions algebra $\mathcal{K} = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, denoted RSWN($\mathcal{K}$), is the ∗–Lie algebra generated by the set*

$$\left\{B_f^-, B_g^+, N_h, \mathbf{1} \quad : \quad f, g, h \in \mathcal{K}\right\},$$

satisfying the following conditions

(i) the commutation relations:

$$[B_f^-, B_g^+] = \frac{c}{2}\langle f, g\rangle \mathbf{1} + N_{\bar{f}g}, \quad f, g, h \in \mathcal{K}, \quad c > 0, \tag{2}$$

$$[N_h, B_f^+] = 2B_{hf}^+ \quad ; \quad [B_f^+, B_g^+] = [N_f, N_g] = 0, \quad f, g, h \in \mathcal{K}, \tag{3}$$

where $\mathbf{1}$ is the central element of this algebra and c is a positive constant,

(ii) the involution property

$$(B_f^+)^* = B_{\bar{f}}^- \quad ; \quad (N_f)^* = N_{\bar{f}} \quad ; \quad \mathbf{1}^* = \mathbf{1},\ f \in \mathcal{K}, \quad f, g, h \in \mathcal{K}, \tag{4}$$

(iii) the maps $f \longmapsto B_f^+, N_f$ are linear and $f \longmapsto B_f^-$ is anti–linear.

As a sort of standardization and also as alignment with what obtained in the one–mode case (see [5],[6],[7] and [8]), we have introduced many changes not only at the notational level, but also in adjustment of some proofs related to this program. Among of them, we cite the domain for which the quadratic exponential vectors are well–defined. See [9] and [4], for more details.

In the same direction, we start in section 2, by recalling the expression of the inner product between two one–mode quadratic exponential vectors, a result previously obtained in [7]. In the section 3, we clarify the transition from the one–mode to the continuous form of the quadratic exponential vector by investigating the connection between the $\mathfrak{sl}_2(\mathbb{C})$–algebra and its continuous extension, the RSWN($\mathcal{K}$)–algebra. This exploration is then extended in the section 4 to a more general class, encompassing the continuous form of quadratic exponential vectors. Although the fundamental result is acknowledged from [10], the associated proof necessitates refinement, prompting us to present a totally different alternative within this work.

2. Review on the one–mode Fock space

It is well–known from [11] and [12], that the complex ∗–Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the elements B^+, B^- and M satisfying the commutation relations

$$[B^-, B^+] = M \quad ; \quad [M, B^\pm] = \pm 2B^\pm \tag{5}$$

and the involution

$$(B^-)^* = B^+ \quad ; \quad M^* = M. \tag{6}$$

Its central extension denoted here by $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, is the $*$–Lie algebra generated by $\mathfrak{sl}_2(\mathbb{C})$ and the central self–adjoint element E, i.e., commuting with all other generators of $\mathfrak{sl}_2(\mathbb{C})$ and satisfying $E^* = E$.

It has been proved in [12] that for all $\mu > 0$, there exists a unique $*$–representation of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, called the Fock representation, realized on a Hilbert space Γ_2 with an orthonormal basis $\{\Phi_n, n \in \mathbb{N}\}$ for which $B^\pm, M$ and E act on the basis $\{\Phi_n,\ n \geq 0\}$ as follows:

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1} \quad , \quad n \in \mathbb{N}, \tag{7}$$

$$B^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1} \quad , \quad n \in \mathbb{N}^*, \tag{8}$$

where $\Phi_{-1} = 0$ and $\Phi_0 =: \Phi$ is the vacuum vector,

$$M\Phi_n = (2n + \mu)\Phi_n \quad , \quad n \in \mathbb{N} \tag{9}$$

$$E\Phi_n = \Phi_n \tag{10}$$

the sequence (ω_n) is uniquely determined to be

$$\{\omega_n = n(\mu + n - 1) \quad , \quad n = 1, 2, \ldots\} \quad , \quad \omega_0 := 1. \tag{11}$$

Note that we identified elements of $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$ with their images under representation and then, the number operator will be defined by its action on the number vector respecting the identity

$$N\Phi_n = 2n\Phi_n \tag{12}$$

or equivalently

$$N = M - \mu E.$$

Definition 2.1. *The space Γ_2 is called the one–mode quadratic Fock space.*

Below, we revisit certain results acquired in [7], as delineated in the subsequent items.

1. For any $n \geq 1$

$$\|B^{+n}\Phi\| = \sqrt{w_n!} = \sqrt{n!(\mu)_n}, \tag{13}$$

where $w_n! = w_1 \cdots w_n, w_0! = 1$ and for any $x \in \mathbb{R}, (x)_n := \prod_{k=1}^{n}(x + k - 1)$.

2. The function ξ given by

$$\xi_\mu(z) = \sum_{n=0}^{+\infty} \sqrt{\frac{(\mu)_n}{n!}} z^n \tag{14}$$

is well–defined for all $z \in \mathbb{C}$ such that $|z| < 1$.

3. For all complex number z such that $|z| < 1$, the one–mode quadratic exponential vector

$$\Phi(z) := e^{zB^+}\Phi = \sum_{n=0}^{+\infty} \frac{z^n}{n!} B^{+n}\Phi \tag{15}$$

is well–defined, where the series converges in Γ_2.

4. The inner product of tow exponential vectors is given by

$$\langle \Phi(z), \Phi(w)\rangle = e^{-\mu \log(1-\bar{z}w)} = (1-\bar{z}w)^{-\mu}, \tag{16}$$

where

$$(1-z)^{-\mu} := \sum_{n=0}^{+\infty} \frac{(\mu)_n}{n!} z^n. \tag{17}$$

and where log is the principal determination of the Logarithm.

