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Asymptotic distributions of the average clustering coefficient and its variant

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Abstract. In network data analysis, summary statistics of a network can provide us with meaningful insight into the structure of the network. The average clustering coefficient is one of the most popular and widely used network statistics. In this paper, we investigate the asymptotic distributions of the average clustering coefficient and its variant of an inhomogeneous random graph. We show that the standardized average clustering coefficient converges in distribution to the standard normal distribution. Interestingly, the variance of the average clustering coefficient exhibits a phase transition phenomenon. The sum of weighted triangles is a variant of the average clustering coefficient. It is recently introduced to detect geometry in a network. We also derive the asymptotic distribution of the sum weighted triangles, which does not exhibit a phase transition phenomenon as the average clustering coefficient. This result signifies the difference between the two summary statistics.

1. Introduction

Network data consists of a set of vertices and a set of edges that represents the connections between vertices. Network data analysis is widely used to study many real-world problems. Amazon and eBay utilize the recommendation network to advertise their online goods [15]. Telecommunication companies optimize the performance of 5G wireless network by studying its topology structure [1]. Academic journals investigate the citation network to study the relationship between papers, authors, and scientific work [20]. Biological network is used to detect gene-gene interactions [13]. Due to the widespread applications, network analysis becomes one of the most proactive research directions.

In network data analysis, a typical task is to understand the structural properties of a network. Visualization is perhaps the most straightforward method to describe the structure of a network. However, larger networks can be difficult to envision and describe. Numerous summary statistics have therefore been proposed to quantify the structures of a network. Based on these statistics, we are able to compare networks or classify them according to properties that they exhibit.

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One commonly used network statistic is the average clustering coefficient, which measures the trend of the vertices of a network to cluster together [25]. The average clustering coefficient was first introduced in social network analysis to quantify the property that friends of a friend tend to be friends [3]. Since then, the average clustering coefficient has been widely used in network analysis. [26] used the average clustering coefficient to identify functional disconnections in patients with borderline personality disorder. [7] applied the average clustering coefficient to examine the dissimilarities of the brain functional networks between endurance runners and healthy controls. [23] applied the average clustering coefficient to investigate the functional brain network development in children. In [5], the average clustering coefficient was employed to compare the brain connectivity networks with generative models of structured networks. In [22], the clustering coefficient was used to quantify the network structure of large networks.

It is shown in [16] that the average clustering coefficient is insufficient to signal the presence of hyperbolic geometry in a network. Therefore, [16] introduces the sum of weighted triangles to detect the geometry in a network. Note that the average clustering coefficient is also a sum of weighted triangles. However, the weights are different from the sum of weighted triangles introduced in [16]. Hence, the sum of weighted triangles is a variant of the average clustering coefficient. The analytical analysis and numeric studies in [16] show that the sum of weighted triangles has high potential for uncovering a hidden network geometry.

One of the important research topics in network analysis is to study the asymptotic properties of summary statistics [4, 11, 27–30]. Especially, subgraph-related quantities have been widely studied. [17] derived the asymptotic distribution of rooted subgraph counts in an inhomogeneous random graph. [8] derived the limit of the average clustering coefficient and asymptotic distribution of subgraph counts of graphs based on exchangeable point processes. The results in [2] can be directly applied to get the limiting distribution of subgraph counts in general homogeneous random graphs. As far as we know, the asymptotic distributions of the average clustering coefficient and the sum of weighted triangles are unknown. In this paper, we study the asymptotic distributions of the average clustering coefficient and the sum of weighted triangles in an inhomogeneous random graph. The results in [2, 8, 17] do not apply to the average clustering coefficient and the sum of weighted triangles. Note that both the average clustering coefficient and the sum of weighted triangles are sums of dependent terms. It is not a trivial task to derive their asymptotic distributions. We prove that the standardized average clustering coefficient and the standardized sum of weighted triangles converge in distribution to the standard normal distribution. Interestingly, the variance of the average clustering coefficient exhibits a phase transition phenomenon. However, the sum of weighted triangles does not present similar phase transition. These results highlight the difference between the two summary statistics.

The rest of the paper is organized as follows. In section 2, we introduce the inhomogeneous random graph model, definitions of the average clustering coefficient and the sum of the weighted triangles, and present the main results. The proof is deferred to section 3.

Notation: We adopt the Bachmann–Landau notation throughout this paper. Let a_n and b_n be two positive sequences. Denote $a_n = \Theta(b_n)$ if $c_1 b_n \leq a_n \leq c_2 b_n$ for some positive constants c_1, c_2 and large n. Denote $a_n = \omega(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. Denote $a_n = O(b_n)$ if $a_n \leq c b_n$ for some positive constants c and large n. Denote $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Let $\mathcal{N}(0,1)$ be the standard normal distribution and X_n be a sequence of random variables. Then $X_n \Rightarrow \mathcal{N}(0,1)$ means X_n converges in distribution to the standard normal distribution as n goes to infinity. Denote $X_n = O_P(a_n)$ if $\frac{X_n}{a_n}$ is bounded in probability for large n. Denote $X_n = o_P(a_n)$ if $\frac{X_n}{a_n}$ converges to zero in probability as n goes to infinity. Let $\mathbb{E}[X_n]$ and $Var(X_n)$ denote the expectation and variance of a random variable X_n respectively. $\mathbb{P}[E]$ denote the probability of an event E. Let $f = f(x)$ be a function. Denote $f^{(k)}(x) = \frac{d^k f}{dx^k}(x)$ for any positive integer k. For positive integer n, denote $[n] = \{1, 2, \ldots, n\}$. Given a finite set E, $|E|$ represents the number of elements in E. Given positive integer t, $\sum_{i_1 \neq i_2 \neq \cdots \neq i_t}$ means summation over all integers $i_1, i_2, \ldots, i_t$ in $[n]$ such that $|\{i_1, i_2, \ldots, i_t\}| = t$. $\sum_{i_1 < i_2 < \cdots < i_t}$ means summation over all integers $i_1, i_2, \ldots, i_t$ in $[n]$ such that $i_1 < i_2 < \cdots < i_t$.

2. Main results

A graph is a mathematical model that consists of a set of nodes (vertices) and a set of edges. Let $\mathcal{V} = [n]$ for any positive integer n. An *undirected* graph on $\mathcal{V}$ is defined as the pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is a set of subsets of $\mathcal{V}$ such that $|e| = 2$ for every $e \in \mathcal{E}$. Each element in $\mathcal{V}$ is called a node or vertex of the graph and each element in $\mathcal{E}$ is called an edge. A graph can be conveniently represented as an adjacency matrix A. In A, $A_{ij} = 1$ if $\{i, j\}$ is an edge, $A_{ij} = 0$ otherwise and $A_{ii} = 0$. Since $\mathcal{G}$ is undirected, the adjacency matrix A is symmetric. The degree d_i of node i is the number of edges connecting it, that is, $d_i = \sum_j A_{ij}$. A graph is said to be random if $A_{ij} (1 \leq i < j \leq n)$ are random.

Definition 2.1. *Let α and β be constants between zero and one, that is, $\alpha, \beta \in (0, 1)$, and $W = \{w_{ij}\}$ be an infinite triangular array such that $w_{ij} \in [\beta, 1]$ $(i < j)$ and $w_{ii} = 0$. Define an inhomogeneous random graph $\mathcal{G}_n(\alpha, \beta, W)$ as*

$$\mathbb{P}(A_{ij} = 1) = n^{-\alpha} w_{ij},$$

where A_{ij} $(1 \leq i < j \leq n)$ are independent and $A_{ij} = A_{ji}$.

