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Generalized ruled surfaces in Myller configuration

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Abstract. In this paper, we introduce a quite big ruled surface family, which is called generalized ruled surfaces with Frenet-type frame in Myller configuration for Euclidean 3-space. This paper especially improves the theory of surfaces with respect to ruled surfaces and presents the relationships between the usual theory of curves and the theory of surfaces with Myller configuration. We investigate some special type ruled surfaces, such as rectifying-type ruled surfaces, osculating-type ruled surfaces, tangent-type ruled surfaces and trajectory ruled surfaces with Frenet-type frame in Myller configuration for E_3. We also give some particular cases of these ruled surfaces, as well. Since the geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of curves in classical Euclidean space, the surface theory of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of surfaces in classical Euclidean space, as well. Then, we establish some numerical examples with some illustrative figures with respect to the ruled surfaces in Myller configuration in order to solidify and concretize the given results.

1. Introduction

The theory of curves has quite a lot of importance and applications in several workframes, such as mathematics, architecture, engineering, etc., and also attracts a lot of researchers. In classical differential geometry, moving frames have been an important concept from the investigation of the Frenet (or Serret-Frenet) frame [13, 44], which is constructed for regular curves with non-zero curvature conditions. In the existing literature, lots of studies have been done and are ongoing with respect to the Frenet frames for regular space curves. In the Euclidean 3-space E_3, every unit speed curve $C : I \to E_3$ can be associated with the orthogonal unit vector fields at each point of the curve C; tangent vector field T, principal normal vector field N and binormal vector field B. The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are called the osculating plane, rectifying plane, and normal plane, respectively [21]. Special curve types named rectifying curves [4, 5, 22–24], normal curves [25–27], and osculating curves [21, 28] have a wide place for several different order spaces such as Euclidean 3 and 4 spaces and Minkowski 3 and 4 spaces with Frenet frame in differential geometry. Curves lying in the normal plane formed by the position vector tangent and principal

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normal are named the osculating curves, the curves lying in the normal plane formed by the position vector principal normal and binormal is named the normal curves, and the curves lying in the rectifying plane formed by the position vectors tangent and binormal are named rectifying curves (cf. [4, 5, 21–28]).

In E_3, one describes some versor fields such as tangent, principal normal, and binormal, alongside some plane fields such as rectifying, normal, and osculating planes along a curve C. By and large, a versor field and a plane field are denoted by $(C, \overline{\xi})$ and (C, π), respectively. The couple $\{(C, \overline{\xi}), (C, \pi)\}$ where $\overline{\xi} \in \pi$ is named a Myller configuration and denoted by $\mathcal{M}(C, \overline{\xi}, \pi)$ in Euclidean space E_3 [42]. Radu Miron completed some studies with respect to this pair $\{(C, \overline{\xi}), (C, \pi)\}$ in 1960 [41]. Provided that the plane π is tangent to the curve C, we have a tangent Myller configuration which is denoted by $\mathfrak{M}_t(C, \overline{\xi}, \pi)$ [40, 42]. Indeed, the geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of curves with classical Frenet frame in E_3 [1)].

Recently, some special curves in a Myller configuration for E_3 such as rectifying-type [40] and Bertrand-type curves [38] were studied by Macsim et al. (see also [39]). Then, Macsim et al. determined the rectifying-type curves with Frenet-type frame in Myller configuration for E_3 due to the special relation that exists between the Frenet-type frame in Myller configuration for E_3 and the classical Frenet frame in E_3. Also, from the natural properties and construction of the Myller configuration, the authors investigated that the rectifying curves with classical Frenet frame in E_3 are one of the special cases of rectifying-type curves with Frenet-type frame in Mller configuration for E_3 [40]. Similarly, Bertrand curves with classical Frenet frame in E_3 are one of the special cases of Bertrand-type curves with Frenet-type frame in Mller configuration for E_3 [38]. Additionally, versor fields along a curve in a four-dimensional Lorentz space were examined by Heroiu [11]. Then, İşbilir and Tosun introduced the osculating-type curves with Frenet-type frame in Myller configuration for E_3 [29]. From the natural construction of Myller configuration, it is said that osculating curves with classical Frenet frame in E_3 are one of the particular cases of osculating-type curves with Frenet-type frame in Myller configuration for E_3. In addition to these, rectifying-type curves with Frenet-type frame in Myller configuration for Euclidean 4-space were introduced by İşbilir and Tosun [30]. Also, generalized Smarandache curves with Frenet-type frame in Myller configuration for E_3 were determined by İşbilir and Tosun [31]. Further, a new type general and interesting frame, which is called the generalized Frenet-type frame in 3-dimensional Lie groups with Myller configurations, includes several special and classical type frames for Euclidean 3-space and 3-dimensional Lie groups in [32] was investigated by İşbilir et al. Also, İşbilir et al. [32] studied some special curves in Lie groups with Myller configuration. In addition to these, Doğan Yazıcı and Tosun determined the quasi-type frame and quasi-type osculating curves in Myller configuration [7]. Moreover, Alkan and Önder studied some special helices in Myller configuration [1] and slant helices and Darboux helices in Myller configuration [2].

On the other hand, surface theory holds quite value, interest, and applications in several work-frames, including architecture, engineering, differential geometry in mathematics, and computer science. Ruled surfaces stand out as among the most popular, intriguing, and aesthetically appealing examples of surfaces. These geometric constructions were determined by French mathematician Gaspard Monge [50]. These types of surfaces have significant geometric relevance in the field of architecture, having been utilized for centuries and continuing to be employed today. A ruled surface is defined as a surface through which a straight line passes at every point, contained within the surface. Due to their validity in architectural and geometrical constructions, particularly in terms of cost and duration considerations, ruled surfaces have found application in numerous architectural projects [10]. Additionally, the exploration of ruled surfaces is extensively covered in the existing body of literature. Numerous studies delve into the properties, classifications, and overall understanding of these types of surfaces, spanning from their initial exploration to the present. It is noteworthy to recognize that one category of ruled surfaces is defined as developable surfaces. The singularities of the tangent developable associated with a regular space curve are thoroughly examined by Cleave [6]. The existing literature is rich with a plethora of investigations in this realm, see [15–20, 37, 46, 48, 49] for shedding further light on the intricacies of ruled surfaces. Also, Kaya and Önder determined the generalized normal ruled surfaces of a curve in E_3 [36]. Then, with the same logic, Önder

[1)] cf. [11, 42]

and Kahraman determined the generalized rectifying ruled surfaces of a curve in E_3 [43]. Also, Kaya et al. investigated the generalized osculating-type ruled surfaces in E_3 [35]. Also, İşbilir et al. [33] and Doğan Yazıcı et al. [8, 9] introduced the rectifying, normal and osculating ruled surfaces of special singular curves, respectively. For a more in-depth understanding, one may consult [10, 14, 45] regarding the concept of ruled surfaces.

This paper is structured as follows. In Section 2, we provide a recap of essential information and backgrounds with respect to both Frenet-type frame in the Myller configuration, ruled surfaces, and surface theory in the Myller configuration. In Section 3, we investigate the generalized ruled surfaces with Frenet-type frame in Myller configuration for Euclidean 3-space E_3. Also, we give some special type ruled surfaces such as rectifying-type ruled surfaces, osculating-type ruled surfaces, tangent-type ruled surfaces and trajectory ruled surfaces with Frenet-type frame in Myller configuration for E_3. Then, we determine some particular cases of these ruled surfaces in terms of the values taken by the invariants, as well. Because the geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of curves in classical Euclidean space, the surface theory of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of surfaces in classical Euclidean space, as well (see also for generalization [11, 42]). In Section 4, we construct some numerical examples with respect to the ruled surfaces in Myller configuration. Then, we give a brief introduction to a new survey with respect to the ruled surfaces in Myller configuration: Trajectory ruled surfaces in Myller configuration in Section 5. Finally, we give conclusions in Section 6, and also, we construct a classification table for examining the special cases and existing literature.

2. Basic concepts

In this section, we remind some backgrounds with respect to the theory of curves and surfaces with Frenet-type frame in Myller configuration for E_3.

Let $(C, \overline{\xi})$ be a versor field and $\overline{r}(s)$ is a position vector of the curve C where s is the arc-length on the curve C. For Frenet-type frame $\mathcal{R}_F = \{P; \overline{\xi}_1, \overline{\xi}_2, \overline{\xi}_3\}$ of versor field, then we can write:

$$\frac{d\overline{r}}{ds} = \rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s), \tag{1}$$

where $\rho_1^2(s) + \rho_2^2(s) + \rho_3^2(s) = 1$. Also, the followings are satisfied:

$$\begin{cases} \overline{\xi}_1'(s) = K_1(s)\overline{\xi}_2(s), \\ \overline{\xi}_2'(s) = -K_1(s)\overline{\xi}_1(s) + K_2(s)\overline{\xi}_3(s), \\ \overline{\xi}_3'(s) = -K_2(s)\overline{\xi}_2(s), \end{cases} \tag{2}$$

where $K_1 > 0$. K_1-curvature and K_2-torsion have the same geometrical interpretation as the curvature and torsion of a curve in E_3. It should be noted that, if $\rho_1(s) = 1$, $\rho_2(s) = 0$ and $\rho_3(s) = 0$, we get Frenet equations of a regular curve in E_3 [42]. The fundamental theorem of invariants for versor field $(C, \overline{\xi})$ is expressed as follows:

Theorem 2.1. ([42]) *If the invariants $K_1(s) > 0, K_2(s)$ and functions $\rho_1(s), \rho_2(s), \rho_3(s)$ with $\rho_1^2(s) + \rho_2^2(s) + \rho_3^2(s) = 1$ are smooth functions for $s \in [a, b]$, then there exist a curve $C : [a, b] \to E_3$ parameterized by arc-length s and a versor field $\overline{\xi}(s)$, $s \in [a, b]$, whose curvature, torsion and the functions $\rho_i(s)$ are $K_1(s), K_2(s)$ and $\rho_i(s), i = 1, 2, 3$. Any two such versor fields $(C, \overline{\xi})$ differ by a proper Euclidean motion.*

Remark 2.2. ([42]) The followings are satisfied.