3. Connection with the re–normalized square of white noise algebra

After [12], it is well–known that the RSWN($\mathcal{K}$)–algebra admits a unique up to unitary isomorphism, $*$–representation characterized by the existence of a cyclic vector Φ satisfying

$$B_f^- \Phi = N_h \Phi = 0 \quad \forall f, h \in \mathcal{K}.$$

Definition 3.1. *The Hilbert space generated by the set*

$$\{B_f^{+n}\Phi \quad : f \in \mathcal{K}, n \in \mathbb{N}\}.$$

is called the quadratic Fock space and denoted by $\Gamma(\mathcal{K})$*.*

It is known from [9] and references therein, that, for all $f \in \mathcal{K}$, the vacuum vector Φ belongs to the domain of the operator $e^{B_f^+}$ and, consequently, the quadratic exponential vector

$$\Phi(f) = e^{B_f^+}\Phi = \sum_{n=0}^{+\infty} \frac{1}{n!} B_f^{+n}\Phi \tag{18}$$

exists, if and only if, $f \in B_\infty(0,1) := \{f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \ : \ \|f\|_\infty < 1\} \subset \mathcal{K}$. For more details one can see [5], [7], [9] and [4].

Since the goal of this paper is to compute the inner product of two exponential vectors $\langle \Phi(f), \Phi(g)\rangle$, it is worth looking at the one–mode case, i.e., when the arguments f and g take the form

$$f = f_I := z\chi_I \quad ; \quad g = g_I := w\chi_I, \tag{19}$$

where $z, w \in \mathbb{C}, |z| < 1, |w| < 1$ and I is a finite measure subset of $\mathbb{R}^d$.
To this goal we shall clarify the connection between $\mathfrak{sl}_2(\mathbb{C})$ and RSWN($\mathcal{K}$).

It is not difficult to see that the commutation relations of $\mathfrak{sl}_2(\mathbb{C})$ appear when we take the test function space to be the complex algebra of the multiple of the characteristic function of the finite measure set $I \subset \mathbb{R}$, i.e., $\mathcal{K}_I = \mathbb{C}\chi_I$. Explicitly, setting

$$B_I^+ := B_{\chi_I}^+ \ ; \ B_I^- := B_{\chi_I}^- \ ; \ N_I := N_{\chi_I} \ ; \ M_I := \mu(I)\mathbf{1} + N_I, \ \ \mu(I) = \frac{c|I|}{2} \tag{20}$$

so we get

$$[B_I^-, B_I^+] = [B_{\chi_I}^-, B_{\chi_I}^+] = \frac{c|I|}{2}\mathbf{1} + N_{\chi_I} = \mu(I)\mathbf{1} + N_I = M_I, \tag{21}$$

$$[M_I, B_I^\pm] = [N_I, B_I^\pm] = [N_{\chi_I}, B_{\chi_I}^\pm] = \pm 2B_{\chi_I}^\pm = \pm 2B_I^\pm, \tag{22}$$

(the other commutation relations vanish) and the involution will be given by

$$(B_I^-)^* = B_I^+ \quad ; \quad N_I^* = N_I \quad ; \quad M_I^* = M_I \quad ; \quad \mathbf{1}^* = \mathbf{1}. \tag{23}$$

Respecting the notations of (19), let us consider $\Phi_I(z) := \Phi(f_I)$ and $\Phi_I(w) := \Phi(g_I)$. Thus

$$\Phi_I(z) := \Phi(z\chi_I) = \sum_{n=0}^{+\infty} \frac{1}{n!} B_{z\chi_I}^{+n}\Phi = \sum_{n=0}^{+\infty} \frac{z^n}{n!} B_{\chi_I}^{+n}\Phi = \sum_{n=0}^{+\infty} \frac{z^n}{n!} B_I^{+n}\Phi.$$

It is now clear from (16) that

$$\begin{aligned}
\Big\langle \Phi(f_I), \Phi(g_I) \Big\rangle &= \Big\langle \Phi_I(z), \Phi_I(w) \Big\rangle = e^{-\mu(I)\log\left(1-\bar{z}w\right)} = e^{-\frac{c}{2}|I|\log\left(1-\bar{z}w\right)} \\
&= e^{-\frac{c}{2}\int_I \log\left(1-\bar{z}w\right)dx} = e^{-\frac{c}{2}\int_{\mathbb{R}^d} \log\left(1-\bar{z}\chi_I(x)w\chi_I(x)\right)dx} \\
&= e^{-\frac{c}{2}\int_{\mathbb{R}^d} \log\left(1-\overline{f_I(x)}g_I(x)\right)dx}.
\end{aligned} \tag{24}$$

Looking at (24) prompts us to consider generalizing it to a wider class of functions. We shall consider the class of functions for which the quadratic exponential vector is well–defined, i.e., the functions in $B_\infty(0,1) \subset \mathcal{K}$.