The random graph $\mathcal{G}_n(\alpha, \beta, W)$ is inhomogeneous because the expected degrees of nodes may be different. Specifically, $\mathbb{E}[d_i] = n^{-\alpha} \sum_k w_{ik}$. In general, $\mathbb{E}[d_i] \neq \mathbb{E}[d_j]$ for distinct nodes i, j. If $w_{ij} = c$ $(1 \leq i < j \leq n)$ for a constant $c \in (0, 1)$, the expected degrees of nodes are the same. In this case, $\mathcal{G}_n(\alpha, \beta, W)$ is homogeneous and it is called the Erdös-Rényi random graph. For convenience, we simply denote it as $\mathcal{G}_n(\alpha)$. Moreover, $\mathcal{G}_n(\alpha, \beta, W)$ is relatively dense due to the assumption $\alpha \in (0, 1)$. In this case, the expected degree of each node diverges as n tends to infinity. The random graph $\mathcal{G}_n(\alpha, \beta, W)$ serves as our benchmark model. It is studied in [27, 29] and includes the inhomogeneous Erdös-Rényi random graphs in [11, 28, 30] as a special case.

2.1. The average clustering coefficient

In social networks, nodes tend to develop highly connected neighborhoods [3, 25]. The average clustering coefficient was first introduced to measure such property of social networks [3, 25]. The clustering coefficient of each node in a graph is the fraction of triangles that actually exist over all possible triangles in its neighborhood. It measures the triangular pattern and the connectivity in a node's neighborhood. A node has a high clustering coefficient if its neighbors tend to be directly connected with each other. The average clustering coefficient of a graph is the mean of the clustering coefficients of all nodes. It measures the extent to which nodes in a graph tend to cluster together. In terms of the adjacency matrix A, the average clustering coefficient is defined as

$$\overline{C}_n = \frac{1}{n} \sum_{i=1}^{n} \frac{t_i}{d_i(d_i - 1)},$$

where $t_i = \sum_{j \neq k} A_{ij} A_{jk} A_{ki}$ and any summation terms with $d_i = 0$ are set to be zero.

Recently, [18] studied the robustness of the average clustering coefficient and [30] obtained the limit of the average clustering coefficient. In this paper, we derive its asymptotic distribution as follows.

Theorem 2.2. *For the inhomogeneous random graph $\mathcal{G}_n(\alpha, \beta, W)$, we have*

$$\frac{\overline{C}_n - \mathbb{E}[\overline{C}_n]}{\sigma_n} \Rightarrow \mathcal{N}(0, 1),$$

where $\sigma_n^2 = \sigma_{1n}^2 + \sigma_{2n}^2$ and

$$\sigma_{1n}^2 = \frac{4}{n^{2+3\alpha}} \sum_{i<j<k} a_{ijk}^2 w_{ij} w_{jk} w_{ki} (1 - n^{-\alpha} w_{ij})(1 - n^{-\alpha} w_{jk})(1 - n^{-\alpha} w_{ki}),$$

$$\begin{aligned} a_{ijk} &= \mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] + \mathbb{E}\left[\frac{1}{d_j(d_j-1)}\right] + \mathbb{E}\left[\frac{1}{d_k(d_k-1)}\right], \\ \sigma_{2n}^2 &= \frac{4}{n^{2+\alpha}} \sum_{i<j} e_{ij}^2 w_{ij}(1-n^{-\alpha}w_{ij}), \\ e_{ij} &= 2n^{-2\alpha} \sum_{k\neq i,j} \mathbb{E}\left[\frac{1}{d_k(d_k-1)}\right] w_{ki}w_{kj} - \frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2} - \frac{\mathbb{E}[t_j](2\mu_j-1)}{\mu_j^2(\mu_j-1)^2} \\ &\quad +2\left(\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] + \mathbb{E}\left[\frac{1}{d_j(d_j-1)}\right]\right) n^{-2\alpha} \sum_{k\neq i,j} w_{ki}w_{kj}, \\ \mu_i &= n^{-\alpha} \sum_j w_{ij}. \end{aligned}$$

Based on Theorem 2.2, the standardized average clustering coefficient converges in distribution to the standard normal distribution. The proof of Theorem 2.2 is not trivial, due to the fact that $n\overline{C}_n$ is a sum of dependent random variables. Our proof strategy is as follows: use the Taylor expansion to expand the summation terms to $k_0 = \lceil 1 + \frac{1}{1-\alpha} \rceil + 2$ order, isolate the leading term and prove the leading term converges in distribution to the standard normal distribution.

Next we apply Theorem 2.2 to the Erdös-Rényi random graph $\mathcal{G}_n(\alpha)$ and get the following corollary.

Corollary 2.3. *For the Erdös-Rényi random graph $\mathcal{G}_n(\alpha)$, we have*

$$\frac{\overline{C}_n - \mathbb{E}[\overline{C}_n]}{\sigma_n} \Rightarrow \mathcal{N}(0,1),$$

where $\sigma_n^2 = \sigma_{1n}^2 + \sigma_{2n}^2$, $\sigma_{1n}^2 = \frac{6}{n^{3-\alpha}}(1+o(1))$ and $\sigma_{2n}^2 = \frac{2}{n^{2+\alpha}}(1+o(1))$. Hence we have

$$\sigma_n^2 = \begin{cases} \frac{6}{n^{3-\alpha}}(1+o(1)), & \text{if } \alpha > \frac{1}{2}, \\ \frac{8}{n^2\sqrt{n}}(1+o(1)), & \text{if } \alpha = \frac{1}{2}, \\ \frac{2}{n^{2+\alpha}}(1+o(1)), & \text{if } \alpha < \frac{1}{2}. \end{cases}$$

According to Corollary 2.3, the average clustering coefficient of the Erdös-Rényi random graph $\mathcal{G}_n(\alpha)$ shows a phase change phenomenon. Given large n, we have

$$\lim_{\alpha\to(\frac{1}{2})^-} \frac{2}{n^{2+\alpha}} = \frac{2}{n^2\sqrt{n}} \neq \frac{8}{n^2\sqrt{n}},$$

$$\lim_{\alpha\to(\frac{1}{2})^+} \frac{6}{n^{3-\alpha}} = \frac{6}{n^2\sqrt{n}} \neq \frac{8}{n^2\sqrt{n}}.$$

As α varies around $\frac{1}{2}$, σ_n^2 does not change continuously as a function of α. In this sense, the scale σ_n^2 exhibits a phase change phenomenon at $\alpha = \frac{1}{2}$.

2.2. The sum of weighted triangles

Many real-world networks have geometric structures and can be accurately modelled by geometric random graphs. Nevertheless, the presence of geometry in an observed network is not always evident. The average clustering coefficient is a standard statistic to indicate the presence of geometry. However, it is shown that the average clustering coefficient fails to detect hyperbolic geometry in a network [16]. Therefore, [16] introduces a novel triangle-based statistic, that is, the sum of weighted triangles. Note that the average clustering coefficient can also be written as a sum of weighted triangles, with the weights different from the sum of weighted triangles introduced in [16]. Hence, the sum of weighted triangles is a variant of the average clustering coefficient. By the analytical analysis and numeric studies in [16], the sum of weighted triangles shows high potential for uncovering hidden network geometry.

In terms of the adjacency matrix A, the sum of the weighted triangles is defined as [16]

$$\mathcal{T}_n = \sum_{i<j<k} \frac{\Delta_{ijk}}{d_i d_j d_k},$$

where $\Delta_{ijk} = A_{ij}A_{jk}A_{ki}$ and the summation term is set to be zero if $d_i d_j d_k = 0$. [16] obtains bounds of $\mathcal{T}_n$ in an inhomogeneous random graph and a geometric inhomogeneous random graph. In this paper, we derive the asymptotic distribution of $\mathcal{T}_n$ in the inhomogeneous random graph $\mathcal{G}_n(\alpha, \beta, W)$.

