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Ulam stability of fractional ψ-Caputo differential equations involving two orders for fractional derivatives

Ayoub Louakar[a,*], Ahmed Kajouni[a], Khalid Hilal[a], Hamid Lmou[a]

[a]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco*

Abstract. In this study, we investigate a class of fractional ψ-Caputo differential equations with nonlocal boundary integral conditions, focusing on the existence and uniqueness of solutions. Our analysis employs standard fixed point theorems, specifically Banach's and Krasnoselskii's fixed point theorems. Additionally, various types of Ulam stability are examined, including Ulam-Hyers stability and Ulam-Hyers-Rassias stability. An illustrative example is provided to demonstrate the validity of the theoretical results.

1. Introduction

As a branch of mathematics, fractional calculus relates to generalizing the concepts of derivation and integration to non-integer orders. Recently, this notion has gained significant importance due to its different uses and applications in science and engineering. (see [8, 15]). As a result, fractional differential equations have gained significant interest because they can model and describe complicated problems in physics, engineering, biology, finance, and signal processing. Lately, there have been several investigations focused on different operators, especially Riemann-Liouville[14, 15], Caputo [3, 19], Hilfer [9], and Hadamard [11, 18].

In [4], Almeida introduced an additional generalization of the fractional differentiable operator, known as the fractional ψ-Caputo operator. (for details, refer to [2, 12]). Recently, fractional boundary value problems have attracted significant attention, in part due to their unique qualitative properties and their uses in various fields. Another fascinating and crucial area of research is their stability analysis. Researchers continue to develop new methods and approaches to better understand the dynamics of such nonlinear fractional differential equations, see [1, 5, 6, 10, 13, 16, 17].

This paper aims to investigate whether the following nonlocal boundary value problem has unique solutions, and if so, whether they are Ulam-Hyers stable:

$$\begin{cases} {}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota)+f_1(\iota,w(\iota))\right)=f_2(\iota,w(\iota)), \quad a\le\iota\le b,\\ w(a)=0,\\ w(b)=\sigma\mathfrak{I}^{\rho;\psi}_{a^+,\iota}w(\varsigma), \quad \sigma>0, \end{cases} \tag{1}$$

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* Corresponding author: Ayoub Louakar
Email addresses: ayoublouakar007@gmail.com (Ayoub Louakar), Ahmed.kajouni@usms.ma (Ahmed Kajouni), hilalkhalid2005@yahoo.fr (Khalid Hilal), hamid.lmou@usms.ma (Hamid Lmou)
ORCID iDs: https://orcid.org/0009-0007-1523-9291 (Ayoub Louakar), https://orcid.org/0000-0001-8484-6107 (Ahmed Kajouni), https://orcid.org/0000-0002-0806-2623 (Khalid Hilal), https://orcid.org/0000-0002-8786-2230 (Hamid Lmou)

where $\Im^{\rho;\psi}_{a^+,\iota}$ and $\mathfrak{D}^{\vartheta;\psi}_{a^+,\iota}$ represent ψ-fractional integrals in order $\rho > 0$ and ψ-Caputo fractional derivative in order $\vartheta \in \{p_1, p_2\}$ respectively. $0 < \vartheta \leq 1, \quad a < \varsigma < b$ and $f_1, f_2 : [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous.
In fact, our discussion of existence and uniqueness makes us use standard fixed-point theorems, including Krasnoselskii's fixed point theorem and Banach's fixed point theorem. Additionally, we consider two different Ulam stability types for problem (1): Ulam-Hyers-Rassias (U-H-R) and Ulam-Hyers stability (U-H).
This work is divided into five sections. Basic definitions of ψ-Caputo fractional calculus, essential lemmas, and certain fixed-point theorems are given in Section 2. During Section 3, we discuss whether or not problem (1) has an existence and uniqueness of solutions. Section 4 discusses how Ulam stability can be achieved in the above problem. Finally, Section 5 demonstrates the utility of our results through an example.

2. Preliminaries

Definition 2.1. *([4]) For $p > 0, w \in \mathbb{L}^1(\Upsilon, \mathbb{R})$, and $\psi \in C^n(\Upsilon, \mathbb{R})$ ($\Upsilon := [a, b]$), the fractional ψ-Riemann-Liouville integral operator of order p of w is represented as*

$$\Im^{p;\psi}_{a^+,\iota} w(\iota) = \frac{1}{\Gamma(p)} \int_a^\iota \psi'(\iota)(\psi(\iota) - \psi(s))^{p-1} w(s) ds, \tag{2}$$

where $\psi'(\iota) > 0$, $\forall \iota \in \Upsilon$.

Definition 2.2. *([4]) For $p > 0$ and $w, \psi \in C^{n-1}(\Upsilon, \mathbb{R})$ with $\psi'(\iota) > 0$, $\forall \iota \in \Upsilon$, the fractional ψ-Caputo derivative operator of order p of w is represented as*

$$\begin{aligned} {}^c\mathfrak{D}^{p;\psi}_{a^+,\iota} w(\iota) &= \Im^{n-p;\psi}_{a^+,\iota} w^{[n]}_\psi(\iota) \\ &= \frac{1}{\Gamma(n-p)} \int_a^\iota \psi'(\iota)(\psi(\iota) - \psi(s))^{n-p-1} w^{[n]}_\psi(s) ds, \end{aligned} \tag{3}$$

where $w^{[n]}_\psi(\iota) = \left(\frac{1}{\psi'(\iota)} \frac{d}{d\iota}\right)^n, \quad n = [p] + 1.$

Lemma 2.3. *([4]) Let $n - 1 < p < n$, $w \in C^n(\Upsilon, \mathbb{R})$, then*

$$\Im^{p;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p;\psi}_{a^+,\iota} w\right)(\iota) = w(\iota) - \sum_{i=0}^{n-1} \frac{w^{[i]}_\psi(a)}{i!} (\psi(\iota) - \psi(a))^i, \tag{4}$$

where $w^{[i]}_\psi(\iota) := \left(\frac{1}{\psi'(\iota)} \frac{d}{d\iota}\right)^i w(\iota)$.