4. Inner product between two quadratic exponential vectors

The goal of this section is to give the inner product between two quadratic exponential vectors. To this goal we consider a measurable subset $I \subset \mathbb{R}^d$ and denote $\Gamma_I(\mathcal{K})$ the closed linear span of the set $\{B_f^{+n}\Phi \ : \ n \in \mathbb{N}, f \in \mathcal{K}, supp(f) \subset I\}$ so the space $\Gamma_2(\mathcal{K}) = \Gamma_{\mathbb{R}^d}(\mathcal{K})$. The vacuum vector in $\Gamma_I(\mathcal{K})$ will be denoted by Φ_I. It was proven in [10], that for all $I_1, ..., I_k \subset \mathbb{R}^d$, $k \geq 1$, such that $I_i \cap I_j = \emptyset, i \neq j$, the operator

$$U_k : \Gamma_{\cup_{i=1}^k I_i}(\mathcal{K}) \longrightarrow \bigotimes_{i=1}^k \Gamma_{I_i}(\mathcal{K})$$

$$\begin{aligned}
& U_k\left(B^+_{f^{(1)}_{m_1}} \cdots B^+_{f^{(1)}_1} \cdots B^+_{f^{(k)}_{m_k}} \cdots B^+_{f^{(k)}_1} \Phi_{\cup_{i=1}^k}\right) \\
= & \left(B^+_{f^{(1)}_{m_1}} \cdots B^+_{f^{(1)}_1} \Phi_{I_1}\right) \otimes \cdots \otimes \left(B^+_{f^{(k)}_{m_k}} \cdots B^+_{f^{(k)}_1} \Phi_{I_k}\right),
\end{aligned} \tag{25}$$

where $f_j^{(i)} \in \mathcal{K}$ such that $supp(f_j^{(i)}) \subset I_i$, is unitary.
Moreover for all $f_j \in \mathcal{K}$ such that $supp(f_j) \subset I_j$ and $f = f_1 + ... + f_k$, one has

$$U_k\left(\Phi(f)\right) = \Phi(f_1) \otimes \cdots \otimes \Phi(f_k). \tag{26}$$

We now proceed into the exposition of our formulation for the inner product of quadratic exponential vectors. This representation serves as an alternative rendition, expanding upon the context of the one–mode scenario. Notably, the proof of this result diverges from that presented in [10], where certain adjustments were deemed necessary.

Theorem 4.1. *For all $f, g \in B_\infty(0,1)$ the inner product of two exponential vectors is given by*

$$\langle \Phi(f), \Phi(g) \rangle = e^{-\frac{c}{2}\int_{\mathbb{R}^d} \log(1-\overline{f(s)}g(s))ds}. \tag{27}$$

To prove this theorem, we need the following lemma

Lemma 4.2. *Let $f, g \in B_\infty(0,1)$ and let $(f_k)_{k\geq 0}, (g_k)_{k\geq 0}$ be two sequences of $B_\infty(0,1)$ such that*

$$\lim_{k\to+\infty} \|f_k - f\| = \lim_{k\to+\infty} \|g_k - g\| = 0,$$

where $\|\cdot\| := \|\cdot\|_\infty + \|\cdot\|_2$. Then we have

$$\lim_{k\to+\infty} \sup_{j\geq 1} \left(\left| \left\langle f_k^j, g_k^j \right\rangle - \left\langle f^j, g^j \right\rangle \right| \right) = 0. \tag{28}$$

Moreover, there exists $M > 0$ such that

$$\left| \langle f_k^j, g_k^j \rangle \right| \leq M, \ \forall j \geq 1, \ \forall k \geq 0. \tag{29}$$

Proof. From hypothesis, we have

$$\|f_k - f\|_2 \leq \|f_k - f\| \longrightarrow 0 \quad ; \quad \|f_k - f\|_\infty \leq \|f_k - f\| \longrightarrow 0, \text{ as } k \to +\infty \tag{30}$$

and similarly

$$\|g_k - g\|_2 \leq \|g_k - g\| \longrightarrow 0 \quad ; \quad \|g_k - g\|_\infty \leq \|g_k - g\| \longrightarrow 0, \text{ as } k \to +\infty. \tag{31}$$

Looking the at right hand side of (30), we deduce that for $\epsilon = \frac{1-\|f\|_\infty}{2} > 0$ there exists $k_f \geq 0$ such that for all $k \geq k_f$

$$\|f_k\|_\infty \leq \|f_k - f\|_\infty + \|f\|_\infty \leq \epsilon + \|f\|_\infty.$$

Thus, using the identity

$$a^j - b^j = (a-b) \sum_{i=0}^{j-1} a^{j-1-i} b^i, \ \ a, b \in \mathbb{C},$$

we get

$$\begin{aligned}
\|f_k^j - f^j\|_2^2 &= \int_{\mathbb{R}^d} \left| (f_k(x))^j - (f(x))^j \right|^2 dx \\
&= \int_{\mathbb{R}^d} \left| f_k(x) - f(x) \right|^2 \left| \sum_{i=0}^{j-1} (f_k(x))^{j-1-i} (f(x))^i \right|^2 dx \\
&\leq \int_{\mathbb{R}^d} \left| f_k(x) - f(x) \right|^2 \left(\sum_{i=0}^{j-1} |f_k(x)|^{j-1-i} |f(x)|^i \right)^2 dx \\
&\leq \int_{\mathbb{R}^d} |f_k(x) - f(x)|^2 \left(\sum_{i=0}^{j-1} \|f_k\|_\infty^{j-1-i} \|f\|_\infty^i \right)^2 dx \\
&= \left(\sum_{i=0}^{j-1} \|f_k\|_\infty^{j-1-i} \|f\|_\infty^i \right)^2 \int_{\mathbb{R}^d} |f_k(x) - f(x)|^2 dx \\
&= \left(\sum_{i=0}^{j-1} \|f_k\|_\infty^{j-1-i} \|f\|_\infty^i \right)^2 \|f_k - f\|_2^2 \\
&\leq \left(\sum_{i=0}^{j-1} (\|f\|_\infty + \epsilon)^{j-1-i} \|f\|_\infty^i \right)^2 \|f_k - f\|_2^2 \\
&= \left(\frac{(\|f\|_\infty + \epsilon)^j - \|f\|_\infty^j}{\epsilon} \right)^2 \|f_k - f\|_2^2.
\end{aligned} \tag{32}$$