Theorem 2.4. *For the inhomogeneous random graph $\mathcal{G}_n(\alpha, \beta, W)$, we have*

$$\frac{\mathcal{T}_n - \mathbb{E}[\mathcal{T}_n]}{v_n} \Rightarrow \mathcal{N}(0,1),$$

where $v_n^2 = v_{1n}^2 + v_{2n}^2$ and

$$\begin{aligned}
v_{1n}^2 &= n^{-3\alpha} \sum_{i<j<k} \frac{w_{ij}(1-n^{-\alpha}w_{ij})w_{jk}(1-n^{-\alpha}w_{jk})w_{ki}(1-n^{-\alpha}w_{ki})}{\mu_i^2\mu_j^2\mu_k^2}, \\
\mu_i &= n^{-\alpha}\sum_j w_{ij}, \\
v_{2n}^2 &= n^{-\alpha}\sum_{i<j}\left(\gamma_{ij} - \frac{\eta_i+\eta_j}{2}\right)^2 w_{ij}(1-n^{-\alpha}w_{ij}), \\
\eta_i &= n^{-3\alpha}\sum_{j\neq k}\frac{w_{ij}w_{jk}w_{ki}}{\mu_i^2\mu_j\mu_k}, \\
\gamma_{ij} &= n^{-2\alpha}\sum_{k\notin\{i,j\}}\frac{w_{jk}w_{ki}}{\mu_i\mu_j\mu_k}.
\end{aligned}$$

Based on Theorem 2.4, the standardized sum of weighted triangles converges in distribution to the standard normal distribution.

Next we apply Theorem 2.4 to a special inhomogeneous random graph. In the definition of $\mathcal{G}_n(\alpha, \beta, W)$, let $w_{ij} = w_i w_j$ for $w_i \in [\beta, 1]$. In this case, $\mathcal{G}_n(\alpha, \beta, W)$ is called rank-1 inhomogeneous random graph.

Corollary 2.5. *For the rank-1 inhomogeneous random graph $\mathcal{G}_n(\alpha, \beta, W)$, we have*

$$\frac{\mathcal{T}_n - \mathbb{E}[\mathcal{T}_n]}{v_n} \Rightarrow \mathcal{N}(0,1),$$

where $v_n^2 = \frac{n^3}{6w^6p_n^3}(1+o(1))$ and $w = \sum_i w_i$. Especially, for the Erdös-Rényi random graph $\mathcal{G}_n(\alpha)$, $v_n^2 = \frac{1}{6n^{3(1-\alpha)}}(1+o(1))$.

Note that

$$\frac{n^3}{6w^6p_n^3} = \frac{1}{6\left(\frac{w}{n}\right)^6 n^{3(1-\alpha)}},$$

and $\frac{w}{n}$ is independent of α. As α varies, v_n^2 changes continuously as a function of α. Hence, the sum of weighted triangles of the rank-1 random graph $\mathcal{G}_n(\alpha, \beta, W)$ does not exhibit a phase change phenomenon. This signifies the difference between the sum of weighted triangles and the average clustering coefficient.

3. Proof of main results

In this section, we provide the detailed proofs of the main results. Denote $\bar{A}_{ij} = A_{ij} - \mu_{ij}$, $p_n = n^{-\alpha}$ and $\mu_{ij} = p_n w_{ij}$ in this section. Before proving Theorem 2.2 and Theorem 2.4, we present two lemmas.

Lemma 3.1. *Let $\delta_n = (\log(np_n))^{-2}$. For the random graph $\mathcal{G}_n(\alpha, \beta, W)$, we have*

$$\mathbb{P}(d_i = k) \le \exp(-np_n\beta(1 + o(1))), \quad k \le \delta_n np_n,$$

uniformly for all $i \in [n]$.

Lemma 3.1 is proved in [28]. Hence we omit the proof.

Lemma 3.2. *For the random graph $\mathcal{G}_n(\alpha, \beta, W)$ and an even positive integer s, we have*

$$\mathbb{E}[(d_i - \mu_i)^s] = O\left((np_n)^{\frac{s}{2}}\right).$$

Proof of Lemma 3.2. Given integer $t \le s$, let $\lambda_{t1}, \lambda_{t2}, \ldots, \lambda_{tt}$ be arbitrary non-negative integers such that $\lambda_{t1} + \lambda_{t2} + \cdots + \lambda_{tt} = s$. Then

$$\begin{aligned}(d_i - \mu_i)^s &= \sum_{j_1,\ldots,j_s \neq i} \bar{A}_{ij_1}\bar{A}_{ij_2}\ldots\bar{A}_{ij_s} \\ &= \sum_{t=1}^{s}\sum_{\lambda_{t1},\ldots,\lambda_{tt}}\sum_{j_1\neq\cdots\neq j_t\neq i} \bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}}. \qquad (1)\end{aligned}$$

It is easy to verify that $\mathbb{E}[\bar{A}_{ij}^k] = 0$ if $k = 1$ and $\mathbb{E}[\bar{A}_{ij}^k] = O(p_n)$ if $k \ge 2$. For distinct indices $j_1, j_2, \ldots, j_t$, $\bar{A}_{ij_1}, \bar{A}_{ij_2}, \ldots, \bar{A}_{ij_t}$ are independent. Hence, we have

$$\mathbb{E}[\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}}] = \begin{cases} O(p_n^t) & \text{if } \lambda_{tl} \ge 2 \text{ for all } l, \\ 0 & \text{if } \lambda_{tl} = 1 \text{ for some } l. \end{cases}$$

When $\lambda_{tl} \ge 2$ for all l, $t \le \frac{s}{2}$. There are at most n^t choices for indices $j_1, j_2, \ldots, j_t$. Hence

$$\sum_{j_1\neq\cdots\neq j_t} \mathbb{E}[\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}}] = O\left((np_n)^{\frac{s}{2}}\right).$$

Then

$$\mathbb{E}[(d_i - \mu_i)^s] = O\left((np_n)^{\frac{s}{2}}\right).$$

□

Lemma 3.3. *For the random graph $\mathcal{G}_n(\alpha, \beta, W)$, we have*

$$\begin{aligned}\mathbb{E}[\overline{C}_n] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[t_i]\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] - \frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i - 1}{\mu_i^2(\mu_i-1)^2}\mu_{ij}(1-\mu_{ij})\mu_{ik}\mu_{jk} \\ &\quad + o\left(\frac{1}{n\sqrt{np_n}} + \frac{\sqrt{p_n}}{n}\right). \qquad (2)\end{aligned}$$

Proof of Lemma 3.3. Recall $t_i = \sum_{j\neq k} A_{ij}A_{jk}A_{ki}$ and let $h(x) = \frac{1}{x(x-1)}$. The k-th derivative of $h(x)$ is equal to

$$h^{(k)}(x) = \left(\frac{1}{x-1} - \frac{1}{x}\right)^{(k)} = k!(-1)^k \frac{\sum_{t=1}^{k+1}\binom{k+1}{t}x^{k+1-t}(-1)^{t+1}}{(x-1)^{k+1}x^{k+1}}. \qquad (3)$$

Let $k_0 = \lceil 1 + \frac{1}{1-\alpha} \rceil + 2$. By the Taylor expansion of $h(x)$, we have

$$\frac{1}{d_i(d_i-1)} = \frac{1}{\mu_i(\mu_i-1)} - \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i) + \sum_{s=2}^{k_0-1} \frac{h^{(s)}(\mu_i)}{s!}(d_i-\mu_i)^s + \frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}, \tag{4}$$

where X_i is between d_i and μ_i, that is, $d_i \le X_i \le \mu_i$ or $\mu_i \le X_i \le d_i$. Then

$$\mathbb{E}[\overline{C}_n] = \frac{1}{n}\sum_i \frac{\mathbb{E}[t_i]}{\mu_i(\mu_i-1)} - \frac{1}{n}\sum_i \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[t_i(d_i-\mu_i)] + \sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[t_i(d_i-\mu_i)^s] + \frac{1}{n}\sum_i \mathbb{E}\left[t_i\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \tag{5}$$