Lemma 2.4. *([12, 15]) Let $p_1, p_2 > 0$, $w \in C([a, b], \mathbb{R})$. Then, $\forall \iota \in \Upsilon$ there is*
(i) $\Im^{p_1;\psi}_{a^+,\iota} \Im^{p_2;\psi}_{a^+,\iota} w(\iota) = \Im^{p_1+p_2;\psi}_{a^+,\iota} w(\iota)$;
(ii) ${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \Im^{p_1;\psi}_{a^+,\iota} w(\iota) = w(\iota)$;
(iii) $\Im^{p_1;\psi}_{a^+,\iota} (\psi(\iota) - \psi(a))^{p_2-1} = \frac{\Gamma(p_2)}{\Gamma(p_2+p_1)} (\psi(\iota) - \psi(a))^{p_2+p_1-1}$;
(iv) $\Im^{p_2;\psi}_{a^+,\iota}(1) = \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)}$;
(v) ${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} (\psi(\iota) - \psi(a))^{p_2-1} = \frac{\Gamma(p_2)}{\Gamma(p_2-p_1)} (\psi(\iota) - \psi(a))^{p_2-p_1-1}$;
(vi) ${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} (\psi(\iota) - \psi(a))^i = 0, \quad \forall i < n$.

Theorem 2.5. *(**Krasnoselskii's fixed-point theorem** [7]) Let $\mathcal{U}$ be a closed, bounded, convex, and nonempty subset of a Banach space. Let $\mathcal{A}_1, \mathcal{A}_2$ a pair of operators such that*
(i) $\mathcal{A}_1 w + \mathcal{A}_2 z \in \mathcal{U}$ once $w, z \in \mathcal{U}$;
(ii) $\mathcal{A}_1$ is compact and continuous;
(iii) $\mathcal{A}_2$ is contraction mapping.
Hence, one exists $x \in \mathcal{U}$, such that $x = \mathcal{A}_1 x + \mathcal{A}_2 x$.

Lemma 2.6. *Let $h_1, h_2 \in C(\Upsilon, \mathbb{R})$. Then, it follows that the problem :*

$$\begin{cases} {}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota) + h_1(\iota)\right) = h_2(\iota), \\ w(a) = 0, \\ w(b) = \sigma \mathfrak{I}^{\rho;\psi}_{a^+,\iota}w(\varsigma), \end{cases} \tag{5}$$

is equivalent to the equation:

$$\begin{aligned} w(\iota) =& \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(\iota) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(\iota) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(b) \\ &-\sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}h_2(\varsigma) + \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}h_1(\varsigma)\Big], \end{aligned} \tag{6}$$

where

$$\Theta = \frac{\sigma\left(\psi(\varsigma)-\psi(a)\right)^{p_2+\rho}}{\Gamma\left(p_2+\rho+1\right)} - \frac{\left(\psi(b)-\psi(a)\right)^{p_2}}{\Gamma\left(p_2+1\right)} \neq 0. \tag{7}$$

Proof. Taking the fractional ψ-integral operator of order p_1 on each side of (5). Then utilizing Lemma 2.3, we arrive at

$${}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota) + h_1(\iota) = \mathfrak{I}^{p_1;\psi}_{a^+,\iota}h_2(\iota) + e_1. \tag{8}$$

Utilizing again Lemma 2.3 and Lemma 2.4, we get by taking fractional ψ-integral operator of order p_2 on each side of (8)

$$w(\iota) = \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(\iota) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(\iota) + e_1\frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + e_2, \tag{9}$$

where e_1 and e_2 are arbitrary constants.
In (9), the boundary condition $w(a) = 0$ leads to $e_2 = 0$, and therefore we get

$$w(\iota) = \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(t) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(\iota) + e_1\frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)}. \tag{10}$$

In addition, if we combine the condition $w(b) = \sigma\mathfrak{I}^{\rho;\psi}_{a^+,\iota}h_2(\varsigma)$ with the value of (10), we obtain

$$e_1 = \frac{1}{\Theta}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(b) - \sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}h_2(\varsigma) + \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}h_1(\varsigma)\Big].$$

We substitute e_1 in (10), we obtain (6).

On the other hand, Suppose w can be the unique solution satisfying (6), taking the fractional ψ-Caputo derivative operator ${}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}$ on both sides of (6), then taking fractional ψ-Caputo derivative operator ${}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}$ again, we otain

$${}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota)\right) = h_2(\iota) - {}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}h_1(\iota).$$

So, it follows

$${}^{c}\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^{c}\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota) + h_1(\iota)\right) = h_2(\iota), \quad \iota \in I.$$

Now, we show that w satisfies the boundary conditions; to do this, we have $w(a) = 0$, and from (6), we get

$$\begin{aligned} \sigma\mathfrak{I}^{\rho;\psi}_{a^+,\iota}w(\varsigma) =& \sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}h_2(\varsigma) - \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}h_1(\varsigma) + \frac{\sigma(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}h_1(b) \\ &-\sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}h_2(\varsigma) + \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}h_1(\varsigma)\Big]. \end{aligned}$$

Based on (7), we get

$$\begin{aligned}\sigma \mathfrak{I}^{\rho;\psi}_{a^+,\iota} w(\varsigma) =& \sigma \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} h_2(\varsigma) - \sigma \mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota} h_1(\varsigma) + \left(1 + \frac{(\psi(b)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)}\right)\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota} h_1(b) \\ & -\sigma \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} h_2(\varsigma) + \sigma \mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota} h_1(\varsigma)\Big] \\ =& \, I^{p_1+p_2;\psi}_{a^+,\iota} h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota} h_1(b) + \frac{(\psi(b)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} h_2(b) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota} h_1(b) \\ & -\sigma \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} h_2(\varsigma) + \sigma \mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota} h_1(\varsigma)\Big] \\ =& \, w(b).\end{aligned}$$