(1) The versor field $\left(C, \overline{\xi}\right)$ determines a ruled surface $S\left(C, \overline{\xi}\right)$.

(2) The surface $S\left(C, \overline{\xi}\right)$ is a cylinder if and only if the invariant $K_1(s)$ vanishes.

(3) The surface $S\left(C, \overline{\xi}\right)$ is with director plane if and only if the invariant $K_2(s) = 0$.

(4) The surface $S\left(C, \overline{\xi}\right)$ is a developing if and only if the invariant $a_3(s)$ vanishes.

Now, let us remind some required notions with respect to the fundamental forms of surfaces for E_3 in Myller configuration [42]:

Let S be a differentiable surface embedded in E_3. With the help of the classical surface theory in E_3, the analytical representation of the surface S is given as with the class C^k where $k \geq 3$ or $k = \infty$:

$$\overline{\omega} = \overline{\omega}(u, v) \quad \text{where} \quad (u, v) \in \mathcal{D}.$$

$\mathcal{D}$ is a simply connected domain in place of the variables (u, v). Moreover, the following vectorial notations are adopted:

$$\overline{\omega}_u = \frac{\partial \overline{\omega}}{\partial u} \quad \text{and} \quad \overline{\omega}_v = \frac{\partial \overline{\omega}}{\partial v},$$

where the condition $\overline{\omega}_u \times \overline{\omega}_v \neq 0$ is satisfied for all $(u, v) \in \mathcal{D}$. By using the following parametric representations, the curve C on the surface S can be denoted as:

$$u = u(t), \quad v = v(t), \quad t \in (t_1, t_2).$$

Then, the curve C is written as follows:

$$\overline{\omega} = \overline{\omega}(u(t), v(t)), \quad t \in (t_1, t_2).$$

Hence, the vector field $d\overline{\omega} = \overline{\omega}_u du + \overline{\omega}_v dv$ is tangent to the curve C at the points $P(t) = P(\overline{\omega}(u(t), v(t))) \in C$. In addition, the vectors $\overline{\omega}_u$ and $\overline{\omega}_v$ are tangent to the parametric lines and $d\overline{\omega}$ is tangent vector to the surface S at the point $P(t)$. The unit normal vector to the surface S is

$$\overline{\nu} = \frac{\overline{\omega}_u \times \overline{\omega}_v}{\|\overline{\omega}_u \times \overline{\omega}_v\|},$$

where $\|\overline{\omega}_u \times \overline{\omega}_v\| \neq 0$ at every point $P(t)$. Also, the first and second fundamental forms of the surface are determined as follows for all $(u, v) \in \mathcal{D}$:

$$I(du, dv) = \langle d\overline{\omega}(u, v), d\overline{\omega}(u, v)\rangle \quad \text{and} \quad II(du, dv) = -\langle d\overline{\omega}(u, v), d\overline{\nu}(u, v)\rangle,$$

and has the following quadratic forms:

$$I(du, dv) = Edu^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad II(du, dv) = Ldu^2 + 2Mdudv + Ndv^2,$$

where the coefficients of the first (E, F, G) and second fundamental form (L, M, N) are written as [42]:

$$\begin{cases} E(u, v) = \langle \overline{\omega}_u, \overline{\omega}_u \rangle, \\ F(u, v) = \langle \overline{\omega}_u, \overline{\omega}_v \rangle, \\ G(u, v) = \langle \overline{\omega}_v, \overline{\omega}_v \rangle, \end{cases} \quad \text{and} \quad \begin{cases} L(u, v) = \dfrac{\det(\overline{\omega}_u, \overline{\omega}_v, \overline{\omega}_{uu})}{\|\overline{\omega}_u \times \overline{\omega}_v\|}, \\ M(u, v) = \dfrac{\det(\overline{\omega}_u, \overline{\omega}_v, \overline{\omega}_{uv})}{\|\overline{\omega}_u \times \overline{\omega}_v\|}, \\ N(u, v) = \dfrac{\det(\overline{\omega}_u, \overline{\omega}_v, \overline{\omega}_{vv})}{\|\overline{\omega}_u \times \overline{\omega}_v\|}. \end{cases}$$

Additionally, the Gauss and mean curvatures are given as [14, 45]:

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}, \tag{3}$$

where the surface is developable (or flat) if and only if $K = 0$ and the surface is minimal if and only if $H = 0$.

Definition 2.3. ([18]) Let J be an open interval or a unit circle $\mathbb{S}^1$. Then, $\varphi : J \to \mathbb{R}^3$ and $v : J \to \mathbb{R}^3 - \{0\}$ be given as smooth functions. A ruled surface in $\mathbb{R}^3$ is the mapping $\psi_{(\varphi,v)} : J \times \mathbb{R} \to \mathbb{R}^3$ determined as $\psi_{(\varphi,v)}(s,u) = \varphi(s) + uv(s)$ where φ is directrix and v is director curve. Additionally, the straight line $u \mapsto \varphi(s) + uv(s)$ is named a ruling.

If $\frac{\partial \psi_{(\varphi,v)}(s,u)}{\partial s} \times \frac{\partial \psi_{(\varphi,v)}(s,u)}{\partial u} = 0$ at any points (s_0, u_0), these points are named singular points of the surface $\psi_{(\varphi,v)}(s,u)$. Otherwise, these points are named regular points. Due to developable surfaces being a type of ruled surface, the following classification can be written. The equation $\det(\varphi'(s), v(s), v'(s)) = 0$ is satisfied if and only if a ruled surface is developable. Moreover, a ruled surface $\psi_{(\varphi,v)}(s,u)$ with $\|v(s)\| = 1$ is cylindrical surface if and only if $v'(s) = 0$, and also non-cylindrical if and only if $v'(s) \neq 0$. A curve $\sigma(s)$ lying on $\psi_{(\varphi,v)}(s,u)$ with the condition $\langle \sigma'(s), v'(s) \rangle = 0$ is striction curve of the surface $\psi_{(\varphi,v)}(s,u)$. The striction curve of the surface $\psi_{(\varphi,v)}(s,u)$ is expressed as follows ([12, 34–36, 43]):

$$\sigma(s) = \varphi(s) - \frac{\langle \varphi'(s), v'(s) \rangle}{\langle v'(s), v'(s) \rangle} v(s).$$

The study of ruled surfaces can be explored in the books [14, 45], as well.

3. Generalized ruled urfaces in Myller configuration

The purpose of this section is to determine generalized ruled surfaces with Frenet-type frame in Myller configuration for E_3. Then, some geometric characterizations and properties, such as being cylindrical, developable, striction curve, and others are presented. Also, we give basic invariants, curvatures, and some classifications. Moreover, some examinations with respect to the Gaussian and mean curvature are given, and the necessary conditions for this surface to be flat and minimal surfaces are obtained, as well.

Definition 3.1. Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3. The ruled surface $\phi_{(\overline{r},\varrho)}(s,u) : I \times \mathbb{R} \to E_3$ is defined as follows:

$$\phi_{(\overline{r},\varrho)}(s,u) = \overline{r}(s) + u\varrho(s), \tag{4}$$

and

- ✻ if $\varrho(s) = \overline{\xi}_1(s)$, then $\phi_{(\overline{r},\varrho)}(s,u)$ is called the $\overline{\xi}_1$-type ruled surface [42],
- ✻ if $\varrho(s) = \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s)$, then $\phi_{(\overline{r},\varrho)}(s,u)$ is called the rectifying-type ruled surface where $\lambda(s)$ and $\mu(s)$ are smooth functions and $\lambda^2(s) + \mu^2(s) = 1$,
- ✻ if $\varrho(s) = \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_2(s)$, then $\phi_{(\overline{r},\varrho)}(s,u)$ is called the osculating-type ruled surface where $\lambda(s)$ and $\mu(s)$ are smooth functions and $\lambda^2(s) + \mu^2(s) = 1$,
- ✻ if $\varrho(s) = {}^{d\overline{r}}/_{ds} = \rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s)$, then $\phi_{(\overline{r},\varrho)}(s,u)$ is called the tangent-type ruled surface where $\rho_1(s), \rho_2(s)$, and $\rho_3(s)$ are smooth functions and $\rho_1^2(s) + \rho_2^2(s) + \rho_3^2(s) = 1$,
- ✻ if $\varrho(s) = b_1(s)\overline{\xi}_1(s) + b_2(s)\overline{\xi}_2(s) + b_3(s)\overline{\xi}_3(s)$, then $\phi_{(\overline{r},\varrho)}(s,u)$ is called the trajectory ruled surface where $b_1(s), b_2(s)$, and $b_3(s)$ are smooth functions and $b_1^2(s) + b_2^2(s) + b_3^2(s) = 1$,

of the curve $\overline{r}(s)$ with Frenet-type frame in Myller configuration for E_3.

In the rest of this paper, we will examine these last four types of special ruled surfaces due to the examination of the first case given in [42].

3.1. Rectifying-type ruled surfaces in Myller configuration

In this part of this study, we scrutinize the rectifying-type ruled surfaces of a curve with Frenet-type frame in Myller configuration for E_3.

Definition 3.2. Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3. The ruled surface $\phi_{(\overline{r},\varrho_1)}(s,u) : I \times \mathbb{R} \to E_3$ is determined as follows:

$$\phi_{(\overline{r},\varrho_1)}(s,u) = \overline{r}(s) + u\varrho_1(s) \quad \text{where} \quad \varrho_1(s) = \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s) \quad \text{and} \quad \lambda^2(s) + \mu^2(s) = 1, \tag{5}$$

and the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is called the rectifying-type ruled surface of the curve $\overline{r}(s)$ with Frenet-type frame in Myller configuration for E_3.