Next, we find the leading term of the first two terms of (5). Note that

$$\begin{aligned}\frac{1}{n}\sum_i \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[t_i(d_i-\mu_i)] &= \frac{1}{n}\sum_{i\neq j\neq k} \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[A_{ij}\bar{A}_{ij}A_{jk}A_{ki}] \\ &\quad + \frac{1}{n}\sum_{i\neq j\neq k} \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[A_{ij}A_{jk}A_{ki}\bar{A}_{ik}] \\ &= \frac{2}{n}\sum_{i\neq j\neq k} \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mu_{ij}(1-\mu_{ij})\mu_{ik}\mu_{jk}. \end{aligned}\tag{6}$$

Then the second term of (5) is equal to the second term of the right-hand side of (2). Now we consider the first term of (5). Taking expectation on both sides of (4) yields

$$\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] = \frac{1}{\mu_i(\mu_i-1)} + \sum_{s=2}^{k_0-1}\frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[(d_i-\mu_i)^s] + \mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \tag{7}$$

Then

$$\frac{1}{n}\sum_i \mathbb{E}[t_i]\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] = \frac{1}{n}\sum_i \frac{\mathbb{E}[t_i]}{\mu_i(\mu_i-1)} + \sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \mathbb{E}[t_i]\frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[(d_i-\mu_i)^s] + \frac{1}{n}\sum_i \mathbb{E}[t_i]\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \tag{8}$$

Hence, by (5), (6) and (8), we have

$$\begin{aligned}\mathbb{E}[\overline{C}_n] &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}[t_i]\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right] - \frac{2}{n}\sum_{i\neq j\neq k} \frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mu_{ij}(1-\mu_{ij})\mu_{ik}\mu_{jk} \\ &\quad + \sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[t_i(d_i-\mu_i)^s] - \sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \mathbb{E}[t_i]\frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[(d_i-\mu_i)^s] \\ &\quad + \frac{1}{n}\sum_i \mathbb{E}\left[t_i\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right] - \frac{1}{n}\sum_i \mathbb{E}[t_i]\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \end{aligned}\tag{9}$$

Now we prove the last four terms of (9) are equal to $o\left(\frac{1}{n\sqrt{np_n}} + \frac{\sqrt{p_n}}{n}\right)$. Firstly, we study the last two terms of (9). Let $\delta_n = (\log(np_n))^{-2}$. Then

$$\mathbb{E}\left[\left|h^{(s)}(X_i)(d_i-\mu_i)^s\right|\right] = \mathbb{E}\left[\left|h^{(s)}(X_i)(d_i-\mu_i)^s\right| I[X_i \le \delta_n np_n]\right]$$

$$+\mathbb{E}\left[\left|h^{(s)}(X_i)(d_i-\mu_i)^s\right| I[X_i > \delta_n n p_n]\right]. \tag{10}$$

For $x > \delta_n n p_n$, it is easy to verify that

$$|h^{(k)}(x)| \quad \leq \quad k!\frac{\sum_{t=1}^{k+1}\binom{k+1}{t}x^{k+1-t}}{(x-1)^{k+1}x^{k+1}} = O\left(\frac{1}{\left(\delta_n n p_n\right)^{k+2}}\right). \tag{11}$$

Then by Lemma 3.2, we get

$$\begin{aligned}\mathbb{E}\left[\left|h^{(s)}(X_i)(d_i-\mu_i)^s\right| I[X_i > \delta_n n p_n]\right] &= O\left(\frac{\sqrt{\mathbb{E}\left[(d_i-\mu_i)^{2s}\right]}}{\left(\delta_n n p_n\right)^{s+2}}\right) \\ &= O\left(\frac{1}{\delta_n^{s+2}\left(n p_n\right)^{\frac{s}{2}+2}}\right).\end{aligned} \tag{12}$$

Note that X_i is between d_i and μ_i, that is, $d_i \leq X_i \leq \mu_i$ or $\mu_i \leq X_i \leq d_i$. Since $c_1 n p_n \leq \mu_i \leq c_2 n p_n$ for some constants $c_1, c_2 > 0$, and $\delta_n = (\log(n p_n))^{-2} = o(1)$, then $\delta_n n p_n < \mu_i$ for large n. If $X_i \leq \delta_n n p_n$, then it is not possible that $\mu_i \leq X_i \leq d_i$. Hence we have $d_i \leq X_i \leq \mu_i$. In this case, we get $d_i \leq \delta_n n p_n$. That is, $X_i \leq \delta_n n p_n$ implies $d_i \leq \delta_n n p_n$. In this case, $|h^{(k)}(x)| = O(1)$. By Lemma 3.1, one has

$$\begin{aligned}&\mathbb{E}\left[\left|h^{(s)}(X_i)(d_i-\mu_i)^s\right| I[X_i \leq \delta_n n p_n]\right] \\ \leq\; & O(1)\mathbb{E}\left[\left|(d_i-\mu_i)^s\right| I[d_i \leq \delta_n n p_n]\right] \\ \leq\; & O(n^s)) \sum_{t=2}^{\delta_n n p_n} \mathbb{P}(d_i = t) \\ =\; & e^{-n p_n \beta(1+o(1))}.\end{aligned} \tag{13}$$

Note that $k_0 \geq 4$. Hence

$$\left|\frac{1}{n}\sum_i \mathbb{E}[t_i]\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]\right| = O\left(\frac{p_n}{\delta_n^{k_0+2}\left(n p_n\right)^{\frac{k_0}{2}}}\right) = o\left(\frac{1}{n\sqrt{n p_n}} + \frac{\sqrt{p_n}}{n}\right). \tag{14}$$

Similarly, we have

$$\begin{aligned}\left|\frac{1}{n}\sum_i \mathbb{E}\left[t_i\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]\right| &\leq \frac{1}{n}\sum_i \frac{1}{k_0!}\sqrt{\mathbb{E}[t_i^2]\mathbb{E}\left[(h^{(k_0)}(X_i))^2(d_i-\mu_i)^{2k_0}\right]} \\ &= o\left(\frac{1}{n\sqrt{n p_n}} + \frac{\sqrt{p_n}}{n}\right).\end{aligned}$$

Hence we get

$$\begin{aligned}&\frac{1}{n}\sum_i \mathbb{E}\left[t_i\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right] - \frac{1}{n}\sum_i \mathbb{E}[t_i]\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right] \\ =\; & o\left(\frac{1}{n\sqrt{n p_n}} + \frac{\sqrt{p_n}}{n}\right).\end{aligned} \tag{15}$$

Now we study the 3rd term and the 4-th term of (9). If $s \geq 4$, then

$$\frac{1}{n}\sum_i \frac{|h^{(s)}(\mu_i)|}{s!}\mathbb{E}[|t_i(d_i-\mu_i)^s|] \quad \leq \quad \frac{1}{n}\sum_i \frac{|h^{(s)}(\mu_i)|}{s!}\sqrt{\mathbb{E}[t_i^2]\mathbb{E}[(d_i-\mu_i)^{2s}]}$$

$$
\begin{aligned}
&= O\left(\frac{p_n}{(np_n)^{\frac{s}{2}}}\right) \\
&= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \qquad (16)
\end{aligned}
$$