Hence, the proof is complete. □

3. Existence and uniqueness results

This part proves the existence as well as the uniqueness of solutions for problem (1). We assume that f_1 and f_2 belong to the Banach space $C(\Upsilon, \mathbb{R})$. Let $\mathcal{U} = \{w : w \in C(\Upsilon, \mathbb{R})\}$ define the Banach space in which each continuous function on Υ into $\mathbb{R}$ is included, associated with $\|w\| = \sup\{|w(\iota)| : \iota \in \Upsilon\}$.
Based on Lemma 2.6, we set the operator $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ to solve problem (1).

$$\begin{aligned}(\mathcal{A}w)(\iota) =& \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} f_2(\iota, w(\iota)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota} f_1(\iota, w(\iota)) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} f_2(b, w(b)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota} f_1(b, w(b)) \\ & -\sigma \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} f_2(\varsigma, w(\varsigma)) + \sigma \mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota} f_1(\varsigma, w(\varsigma))\Big].\end{aligned} \tag{11}$$

As we can see, the existence of an operator's $\mathcal{A}$ fixed point ensures that problem (1) has a solution.
In order to make computation easier, we introduce the following notation:

$$\Lambda_1 = \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)}\left[\frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_2+p_1+1)} + |\sigma|\frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_2+p_1+\rho+1)}\right], \tag{12}$$

$$\Lambda_2 = \frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)}\left[\frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + |\sigma|\frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)}\right]. \tag{13}$$

Using theorem 2.5, we now discuss the existence result.

Theorem 3.1. *Let $f_1, f_2 : \Upsilon \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions achieving the following requirements:*
(H1) $|f_2(\iota, z) - f_2(\iota, w)| \leq K_1\|z - w\|$, and $|f_1(\iota, z) - f_1(\iota, w)| \leq K_2\|z - w\|$ for all $\iota \in \Upsilon$, every $z, w \in \mathbb{R}$ and $K > 0$, with $K\Lambda_2 < 1$ and $K = \max\{K_1, K_2\}$.
(H2) Further, Assume there are continuous, nonnegative functions $\omega_1, \omega_2 \in C(\Upsilon, \mathbb{R})$ such that

$$|f_2(\iota, w)| \leq \omega_1(\iota) \text{ and } |f_1(\iota, w)| \leq \omega_2(\iota), \forall(\iota, w) \in \Upsilon \times \mathbb{R} \text{ and } \omega = \max\{\omega_1, \omega_2\}.$$

Hence, there is at least one solution to problem (1) on Υ.

Proof. For a positive number η, let $B_\eta = \{w \in \mathcal{U} : \|w\| \leqslant \eta\}$, where $\eta \geq \|\omega\|(\Lambda_1 + \Lambda_2)$, and on the bounded set B_η we divide $\mathcal{A}$ into two operators $\mathcal{A}_1$ and $\mathcal{A}_2$ where $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ by

$$\begin{aligned}(\mathcal{A}_1 w)(\iota) =& \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} f_2(\iota, w(\iota)) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)}\Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} f_2(b, w(b)) \\ & -\sigma \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} f_2(\varsigma, w(\varsigma))\Big],\end{aligned} \tag{14}$$

and

$$
\begin{aligned}
(\mathcal{A}_2 w)(\iota) = &-\Im^{p_2;\psi}_{a^+,\iota} f_1(\iota, w(\iota)) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)} \Big[-\Im^{p_2;\psi}_{a^+,\iota} f_1(b, w(b)) \\
&+\sigma \Im^{p_2+\rho;\psi}_{a^+,\iota} f_1(\varsigma, w(\varsigma))\Big].
\end{aligned}
\tag{15}
$$

Let $w, z \in B_\eta$. Then, we have

$$
\begin{aligned}
\|\mathcal{A}_1 w + \mathcal{A}_2 z\| \le & \sup_{\iota\in[a,b]} \Big\{ \Im^{p_1+p_2;\psi}_{a^+,\iota} |f_2(\iota, w(\iota))| + \Im^{p_2;\psi}_{a^+,\iota} |f_1(\iota, z(\iota))| \\
&+ \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im^{p_1+p_2;\psi}_{a^+,\iota} |f_2(b, w(b))| + |\sigma| \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} |f_2(\varsigma, w(\varsigma))| + \Im^{p_2;\psi}_{a^+,\iota} |f_1(b, z(b))| \\
&+ |\sigma| \Im^{p_2+\rho;\psi}_{a^+,\iota} |f_1(\varsigma, z(\varsigma))| \Big] \Big\} \\
\le & \|\omega\| \sup_{\iota\in[a,b]} \Bigg\{ \frac{(\psi(\iota)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \Bigg[\frac{(\psi(\iota)-\psi(a))^{p_1+p_2}}{\Gamma(p_2+p_1+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_2+p_1+\rho+1)} \Bigg] \\
&+ \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \Bigg[\frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)} |\sigma| \Bigg] \Bigg\} \\
\le & \|\omega\| (\Lambda_1 + \Lambda_2) \le \eta.
\end{aligned}
$$

This shows that $\mathcal{A}_1 w + \mathcal{A}_2 z \in B_\eta$.
Next, we prove that $\mathcal{A}_2$ will be a contraction mapping. Let $w, z \in \mathcal{U}$ and $\iota \in \Upsilon$. Then, under assumption (H1), we obtain

$$
\begin{aligned}
\|\mathcal{A}_2 w - \mathcal{A}_2 z\| \le & \sup_{\iota\in[a,b]} \Big\{ \Im^{p_2;\psi}_{a^+,\iota} |f_1(\iota, w(\iota)) - f_1(\iota, z(\iota))| \\
&+ \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im^{p_2;\psi}_{a^+,\iota} |f_1(b, w(b)) - f_1(b, z(b))| + |\sigma| \Im^{p_2+\rho;\psi}_{a^+,\iota} |f_1(\varsigma, w(\varsigma)) - f_1(\varsigma, z(\varsigma))| \Big] \Big\} \\
\le & K\|w-z\| \sup_{\iota\in[a,b]} \Bigg\{ \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Bigg[\frac{(\psi(\iota)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)} \Bigg] \Bigg\} \\
\le & K\Lambda_2 \|w-z\|.
\end{aligned}
$$