Applying the mean value theorem to the function $\gamma : x \longmapsto x^j$ on the interval $\Big[\|f\|_\infty, \|f\|_\infty + \epsilon\Big]$, we deduce that there exists $\delta \in (0, \epsilon)$, such that

$$\begin{aligned} \frac{\Big(\|f\|_\infty + \epsilon\Big)^j - \|f\|_\infty^j}{\epsilon} &= \gamma'\Big(\|f\|_\infty + \delta\Big) = j\Big(\|f\|_\infty + \delta\Big)^{j-1} \\ &\le j\Big(\|f\|_\infty + \epsilon\Big)^{j-1} = j\left(\frac{1 + \|f\|_\infty}{2}\right)^{j-1} = \rho(j), \end{aligned} \tag{33}$$

where $\rho : [1, +\infty) \ni x \longmapsto \rho(x) := x\eta^{x-1}$ with $\eta = \frac{1+\|f\|_\infty}{2} \in [1/2, 1)$.
A simple study of the function ρ shows that $\sup_{x\ge 1} \rho(x) = -\frac{1}{e\eta \ln(\eta)}$. Consequently, (33) becomes

$$\frac{\Big(\|f\|_\infty + \epsilon\Big)^j - \|f\|_\infty^j}{\epsilon} \le -\frac{1}{e\eta \ln(\eta)} = \frac{-2}{(1 + \|f\|_\infty)e \ln\Big(\frac{1+\|f\|_\infty}{2}\Big)} =: M_f > 0\cdot$$

from which the inequality (32), gives

$$\|f_k^j - f^j\|_2 \le M_f \|f_k - f\|_2, \ \forall k \ge k_f, \ \forall j \ge 1. \tag{34}$$

Similarly, we get

$$\|g_k^j - g^j\|_2 \le M_g \|g_k - g\|_2, \ \forall k \ge k_g, \ \forall j \ge 1. \tag{35}$$

Since $\|g\|_\infty < 1$, then $\|g^j\|_2 \le \|g\|_2$ for any $j \ge 1$. It follows, with help of (35), that

$$\|g_k^j\|_2 \le \|g_k^j - g^j\|_2 + \|g^j\|_2 \le M_g \|g_k - g\|_2 + \|g\|_2, \ \forall j \ge 1, \ \forall k \ge k_g. \tag{36}$$

Combining (34) and (35) with (36), we deduce that any $k \ge \max(k_f, k_g)$,

$$\begin{aligned} \Big|\Big\langle f_k^j, g_k^j\Big\rangle - \Big\langle f^j, g^j\Big\rangle\Big| &= \Big|\Big\langle f_k^j - f^j, g_k^j\Big\rangle + \Big\langle f^j, g_k^j - g^j\Big\rangle\Big| \\ &\le \Big|\Big\langle f_k^j - f^j, g_k^j\Big\rangle\Big| + \Big|\Big\langle f^j, g_k^j - g^j\Big\rangle\Big| \\ &\le \Big\|f_k^j - f^j\Big\|_2 \cdot \Big\|g_k^j\Big\|_2 + \Big\|f^j\Big\|_2 \cdot \Big\|g_k^j - g^j\Big\|_2 \\ &\le \Big\|f_k^j - f^j\Big\|_2 \Big(M_g\|g_k - g\|_2 + \|g\|_2\Big) + \Big\|f\Big\|_2 \Big\|g_k^j - g^j\Big\|_2 \\ &\le M_f \Big\|f_k - f\Big\|_2 \Big(M_g\|g_k - g\|_2 + \|g\|_2\Big) + M_g\Big\|f\Big\|_2 \Big\|g_k - g\Big\|_2 . \end{aligned} \tag{37}$$

Taking the limit of (37) as $k \to \infty$ using the left hand sides of (30) and (31), one obtains

$$\lim_{k\to+\infty} \sup_{j\ge 1} \Big(\Big|\Big\langle f_k^j, g_k^j\Big\rangle - \Big\langle f^j, g^j\Big\rangle\Big|\Big) = 0.$$

For the the second part of the proof, since $\Big\|f_k\Big\|_\infty < 1$ and $\Big\|g_k\Big\|_\infty < 1$, then

$$\Big\|f_k^j\Big\|_2 \le \Big\|f_k\Big\|_2 \quad ; \quad \Big\|g_k^j\Big\|_2 \le \Big\|g_k\Big\|_2.$$

Consequently, the Cauchy Shwartz inequality leads to

$$\Big|\Big\langle f_k^j, g_k^j\Big\rangle\Big| \le \Big\|f_k^j\Big\|_2 \Big\|g_k^j\Big\|_2 \le \Big\|f_k\Big\|_2 \Big\|g_k\Big\|_2 \forall j \ge 1. \tag{38}$$

But the sequences $\Big(\Big\|f_k\Big\|_2\Big)_{k\ge 0}$ and $\Big(\Big\|g_k\Big\|_2\Big)_{k\ge 0}$ are convergent then they are bounded. So their product is also bounded. We conclude from (38) that there is $M > 0$ such that (29) is satisfied. □

Proof. (of Theorem 4.1)
The strategy of the proof will be as follows:
We start by proving the result (27) for step functions in $B_\infty(0,1)$. Then, by an approximation procedure, we prove that (27) still true for any functions $f,g \in B_\infty(0,1)$. The proof will be divided in four steps.