Special Cases 3.3. *According to the values of the functions $\lambda(s), \mu(s)$ and $\rho_i(s)$ for $i = 1, 2, 3$, the followings are satisfied:*

* *If $\lambda(s) = 1$ and $\mu(s) = 0$, $\overline{\xi}_1$-type ruled surface with Frenet-type frame in Myller configuration for E_3 is obtained [42].*
* *If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, rectifying ruled surface with Frenet frame in E_3 is obtained [43].*
* *If $\lambda(s) = 1$, $\mu(s) = 0$ and $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$, tangent developable surface with Frenet frame in E_3 is obtained [14].*

Theorem 3.4. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. The surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is not a regular surface if and only if*

$$\begin{cases} \mu(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right) = 0, \\ \lambda(s)\rho_3(s) - \mu(s)\rho_1(s) + u\left(\lambda(s)\mu'(s) - \lambda'(s)\mu(s)\right) = 0, \\ \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right) = 0. \end{cases} \tag{6}$$

Proof. Let us take the partial derivatives of the equation (5) with respect to the parameter s and u, and by using the equation (2), we get:

$$\frac{\partial\phi_{(\overline{r},\varrho_1)}(s,u)}{\partial s} = \left(\rho_1(s) + u\lambda'(s)\right)\overline{\xi}_1(s) + \left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right)\overline{\xi}_2(s) + \left(\rho_3(s) + u\mu'(s)\right)\overline{\xi}_3(s), \tag{7}$$

$$\frac{\partial\phi_{(\overline{r},\varrho_1)}(s,u)}{\partial u} = \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s). \tag{8}$$

Taking the cross product of the equations (7) and (8), we have:

$$\begin{aligned} \frac{\partial\phi_{(\overline{r},\varrho_1)}(s,u)}{\partial s} \times \frac{\partial\phi_{(\overline{r},\varrho_1)}(s,u)}{\partial u} &= \left[\mu(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right)\right]\overline{\xi}_1(s) \\ &\quad + \left[\lambda(s)\left(\rho_3(s) + u\mu'(s)\right) - \mu(s)\left(\rho_1(s) + u\lambda'(s)\right)\right]\overline{\xi}_2(s) \\ &\quad + \left[-\lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right)\right]\overline{\xi}_3 \\ &= 0. \end{aligned}$$

Hence, we get the desired equation (6). □

Then, we can write the set of singular points of the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ as follows:

$$S_{\phi_{(\overline{r},\varrho_1)}(s,u)} = \left\{ (s,u) \; : \; \begin{array}{c} \mu(s)\rho_2(s) + u\mu(s)\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right) = 0, \\ \lambda(s)\rho_3(s) - \mu(s)\rho_1(s) + u\left(\lambda(s)\mu'(s) - \lambda'(s)\mu(s)\right) = 0, \\ \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right) = 0. \end{array} \right\}.$$

Theorem 3.5. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. The following statements are satisfied:*

(1) *The surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is cylindrical if and only if*

$$\lambda'(s) = 0, \quad \lambda(s)K_1(s) - \mu(s)K_2(s) = 0 \quad \text{and} \quad \mu'(s) = 0.$$

(2) *The surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable if and only if*

$$(\lambda(s)\rho_3(s) - \mu(s)\rho_1(s))\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right) + \rho_2(s)\left(\lambda'(s)\mu(s) - \mu'(s)\lambda(s)\right) = 0.$$

Proof. (1) The surface $\phi_{(\overline{r},\varrho)}(s,u)$ is cylindrical if and only if

$$\begin{aligned}\varrho_1'(s) =& (\lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s))'\\ =& \lambda'(s)\overline{\xi}_1(s) + \lambda(s)\overline{\xi}_1'(s) + \mu'(s)\overline{\xi}_3(s) + \mu(s)\overline{\xi}_3'(s)\\ =& \lambda'(s)\overline{\xi}_1(s) + (\lambda(s)K_1(s) - \mu(s)K_2(s))\,\overline{\xi}_2(s) + \mu'(s)\overline{\xi}_3(s)\\ =& 0.\end{aligned}$$

The desired result can be seen easily.

(2) The surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable if and only if the equation $\det(d\overline{r}/ds, \varrho_1(s), \varrho_1'(s)) = 0$ is satisfied. We get:

(⇒) Let the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ be a developable surface.

$$\begin{aligned}\det(d\overline{r}/ds, \varrho_1(s), \varrho_1'(s)) =& \det\begin{pmatrix}\rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s), \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s),\\ \lambda'(s)\overline{\xi}_1(s) + (\lambda(s)K_1(s) - \mu(s)K_2(s))\,\overline{\xi}_2(s) + \mu'(s)\overline{\xi}_3(s)\end{pmatrix}\\ =& (\lambda(s)\rho_3(s) - \mu(s)\rho_1(s))\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right) + \rho_2(s)\left(\lambda'(s)\mu(s) - \mu'(s)\lambda(s)\right)\\ =& 0.\end{aligned}$$

Therefore, we get the desired result.

(⇐) It is clear.

The proof is finished. □

Corollary 3.6. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. If the rectifying-type ruled surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is cylindrical with the condition $K_2(s) \neq 0$, then the base curve is helix (cf. [1–3, 40]).*

Corollary 3.7. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$.*

- ✻ *If $\rho_1^2(s) + \rho_3^2(s) = 1$, $\rho_2(s) = 0$ and $\lambda(s)K_1(s) - \mu(s)K_2(s) = 0$, then the rectifying-type ruled surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable.*
- ✻ *If $\rho_1(s) = \rho_3(s) = 0$, $\rho_2(s) = 1$ and $\lambda(s), \mu(s)$ are constants, then the rectifying-type ruled surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable.*
- ✻ *If $\lambda(s)K_1(s) - \mu(s)K_2(s) = 0$ and $\lambda(s), \mu(s)$ are constants, then the rectifying-type ruled surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable.*
- ✻ *If the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is cylindrical, then $\phi_{(\overline{r},\varrho_1)}(s,u)$ is developable. The converse of this statement is not always true. Namely, if the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is a developable surface, it does not necessarily have to be cylindrical.*

Theorem 3.8. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. The base curve $\overline{r}(s)$ of the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is its striction curve if and only if $\rho_1(s)\lambda(s) + \rho_3(s)\mu(s) = 0$.*

Proof. The striction curve of the surface $\phi_{(\overline{r},\varrho)}(s,u)$ is written as follows:

$$\sigma(s) = \overline{r}(s) - \frac{\langle d\overline{r}/ds, \varrho_1(s)\rangle}{\langle \varrho_1'(s), \varrho_1'(s)\rangle}\varrho_1(s)$$

and we get:

$$\begin{aligned}\langle d\overline{r}/ds, \varrho_1(s)\rangle &= \langle \rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s), \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_3(s)\rangle \\ &= \rho_1(s)\lambda(s) + \rho_3(s)\mu(s).\end{aligned}$$

Also,

$$\begin{aligned}\langle \varrho_1'(s), \varrho_1'(s)\rangle &= \left\langle \begin{matrix}\lambda'(s)\overline{\xi}_1(s) + (\lambda(s)K_1(s) - \mu(s)K_2(s))\,\overline{\xi}_2(s) + \mu'(s)\overline{\xi}_3(s), \\ \lambda'(s)\overline{\xi}_1(s) + (\lambda(s)K_1(s) - \mu(s)K_2(s))\,\overline{\xi}_2(s) + \mu'(s)\overline{\xi}_3(s)\end{matrix}\right\rangle \\ &= \left(\lambda'(s)\right)^2 + (\lambda(s)K_1(s) - \mu(s)K_2(s))^2 + \left(\mu'(s)\right)^2.\end{aligned}$$

Then, we have:

$$\sigma(s) = \overline{r}(s) - \frac{\rho_1(s)\lambda(s) + \rho_3(s)\mu(s)}{(\lambda'(s))^2 + (\lambda(s)K_1(s) - \mu(s)K_2(s))^2 + (\mu'(s))^2}\varrho_1(s). \tag{9}$$

Hence, we can see that the base curve $\overline{r}(s)$ of the surface $\phi_{(\overline{r},\varrho)}(s,u)$ is its striction curve if and only if $\rho_1(s)\lambda(s) + \rho_3(s)\mu(s) = 0$. □

The normal field of the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ is given as follows:

$$\overline{\nu} = \frac{x(s,u)\overline{\xi}_1(s) + y(s,u)\overline{\xi}_2(s) - z(s,u)\overline{\xi}_3(s)}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}},$$

where

$$\begin{cases} x(s,u) = \mu(s)\left(\rho_2(s) + u\left(\lambda(s)K_1 - \mu(s)K_2(s)\right)\right), \\ y(s,u) = \lambda(s)\left(\rho_3(s) + u\mu'(s)\right) - \mu(s)\left(\rho_1(s) + u\lambda'(s)\right), \\ z(s,u) = \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right)\right). \end{cases}$$

Additionally, the invariants of the surface $\phi_{(\overline{r},\varrho_1)}(s,u)$ are given as:

$$E = \Psi_1^2(s,u) + \Psi_2^2(s,u) + \Psi_3^2(s,u), \tag{10a}$$

$$F = \lambda(s)\Psi_1(s,u) + \mu(s)\Psi_3(s,u), \tag{10b}$$

$$G = \lambda^2(s) + \mu^2(s), \tag{10c}$$

$$L = \frac{\begin{matrix}\left(\Psi_1'(s,u) - K_1(s)\Psi_2(s,u)\right)x(s,u) + \left(\Psi_1(s,u)K_1(s,u) + \Psi_2'(s,u) - K_2(s)\Psi_3(s,u)\right)y(s,u) \\ -\left(K_2(s)\Psi_2(s) + \Psi_3'(s,u)\right)z(s,u)\end{matrix}}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}}, \tag{10d}$$

$$M = \frac{\lambda'(s)x(s,u) + (\lambda(s)K_1(s) - \mu(s)K_2(s))\,y(s,u) - \mu'(s)z(s,u)}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}}, \tag{10e}$$

$$N = 0, \tag{10f}$$

where

$$\begin{cases}\Psi_1(s,u) = \rho_1(s) + u\lambda^{'}(s),\\ \Psi_2(s,u) = \rho_2(s) + u\left(\lambda(s)K_1(s) - \mu(s)K_2(s)\right),\\ \Psi_3(s,u) = \rho_3(s) + u\mu^{'}(s).\end{cases}$$

The following statements should be written:

- ✻ Throughout this study, the notation prime represents the derivation of the functions, and it does not represent a derivation based on u unless otherwise stated.
- ✻ For the sake of brevity, we use the followings from the Theorem 3.9:

  $$\lambda(s) = \lambda, \quad \mu(s) = \mu, \quad K_i(s) = K_i, \quad x(s,u) = x, \quad y(s,u) = y, \quad z(s,u) = z, \quad \Psi_j(s,u) = \Psi_j,$$

  where $i = 1,2$ and $j = 1,2,3$.