Thus, $\mathcal{A}_2$ is a contraction mapping according to assumption $K\Lambda_2 < 1$.
From the continuity of f_2, it follows that $\mathcal{A}_1$ is continuous. Hence, $\mathcal{A}_1$ is uniformly bounded on B_η; that is,

$$
\begin{aligned}
\|\mathcal{A}_1 w\| \le & \sup_{\iota\in[a,b]} \Big\{ \Im^{p_1+p_2;\psi}_{a^+,\iota} |f_2(\iota, w(\iota))| + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im^{p_1+p_2;\psi}_{a^+,\iota} |f_2(b, w(b))| + |\sigma| \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} |f_2(\varsigma, w(\varsigma))| \Big] \Big\} \\
\le & \|\omega\| \sup_{\iota\in[a,b]} \Bigg\{ \frac{(\psi(\iota)-\psi(a))^{p_1+p_2}}{\Gamma(1+p_1+p_2)} + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \Bigg[\frac{(\psi(\iota)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} \Bigg] \Bigg\} \\
\le & \|\omega\| \Lambda_1.
\end{aligned}
$$

We conclude by proving that $\mathcal{A}_1$ is compact. To this end, define $\sup_{(\iota,w)\in[a,b]\times B_\eta} |f_2(\iota, w)| = \bar{f_2} < \infty$, and let

$a \leq \iota_1 < \iota_2 \leq b$. Then, it follows that

$$
\begin{aligned}
&|(\mathcal{A}_1 w)(\iota_2) - (\mathcal{A}_1 w)(\iota_1)| \\
&= \frac{1}{\Gamma(p_1+p_2)} \Big| \int_a^{\iota_1} \left[(\psi(\iota_2)-\psi(s))^{p_1+p_2-1} - (\psi(\iota_1)-\psi(s))^{p_1+p_2-1}\right] \psi'(s) f_2(s,w(s))ds \\
&+ \int_{\iota_1}^{\iota_2} \psi'(s)(\psi(\iota_2)-\psi(s))^{p_1+p_2-1} f_2(s,w(s))ds \Big| \\
&+ \frac{(\psi(\iota_2)-\psi(a))^{p_2} - (\psi(\iota_1)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[|\sigma|\bar{f}_2 \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} + \bar{f}_2 \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\right] \\
&\leq \frac{\bar{f}_2}{\Gamma(p_1+p_2+1)} \left[2(\psi(\iota_2)-\psi(\iota_1))^{p_2+p_1} + (\psi(\iota_2)-\psi(a))^{p_2+p_1} - (\psi(\iota_1)-\psi(a))^{p_2+p_1}\right] \\
&+ \frac{(\psi(\iota_2)-\psi(a))^{p_2} - (\psi(\iota_1)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[|\sigma|\bar{f}_2 \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} + \bar{f}_2 \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)}\right] \longrightarrow 0
\end{aligned}
$$

as $\iota_2 - \iota_1 \longrightarrow 0$,
independent of $w \in B_\eta$. Therefore, $\mathcal{A}_1$ is equicontinuous. This means that $\mathcal{A}_1$ is relatively compact on B_η. As a result, the Arzela-Ascoli hypothesis is fulfilled, $\mathcal{A}_1$ is compact on B_η. Hence, our hypothesis for Theorem 2.5 holds, which leads to at least one solution to problem (1) on Υ. □

Using Banach's fixed-point theorem, we will prove the uniqueness of the result.

Theorem 3.2. *suppose that (H1) is valid. If the constants Λ_1 and Λ_2, defined by (12) and (13), respectively, satisfy*

$$K(\Lambda_1 + \Lambda_2) < 1. \tag{16}$$

Hence, the solution to problem (1) is unique on Υ.

Proof. As a first step, let us show that $\mathcal{A}$ defined by (11) satisfies $\mathcal{A}B_\tau \subset B_\tau$, where $B_\tau = \{w \in \mathcal{U} : \|w\| \leq \tau\}$ with $\tau \geq \frac{N(\Lambda_1+\Lambda_2)}{1-K(\Lambda_1+\Lambda_2)}$), $\sup_{\iota\in[a,b]} |f_2(\iota,0)| = N_1 < \infty$, $\sup_{\iota\in[a,b]} |f_1(\iota,0)| = N_2 < \infty$ and $N = \max\{N_1, N_2\}$.
For any $w \in B_\tau$, we have

$$
\begin{aligned}
\|\mathcal{A}w\| &\leq \sup_{\iota\in[a,b]} \Big\{ \Im_{a^+,\iota}^{p_1+p_2;\psi} |f_2(\iota,w(\iota))| + \Im_{a^+,\iota}^{p_2;\psi} |f_1(\iota,w(\iota))| \\
&+ \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} |f_2(b,w(b))| + |\sigma| \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} |f_2(\varsigma,w(\varsigma))| + \Im_{a^+,\iota}^{p_2;\psi} |f_1(b,w(b))| \\
&+ |\sigma| \Im_{a^+,\iota}^{p_2+\rho;\psi} |f_1(\varsigma,w(\varsigma))| \Big] \Big\} \\
&\leq \sup_{\iota\in[a,b]} \Big\{ \Im_{a^+,\iota}^{p_1+p_2;\psi} (|f_2(\iota,w(\iota)) - f_2(\iota,0)| + |f_2(\iota,0)|) + \Im_{a^+,\iota}^{p_2;\psi} (|f_1(\iota,w(\iota)) - f_1(\iota,0)| + |f_1(\iota,0)|) \\
&+ \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} (|f_2(b,w(b)) - f_2(b,0)| + |f_2(b,0)|) \\
&+ |\sigma| \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} (|f_2(\varsigma,w(\varsigma)) - f_2(\varsigma,0)| + |f_2(\varsigma,0)|) + \Im_{a^+,\iota}^{p_2;\psi} (|f_1(b,w(b) - f_1(b,0)| + |f_1(b,0)|) \\
&+ |\sigma| \Im_{a^+,\iota}^{p_2+\rho;\psi} (|f_1(\varsigma,w(\varsigma)) - f_1(\varsigma,0)| + |f_1(\varsigma,0)|) \Big] \Big\} \\
&\leq (K\|w\| + N) \Big\{ \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} \right] \\
&+ \frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)} \right] \Big\}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\|\mathcal{A}w\| &\leq (K\|w\| + N)(\Lambda_1 + \Lambda_2) \\ &\leq K\tau(\Lambda_1 + \Lambda_2) + N(\Lambda_1 + \Lambda_2) \\ &\leq \tau.\end{aligned}$$