Step.1
According to the notations adapted in (20), setting $B_i^{\pm} := B_{I_i}^{\pm}$ and $\Phi_i(z) := e^{zB_i^+}\Phi$, where $\{I_i : \ i = 1,..,k\}$ is a family of finite measure disjoint sets in $\mathbb{R}^d$. Thus

$$\|B_i^{+n}\Phi\| = \sqrt{n!(\mu_i)_n}\,, \tag{39}$$

and for all $z_i, w_i \in D(0,1) := \{z \in \mathbb{C} : \ |z| < 1\}$,

$$\Big\langle \Phi_i(z_i), \Phi_i(w_i)\Big\rangle = e^{-\mu_i \log(1-\overline{z}_i w_i)}\ ,\ \mu_i := \mu(I_i) = \frac{c|I_i|}{2}. \tag{40}$$

Now, we consider two step functions $f = \sum_{j=1}^k u_j \chi_{I_j}$ and $g = \sum_{j=1}^k v_j \chi_{I_j}$ such that $u_j, v_j \in D(0,1)$ for all $j = 1,...,k$. Then from (26), we deduce

$$\begin{aligned}
\Big\langle \Phi(f), \Phi(g)\Big\rangle &= \Big\langle U_k\left(\Phi(f)\right), U_k\left(\Phi(g)\right)\Big\rangle = \Big\langle \bigotimes_{j=1}^k \Phi\Big(u_j\chi_{I_j}\Big), \bigotimes_{j=1}^k \Phi\Big(v_j\chi_{I_j}\Big)\Big\rangle \\
&= \Big\langle \bigotimes_{j=1}^k \Phi_j(u_j), \bigotimes_{j=1}^k \Phi_j(v_j)\Big\rangle = \prod_{j=1}^k \Big\langle \Phi_j(u_j), \Phi_j(v_j)\Big\rangle \\
&= \prod_{j=1}^k e^{-\mu_j \log(1-\overline{u}_j v_j)} = e^{-\frac{c}{2}\sum_{j=1}^k |I_j|\log(1-\overline{u}_j v_j)}.
\end{aligned}$$

But we have

$$\begin{aligned}
\sum_{j=1}^k |I_j|\log(1-\overline{u}_j v_j) &= \sum_{j=1}^k \int_{\mathbb{R}^d} \log(1-\overline{u}_j v_j)\chi_{I_j}(x)dx \\
&= \int_{\mathbb{R}^d} \Big(\sum_{j=1}^k \log(1-\overline{u}_j v_j)\chi_{I_j}(x)\Big)dx \\
&= \int_{\mathbb{R}^d} \log\left(1 - \Big(\sum_{j=1}^k \overline{u}_j\chi_{I_j}(x)\Big)\Big(\sum_{j=1}^k v_j\chi_{I_j}(x)\Big)\right)dx \\
&= \int_{\mathbb{R}^d} \log\Big(1 - \overline{f(x)}g(x)\Big)\,dx.
\end{aligned}$$

Consequently, for all step functions f and g in $B_\infty(0,1)$, one has

$$\Big\langle \Phi(f), \Phi(g)\Big\rangle = e^{-\frac{c}{2}\int_{\mathbb{R}^d}\log\left(1-\overline{f(x)}g(x)\right)dx}. \tag{41}$$

This proves that (27) is true for step functions in $B_\infty(0,1)$.

Step.2
Let $f,g \in B_\infty(0,1)$. Since the space of step functions $\mathcal{T}_{step}$

$$\mathcal{T}_{step} := \left\{ f = \sum_{j=1}^k u_j\chi_{I_j} : |I_j| < +\infty, u_j \in \mathbb{C}, k \geq 1 \right\}$$

is dense in $\mathcal{K}$ for the norm $\|\cdot\| = \|\cdot\|_2 + \|\cdot\|_\infty$, there exist two sequences of step functions $(f_k)_k$ and $(g_k)_k$ such that

$$\lim_{k\to+\infty} \|f_k - f\| = \lim_{k\to+\infty} \|g_k - g\| = 0. \tag{42}$$

Note that we can usually assume that $f_k, g_k \in B_\infty(0,1)$. Consequently, the sequences $(f_k)_k$ and $(g_k)_k$ satisfy the conditions of Lemma 4.2.
In this step, we shall prove

$$\lim_{k\to+\infty} \left\langle \Phi(f_k), \Phi(g_k)\right\rangle = \left\langle \Phi(f), \Phi(g)\right\rangle. \tag{43}$$

From the definition of the quadratic exponential vectors (18), we know that

$$\begin{aligned}\left\langle \Phi(f_k), \Phi(g_k)\right\rangle &= \left\langle \sum_{n=0}^{+\infty} \frac{1}{n!} B_{f_k}^{+n}\Phi, \sum_{n=0}^{+\infty} \frac{1}{n!} B_{g_k}^{+n}\Phi \right\rangle \\ &= \sum_{n=0}^{+\infty} \frac{1}{n!^2} \left\langle B_{f_k}^{+n}\Phi, B_{g_k}^{+n}\Phi\right\rangle = \sum_{n=0}^{+\infty} v_{n,k}\end{aligned}$$

and similarly,

$$\left\langle \Phi(f), \Phi(g)\right\rangle = \sum_{n=0}^{+\infty} \frac{1}{n!^2} \left\langle B_f^{+n}\Phi, B_g^{+n}\Phi\right\rangle = \sum_{n=0}^{+\infty} v_n\,,$$

where

$$v_{n,k} = \frac{\left\langle B_{f_k}^{+n}\Phi, B_{g_k}^{+n}\Phi\right\rangle}{n!^2} \quad \text{and} \quad v_n = \frac{\left\langle B_f^{+n}\Phi, B_g^{+n}\Phi\right\rangle}{n!^2}.$$