Theorem 3.9. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. The Gaussian and mean curvature of the surface are as follows, respectively:*

$$K = -\frac{\left(\lambda^{'}x + (\lambda K_1 - \mu K_2)\,y - \mu^{'}z\right)^2}{(x^2+y^2+z^2)\left[\mu^2\left(\Psi_1^2+\Psi_2^2\right) + \lambda^2\left(\Psi_2^2+\Psi_3^2\right) - 2\lambda\mu\Psi_1\Psi_3\right]},$$

$$H = \frac{\begin{pmatrix}\left(\lambda^2+\mu^2\right)\left[\left(\Psi_1^{'} - K_1\Psi_2\right)x + \left(\Psi_1K_1 + \Psi_2^{'} - K_2\Psi_3\right)y - \left(K_2\Psi_2 + \Psi_3^{'}\right)z\right]\\ -2\left(\lambda\Psi_1 + \mu\Psi_3\right)\left[\lambda^{'}x + (\lambda K_1 - \mu K_2)\,y - \mu^{'}z\right]\end{pmatrix}}{2\sqrt{x^2+y^2+z^2}\left[\mu^2\left(\Psi_1^2+\Psi_2^2\right) + \lambda^2\left(\Psi_2^2+\Psi_3^2\right) - 2\lambda\mu\Psi_1\Psi_3\right]}.$$

Proof. The proof is straightforward by using the equations (3) and (10a)-(10f). □

Corollary 3.10. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_1)}(s,u)$ is the rectifying-type ruled surface of the curve $\overline{r}(s)$. The following statements are given:*

- ✻ *The surface $\phi_{(\overline{r},\varrho_1)}$ is flat (developable) if and only if*

  $$\lambda^{'}x + (\lambda K_1 - \mu K_2)\,y - \mu^{'}z = 0.$$

- ✻ *The surface $\phi_{(\overline{r},\varrho_1)}$ is a minimal surface if and only if*

  $$\begin{pmatrix}\left(\lambda^2+\mu^2\right)\left[\left(\Psi_1^{'} - K_1\Psi_2\right)x + \left(\Psi_1K_1 + \Psi_2^{'} - K_2\Psi_3\right)y - \left(K_2\Psi_2 + \Psi_3^{'}\right)z\right]\\ -2\left(\lambda\Psi_1 + \mu\Psi_3\right)\left[\lambda^{'}x + (\lambda K_1 - \mu K_2)\,y - \mu^{'}z\right]\end{pmatrix} = 0.$$

3.2. Osculating-type ruled surfaces in Myller configuration

The aim of this subsection is to introduce the osculating-type ruled surfaces of a curve with Frenet-type frame in Myller configuration for E_3.

Definition 3.11. Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3. The ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u) : I \times \mathbb{R} \to E_3$ defined as follows:

$$\phi_{(\overline{r},\varrho_2)}(s,u) = \overline{r}(s) + u\varrho_2(s) \quad \text{where} \quad \varrho_2(s) = \lambda(s)\overline{\xi}_1 + \mu(s)\overline{\xi}_2 \quad \text{and} \quad \lambda^2(s) + \mu^2(s) = 1, \tag{11}$$

and the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is called the osculating-type ruled surface of the curve $\overline{r}(s)$ with Frenet-type frame in Myller configuration for E_3.

Special Cases 3.12. *According to the values of the functions $\lambda(s)$, $\mu(s)$ and $\rho_i(s)$ for $i = 1, 2, 3$, the followings hold:*

- ✻ *If $\lambda(s) = 1$, $\mu(s) = 0$ and $\rho_1(s) = 1, \rho_2(s) = 0, \rho_3(s) = 0$, tangent developable surface with Frenet frame in E_3 is obtained [14].*
- ✻ *If $\rho_1(s) = 1, \rho_2(s) = 0, \rho_3(s) = 0$, osculating-type ruled surface with Frenet frame in E_3 is obtained [35].*
- ✻ *If $\lambda(s) = 1$ and $\mu(s) = 0$, $\overline{\xi}_1(s)$-type ruled surface with Frenet-type frame in Myller configuration for E_3 is obtained [42].*

Theorem 3.13. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$. The surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is not a regular surface if and only if*

$$\begin{cases} \mu(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right) = 0, \\ \lambda(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right) = 0, \\ \mu(s)\left(\rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right)\right) - \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) + \mu'(s)\right)\right) = 0. \end{cases} \tag{12}$$

Proof. Taking the partial derivatives of the equation (11) according to the parameter s and u, and by the equation (2), we have:

$$\frac{\partial \phi_{(\overline{r},\varrho_2)}(s,u)}{\partial s} = \left(\rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right)\right)\overline{\xi}_1(s) + \left(\rho_2(s) + u\left(\lambda(s)K_1(s) + \mu'(s)\right)\right)\overline{\xi}_2(s) + \left(\rho_3(s) + u\mu(s)K_2(s)\right)\overline{\xi}_3(s), \tag{13}$$

$$\frac{\partial \phi_{(\overline{r},\varrho_2)}(s,u)}{\partial u} = \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_2(s). \tag{14}$$

By the cross product of the equations (13) and (14), we have:

$$\frac{\partial \phi_{(\overline{r},\varrho_2)}(s,u)}{\partial s} \times \frac{\partial \phi_{(\overline{r},\varrho_2)}(s,u)}{\partial u} = \left(-\mu(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right)\right)\overline{\xi}_1(s) + \left(\rho_3(s) + u\mu(s)K_2(s)\right)\lambda(s)\overline{\xi}_2(s) + \left[\mu(s)\left(\rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right)\right) - \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) + \mu'(s)\right)\right)\right]\overline{\xi}_3.$$

Hence, we obtain the equation (12). □

Then, we get the set of singular points of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ as follows:

$$S_{\phi_{(\overline{r},\varrho_2)}(s,u)} = \left\{ (s,u) \ : \ \begin{array}{l} \mu(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right) = 0, \\ \lambda(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right) = 0, \\ \mu(s)\left(\rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right)\right) - \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) + \mu'(s)\right)\right) = 0. \end{array} \right\}.$$

Theorem 3.14. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$. The followings are presented:*

(1) *The osculating-type surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ with Frenet-type frame in Myller configuration is cylindrical if and only if*

$$\lambda'(s) - K_1(s)\mu(s) = 0, \quad \lambda(s)K_1(s) + \mu'(s) = 0 \quad \textit{and} \quad \mu(s)K_2(s) = 0.$$

(2) *The osculating-type surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ with Frenet-type frame in Myller configuration is developable if and only if*

$$\rho_1(s)\mu^2(s)K_2(s) + \lambda(s)\left(\lambda(s)K_1(s) + \mu'(s)\right)\rho_3(s) - \rho_3(s)\mu(s)\left(\lambda'(s) - K_1(s)\mu(s)\right) - \mu(s)K_2(s)\rho_2(s)\lambda(s) = 0.$$

Proof. Let $\overline{r}(s)$ is a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$.

(1) The surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is cylindrical if and only if $\varrho_2^{'}(s) = 0$.

$$\begin{aligned}\varrho_2^{'}(s) &= \Big(\lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_2(s)\Big)^{'} \\ &= \lambda^{'}(s)\overline{\xi}_1(s) + \lambda(s)\overline{\xi}_1^{'}(s) + \mu^{'}(s)\overline{\xi}_2(s) + \mu(s)\overline{\xi}_2^{'}(s) \\ &= \Big(\lambda^{'}(s) - K_1(s)\mu(s)\Big)\overline{\xi}_1(s) + \Big(\lambda(s)K_1(s) + \mu^{'}(s)\Big)\overline{\xi}_2(s) + \mu(s)K_2(s)\overline{\xi}_3(s).\end{aligned}$$

We prove the desired expression.

(2) The surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is developable if and only if the equation $\det\Big({}^{d\overline{r}}/_{ds}, \varrho_2(s), \varrho_2^{'}(s)\Big) = 0$ is satisfied. We can write the following:

(⇒) Let the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ be a developable surface.

$$\begin{aligned}\det\Big({}^{d\overline{r}}/_{ds}, \varrho_2(s), \varrho_2^{'}(s)\Big) &= \det\begin{pmatrix}\rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s), \\ \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_2(s), \\ \Big(\lambda^{'}(s) - K_1(s)\mu(s)\Big)\overline{\xi}_1(s) + \Big(\lambda(s)K_1(s) + \mu^{'}(s)\Big)\overline{\xi}_2(s) + \mu(s)K_2(s)(s)\overline{\xi}_3(s)\end{pmatrix} \\ &= \rho_1(s)\mu^2(s)K_2(s) + \lambda(s)\Big(\lambda(s)K_1(s) + \mu^{'}(s)\Big)\rho_3(s) \\ &\quad - \rho_3(s)\mu(s)\Big(\lambda^{'}(s) - K_1(s)\mu(s)\Big) - \mu(s)K_2(s)\rho_2(s)\lambda(s) \\ &= 0.\end{aligned}$$

Therefore, we get the desired result.