Thus, $\mathcal{A}B_\tau \subset B_\tau$.
Next, let $w, z \in \mathcal{U}$. Then, for $\iota \in \Upsilon$, we have

$$\begin{aligned}\|\mathcal{A}w - \mathcal{A}z\| \leq& \sup_{\iota\in[a,b]} \Big\{ \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota} \left(|f_2(\iota, w(\iota)) - f_2(\iota, z(\iota))|\right) + \mathfrak{I}^{p_2;\psi}_{a^+,\iota} \left(|f_1(\iota, w(\iota)) - f_1(\iota, z(\iota))|\right) \\ &+ \frac{(\psi(\iota) - \psi(a))^{p_2}}{|\Theta|\Gamma(p_2 + 1)} \Big[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,p_2} \left(|f_2(b, w(b)) - f_2(b, z(b))|\right) + |\sigma| \mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota} \left(|f_2(\varsigma, w(\varsigma)) - f_2(\varsigma, z(\varsigma))|\right) \\ &+ \mathfrak{I}^{p_2;\psi}_{a^+,\iota} \left(|f_1(b, w(b)) - f_1(b, z(b))|\right) + |\sigma| \mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota} \left(|f_1(\varsigma, w(\varsigma)) - f_1(\varsigma, z(\varsigma))|\right) \Big] \Big\} \\ \leq& K\|w - z\| \sup_{\iota\in[a,b]} \left\{ \frac{(\psi(\iota) - \psi(a))^{p_1+p_2}}{\Gamma(p_1 + p_2 + 1)} + \frac{(\psi(\iota) - \psi(a))^{p_2}}{|\Theta|\Gamma(p_2 + 1)} \cdot \left[\frac{(\psi(\iota) - \psi(a))^{p_1+p_2}}{\Gamma(p_1 + p_2 + 1)} + |\sigma| \frac{(\psi(\varsigma) - \psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1 + p_2 + \rho + 1)} \right] \right. \\ &\left. + \frac{(\psi(\iota) - \psi(a))^{p_2}}{\Gamma(p_2 + 1)} + \frac{(\psi(\iota) - \psi(a))^{p_2}}{|\Theta|\Gamma(p_2 + 1)} \cdot \left[\frac{(\psi(\iota) - \psi(a))^{p_2}}{\Gamma(p_2 + 1)} + |\sigma| \frac{(\psi(\varsigma) - \psi(a))^{p_2+\rho}}{\Gamma(p_2 + \rho + 1)} \right] \right\} \\ \leq& K\|w - z\| \left\{ \frac{(\psi(b) - \psi(a))^{p_1+p_2}}{\Gamma(p_2 + p_1 + 1)} + \frac{(\psi(b) - \psi(a))^{p_2}}{|\Theta|\Gamma(p_2 + 1)} \cdot \left[\frac{(\psi(b) - \psi(a))^{p_2+p_1}}{\Gamma(p_1 + p_2 + 1)} + |\sigma| \frac{(\psi(\varsigma) - \psi(a))^{p_2+p_1+\rho}}{\Gamma(p_1 + p_2 + \rho + 1)} \right] \right. \\ &\left. + \frac{(\psi(b) - \psi(a))^{p_2}}{\Gamma(p_2 + 1)} + \frac{(\psi(b) - \psi(a))^{p_2}}{|\Theta|\Gamma(p_2 + 1)} \cdot \left[\frac{(\psi(b) - \psi(a))^{p_2}}{\Gamma(p_2 + 1)} + |\sigma| \frac{(\psi(\varsigma) - \psi(a))^{p_2+\rho}}{\Gamma(p_2 + \rho + 1)} \right] \right\} \\ \leq& K(\Lambda_1 + \Lambda_2)\|w - z\|,\end{aligned}$$

which implies $\|\mathcal{A}w - \mathcal{A}z\| \leq K(\Lambda_1 + \Lambda_2)\|w - z\|$. As $K(\Lambda_1 + \Lambda_2) < 1$. As a result, the operator $\mathcal{A}$ possesses a fixed point that is the unique solution to problem (1) according to the Banach's fixed-point theorem.
□

4. Ulam stability results

In this part, we discuss the stability of problem (1) in terms of U-H, U-H-R, and their generalized forms. Let $\varepsilon > 0$, $\phi \in C(\Upsilon, \mathbb{R})$ and consider:

$$\left| {}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota} z(\iota) + f_1(\iota, z(\iota)) \right) - f_2(\iota, z(\iota)) \right| \leq \varepsilon, \quad \iota \in \Upsilon, \tag{17}$$

$$\left| {}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota} z(\iota) + f_1(\iota, z(\iota)) \right) - f_2(\iota, z(\iota)) \right| \leq \phi(\iota), \quad \iota \in \Upsilon, \tag{18}$$

$$\left| {}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota} z(\iota) + f_1(\iota, z(\iota)) \right) - f_2(\iota, z(\iota)) \right| \leq \varepsilon\phi(\iota), \quad \iota \in \Upsilon. \tag{19}$$