Consequently, proving (43), it is equivalent to prove

$$\lim_{k\to+\infty} \sum_{n=0}^{\infty} v_{n,k} = \sum_{n=0}^{\infty} v_n, \tag{44}$$

for which we shall apply the dominated discrete convergence theorem 5.1 (D.D.C.T later) for $(v_{n,k})$ and (v_n), as soon as they satisfy the underlying conditions. Thus, to obtain (44) and consequently (43), the only thing that remains is to check conditions **(i)** and **(ii)** of the D.D.C.T.

(i) Using Identity (8) of [9], we deduce that

$$\begin{aligned} v_{n,k} &= \frac{1}{n!^2} \left\langle B_{f_k}^{+n}\Phi, B_{g_k}^{+n}\Phi\right\rangle \\ &= \frac{c}{2n} \sum_{j=1}^{n} \langle f_k^j, g_k^j\rangle \frac{\left\langle B_{f_k}^{+(n-j)}\Phi, B_{g_k}^{+(n-j)}\Phi\right\rangle}{(n-j)!^2} \\ &= \frac{c}{2n} \sum_{j=1}^{n} \langle f_k^j, g_k^j\rangle v_{n-j,k}, \ n \geq 1.\end{aligned}$$

Similarly,

$$v_n = \frac{c}{2n} \sum_{j=1}^{n} \langle f^j, g^j\rangle v_{n-j}, \ n \geq 1.$$

By induction on n, we shall prove the property

$$\mathbf{P}_n : \quad \lim_{k\to+\infty} v_{n,k} = v_n$$

which is **(i)**.

The case $n = 0$ is trivial because $v_{0,k} = \langle \Phi, \Phi \rangle = v_0$.
Assuming by induction the property

$$\mathbf{P}_{n-1]}: \quad \lim_{k\to+\infty} v_{j,k} = v_j, \quad 0 \le j \le n-1.$$

First, we have

$$\begin{aligned} \left|v_{n,k} - v_n\right| &= \left|\frac{c}{2n}\sum_{j=1}^{n}\Big(\langle f_k^j, g_k^j\rangle v_{n-j,k} - \langle f^j, g^j\rangle v_{n-j}\Big)\right| \\ &\le \frac{c}{2n}\sum_{j=1}^{n}\left|\langle f_k^j, g_k^j\rangle v_{n-j,k} - \langle f^j, g^j\rangle v_{n-j}\right| \\ &\le \frac{c}{2n}\sum_{j=1}^{n}\Big(\left|\langle f_k^j, g_k^j\rangle\right| \cdot \left|v_{n-j,k} - v_{n-j}\right| \\ &+ \left|\langle f_k^j, g_k^j\rangle - \langle f^j, g^j\rangle\right| \cdot \left|v_{n-j}\right|\Big) \end{aligned} \tag{45}$$

Now let us consider $\epsilon > 0$.
From Inequality (29) in Lemma 4.2, there exists $M > 0$ such that

$$\left|\left\langle f_k^j, g_{k'}^j\right\rangle\right| \le M, \ \forall k \ge 0 \text{ and } \forall j \ge 1. \tag{46}$$

The induction hypothesis implies that for any $j = 1, .., n$, there exists $k_j \ge 0$, such that for all $k \ge k_j$,

$$\left|v_{n-j,k} - v_{n-j}\right| \le \frac{\epsilon}{cM}.$$

Thus for any $k \ge K_n = \max_{1\le j\le n} k_j$ and for any $j = 1, .., n$, one has

$$\left|v_{n-j,k} - v_{n-j}\right| \le \frac{\epsilon}{cM}. \tag{47}$$

From Identity (28) in Lemma 4.2, there exists $L_n \ge 0$ such that for $k \ge L_n$,

$$\left|\langle f_k^j, g_k^j\rangle - \langle f^j, g^j\rangle\right| \le \frac{\epsilon}{cV_n} \tag{48}$$

where $V_n := \max_{1\le j\le n} \left|v_{n-j}\right|$.
Now injecting (46), (47) and (48) in the inequality (45), we obtain

$$\left|v_{n,k} - v_n\right| \le \frac{c}{2n}\sum_{j=1}^{n}\Big(M\frac{\epsilon}{cM} + \frac{\epsilon}{cV_n}V_n\Big) = \frac{c}{2n}\sum_{j=1}^{n}\Big(\frac{\epsilon}{c} + \frac{\epsilon}{c}\Big) = \frac{c}{2n}\sum_{j=1}^{n}\frac{2\epsilon}{c} = \epsilon$$

for any $k \ge N_n := \max(K_n, L_n)$.
This proves that the property $\mathbf{P}_n$ is true and hence the condition **(i)** is well satisfied.