(⇐) It is clear.

□

Corollary 3.15. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$.*

- ✻ *If $\rho_1^2(s) + \rho_2^2(s) = 1, \rho_3(s) = 0$ and $K_2(s) = 0$, then the osculating-type ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is developable.*
- ✻ *If $\rho_1(s) = 1, \rho_2(s) = \rho_3(s) = 0$ and $K_2(s) = 0$, then the osculating-type ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is developable.*
- ✻ *If $\rho_1(s) = 0, \rho_2(s) = 1, \rho_3(s) = 0$ and $K_2(s) = 0$, then the osculating-type ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is developable.*
- ✻ *If $K_1(s) = \lambda^{'}(s)\mu(s) - \lambda(s)\mu^{'}(s)$ when $\rho_1(s) = \rho_2(s) = 0, \rho_3(s) = 1$ and the functions $\lambda(s), \mu(s)$ are not constant, then the osculating-type ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is developable.*

Theorem 3.16. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$. The base curve $\overline{r}(s)$ of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is its striction curve if and only if $\rho_1(s)\lambda(s) + \rho_2(s)\mu(s) = 0$.*

Proof. The striction curve of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is written as follows:

$$\sigma(s) = \overline{r}(s) - \frac{\langle {}^{d\overline{r}}/_{ds}, \varrho_2(s)\rangle}{\Big\langle \varrho_2^{'}(s), \varrho_2^{'}(s)\Big\rangle}\varrho_2(s).$$

Then, we have

$$\begin{aligned}\langle {}^{d\overline{r}}/_{ds}, \varrho_2(s)\rangle &= \langle \rho_1(s)\overline{\xi}_1(s) + \rho_2(s)\overline{\xi}_2(s) + \rho_3(s)\overline{\xi}_3(s), \lambda(s)\overline{\xi}_1(s) + \mu(s)\overline{\xi}_2(s)\rangle \\ &= \rho_1(s)\lambda(s) + \rho_2(s)\mu(s)\end{aligned}$$

and

$$\langle \varrho_2'(s), \varrho_2'(s) \rangle = \left\langle \begin{array}{l} \left(\lambda'(s) - K_1(s)\mu(s)\right)\overline{\xi}_1(s) + \left(\lambda(s)K_1(s) + \mu'(s)\right)\overline{\xi}_2(s) + \mu(s)K_2(s)\overline{\xi}_3(s), \\ \left(\lambda'(s) - K_1(s)\mu(s)\right)\overline{\xi}_1(s) + \left(\lambda(s)K_1(s) + \mu'(s)\right)\overline{\xi}_2(s) + \mu(s)K_2(s)\overline{\xi}_3(s) \end{array} \right\rangle$$
$$= \left(\lambda'(s) - K_1(s)\mu(s)\right)^2 + \left(\lambda(s)K_1(s) + \mu'(s)\right)^2 + \left(\mu(s)K_2(s)\right)^2.$$

Therefore, we obtain the following equation

$$\sigma(s) = \overline{r}(s) - \frac{\rho_1(s)\lambda(s) + \rho_2(s)\mu(s)}{\left(\lambda'(s) - K_1(s)\mu(s)\right)^2 + \left(\lambda(s)K_1(s) + \mu'(s)\right)^2 + \left(\mu(s)K_2(s)\right)^2}\varrho_2(s).$$

Consequently, we can see that the base curve $\overline{r}(s)$ of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is its striction curve if and only if $\rho_1(s)\lambda(s) + \rho_2(s)\mu(s) = 0$. □

In addition to these, the normal vector of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ is calculated as follows:

$$\overline{\nu} = \frac{-x(s,u)\overline{\xi}_1(s) + y(s,u)\overline{\xi}_2(s) + z(s,u)\overline{\xi}_3(s)}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}},$$

where

$$\begin{cases} x(s,u) = \mu(s)\left(\rho_3(s) + u\mu(s)K_2(s)\right), \\ y(s,u) = \left(\rho_3(s) + u\mu(s)K_2(s)\right)\lambda(s), \\ z(s,u) = \mu(s)\left(\rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right)\right) - \lambda(s)\left(\rho_2(s) + u\left(\lambda(s)K_1(s) + \mu'(s)\right)\right). \end{cases}$$

Additionally, the basic invariants of the surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ are given as follows:

$$E = \Psi_1^2(s,u) + \Psi_2^2(s,u) + \Psi_3^2(s,u), \tag{15a}$$
$$F = \Psi_1(s,u)\lambda(s) + \Psi_2(s,u)\mu(s), \tag{15b}$$
$$G = \lambda^2(s) + \mu^2(s), \tag{15c}$$
$$L = \frac{\begin{array}{l}\left(K_1(s)\Psi_2(s,u) - \Psi_1'(s,u)\right)x(s,u) + \left(\Psi_1(s,u)K_1(s) + \Psi_2'(s,u) - \Psi_3(s,u)K_2(s)\right)y(s,u) \\ + \left(K_2(s)\Psi_2(s) + \Psi_3'(s,u)\right)z(s,u)\end{array}}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}}, \tag{15d}$$
$$M = \frac{\left(K_1(s)\mu(s) - \lambda'(s)\right)x(s,u) + \left(\lambda(s)K_1(s) + \mu'(s)\right)y(s,u) + \mu(s)K_2(s)z(s,u)}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}}, \tag{15e}$$
$$N = 0, \tag{15f}$$

where

$$\begin{cases} \Psi_1(s,u) = \rho_1(s) + u\left(\lambda'(s) - K_1(s)\mu(s)\right), \\ \Psi_2(s,u) = \rho_2(s) + u\left(\mu'(s) + K_1(s)\lambda(s)\right), \\ \Psi_3(s,u) = \rho_3(s) + u\mu(s)K_2(s). \end{cases}$$

Theorem 3.17. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$. The Gaussian and mean curvature of the surface are given as follows, respectively:*

$$K = -\frac{\left(\left(K_1\mu - \lambda'\right)x + \left(\lambda K_1 + \mu'\right)y + \mu K_2 z\right)^2}{\left(x^2 + y^2 + z^2\right)\left[\lambda^2\left(\Psi_2^2 + \Psi_3^2\right) + \mu^2\left(\Psi_1^2 + \Psi_3^2\right) - 2\lambda\mu\Psi_1\Psi_2\right]},$$

$$H=\frac{\begin{pmatrix}\left(\lambda^2+\mu^2\right)\left[\left(K_1\Psi_2-\Psi_1'\right)x+\left(\Psi_1K_1+\Psi_2'-\Psi_3K_2\right)y+\left(K_2\Psi_2+\Psi_3'\right)z\right]\\-2\left(\Psi_1\lambda+\Psi_2\mu\right)\left[\left(K_1\mu-\lambda'\right)x+\left(\lambda K_1+\mu'\right)y+\mu K_2z\right]\end{pmatrix}}{2\sqrt{x^2+y^2+z^2}\left[\lambda^2\left(\Psi_2^2+\Psi_3^2\right)+\mu^2\left(\Psi_1^2+\Psi_3^2\right)-2\lambda\mu\Psi_1\Psi_2\right]}.$$

Proof. The proof is straightforward with the equations (3) and (15a)-(15f). □

Corollary 3.18. *Let $\overline{r}(s):I\to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\overline{\varrho}_2)}(s,u)$ is the osculating-type ruled surface of the curve $\overline{r}(s)$. The followings can be given:*

- ✻ *The surface $\phi_{(\overline{r},\overline{\varrho}_2)}(s,u)$ is flat (developable) if and only if*

$$\left(K_1\mu-\lambda'\right)x+\left(\lambda K_1+\mu'\right)y+\mu K_2z=0.$$

- ✻ *The surface $\phi_{(\overline{r},\overline{\varrho}_2)}(s,u)$ is a minimal surface if and only if*

$$\begin{aligned}&\left(\lambda^2+\mu^2\right)\left[\left(K_1\Psi_2-\Psi_1'\right)x+\left(\Psi_1K_1+\Psi_2'-\Psi_3K_2\right)y+\left(K_2\Psi_2+\Psi_3'\right)z\right]\\&-2\left(\Psi_1\lambda+\Psi_2\mu\right)\left[\left(K_1\mu-\lambda'\right)x+\left(\lambda K_1+\mu'\right)y+\mu K_2z\right]=0.\end{aligned}$$

3.3. Tangent-type ruled surfaces in Myller configuration

In this part, we investigate the tangent-type ruled surfaces of a curve with Frenet-type frame in Myller configuration for E_3. In this subsection, we do not write proofs of all of the theorems for the sake of brevity. Since all proofs can be shown by using the same way in the previous two subsections.

Definition 3.19. Let $\overline{r}(s):I\to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3. The ruled surface $\phi_{(\overline{r},\varrho_3)}(s,u):I\times\mathbb{R}\to E_3$ defined as follows:

$$\phi_{(\overline{r},\varrho_3)}(s,u)=\overline{r}(s)+u\varrho_3(s)\quad\text{where}\quad\varrho_3(s)=\frac{d\overline{r}}{ds}=\rho_1(s)\overline{\xi}_1(s)+\rho_2(s)\overline{\xi}_2(s)+\rho_3(s)\overline{\xi}_3(s),\tag{16}$$

and the surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is called the tangent-type ruled surface of the curve $\overline{r}(s)$ with Frenet-type frame in Myller configuration for E_3.