Definition 4.1. *([1, 10]) Problem (1) is U-H stable if a real number $c_{f_2} > 0$ exists such that, for all $\varepsilon > 0$ and all $z \in C(\Upsilon, \mathbb{R})$ solution of (17), there is a solution $w \in C(\Upsilon, \mathbb{R})$ to the problem (1) with*

$$\|z - w\| \leq \varepsilon c_{f_2}, \quad \iota \in \Upsilon.$$

Definition 4.2. *([1, 10]) Problem (1) is generalized U-H stable if a continuous function c_{f_2} on $\mathbb{R}$ exist with $c_{f_2}(0) = 0$ such that, for all $\varepsilon > 0$ and all $z \in C(\Upsilon, \mathbb{R})$ solution of (17), there is a solution $w \in C(\Upsilon, \mathbb{R})$ to the problem (1) with*

$$\|z - w\| \leq c_{f_2}(\varepsilon), \quad \iota \in \Upsilon.$$

Definition 4.3. *([1, 10]) Problem (1) is U-H-R stable in terme to* ϕ *if a real number* $c_{f_2,\phi} > 0$ *exists such that, for all* $\varepsilon > 0$ *and all* $z \in C(\Upsilon, \mathbb{R})$ *solution of (19), there is a solution* $w \in C(\Upsilon, \mathbb{R})$ *to the problem (1) with*

$$\|z - w\| \leq \varepsilon c_{f_2,\phi}\phi(\iota), \quad \iota \in \Upsilon.$$

Definition 4.4. *([1, 10]) Problem (1) is generalized U-H-R stable in terme to* ϕ *if a real number* $c_{f_2,\phi} > 0$ *exists such that, for all* $\varepsilon > 0$ *and all* $z \in C(\Upsilon, \mathbb{R})$ *solution of (18), there is a solution* $w \in C(\Upsilon, \mathbb{R})$ *to the problem (1) with*

$$\|z - w\| \leq c_{f_2,\phi}\phi(\iota), \quad \iota \in \Upsilon.$$

Remark 4.5. *([1]) A continious function z on* Υ *is a solution of (17) if and only if a function* $g \in C(\Upsilon, \mathbb{R})$ *(which depends on solution z) exists such that*
(i) $|g(\iota)| \leq \varepsilon, \quad \iota \in \Upsilon$;
(ii) ${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota}z(\iota) + f_1(\iota, z(\iota))\right) = f_2(\iota, z(\iota)) + g(\iota), \quad \iota \in \Upsilon.$

Remark 4.6. *([1]) A continious function z on* Υ *is a solution of (19) if and only if a function* $g \in C(\Upsilon, \mathbb{R})$ *(which depends on solution z) exists such that*
(i) $|g(\iota)| \leq \varepsilon\phi(\iota), \quad \iota \in \Upsilon$;
(ii) ${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota}z(\iota) + f_1(\iota, z(\iota))\right) = f_2(\iota, z(\iota)) + g(\iota), \quad \iota \in \Upsilon.$

First, let's discuss the (U-H) stability.

Theorem 4.7. *Assume both* $f_2, f_1 : \Upsilon \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous, (H1) holds and* $1 - K(\Lambda_1 + \Lambda_2) \neq 0$*. Then, the problem (1) is U-H and generalized U-H stable.*

Proof. Let $z \in C(\Upsilon, \mathbb{R})$ as the solution to (17) and $w \in C(\Upsilon, \mathbb{R})$ as the unique solution to the problem:

$$\begin{cases} {}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota}w(\iota) + f_1(\iota, w(\iota))\right) = f_2(\iota, w(\iota)), \quad a \leq \iota \leq b, \\ w(a) = 0, \\ w(b) = \sigma\mathfrak{I}^{\rho;\psi}_{a^+,\iota}w(\varsigma), \quad \sigma > 0. \end{cases} \tag{20}$$

According to Lemma 2.6, we have

$$\begin{aligned} w(\iota) =& \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}f_2(\iota, w(\iota)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}f_1(\iota, w(\iota)) + \frac{(\psi(\iota) - \psi(a))^{p_2}}{\Theta\Gamma(p_2 + 1)}\left[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}f_2(b, w(b)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}f_1(b, w(b))\right. \\ &\left. -\sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}f_2(\varsigma, w(\varsigma)) + \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}f_1(\varsigma, w(\varsigma))\right]. \end{aligned}$$

By assuming that z is a solution of (17). Hence, based on Remark 4.5, the solution of the equation

$${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota}\left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota}z(\iota) + f_1(\iota, z(\iota))\right) = f_2(\iota, z(\iota)) + g(\iota), \quad \iota \in [a, b],$$

can be formulated as follows:

$$\begin{aligned} z(\iota) =& \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}f_2(\iota, z(\iota)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}f_1(\iota, z(\iota)) + \frac{(\psi(\iota) - \psi(a))^{p_2}}{\Theta\Gamma(p_2 + 1)}\left[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}f_2(b, z(b)) - \mathfrak{I}^{p_2;\psi}_{a^+,\iota}f_1(b, z(b))\right. \\ &\left. -\sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}f_2(\varsigma, z(\varsigma)) + \sigma\mathfrak{I}^{p_2+\rho;\psi}_{a^+,\iota}f_1(\varsigma, z(\varsigma))\right] + \mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}g(\iota) \\ &+ \frac{(\psi(\iota) - \psi(a))^{p_2}}{\Theta\Gamma(p_2 + 1)}\left[\mathfrak{I}^{p_1+p_2;\psi}_{a^+,\iota}g(b) - \sigma\mathfrak{I}^{p_1+p_2+\rho;\psi}_{a^+,\iota}g(\varsigma)\right]. \end{aligned}$$