(ii) We know that $m_0 := \max(\|f\|_\infty, \|g\|_\infty) < 1$. Setting $\epsilon = \frac{1-m_0}{2}$. Then from (42), there exists $k_0 \ge 0$ such that

$$\forall k \ge k_0, \ \ \|f_k - f\| \le \epsilon \quad \text{and} \quad \|g_k - g\| \le \epsilon.$$

Thus for all $k \geq k_0$, we have

$$\|f_k\|_\infty \leq \|f_k - f\|_\infty + \|f\|_\infty \leq \|f_k - f\| + m_0 \leq \epsilon + m_0 = \frac{1+m_0}{2}. \tag{49}$$

Similarly,

$$\|g_k\|_\infty \leq \frac{1+m_0}{2}. \tag{50}$$

Also for all $k \geq k_0$, we have

$$\|f_k\|_2 \leq \|f_k - f\|_2 + \|f\|_2 \leq \|f_k - f\| + \|f\|_2 \leq \epsilon + \|f\|_2.$$

and similarly

$$\|g_k\|_2 \leq \epsilon + \|g\|_2.$$

Setting $b = \epsilon + \max(\|f\|_2, \|g\|_2)$, the above inequalities become

$$\|f_k\|_2 \leq b \quad ; \quad \|g_k\|_2 \leq b. \tag{51}$$

Now let us recall the inequality mentioned in [9], which states the following:
For all $n \geq 1$ and $f \in \mathcal{K}$

$$\frac{\|B_h^{+n}\Phi\|}{n!} \leq \prod_{j=1}^{n}\left(\|h\|_\infty^2 + \frac{C(h)}{j}\right)^{\frac{1}{2}}, \tag{52}$$

where $C(h) := \frac{c}{2}\|h\|_2^2 - \|h\|_\infty^2$.
From inequalities (49), (50) and (51), we deduce that

$$C(f_k) = \frac{c}{2}\|f_k\|_2^2 - \|f_k\|_\infty^2 \leq \frac{c}{2}b^2 + \left(\frac{1+m_0}{2}\right)^2 =: \nu$$

and similarly,

$$C(g_k) \leq \nu.$$

Consequently, the inequality (52) applied to f_k and g_k leads to

$$\frac{\|B_{f_k}^{+n}\Phi\|}{n!} \leq \prod_{j=1}^{n}\left(\|f_k\|_\infty^2 + \frac{\nu}{j}\right)^{\frac{1}{2}} \leq \prod_{j=1}^{n}\left(\left(\frac{1+m_0}{2}\right)^2 + \frac{\nu}{j}\right)^{\frac{1}{2}}, \ k \geq k_0 \tag{53}$$

and

$$\frac{\|B_{g_k}^{+n}\Phi\|}{n!} \leq \prod_{j=1}^{n}\left(\left(\frac{1+m_0}{2}\right)^2 + \frac{\nu}{j}\right)^{\frac{1}{2}}, \ k \geq k_0. \tag{54}$$

Using the Cauchy-Shwartz inequality together with (53) and (54), one obtains

$$\begin{aligned} \left|v_{n,k}\right| &= \frac{\left|\left\langle B_{f_k}^{+n}\Phi, B_{g_k}^{+n}\Phi\right\rangle\right|}{n!^2} \leq \frac{\|B_{f_k}^{+n}\Phi\|}{n!}\frac{\|B_{g_k}^{+n}\Phi\|}{n!} \\ &\leq \prod_{j=1}^{n}\left(\frac{(1+m_0)^2}{4} + \frac{\nu}{j}\right)^{\frac{1}{2}}\left(\frac{(1+m_0)^2}{4} + \frac{\nu}{j}\right)^{\frac{1}{2}} \\ &= \prod_{j=1}^{n}\left(\frac{(1+m_0)^2}{4} + \frac{\nu}{j}\right) =: q_n \ \forall k \geq k_0. \end{aligned}$$

To show that $\sum_{n=1}^{+\infty} q_n < +\infty$, we use the D'Alembert criterion. In fact, one has

$$\frac{q_{n+1}}{q_n} = \frac{(1+m_0)^2}{4} + \frac{\nu}{n+1} \longrightarrow \frac{(1+m_0)^2}{4} < 1, \text{ (as } n \to +\infty).$$

which implies that **(ii)** is satisfied.

Step.3

In this step, we prove that

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} \log(1 - \overline{f_k(s)} g_k(s))ds = \int_{\mathbb{R}^d} \log(1 - \overline{f(s)} g(s))ds \tag{55}$$

To this goal we apply the Generalized Lebesgue Dominated Convergence Theorem (see Appendix: Theorem 6.1).

In view of the notations of Theorem 6.1, we take $\phi_k(x) = \log(1 - \overline{f_k(x)} g_k(x))$ and $\phi(x) = \log(1 - \overline{f(x)} g(x))$.

(i) Since $\|f_k - f\|_\infty \le \|f_k - f\| \longrightarrow 0$ and $\|g_k - g\|_\infty \le \|g_k - g\| \longrightarrow 0$ as $(k \longrightarrow +\infty)$, then for sufficiently $\epsilon > 0$, there exists $K \ge 1$ from which

$$|f_k(x)g_k(x)| \le \|f_k\|_\infty \|g_k\|_\infty \le (\|f\|_\infty + \epsilon)(\|g\|_\infty + \epsilon) =: C(f, g, \epsilon) < 1. \tag{56}$$

Using the inequality

$$|\log(1-b) - \log(1-a)| \le \frac{|b-a|}{1 - \max(|a|, |b|)}, \quad a, b \in \mathbb{C}, \ |a|, |b| < 1 \tag{57}$$

for $a = 0$ and $b = f_k(x)g_k(x)$, we get

$$\begin{aligned} |\phi_k(x)| &= |\log(1 - f_k(x)g_k(x))| \le \frac{|f_k(x)g_k(x)|}{1 - |f_k(x)g_k(x)|} \\ &\le \frac{|f_k(x)g_k(x)|}{1 - C(f,g,\epsilon)} =: \psi_k(x), \ a.e. x \in \mathbb{R}^d, k \ge K. \end{aligned}$$