Special Cases 3.20. *According to the values $\rho_i(s)$ for $i=1,2,3$, the followings are written:*

- ✻ *If we take $\rho_1(s)=1,\rho_2(s)=\rho_3(s)=0$, we get the $\overline{\xi}_1(s)$-type ruled surface with Frenet-type frame in Myller configuration for E_3 (cf. [42]) and tangent developable surface in E_3 [14].*
- ✻ *If we take $\rho_1^2(s)+\rho_3^2(s)=1$, $\rho_2(s)=0$, we get the rectifying-type ruled surface (cf. Definition 3.2 in Subsection 3.1).*
- ✻ *If we take $\rho_1^2(s)+\rho_2^2(s)=1$, $\rho_3(s)=0$, we get the osculating-type ruled surface (cf. Definition 3.11 in Subsection 3.2).*

Theorem 3.21. *Let $\overline{r}(s):I\to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_3)}(s,u)$ is the tangent-type ruled surface of the curve $\overline{r}(s)$. The surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is not a regular surface if and only if*

$$\begin{cases}\rho_3(s)\left(\rho_2(s)+u\left(\rho_1(s)K_1(s)+\rho_2'(s)-\rho_3(s)K_2(s)\right)\right)-\rho_2(s)\left(\rho_3(s)+u\left(\rho_2(s)K_2(s)+\rho_3'(s)\right)\right)=0,\\\rho_1(s)\left(\rho_3(s)+u\left(\rho_2(s)K_2(s)+\rho_3'(s)\right)\right)-\rho_3(s)\left(\rho_1(s)+u\left(\rho_1'(s)-\rho_2(s)K_1(s)\right)\right)=0,\\\rho_2(s)\left(\rho_1(s)+u\left(-\rho_2(s)K_1(s)+\rho_1'(s)\right)\right)-\rho_1(s)\left(\rho_2(s)+u\left(\rho_2'(s)+\rho_1(s)K_1(s)-K_2(s)\rho_3(s)\right)\right)=0.\end{cases}\tag{17}$$

Then, we get the set of singular points of the surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ as follows:

$$S_{\phi_{(\overline{r},\varrho_3)}(s,u)} = \left\{ (s,u) : \begin{array}{l} \rho_3(s)\left(\rho_2(s) + u\left(\rho_1(s)K_1(s) + \rho_2'(s) - \rho_3(s)K_2(s)\right)\right) - \rho_2(s)\left(\rho_3(s) + u\left(\rho_2(s)K_2(s) + \rho_3'(s)\right)\right) = 0, \\ \rho_1(s)\left(\rho_3(s) + u\left(\rho_2(s)K_2(s) + \rho_3'(s)\right)\right) - \rho_3(s)\left(\rho_1(s) + u\left(\rho_1'(s) - \rho_2(s)K_1(s)\right)\right) = 0, \\ \rho_2(s)\left(\rho_1(s) + u\left(-\rho_2(s)K_1(s) + \rho_1'(s)\right)\right) - \rho_1(s)\left(\rho_2(s) + u\left(\rho_2'(s) + \rho_1(s)K_1(s) - K_2(s)\rho_3(s)\right)\right) = 0. \end{array} \right\}.$$

Theorem 3.22. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_3)}(s,u)$ is the tangent-type ruled surface of the curve $\overline{r}(s)$. The followings are written:*

(1) *The tangent-type surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ with Frenet-type frame in Myller configuration is cylindrical if and only if*

$$\rho_1'(s) - \rho_2(s)K_1(s) = 0, \quad \rho_1(s)K_1(s) + \rho_2'(s) - \rho_3(s)K_2(s) = 0 \quad \textit{and} \quad \rho_2(s)K_2(s) + \rho_3'(s) = 0.$$

(2) *The tangent-type surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ with Frenet-type frame in Myller configuration is developable for every $s \in I$.*

Theorem 3.23. *Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_3)}(s,u)$ is the tangent-type ruled surface of the curve $\overline{r}(s)$. The base curve $\overline{r}(s)$ of the surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is never its striction curve.*

Additionally, the normal vector of the surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is written as:

$$\overline{\nu} = \frac{x(s,u)\overline{\xi}_1(s) + y(s,u)\overline{\xi}_2(s) + z(s,u)\overline{\xi}_3(s)}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}},$$

where

$$\begin{cases} x(s,u) = \rho_3(s)\left(\rho_2(s) + u\left(\rho_1(s)K_1(s) + \rho_2'(s) - \rho_3(s)K_2(s)\right)\right) - \rho_2(s)\left(\rho_3(s) + u\left(\rho_2(s)K_2(s) + \rho_3'(s)\right)\right), \\ y(s,u) = \rho_1(s)\left(\rho_3(s) + u\left(\rho_2(s)K_2(s) + \rho_3'(s)\right)\right) - \rho_3(s)\left(\rho_1(s) + u\left(\rho_1' - \rho_2(s)K_1(s)\right)\right), \\ z(s,u) = \rho_2(s)\left(\rho_1(s) + u\left(\rho_1'(s) - \rho_2(s)K_1(s)\right)\right) - \rho_1(s)\left(\rho_2(s) + u\left(\rho_2'(s) + \rho_1(s)K_1(s) - \rho_3(s)K_2(s)\right)\right). \end{cases}$$

Moreover, the basic invariants of the surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ are written as follows:

$$E = \Psi_1^2(s,u) + \Psi_2^2(s,u) + \Psi_3^2(s,u), \tag{18a}$$

$$F = \Psi_1(s,u)\rho_1(s) + \Psi_2(s,u)\rho_2(s) + \Psi_3(s,u)\rho_3(s), \tag{18b}$$

$$G = 1, \tag{18c}$$

$$L = \frac{\begin{array}{l}\left(\Psi_1'(s,u) - K_1(s)\Psi_2(s,u)\right)x(s,u) + \left(\Psi_2'(s,u) - K_2(s)\Psi_3(s,u) + K_1(s)\Psi_1(s,u)\right)y(s,u) \\ + \left(K_2(s)\Psi_2(s,u) + \Psi_3'(s,u)\right)z(s,u)\end{array}}{\sqrt{x^2(s,u) + y^2(s,u) + z^2(s,u)}}, \tag{18d}$$

$$M = 0, \tag{18e}$$

$$N = 0, \tag{18f}$$

where

$$\begin{cases} \Psi_1(s,u) = \rho_1(s) + u\left(\rho_1'(s) - \rho_2(s)K_1(s)\right), \\ \Psi_2(s,u) = \rho_2(s) + u\left(\rho_1(s)K_1(s) + \rho_2'(s) - \rho_3(s)K_2(s)\right), \\ \Psi_3(s,u) = \rho_3(s) + u\left(\rho_2(s)K_2(s) + \rho_3'(s)\right). \end{cases}$$

Theorem 3.24. *Let $\overline{r}(s) : I \rightarrow E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_3)}(s,u)$ is the tangent-type ruled surface of the curve $\overline{r}(s)$. The Gaussian and mean curvature of the surface are written as follows, respectively:*

$$K = 0,$$

$$H = \frac{\left(\Psi_1' - K_1\Psi_2\right)x + \left(\Psi_2' - K_2\Psi_3 + K_1\Psi_1\right)y + \left(K_2\Psi_2 + \Psi_3'\right)z}{2\sqrt{x^2+y^2+z^2}\left(\Psi_1^2 + \Psi_2^2 + \Psi_3^2 - (\Psi_1\rho_1 + \Psi_2\rho_2 + \Psi_3\rho_3)^2\right)}.$$

Corollary 3.25. *Let $\overline{r}(s) : I \rightarrow E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3 and $\phi_{(\overline{r},\varrho_3)}(s,u)$ is the tangent-type ruled surface of the curve $\overline{r}(s)$. The followings can be obtained:*

- ✻ *The surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is flat (developable).*
- ✻ *The surface $\phi_{(\overline{r},\varrho_3)}(s,u)$ is a minimal surface if and only if*

$$\left(\Psi_1' - K_1\Psi_2\right)x + \left(\Psi_2' - K_2\Psi_3 + K_1\Psi_1\right)y + \left(K_2\Psi_2 + \Psi_3'\right)z = 0.$$

4. Example

In this section, we construct a numerical example in order to improve the given theory.
Thanks to the studies [47] and [31], we get the following example:

Example 4.1. Considering the following versor fields and invariants:

$$\begin{cases} \overline{\xi}_1(s) = \left(-\frac{8}{10}\sin s, -\cos s, \frac{6}{10}\sin s\right), \\ \overline{\xi}_2(s) = \left(-\frac{8}{10}\cos s, \sin s, \frac{6}{10}\cos s\right), \\ \overline{\xi}_3(s) = \left(-\frac{6}{10}, 0, -\frac{8}{10}\right), \end{cases} \quad \text{and} \quad \begin{cases} K_1(s) = 1, \\ K_2(s) = 0, \end{cases}$$

and choosing $\rho_1(s) = \sin s$, $\rho_2(s) = \cos s$, $\rho_3(s) = 0$, we get:

$$\overline{r}(s) = \left(-\frac{8s}{10}, 1, \frac{6s}{10}\right). \tag{19}$$

In the following Figure 1, the curve $\overline{r}(s)$ can be seen for $s \in [0, 2\pi]$:

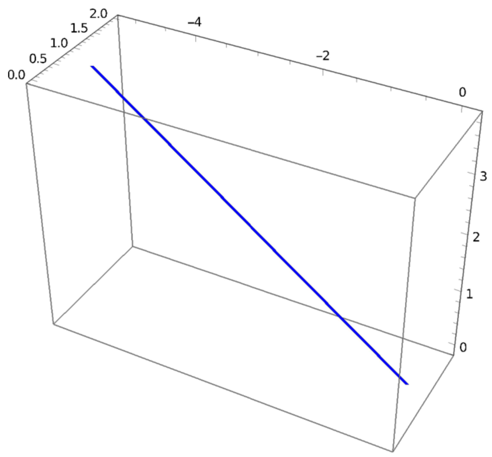


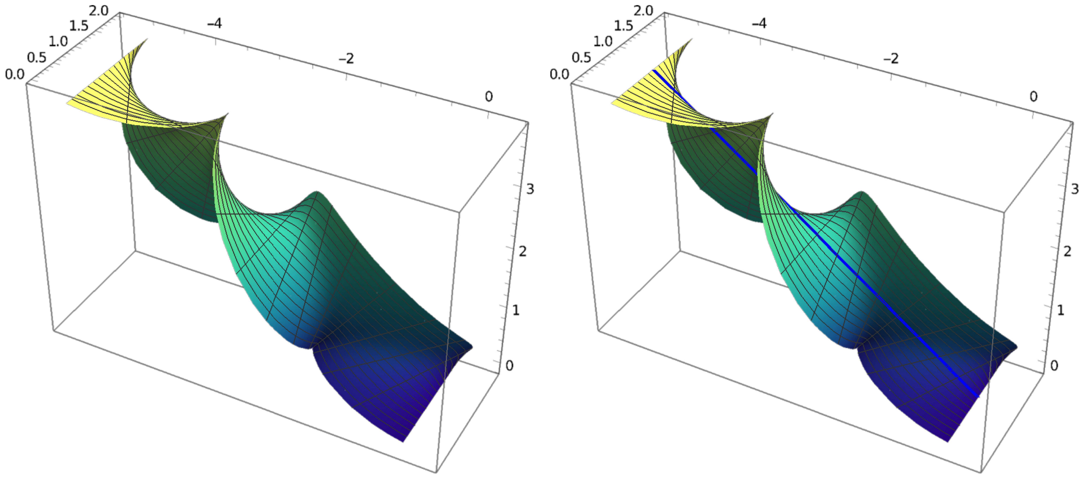
Figure 1: $\overline{r}(s)$ in the equation (19)

✻ If we choose $\lambda(s) = \cos s$ and $\mu(s) = \sin s$, we have the following types ruled surfaces:

⋆ **The rectifying-type ruled surface** is written as:

$$\phi_{(\bar{r},\varrho_1)}(s,u) = \left(-\frac{8s}{10} + u\left(-\frac{8}{10}\cos s \sin s - \frac{6}{10}\sin s\right), 1 - u\cos^2 s, \frac{6s}{10} + u\left(\frac{6}{10}\cos s \sin s - \frac{8}{10}\sin s\right)\right).$$

According to the materials, the surface $\phi_{(\bar{r},\varrho_1)}(s,u)$ is not cylindrical, and is developable where $s = n\pi$,
$n \in \mathbb{Z}$. The base curve $\bar{r}(s)$ of the surface $\phi_{(\bar{r},\varrho_1)}(s,u)$ is its striction curve if $s = n\pi$ or $s = n\pi + \pi/2, n \in \mathbb{Z}$. The following Figure 2 is drawn as $s \in [0, 2\pi]$ and $u \in [-1, 1]$:



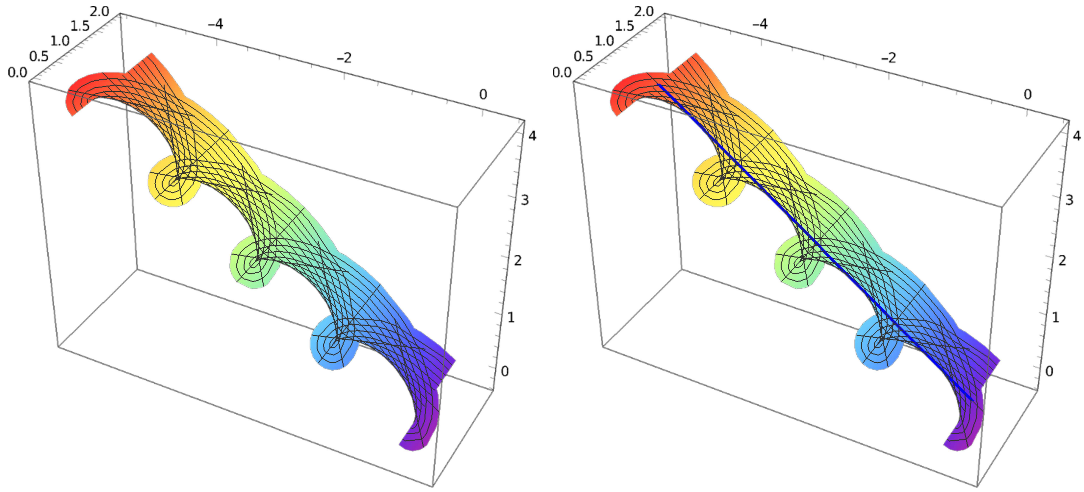
(a) $\phi_{(\bar{r},\varrho_1)}(s,u)$ (b) $\bar{r}(s)$ and $\phi_{(\bar{r},\varrho_1)}(s,u)$

Figure 2: $\phi_{(\bar{r},\varrho_1)}(s,u)$ with the curve $\bar{r}(s)$

⋆ **The osculating-type ruled surface** is written as:

$$\phi_{(\bar{r},\varrho_2)}(s,u) = \left(-\frac{8s}{10} - \frac{8u}{5}\cos s \sin s, 1 + ucos2s, \frac{6s}{10} + u\frac{6}{5}\cos s \sin s\right).$$

Then, the surface $\phi_{(\bar{r},\varrho_2)}(s,u)$ is cylindrical developable. The base curve $\bar{r}(s)$ of the surface $\phi_{(\bar{r},\varrho_2)}(s,u)$ is its striction curve if $s = n\pi$ or $s = n\pi + \pi/2, n \in \mathbb{Z}$. The following Figure 3 is drawn as $s \in [0, 2\pi]$ and $u \in [-1, 1]$:



(a) $\phi_{(\overline{r},\varrho_2)}(s,u)$ (b) $\overline{r}$ and $\phi_{(\overline{r},\varrho_2)}(s,u)$

Figure 3: $\phi_{(\overline{r},\varrho_2)}(s,u)$ with the curve $\overline{r}(s)$

- ⋆ Since the function $\rho_3(s) = 0$, we get the osculating-type ruled surface $\phi_{(\overline{r},\varrho_2)}(s,u)$ as special case of the tangent-type ruled surface.

5. A brief introduction and a motivation for a new survey: Trajectory ruled surfaces in Myller configuration

Now, we want to determine the other type ruled surface without detailed geometrical analysis and interpretations. We only give the definition and some special cases of them, which are called trajectory ruled surfaces with Frenet-type frame in Myller configuration for E_3.

Definition 5.1. Let $\overline{r}(s) : I \to E_3$ be a regular curve with Frenet-type frame in Myller configuration for E_3. The ruled surface $\phi_{(\overline{r},\varrho_4)}(s,u) : I \times \mathbb{R} \to E_3$ defined as follows:

$$\phi_{(\overline{r},\varrho_4)}(s,u) = \overline{r}(s) + u\varrho_4(s) \quad \text{where} \quad \varrho_4(s) = b_1(s)\overline{\xi}_1 + b_2(s)\overline{\xi}_2 + b_3(s)\overline{\xi}_3, \tag{20}$$

where $b_1^2(s) + b_2^2(s) + b_3^2(s) = 1$ and the surface $\phi_{(\overline{r},\varrho_4)}(s,u)$ is called the trajectory ruled surface of the curve $\overline{r}(s)$ with Frenet-type frame in Myller configuration for E_3.

- ✻ If we take $b_1(s) = 1, b_2(s) = 0, b_3(s) = 0$, we get $\overline{\xi}_1$-type ruled surface with Frenet-type frame in Myller configuration for E_3 [42].
- ✻ If we take $b_1(s) = 1, b_2(s) = 0, b_3(s) = 0$ and $\rho_1(s) = 1, \rho_2(s) = 0, \rho_3(s) = 0$, we get the tangent developable surface with Frenet frame in E_3 [14].

For the sake of brevity, we do not examine this type ruled surface in detail like in the previous section. In the future study, we intend to examine this type ruled surface in detail.

6. Conclusions

In this paper, we determine the ruled surface family, which is called generalized ruled surfaces with Frenet-type frame in Myller configuration for E_3. Then, we examined some special type ruled surfaces such as rectifying-type ruled surfaces, osculating-type ruled surfaces and tangent-type ruled surfaces with

Frenet-type frame in Myller configuration for E_3. Moreover, we obtained some particular cases of these ruled surfaces, as well. The surface theory of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of surfaces in classical Euclidean space due to the geometry of versor fields along a curve with Frenet-type frame in Myller configuration for E_3 is a generalization of the usual theory of curves in classical Euclidean space. Further, we construct some numerical examples with respect to these surfaces.