Therfore,

$$\begin{aligned}
\|z-w\| \leq & \sup_{\iota\in[a,b]} \Big\{ \Im^{p_1+p_2;\psi}_{a^+,\iota} \left(|f_2(\iota,w(\iota)) - f_2(\iota,z(\iota))|\right) + \Im^{p_2;\psi}_{a^+,\iota} \left(|f_1(\iota,w(\iota)) - f_1(\iota,z(\iota))|\right) \\
& + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im^{p_1+p_2;\psi}_{a^+,\iota} \left(|f_2(b,w(b)) - f_2(b,z(b))|\right) + |\sigma| \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} \left(|f_2(\varsigma,w(\varsigma)) - f_2(\varsigma,z(\varsigma))|\right) \\
& + \Im^{p_2;\psi}_{a^+,\iota} \left(|f_1(b,w(b)) - f_1(b,z(b))|\right) + |\sigma| \Im^{p_2+\rho;\psi}_{a^+,\iota} \left(|f_1(\varsigma,w(\varsigma)) - f_1(\varsigma,z(\varsigma))|\right) \Big] + \Im^{p_1+p_2;\psi}_{a^+,\iota} |g(\iota)| \\
& + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im^{p_1+p_2;\psi}_{a^+,\iota} |g(b)| + |\sigma| \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} |g(\varsigma)| \Big] \Big\} \\
\leq & K\|z-w\| \left\{ \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} \right] \right. \\
& \left. + \frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)} \right] \right\} \\
& + \varepsilon \left\{ \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} \right] \right\} \\
\leq & (\Lambda_2+\Lambda_1) K \|z-w\| + \varepsilon \Lambda_1 \\
\leq & K\|z-w\|(\Lambda_2+\Lambda_1) + \varepsilon\Lambda_1.
\end{aligned}$$

Then, $\|z-w\| \leq K(\Lambda_1+\Lambda_2)\|z-w\| + \varepsilon\Lambda_1$. Consequently, $\|z-w\| \leq \varepsilon \frac{\Lambda_1}{1-K(\Lambda_1+\Lambda_2)} \leq \varepsilon c_{f_2}$, $\iota \in \Upsilon$, where $c_{f_2} = \frac{\Lambda_1}{1-K(\Lambda_1+\Lambda_2)}$. Thus, the problem (1) is U-H stable. Additionally, using $\Phi_{f_2}(\varepsilon) = c_{f_2}\varepsilon$, $\Phi_{f_2}(0) = 0$ shows the generalized U-H stability of the solution to (1). □

Now, we can state the U–H–R stability.

Theorem 4.8. *Suppose both $f_2, f_1 : \Upsilon \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, (H1) holds, $1 - K(\Lambda_1+\Lambda_2) \neq 0$. Additionally, one has the following hypothesis:*
(H3): $\phi : \Upsilon \to \mathbb{R}$ satisfy the properities

$$\begin{cases} \Im^{p_1+p_2;\psi}_{a^+,\iota} \phi(b) \leq \phi(\iota), & \iota \in \Upsilon, \\ \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} \phi(\varsigma) \leq \phi(\iota), & \iota \in \Upsilon. \end{cases}$$

Then, the problem (1) is U-H-R and generalized U-H-R stable.

Proof. Let $z \in C(\Upsilon,\mathbb{R})$ as the solution to (19) and $w \in C(\Upsilon,\mathbb{R})$ as the unique solution to the problem:

$$\begin{cases} {}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota} w(\iota) + f_1(\iota,w(\iota)) \right) = f_2(\iota,w(\iota)), & a \leq \iota \leq b, \\ w(a) = 0, \\ w(b) = \sigma \Im^{\rho;\psi}_{a^+,\iota} w(\varsigma), \ \sigma > 0. \end{cases} \tag{21}$$

According to Lemma 2.6, we obtain

$$\begin{aligned} w(\iota) = & \Im^{p_1+p_2;\psi}_{a^+,\iota} f_2(\iota,w(\iota)) - \Im^{p_2;\psi}_{a^+,\iota} f_1(\iota,w(\iota)) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)} \Big[\Im^{p_1+p_2;\psi}_{a^+,\iota} f_2(b,w(b)) - \Im^{p_2;\psi}_{a^+,\iota} f_1(b,w(b)) \\ & - \sigma \Im^{p_1+p_2+\rho;\psi}_{a^+,\iota} f_2(\varsigma,w(\varsigma)) + \sigma \Im^{p_2+\rho;\psi}_{a^+,\iota} f_1(\varsigma,w(\varsigma)) \Big]. \end{aligned}$$

By assuming that z is a solution of (19). Hence, based on Remark 4.6, the solution of the equation

$${}^c\mathfrak{D}^{p_1;\psi}_{a^+,\iota} \left({}^c\mathfrak{D}^{p_2;\psi}_{a^+,\iota} z(t) + f_1(\iota,z(\iota)) \right) = f_2(\iota,z(\iota)) + g(\iota), \quad \iota \in [a,b],$$

can be formulated as follows:

$$\begin{aligned} z(\iota) =& \Im_{a^+,\iota}^{p_1+p_2;\psi} f_2(\iota, z\iota)) - \Im_{a^+,\iota}^{p_2;\psi} f_1(\iota, z(\iota)) + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} f_2(b, z(b)) - \Im_{a^+,\iota}^{p_2;\psi} f_1(b, z(b)) \\ & -\sigma \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} f_2(\varsigma, z(\varsigma)) + \sigma \Im_{a^+,\iota}^{p_2+\rho;\psi} f_1(\varsigma, z(\varsigma)) \Big] + \Im_{a^+,\iota}^{p_1+p_2;\psi} g(\iota) \\ & + \frac{(\psi(\iota)-\psi(a))^{p_2}}{\Theta\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} g(b) - \sigma \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} g(\varsigma) \Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|z-w\| \leq & \sup_{\iota\in[a,b]} \Big\{ \Im_{a^+,\iota}^{p_1+p_2;\psi} \left(|f_2(\iota, w(\iota)) - f_2(\iota, z(\iota))|\right) + \Im_{a^+,\iota}^{p_2;\psi} \left(|f_1(\iota, w(\iota)) - f_1(\iota, z(\iota))|\right) \\ & + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} \left(|f_2(b, w(b)) - f_2(b, z(b))|\right) + |\sigma| \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} \left(|f_2(\varsigma, w(\varsigma)) - f_2(\varsigma, z(\varsigma))|\right) \\ & + \Im_{a^+,\iota}^{p_2;\psi} \left(|f_1(b, w(b)) - f_1(b, z(b))|\right) + |\sigma| \Im_{a^+,\iota}^{p_2+\rho;\psi} \left(|f_1(\varsigma, w(\varsigma)) - f_1(\varsigma, z(\varsigma))|\right) \Big] + \Im_{a^+,\iota}^{p_1+p_2;\psi} |g(\iota)| \\ & + \frac{(\psi(\iota)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} |g(b)| + |\sigma| \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} |g(\varsigma)| \Big] \Big\} \\ \leq & K\|z-w\| \Big\{ \frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_1+p_2}}{\Gamma(p_1+p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_1+p_2+\rho}}{\Gamma(p_1+p_2+\rho+1)} \right] \\ & + \frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \cdot \left[\frac{(\psi(b)-\psi(a))^{p_2}}{\Gamma(p_2+1)} + |\sigma| \frac{(\psi(\varsigma)-\psi(a))^{p_2+\rho}}{\Gamma(p_2+\rho+1)} \right] \Big\} \\ & + \Im_{a^+,\iota}^{p_1+p_2;\psi} |g(b)| + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \Big[\Im_{a^+,\iota}^{p_1+p_2;\psi} |g(b)| + |\sigma| \Im_{a^+,\iota}^{p_1+p_2+\rho;\psi} |g(\varsigma)| \Big]. \end{aligned}$$

By utilizing (H3) and Remark 4.6, we have

$$\begin{aligned} \|z-w\| &\leq K\|z-w\|(\Lambda_1+\Lambda_2) + \varepsilon\phi(\iota) + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} \left[\varepsilon\phi(\iota) + |\sigma|\varepsilon\phi(\iota) \right] \\ &\leq K(\Lambda_1+\Lambda_2)\|z-w\| + \left[1 + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} (1+|\sigma|) \right] \varepsilon\phi(\iota) \\ &\leq c_{f_2,\phi}\varepsilon\phi(\iota), \text{ where} \end{aligned}$$

$$c_{f_2,\phi} = \frac{\left[1 + \frac{(\psi(b)-\psi(a))^{p_2}}{|\Theta|\Gamma(p_2+1)} (1+|\sigma|) \right]}{1 - K(\Lambda_1+\Lambda_2)}.$$

Conequently, the problem (1) is U-H-R stable. Furthermore, it is generalized U-H-R stable when we set $\varepsilon = 1$. □

5. Example

Consider the following problem:

$$\begin{cases} {}^c\mathfrak{D}_{0^+,\iota}^{p_1;\psi} \left({}^c\mathfrak{D}_{0^+,\iota}^{p_2;\psi} w(\iota) + f_1(\iota, w(\iota)) \right) = f_2(\iota, w(\iota)), \quad 0 \leq \iota \leq \frac{3}{4}, \\ w(0) = 0, \\ w(\frac{3}{4}) = \sigma \Im_{0^+,\iota}^{\rho;\psi} w(\varsigma), \ \sigma > 0, \end{cases} \tag{22}$$

where $p_1 = \frac{1}{4}$, $p_2 = \frac{3}{4}$, $\sigma = \frac{5}{6}$, $\rho = \frac{2}{3}$, $\varsigma = \frac{1}{2}$, and $\psi(\iota) = \iota$. Let

$$f_2(\iota, w) = \frac{e^{-\iota}}{11\sqrt{44}}\left(\frac{\iota^2}{5} + \cos w\right),$$
$$f_1(\iota, w) = \frac{1}{21(\iota+1)^2}(1 + \sin w).$$

f_2 and f_1 satisfy (H1) and (H2) of theorem 3.1. Additionally, we find $K = 0.0476190476$, $\Lambda_1 = 2.040788437$, $\Lambda_2 = 2.446102709$, $K\Lambda_2 \approx 0,1164810813 < 1$, and $K(\Lambda_1 + \Lambda_2) \approx 0,2136614831 < 1$. Theorem 3.2 shows that the solution to problem (22) on $\left[0, \frac{3}{4}\right]$ is unique. Furthermore, we find that $c_{f_2} = \frac{\Lambda_1}{1-K(\Lambda_1+\Lambda_2)} \approx 2,595305194 > 0$. Hence, according to Theorem 4.7, the problem (22) is U–H and generalized U–H stable.

Next, by selecting $\phi(\iota) = -0.1\iota + 1$ for any $\iota \in \left[0, \frac{3}{4}\right]$, we have

$$\begin{cases} \mathfrak{I}_{0^+,\iota}^{p_1+p_2;\psi}\phi(\frac{3}{4}) = \mathfrak{I}_{0^+,\iota}^{1;\psi}\phi(\frac{3}{4}) \le \phi(\iota), & \iota \in [0, \frac{3}{4}], \\ \mathfrak{I}_{0^+,\iota}^{p_1+p_2+\rho;\psi}\phi(\frac{3}{4}) = \mathfrak{I}_{0^+,\iota}^{\frac{5}{3};\psi}\phi(\frac{1}{2}) \le \phi(\iota), & \iota \in [0, \frac{3}{4}]. \end{cases}$$

Therefore, condition (H3) is satisfied, and Theorem 4.8 leads us to the conclusion that problem (22) is both U-H-R and generalized U-H-R stable.

6. Conclusion

In this paper, we examined the existence and uniqueness of solutions to a class of fractional ψ-Caputo differential equations. The results were established using fixed point theory. Additionally, various types of Ulam stability for problem (1) were studied. Finally, an example was provided to illustrate the validity of the theoretical findings.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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