Similarly, for $s \geq 4$, one has

$$
\frac{1}{n}\sum_{i}\frac{|h^{(s)}(\mu_i)|}{s!}\mathbb{E}[t_i]\mathbb{E}[|(d_i-\mu_i)^s|] = o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \qquad (17)
$$

For $s = 2$, we have

$$
\begin{aligned}
\frac{1}{2n}\sum_{i}h^{(2)}(\mu_i)\mathbb{E}[t_i(d_i-\mu_i)^2] &= \frac{1}{2n}\sum_{i\neq j\neq k}h^{(2)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{ij}\bar{A}_{ik}] \\
&\quad +\frac{1}{2n}\sum_{i\neq j\neq k\neq s}h^{(2)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{is}^2] \\
&\quad +\frac{1}{2n}\sum_{i\neq j\neq k}h^{(2)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{ij}^2] \\
&\quad +\frac{1}{2n}\sum_{i\neq j\neq k}h^{(2)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{ik}^2] \\
&= O\left(\frac{1}{n(np_n)}\right)+\frac{1}{2n}\sum_{i\neq j\neq k\neq s}h^{(2)}(\mu_i)\mu_{ij}\mu_{ik}\mu_{jk}\mu_{is}(1-\mu_{is}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\frac{1}{2n}\sum_{i}h^{(2)}(\mu_i)\mathbb{E}[t_i]\mathbb{E}[(d_i-\mu_i)^2] \\
=\ &\frac{1}{2n}\sum_{i\neq j\neq k\neq s}h^{(2)}(\mu_i)\mu_{ij}\mu_{ik}\mu_{jk}\mu_{is}(1-\mu_{is})+\frac{1}{2n}\sum_{i\neq j\neq k}h^{(2)}(\mu_i)\mu_{ij}\mu_{ik}\mu_{jk}\mu_{ij}(1-\mu_{ij}) \\
&+\frac{1}{2n}\sum_{i\neq j\neq k}h^{(2)}(\mu_i)\mu_{ij}\mu_{ik}\mu_{jk}\mu_{ik}(1-\mu_{ik}) \\
=\ &\frac{1}{2n}\sum_{i\neq j\neq k\neq s}h^{(2)}(\mu_i)\mu_{ij}\mu_{ik}\mu_{jk}\mu_{is}(1-\mu_{is})+O\left(\frac{p_n}{n(np_n)}\right).
\end{aligned}
$$

Hence,

$$
\frac{1}{2n}\sum_{i}h^{(2)}(\mu_i)\mathbb{E}[t_i(d_i-\mu_i)^2]-\frac{1}{2n}\sum_{i}h^{(2)}(\mu_i)\mathbb{E}[t_i]\mathbb{E}[(d_i-\mu_i)^2] = O\left(\frac{1}{n(np_n)}\right). \qquad (18)
$$

Let $s = 3$. In this case, $|\mathbb{E}[(d_i-\mu_i)^3]| = O(np_n)$. Then

$$
\frac{1}{6n}\sum_{i}\left|h^{(3)}(\mu_i)\mathbb{E}[t_i]\mathbb{E}[(d_i-\mu_i)^3]\right| = O\left(\frac{1}{n(np_n)}\right).
$$

Moreover,

$$
\frac{1}{6n}\sum_{i}h^{(3)}(\mu_i)\mathbb{E}[t_i(d_i-\mu_i)^3] = \frac{1}{6n}\sum_{i\neq j\neq k, r\neq s\neq t}h^{(3)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{ir}\bar{A}_{is}\bar{A}_{it}]
$$

$$
\begin{aligned}
&+\frac{1}{6n}\sum_{i\neq j\neq k\neq s} h^{(3)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{is}^2\bar{A}_{ij}] \\
&+\frac{1}{6n}\sum_{i\neq j\neq k,s} h^{(3)}(\mu_i)\mathbb{E}[A_{ij}A_{ik}A_{jk}\bar{A}_{is}^3] \\
= \;& O\left(\frac{1}{n(np_n)}\right).
\end{aligned}
$$

Hence,

$$
\frac{1}{6n}\sum_i h^{(3)}(\mu_i)\mathbb{E}[t_i(d_i-\mu_i)^3] - \frac{1}{6n}\sum_i h^{(3)}(\mu_i)\mathbb{E}[t_i]\mathbb{E}[(d_i-\mu_i)^3] \;=\; O\left(\frac{1}{n(np_n)}\right). \tag{19}
$$

Combining (16), (17), (18) and (19) yields

$$
\begin{aligned}
&\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[t_i(d_i-\mu_i)^s] - \sum_{s=2}^{k_0-1}\frac{1}{n}\sum_i \mathbb{E}[t_i]\frac{h^{(s)}(\mu_i)}{s!}\mathbb{E}[(d_i-\mu_i)^s] \\
=\;& o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right)+O\left(\frac{1}{n(np_n)}\right) \\
=\;& o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right).
\end{aligned} \tag{20}
$$

By (9), (15) and (20), the proof is complete. □

Lemma 3.4. *Let* $a_i = \mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]$, $a_{ijk} = a_i + a_j + a_k$ *and*

$$
\sigma_{1n}^2 = \frac{4}{n^2}\sum_{i<j<k} a_{ijk}^2\mu_{ij}(1-\mu_{ij})\mu_{jk}(1-\mu_{jk})\mu_{ki}(1-\mu_{ki}).
$$

$$
\sigma_{2n}^2 = \frac{4}{n^2}\sum_{i<j} e_{ij}^2\mu_{ij}(1-\mu_{ij}).
$$

For any fixed constants λ_1, λ_2 *with* $\lambda_1^2+\lambda_2^2=1$, *we have*

$$
\frac{\lambda_1\frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} + \lambda_2\frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij}}{\sqrt{\lambda_1^2\sigma_{1n}^2+\lambda_2^2\sigma_{2n}^2}} \Rightarrow \mathcal{N}(0,1).
$$

To prove Lemma 3.4, we need the following proposition.

Proposition 3.5 ([14]). *Suppose that for every* $n\in\mathbb{N}$ *and* $k_n\to\infty$ *the random variables* $X_{n,1},\ldots,X_{n,k_n}$ *are a martingale difference sequence relative to an arbitrary filtration* $\mathcal{F}_{n,1}\subset\mathcal{F}_{n,2}\subset\ldots\subset\mathcal{F}_{n,k_n}$. *If (I)* $\sum_{i=1}^{k_n}\mathbb{E}(X_{n,i}^2|\mathcal{F}_{n,i-1})\to 1$ *in probability, (II)* $\sum_{i=1}^{k_n}\mathbb{E}(X_{n,i}^2 I[|X_{n,i}|>\epsilon]|\mathcal{F}_{n,i-1})\to 0$ *in probability for every* $\epsilon>0$, *then* $\sum_{i=1}^{k_n}X_{n,i}\to N(0,1)$ *in distribution.*

Proof of Lemma 3.4: We employ Proposition 3.5 to prove Lemma 3.4. Let $\sigma_n^2=\lambda_1^2\sigma_{1n}^2+\lambda_2^2\sigma_{2n}^2$,

$$
Y_t = \frac{\lambda_1\frac{2}{n}\sum_{1\leq i<j<k\leq t} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} + \lambda_2\frac{2}{n}\sum_{i<j\leq t} e_{ij}\bar{A}_{ij}}{\sigma_n},
$$

for $3 \leq t \leq n$, and $Y_2 = 0$. Then $\{Y_t\}_{t=2}^n$ is a martingale and

$$Y_n = \frac{\lambda_1 \frac{2}{n} \sum_{1\leq i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} + \lambda_2 \frac{2}{n} \sum_{i<j} e_{ij}\bar{A}_{ij}}{\sigma_n}.$$

Let $X_t = Y_t - Y_{t-1}$ and $F_t = \{A_{ij}, 1 \leq i < j \leq t\}$ for $t \geq 3$. It is easy to verify that

$$X_t = \frac{\lambda_1 \frac{2}{n} \sum_{1\leq i<j<t} a_{ijt}\bar{A}_{ij}\bar{A}_{jt}\bar{A}_{ti} + \lambda_2 \frac{2}{n} \sum_{i<t} e_{it}\bar{A}_{it}}{\sigma_n},$$

and $\mathbb{E}[X_t|F_{t-1}] = 0$. Hence, $\{X_t\}_{t=3}^n$ is a martingale difference.