(ii) Now from (56), we deduce that

$$\sup_{x\in\mathbb{R}^d} \Big(|f_k(x)g_k(x)|, |f(x)g(x)| \Big) \le C(f, g, \epsilon).$$

Thus with help of (57), we deduce that for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} \Big|\phi_k(x) - \phi(x)\Big| &= \Big|\log(1 - f_k(x)g_k(x)) - \log(1 - f(x)g(x))\Big| \\ &\le \frac{|f_k(x)g_k(x) - f(x)g(x)|}{1 - \max\Big(|f_k(x)g_k(x)|, |f(x)g(x)|\Big)} \\ &\le \frac{|f_k(x)g_k(x) - f(x)g(x)|}{1 - C(f,g,\epsilon)} \\ &\le \frac{|f_k(x) - f(x)| \cdot |g_k(x)| + |f(x)| \cdot |g_k(x) - g(x)|}{1 - C(f,g,\epsilon)} \longrightarrow 0 \ (k \to +\infty). \end{aligned}$$

Moreover it is clear that ψ_k converges pointwise to the function

$$\psi(x) = \frac{|f(x)g(x)|}{1 - C(f,g,\epsilon)}.$$

(iii) We know that there exists $K > 0$ such that for all $k \geq K$,

$$\|g_k\|_\infty \leq \|g\|_\infty + 1.$$

Thus for all $k \geq K$, we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} \psi_k(x)dx - \int_{\mathbb{R}^d} \psi(x)dx\Big| = \Big|\int_{\mathbb{R}^d} (\psi_k(x) - \psi(x))\, dx\Big| \\
&\leq \int_{\mathbb{R}^d} \Big|\psi_k(x) - \psi(x)\Big|dx \\
&= \frac{1}{1 - C(f,g,\epsilon)} \int_{\mathbb{R}^d} |f_k(x)g_k(x) - f(x)g(x)|dx \\
&\leq \frac{1}{1 - C(f,g,\epsilon)} \int_{\mathbb{R}^d} \Big(|f_k(x) - f(x)||g_k(x)| + |f(x)||g_k(x) - g(x)|\Big)dx \\
&\leq \frac{1}{1 - C(f,g,\epsilon)}\Big(\|f_k - f\|_2\|g_k\|_\infty + \|f\|_\infty\|g_k - g\|_2\Big) \\
&\leq \frac{1}{1 - C(f,g,\epsilon)}\Big(\|f_k - f\|_2(\|g\|_\infty + 1) + \|f\|_\infty\|g_k - g\|_2\Big) \longrightarrow 0,\ (k \to +\infty).
\end{aligned}$$

Step.4
We know from (41), that

$$\Big\langle \Phi(f_k), \Phi(g_k) \Big\rangle = e^{-\frac{c}{2}\int_{\mathbb{R}^d} \log\left(1 - \overline{f_k(x)}g_k(x)\right)dx}. \tag{58}$$

Thus taking the limits as $k \to +\infty$ in (58) and using the equations (43) and (55), we conclude that (27) is valid for all functions in $B_\infty(0,1) \subset \mathcal{K}$. □

Appendices

5. Discrete Dominated Convergence Theorem

Theorem 5.1. *If $(v_{n,k})_{n,k\geq 0}$ and $(v_n)_{n\geq 0}$ are two sequences in $\mathbb{C}$ such that:*

- **(i)** $\lim_{k\to+\infty} v_{n,k} = v_n \ \ \forall n \geq 0$;
- **(ii)** *there exists a sequence $(q_n)_n$ such that $|v_{n,k}| \leq q_n \ \ \forall k \geq k_0$ for some integer k_0 and $\sum_{n=0}^{+\infty} q_n < +\infty$.*

Then the following series converge and we have

$$\lim_{k\to+\infty} \sum_{n=0}^{+\infty} v_{n,k} = \sum_{n=0}^{+\infty} v_n.$$

Note that the D.D.C theorem is a discrete form of the dominated convergence theorem where the measure is the discrete measure

$$\mu(A) := \sum_{n=0}^{+\infty} \delta_n(A), \ \ A \in \mathcal{B}(\mathbb{R}),$$

with $\mathcal{B}(\mathbb{R})$ is the Borel sigma algebra and δ_n is the Dirac measure at $n \in \mathbb{N}$.

6. Generalized Lebesgue Dominated Convergence Theorem

Theorem 6.1. *([13])*
Let $(\phi_k(x))_{k\geq 1}$ be a sequence of Lebesgue measurable functions defined on a Lebesgue measurable set E, and let $(\psi_k(x))_{k\geq 1}$ be a sequence of nonnegative Lebesgue measurable functions defined on E. Suppose that:

- **(i)** $|\phi_k(x)| \leq \psi_k(x)$ *for almost all* $x \in E$ *and for all* $k \geq K$ *form some* $K \geq 1$.
- **(ii)** $(\phi_k(x))_{k\geq 1}$ *converges pointwise almost everywhere to* $\phi(x)$ *and* $(\psi_k(x))_{k\geq 1}$ *converges pointwise almost everywhere to* $\psi(x)$.
- **(iii)** $\lim_{k\to+\infty} \int_E \psi_k = \int_E \psi < +\infty$.

Then ϕ *is Lebesgue integrable on E and* $\lim_{k\to+\infty} \int_E \phi_k = \int_E \phi$.

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