To better analyze the construction of special surfaces and situations, we can give the following Table 1:

Table 1: Classifications of Generalized Ruled Surfaces with Frenet-Type Frame in Myller Configuration for E_3

Some Special Types of the Generalized Ruled Surface $\phi_{(\overline{r},\varrho)}(s,u) = \overline{r}(s) + u\varrho(s)$						
According to the values of ρ_i, b_i for $i = 1,2,3$ and λ and μ.	$\rho_1 = 1$ $\rho_2 = 0$ $\rho_3 = 0$	$\lambda = 1$ $\mu = 0$	$\lambda = 1$ $\mu = 0$ $\rho_1 = 1$ $\rho_2 = 0$ $\rho_3 = 0$	$\rho_1^2 + \rho_2^2 = 1$ $\rho_3 = 0$	$\rho_1^2 + \rho_3^2 = 1$ $\rho_2 = 0$	$b_1 = 1$ $b_2 = 0$ $b_3 = 0$
$\varrho = \overline{\xi}_1$ $\overline{\xi}_1$-type RSFTF in MC for E_3 [42]	Tangent developable RSFF in E_3 [14]					
$\varrho = \lambda\overline{\xi}_1 + \mu\overline{\xi}_3$ Rectifying-type RSFTF in MC for E_3	General rectifying RSFF in E_3 [43]	$\overline{\xi}_1$-type RSFTF in MC for E_3[42]	Tangent developable RSFF in E_3 [14]			
$\varrho = \lambda\overline{\xi}_1 + \mu\overline{\xi}_2$ Osculating-type RSFTF in MC for E_3	Osculating-type RSFF in E_3 [35]	$\overline{\xi}_1$-type RSFTF in MC for E_3 [42]	Tangent developable RSFF in E_3 [14]			
$\varrho = \rho_1\overline{\xi}_1 + \rho_2\overline{\xi}_2 + \rho_3\overline{\xi}_3$ Tangent-type RSFTF in MC for E_3	$\overline{\xi}_1$-type RSFTF in MC for E_3 [42] and tangent developable RSFF in E_3 [14]			Osculating-type RSFTF for E_3	Rectifying-type RSFTF in MC for E_3	
$\varrho = b_1\overline{\xi}_1 + b_2\overline{\xi}_2 + b_3\overline{\xi}_3$ Trajectory RSFTF in MC for E_3	Tangent developable RSFF in E_3 with $b_1 = 1$, $b_2 = 0$, $b_3 = 0$ [14]					$\overline{\xi}_1$-type RSFTF with MC for E_3 [42] and tangent developable RSFF in E_3 [14]

* RSFF: Ruled surfaces with Frenet frame
* RSFTF in MC: Ruled surfaces with Frenet-type frame in Myller configuration

References

[1] A. Alkan, M. Önder, *Some special helices in Myller configuration*, arXiv preprint, (2023), 12 pages, `https://doi.org/10.48550/arXiv.2305.07548`.

[2] A. Alkan, M. Önder, *Slant helices and Darboux helices in Myller configuration*, arXiv preprint, (2023), 14 pages, `https://doi.org/10.48550/arXiv.2307.06394`.

[3] M. Barros, *General helices and a theorem of Lancret*, Proc. Amer. Math. Soc. **125** (1997), 1503–1509.

[4] B.-Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly **110** (2003), 147–152.

[5] B.-Y. Chen, F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Acad. Sinica **33** (2005), 77–90.

[6] J. P. Cleave, *The form of the tangent-developable at points of zero torsion on space curves*, Math. Proc. Cambridge Philos. Soc. **88** (1980), 403–407.

[7] B. Doğan Yazıcı, M. Tosun, *Quasi-type frame and quasi-type osculating curves in Myller configuration*, Filomat **38** (2024), 2813–2820.

[8] B. Doğan Yazıcı, Z. İşbilir, M. Tosun, *Generalized normal ruled surfaces of Frenet-type framed curves*, Session of Mathematical Communications (SCM-2022), Constanţa, România, (2022).

[9] B. Doğan Yazıcı, Z. İşbilir, M. Tosun, *Generalized osculating-type ruled surfaces of singular curves*, Math. Method. Appl. Sci. **46** (2023), 8532–8545.

[10] M. Emmer, *Imagine Math: Between Culture and Mathematics*, Springer, New York, 2012.

[11] B. Heroiu, *Versor fields along a curve in a four dimensional Lorentz space*, J. Adv. Math. Stud. **4** (2011), 49–57.

[12] S. I. Honda, *Rectifying developable surfaces of framed base curves and framed helices*, In Singularities in Generic Geometry, Mathematical Society of Japan **78** (2018), 273–292.

[13] F. Frenet, *Sur les courbes á double courbure*, J. Math. Pures Appl. (1852), 437–447.

[14] A. Gray, E. Abbena, S. Salamon, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, Chapman and Hall/CRC, 2017.

[15] G. Ishikawa, *Singularities of tangent surfaces and generalised frontals*, **1948** (2015), 168–182.

[16] G. Ishikawa, *Topological classification of the tangent developable of space curves*, Hokkaido Univ. Preprint Series **341** (1996).

[17] S. Izumiya, N. Takeuchi, *Geometry of ruled surfaces*, Applicable Mathematics in the Golden Age (edited by J. C. Misra), Narosa Publishing House, New Delhi, (2003), 305–338.

[18] S. Izumiya, N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28** (2004), 153–163.

[19] S. Izumiya, H. Katsumi, T. Yamasaki, *The rectifying developable and the spherical Darboux image of a space curve*, Banach Center Publications **50** (1999), 137–149.

[20] S. Izumiya, N. Takeuchi, *Special curves and ruled surfaces*, Beiträge zur Algebra und Geometrie **44** (2003), 203–212.

[21] K. İlarslan, E. Nešović, *Some characterizations of osculating curves in the Euclidean spaces*, Demonstr. Math. **41** (2008), 931–940.

[22] K. İlarslan, E. Nešović, M. Petrovic-Torgasev, *Some characterizations of rectifying curves in the Minkowski 3-space*, Novi Sad J. Math. **33** (2003), 23–32.

[23] K. İlarslan, E. Nešović, *Some characterizations of rectifying curves in the Euclidean space E^4*, Turk. J. Math. **32** (2008), 21–30.

[24] K. İlarslan, E. Nešović, *On rectifying curves as centrodes and extremal curves in the Minkowski 3-space*, Novi Sad J. Math. **37** (2007), 53–64.

[25] K. İlarslan, *Spacelike normal curves in Minkowski space $\mathbb{E}^3_1$*, Turk. J. Math. **29**(2005), 53–63.

[26] K. İlarslan, E. Nešović, *Spacelike and timelike normal curves in Minkowski space-time*, Publ. Inst. Math. **85(99)** (2009), 111–118.

[27] K. İlarslan, E. Nešović, *Timelike and null normal curves in Minkowski space $\mathbb{E}^3_1$*, Indian J. Pure Appl. Math. **35** (2004), 881–888.

[28] K. İlarslan, E. Nešović, *The first kind and the second kind osculating curves in Minkowski space-time*, Compt. Rend. Acad. Bulg. Sci. **62** (2009), 677–686.

[29] Z. İşbilir, M. Tosun, *On generalized osculating-type curves in Myller configuration*, An. Şt. Univ. Ovidius Constanţa, **32** (2024), 85–97.

[30] Z. İşbilir, M. Tosun, *A new insight on rectifying-type curves in Euclidean 4-space*, Int. Electron. J. Geom. **16** (2023), 644–652.

[31] Z. İşbilir, M. Tosun, *Generalized Smarandache curves with Frenet-type frame*, Honam Math. J. **46** (2024), 181–197.

[32] Z. İşbilir, B. Doğan Yazıcı, M. Tosun, *On special curves in Lie groups with Myller configuration*, Math. Method. Appl. Sci. **47** (2024), 11693–11708.

[33] Z. İşbilir, B. Doğan Yazıcı, M. Tosun, *Generalized rectifying ruled surfaces of special singular curves*, An. Şt. Univ. Ovidius Constanţa **31** (2023), 177–206.

[34] A. Karger, J. Novák, *Space Kinematics and Lie Groups*, STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.

[35] O. Kaya, T. Kahraman, M. Önder, *Osculating-type ruled surface in the Euclidean 3-space*, FU. Math. Inform. **36** (2021), 939–959.

[36] O. Kaya, M. Önder, *Generalized normal ruled surface of a curve in the Euclidean 3-space*, Acta Univ. Sapientiae, Mathematica **13** (2021), 217–238.

[37] Y. Li, S. Liu, Z. Wang, *Tangent developables and Darboux developables of framed curves*, Topol. Appl. **301** (2021), 107526

[38] G. F. Macsim, A. Mihai, A. Olteanu, *Special curves in a Myller configuration*, Proceedings of the 16th Workshop on Mathematics, Computer Science and Technical Education, Department of Mathematics and Computer Science, **2** (2019), 78–84.

[39] G. Macsim, A. Mihai, A. Olteanu, *Curves in a Myller configuration*, International Conference on Applied and Pure Mathematics (ICAPM 2017), Iaşi, November 2-5, 2017.

[40] G. Macsim, A. Mihai, A. Olteanu, *On rectifying-type curves in a Myller configuration*, Bull. Korean Math. Soc. **56** (2019), 383–390.

[41] R. Miron, *Geometria unor configuraţii Myller*, Analele Şt. Univ. **VI**(3) (1960).

[42] R. Miron, *The Geometry of Myller Configurations, Applications to Theory of Surfaces and Nonholonomic Manifolds*, Romanian Academy, 2010.

[43] M. Önder, T. Kahraman, *On rectifying ruled surfaces*, Kuwait J. Sci. **47**(4) (2020), 1–11.

[44] J. A. Serret, *Sur quelques formules relatives á la théorie des courbes á double courbure*, J. Math. Pures Appl. (1851), 193–207.

[45] T. Shifrin, *Differential Geometry: A First Course in Curves and Surfaces*, University of Georgia, Preliminary Version, 2008.

[46] M. A. Solimon, H. N. Abd-Ellah, S. A. Hassan, S. Q. Saleh, *Darboux ruled surfaces with Pointwise 1-type Gauss map*, Sohag J. Sci. **2** (2017), 1–8.

[47] E. Solouma, *Characterization of Smarandache trajectory curves of constant mass point particles as they move along the trajectory curve via PAF*, Bull. Math. Anal. Appl. **13**(4) (2021), 14–30.

[48] E. Solouma, I. Al-Dayel, M. A. Khan, M. Abdelkawy, *Investigation of special type-*Π *Smarandache ruled surfaces due to rotation minimizing Darboux frame in* E^3, Symmetry **15** (2023), 2207.

[49] E. Solouma, I. Al-Dayel, M. A. Khan, Y. A. A. Lazer, *Characterization of imbricate-ruled surfaces via rotation minimizing Darboux frame in Minkowski 3-space* E_1^3, AIMS Mathematics **9** (2024), 13028–13042.

[50] N. Ulucan, M. Akyiğit, *Offset ruled surface in Euclidean space with density*, An. Şt. Univ. Ovidius Constanţa **29** (2021), 219–233.