Now we verify the two conditions in Proposition 3.5. Firstly, we prove condition (I) is satisfied. Note that A_{ij}, A_{kl} are independent if $\{i, j\} \neq \{k, l\}$. Then $\mathbb{E}\left[\bar{A}_{ti}\middle|F_{t-1}\right] = 0$ and

$$\mathbb{E}\left[\sum_{1\leq i<j<k=t} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\middle|F_{t-1}\right] = \sum_{1\leq i<j<k=t} a_{ijk}\bar{A}_{ij}\mathbb{E}\left[\bar{A}_{jt}\bar{A}_{ti}\middle|F_{t-1}\right] = 0, \tag{21}$$

$$\mathbb{E}\left[\sum_{i<t} e_{it}\bar{A}_{it}\middle|F_{t-1}\right] = \sum_{i<t} e_{it}\mathbb{E}\left[\bar{A}_{it}\middle|F_{t-1}\right] = 0. \tag{22}$$

By (21) and (22), it is easy to verify that

$$\begin{aligned}
&\mathbb{E}\Big[Y_t|F_{t-1}\Big] \\
=\ & \frac{\lambda_1 \frac{2}{n}\mathbb{E}\left[\sum_{1\leq i<j<k\leq t} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\middle|F_{t-1}\right] + \lambda_2 \frac{2}{n}\mathbb{E}\left[\sum_{i<j\leq t} e_{ij}\bar{A}_{ij}\middle|F_{t-1}\right]}{\sigma_n} \\
=\ & \frac{\lambda_1 \frac{2}{n}\mathbb{E}\left[\sum_{1\leq i<j<k\leq t-1} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\middle|F_{t-1}\right] + \lambda_1 \frac{2}{n}\mathbb{E}\left[\sum_{1\leq i<j<k=t} a_{ijk}\bar{A}_{ij}\bar{A}_{jt}\bar{A}_{ti}\middle|F_{t-1}\right]}{\sigma_n} \\
&+\frac{\lambda_2 \frac{2}{n}\mathbb{E}\left[\sum_{i<j\leq t-1} e_{ij}\bar{A}_{ij}\middle|F_{t-1}\right] + \lambda_2 \frac{2}{n}\sum_{i<j=t} e_{ij}\mathbb{E}\left[\bar{A}_{it}\middle|F_{t-1}\right]}{\sigma_n} \\
=\ & \frac{\lambda_1 \frac{2}{n}\sum_{1\leq i<j<k\leq t-1} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} + \lambda_2 \frac{2}{n}\sum_{i<j\leq t-1} e_{ij}\bar{A}_{ij}}{\sigma_n} \\
=\ & Y_{t-1}.
\end{aligned} \tag{23}$$

By the property of conditional expectation, one has

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=3}^n \mathbb{E}\Big[X_t^2|F_{t-1}\Big]\right] &= \mathbb{E}\left[\sum_{t=3}^n \mathbb{E}\Big[(Y_t^2 - 2Y_tY_{t-1} + Y_{t-1}^2)\Big|F_{t-1}\Big]\right] \\
&= \mathbb{E}\left[\sum_{t=3}^n \mathbb{E}\Big[(Y_t^2 - Y_{t-1}^2)|F_{t-1}\Big]\right] = \mathbb{E}[Y_n^2].
\end{aligned} \tag{24}$$

Now we show $\mathbb{E}[Y_n^2] = 1$. Straightforward calculation yields

$$\begin{aligned}
Y_n^2 &= \frac{\lambda_1^2 \frac{4}{n^2}\left(\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\right)^2 + \lambda_2^2 \frac{4}{n^2}\left(\sum_{i<j} e_{ij}\bar{A}_{ij}\right)^2}{\sigma_n^2} \\
&+\frac{\lambda_1\lambda_2 \frac{8}{n^2}\sum_{i<j<k, i_1<j_1} a_{ijk}e_{i_1j_1}\bar{A}_{i_1j_1}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\sigma_n^2}.
\end{aligned} \tag{25}$$

Since $i < j < k, i_1 < j_1$, then there exists $(i_2, j_2) \in \{(i,j),(j,k),(k,i)\}$ such that $\{i_2, j_2\} \neq \{i_1, j_1\}$. Then

$$\mathbb{E}\left[\bar{A}_{i_1j_1}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\right] = 0. \tag{26}$$

Moreover,

$$\mathbb{E}\left[\left(\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\right)^2\right] = \sum_{\substack{i<j<k\\ i_1<j_1<k_1}} a_{ijk}a_{i_1j_1k_1}\mathbb{E}\left[\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\bar{A}_{i_1j_1}\bar{A}_{j_1k_1}\bar{A}_{k_1i_1}\right]. \tag{27}$$

If $(i,j) \notin \{(i_1,j_1),(j_1,k_1),(k_1,i_1)\}$, then

$$\mathbb{E}\left[\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\bar{A}_{i_1j_1}\bar{A}_{j_1k_1}\bar{A}_{k_1i_1}\right] = \mathbb{E}\left[\bar{A}_{ij}\right]\mathbb{E}\left[\bar{A}_{jk}\bar{A}_{ki}\bar{A}_{i_1j_1}\bar{A}_{j_1k_1}\bar{A}_{k_1i_1}\right] = 0.$$

Hence $(i,j) \in \{(i_1,j_1),(j_1,k_1),(k_1,i_1)\}$. Similarly, we can get $(j,k) \in \{(i_1,j_1),(j_1,k_1),(k_1,i_1)\}$ and $(k,i) \in \{(i_1,j_1),(j_1,k_1),(k_1,i_1)\}$. Since $i < j < k$ and $i_1 < j_1 < k_1$, then $\{(i,j),(j,k),(k,i)\} = \{(i_1,j_1),(j_1,k_1),(k_1,i_1)\}$ and $i = i_1, j = j_1, k = k_1$. Then by (27) we have

$$\mathbb{E}\left[\left(\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\right)^2\right] = \sum_{i<j<k} a_{ijk}^2\mathbb{E}\left[\bar{A}_{ij}^2\bar{A}_{jk}^2\bar{A}_{ki}^2\right] = \frac{n^2}{4}\sigma_{1n}^2. \tag{28}$$

Similarly one get

$$\mathbb{E}\left[\left(\sum_{i<j} e_{ij}\bar{A}_{ij}\right)^2\right] = \frac{n^2}{4}\sigma_{2n}^2. \tag{29}$$

Recall that $\sigma_n^2 = \lambda_1^2\sigma_{1n}^2 + \lambda_2^2\sigma_{2n}^2$. Combining (24),(25),(26), (28) and (29) yields

$$\mathbb{E}\left[\sum_{t=3}^{n}\mathbb{E}\left[X_t^2|F_{t-1}\right]\right] = \mathbb{E}[Y_n^2] = 1. \tag{30}$$

To prove condition (I) holds, we only need to show that

$$\mathbb{E}\Big(\sum_{t=3}^{n}\mathbb{E}[X_t^2|F_{t-1}]\Big)^2 = 1 + o(1).$$

For convenience, let $\sigma_{ij}^2 = \mu_{ij}(1-\mu_{ij})$. Given $t \in \{3,4,\dots,n\}$, one has

$$\begin{aligned}
\mathbb{E}[X_t^2|F_{t-1}] &= \frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{\substack{1\le i_1<j_1<t\\ 1\le i_2<j_2<t}} a_{i_1j_1t}a_{i_2j_2t}\mathbb{E}\Big[\bar{A}_{i_1j_1}\bar{A}_{j_1t}\bar{A}_{ti_1}\bar{A}_{i_2j_2}\bar{A}_{j_2t}\bar{A}_{ti_2}|F_{t-1}\Big]\\
&\quad+\frac{4\lambda_2^2}{n^2\sigma_n^2}\sum_{\substack{1\le i_1<t\\ 1\le i_2<t}} e_{i_1t}e_{i_2t}\mathbb{E}\Big[\bar{A}_{i_1t}\bar{A}_{i_2t}|F_{t-1}\Big]\\
&\quad+\frac{8\lambda_1\lambda_2}{n^2\sigma_n^2}\sum_{\substack{1\le i_1<j_1<t\\ 1\le i_2<t}} a_{i_1j_1t}e_{i_2t}\mathbb{E}\Big[\bar{A}_{i_1j_1}\bar{A}_{j_1t}\bar{A}_{ti_1}\bar{A}_{i_2t}|F_{t-1}\Big]\\
&= \frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{1\le i<j<t} a_{ijt}^2\bar{A}_{ij}^2\sigma_{jt}^2\sigma_{it}^2 + \frac{4\lambda_2^2}{n^2\sigma_n^2}\sum_{1\le i<t} e_{it}^2\sigma_{it}^2
\end{aligned}$$

Then we have

$$\sum_{t=3}^{n}\mathbb{E}[X_t^2|F_{t-1}] = \frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{1\leq i<j<t\leq n}a_{ijt}^2\bar{A}_{ij}^2\sigma_{jt}^2\sigma_{it}^2+\frac{\lambda_2^2\sigma_{2n}^2}{\sigma_n^2}.$$

Note that A_{ij} $(1\leq i<j\leq n)$ are independent,

$$\mathbb{E}\Big[\bar{A}_{ij}^2(A_{kl}-\mu_{kl})^2\Big]=O(p_n), \qquad if\ \{i,j\}=\{k,l\},$$

and

$$\mathbb{E}\Big[\bar{A}_{ij}^2(A_{kl}-\mu_{kl})^2\Big]=\sigma_{ij}^2\sigma_{kl}^2, \qquad if\ |\{i,j\}\cap\{k,l\}|\leq 1.$$

Then we have

$$\begin{aligned}
&\mathbb{E}\left[\Big(\frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{1\leq i<j<t}a_{ijt}^2\bar{A}_{ij}^2\sigma_{jt}^2\sigma_{it}^2\Big)^2\right]\\
=\ &\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n}}a_{ijt}^2a_{kls}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{ks}^2\sigma_{ls}^2\mathbb{E}[\bar{A}_{ij}^2\bar{A}_{kl}^2] \qquad (31)\\
=\ &\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\|\{i,j\}\cap\{k,l\}|\leq 1}}a_{ijt}^2a_{kls}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{ks}^2\sigma_{ls}^2\mathbb{E}[\bar{A}_{ij}^2\bar{A}_{kl}^2]+\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\\{i,j\}=\{k,l\}}}a_{ijt}^2a_{kls}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{ks}^2\sigma_{ls}^2\mathbb{E}[\bar{A}_{ij}^4]\\
=\ &\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\|\{i,j\}\cap\{k,l\}|\leq 1}}a_{ijt}^2a_{klt}^2\sigma_{ij}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{kl}^2\sigma_{ks}^2\sigma_{ls}^2+\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\\{i,j\}=\{k,l\}}}a_{ijt}^2a_{klt}^2\mathbb{E}[\bar{A}_{ij}^4]\sigma_{jt}^2\sigma_{it}^2\sigma_{ks}^2\sigma_{ls}^2\\
=\ &\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\|\{i,j\}\cap\{k,l\}|\leq 1}}a_{ijt}^2a_{klt}^2\sigma_{ij}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{kl}^2\sigma_{ks}^2\sigma_{ls}^2+\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\\{i,j\}=\{k,l\}}}a_{ijt}^2a_{klt}^2\sigma_{ij}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{kl}^2\sigma_{ks}^2\sigma_{ls}^2\\
&+\frac{16\lambda_1^4}{n^4\sigma_n^4}\sum_{\substack{1\leq i<j<t\leq n\\1\leq k<l<s\leq n\\\{i,j\}=\{k,l\}}}a_{ijt}^2a_{klt}^2\Big(\mathbb{E}[\bar{A}_{ij}^4]\sigma_{jt}^2\sigma_{it}^2\sigma_{ks}^2\sigma_{ls}^2-\sigma_{ij}^2\sigma_{jt}^2\sigma_{it}^2\sigma_{kl}^2\sigma_{ks}^2\sigma_{ls}^2\Big)\\
=\ &\frac{\lambda_1^4\sigma_{1n}^4}{\sigma_n^4}+O\Big(\frac{1}{(np_n)^4}\Big). \qquad (32)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\Big(\sum_{t=1}^{n}\mathbb{E}[X_{n,t}^2|F_{t-1}]\Big)^2 &= \mathbb{E}\left[\Big(\frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{1\leq i<j<t}a_{ijt}^2\bar{A}_{ij}^2\sigma_{jt}^2\sigma_{it}^2\Big)^2\right]+\left(\frac{\lambda_2^2\sigma_{2n}^2}{\sigma_n^2}\right)^2\\
&\quad+2\left(\frac{\lambda_2^2\sigma_{2n}^2}{\sigma_n^2}\right)\frac{4\lambda_1^2}{n^2\sigma_n^2}\sum_{1\leq i<j<t\leq n}a_{ijt}^2\mathbb{E}[\bar{A}_{ij}^2]\sigma_{jt}^2\sigma_{it}^2\\
&= \frac{\lambda_1^4\sigma_{1n}^4}{\sigma_n^4}+\left(\frac{\lambda_2^2\sigma_{2n}^2}{\sigma_n^2}\right)^2+2\left(\frac{\lambda_1^2\sigma_{1n}^2}{\sigma_n^2}\right)\left(\frac{\lambda_2^2\sigma_{2n}^2}{\sigma_n^2}\right)+o(1)\\
&= 1+o(1).
\end{aligned}$$

Now we check condition (*II*) in Proposition 3.5. Let ϵ be a fixed positive constant. By the Cauchy-Schwarz inequality and Markov's inequality, we have

$$\begin{aligned}
& \mathbb{E}\Big[\sum_{t=3}^{n}\mathbb{E}\Big[X_t^2 I[|X_t|>\epsilon|F_{t-1}\Big]\Big] \\
\leq\ & \mathbb{E}\Big[\sum_{t=3}^{n}\sqrt{\mathbb{E}\Big[X_t^4|F_{t-1}\Big]\mathbb{P}[|X_t|>\epsilon|F_{t-1}\Big]}\Big] \\
\leq\ & \mathbb{E}\Bigg[\frac{4}{\epsilon^2 n^4\sigma_n^4}\sum_{t=1}^{n}\mathbb{E}\Big[\Big(\lambda_1\sum_{1\leq i<j<t}a_{ijt}\bar{A}_{ij}\bar{A}_{jt}\bar{A}_{ti}+\lambda_2\sum_{i<t}e_{it}\bar{A}_{it}\Big)^4\Big|F_{t-1}\Big]\Bigg] \\
\leq\ & \frac{32}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{\substack{1\leq i_1<j_1<t\\1\leq i_2<j_2<t\\1\leq i_3<j_3<t\\1\leq i_4<j_4<t}}\lambda_1^4 a_{i_1j_1t}a_{i_2j_2t}a_{i_3j_3t}a_{i_4j_4t}\mathbb{E}\Big[\bar{A}_{i_1j_1}\bar{A}_{j_1t}\bar{A}_{ti_1}\bar{A}_{i_2j_2}\bar{A}_{j_2t}\bar{A}_{ti_2} \\
& \times\bar{A}_{i_3j_3}\bar{A}_{j_3t}\bar{A}_{ti_3}\bar{A}_{i_4j_4}\bar{A}_{j_4t}\bar{A}_{ti_4}\Big] \qquad (33)\\
& +\frac{32}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{\substack{1\leq i_1<t\\1\leq i_2<t\\1\leq i_3<t\\1\leq i_4<t}}\lambda_2^4 e_{i_1t}e_{i_2t}e_{i_3t}e_{i_4t}\mathbb{E}\Big[\bar{A}_{i_1t}\bar{A}_{i_2t}\bar{A}_{i_3t}\bar{A}_{i_4t}\Big] \\
\leq\ & \frac{32C}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{\substack{1\leq i_1<j_1<t\\1\leq i_2<j_2<t}}a_{i_1j_1t}^2 a_{i_2j_2t}^2\mathbb{E}\Big[\bar{A}_{i_1j_1}^2\bar{A}_{j_1t}^2\bar{A}_{i_1t}^2\bar{A}_{i_2j_2}^2\bar{A}_{j_2t}^2\bar{A}_{ti_2}^2\Big] \qquad (34)\\
& +\frac{32C}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{1\leq i_1<j_1<t}a_{i_1j_1t}^4\mathbb{E}\Big[\bar{A}_{i_1j_1}^4\bar{A}_{j_1t}^4\bar{A}_{i_1t}^4\Big] \\
& +\frac{32}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{\substack{1\leq i_1<t\\1\leq i_2<t}}\lambda_2^4 e_{i_1t}^2 e_{i_2t}^2\mathbb{E}\Big[\bar{A}_{i_1t}^2\bar{A}_{i_2t}^2\Big]+\frac{32}{\epsilon^2 n^4\sigma_n^4}\sum_{t=3}^{n}\sum_{1\leq i_1<t}\lambda_2^4 e_{i_1t}^4\mathbb{E}\Big[\bar{A}_{i_1t}^4\Big] \\
=\ & O\Big(\frac{n^5p_n^6}{\epsilon^2(np_n)^8n^4\sigma_n^4}\Big)+O\Big(\frac{n^3p_n^3}{\epsilon^2(np_n)^8n^4\sigma_n^4}\Big)+O\Big(\frac{p_n^2}{\epsilon^2(np_n)^8n^5\sigma_n^4}\Big) \\
=\ & o(1).
\end{aligned}$$

Then the desired result follows from Proposition 3.5.
□

3.1. *Proof of Theorem 2.2*

By Lemma 3.3, straightforward calculation yields

$$\overline{C}_n-\mathbb{E}[\overline{C}_n]=Y_{1n}+Y_{2n}+Y_{3n}+o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right), \qquad (35)$$

where

$$\begin{aligned}
Y_{1n} &= \frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right], \qquad (36)\\
Y_{2n} &= \frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\left(\frac{1}{d_i(d_i-1)}-\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\right)
\end{aligned}$$

$$+\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[\bar{A}_{ij}^2]\mu_{ik}\mu_{jk}, \quad (37)$$

$$Y_{3n} = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[t_i]\left(\frac{1}{d_i(d_i-1)}-\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\right). \quad (38)$$

Next we find the order of Y_{1n}, Y_{2n} and Y_{3n}.

(a) Order of Y_{3n}. Firstly, we find the order of Y_{3n}. By (4) and (7), one has

$$\begin{aligned}&\frac{1}{d_i(d_i-1)}-\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\\ =&-\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)+\sum_{s=2}^{k_0-1}\frac{h^{(s)}(\mu_i)}{s!}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big]\\ &+\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}-\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \end{aligned} \quad (39)$$

By (38) and (39), we can express Y_{3n} as follows.

$$\begin{aligned}Y_{3n} =& -\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)+\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_{i=1}^{n}\frac{h^{(s)}(\mu_i)\mathbb{E}[t_i]}{s!}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big]\\ &+\frac{1}{n}\sum_{i=1}^{n}\frac{h^{(k_0)}(X_i)\mathbb{E}[t_i]}{k_0!}(d_i-\mu_i)^{k_0}-\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[t_i]\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]. \end{aligned} \quad (40)$$

Now we find the order of each term in (40) of Y_{3n}. To this end, we will find the first order absolute moment or the second moment of each term and then use Markov's inequality to get an upper bound.

We firstly find the order of the first term in (40). Recall that $A_{ij}(1\leq i<j\leq n)$ are independent and $\mathbb{E}[A_{ij}]=\mu_{ij}$. Hence, for indices $i,j(i\neq j)$ and $k,l(k\neq l)$, we have

$$\mathbb{E}[\bar{A}_{ij}\bar{A}_{kl}]=\begin{cases}\mu_{ij}(1-\mu_{ij}) & \text{if } \{i,j\}=\{k,l\},\\ 0 & \text{if } \{i,j\}\neq\{k,l\}.\end{cases} \quad (41)$$

The first term in (40) is equal to

$$-\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)=-\frac{1}{n}\sum_{i\neq j}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}. \quad (42)$$

By (41) and (42), we have

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\right)^2\right]\\ =&\frac{1}{n^2}\sum_{i\neq j,s\neq t}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}\frac{\mathbb{E}[t_s](2\mu_s-1)}{\mu_s^2(\mu_s-1)^2}\mathbb{E}\left[\bar{A}_{ij}\bar{A}_{st}\right]\\ =&\frac{1}{n^2}\sum_{i\neq j}^{n}\frac{(\mathbb{E}[t_i](2\mu_i-1))^2}{\mu_i^4(\mu_i-1)^4}\mathbb{E}\left[\bar{A}_{ij}^2\right]\\ &+\frac{1}{n^2}\sum_{i\neq j}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}\frac{\mathbb{E}[t_j](2\mu_j-1)}{\mu_j^2(\mu_j-1)^2}\mathbb{E}\left[\bar{A}_{ij}^2\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i\neq j}^{n}\frac{(\mathbb{E}[t_i](2\mu_i-1))^2}{\mu_i^4(\mu_i-1)^4}\mu_{ij}(1-\mu_{ij}) + \frac{1}{n^2}\sum_{i\neq j}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}\frac{\mathbb{E}[t_j](2\mu_j-1)}{\mu_j^2(\mu_j-1)^2}\mu_{ij}(1-\mu_{ij}) \\
&= \Theta\left(\frac{p_n}{n^2}\right). \qquad (43)
\end{aligned}
$$

Here the last equality follows from the facts that $\beta p_n \leq \mu_{ij} \leq p_n$, $\beta np_n \leq \mu_i \leq np_n$ and $\mathbb{E}[t_i] = \Theta(n^2p_n^3)$. By (43) and Markov's inequality, the first term of Y_{3n} is of order $\frac{\sqrt{p_n}}{n}$. That is

$$
-\frac{1}{n}\sum_{i=1}^{n}\frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i) = O_P\left(\frac{\sqrt{p_n}}{n}\right). \qquad (44)
$$

Now we show the last two terms of Y_{3n} are $o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right)$. Recall that $k_0 = \lceil 1+\frac{1}{1-\alpha}\rceil + 2$ in the proof of Lemma 3.3. Then $k_0 > \max\{1+\frac{1}{1-\alpha}, 3\}$. By (10), (12) and (13), we get

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[t_i]\left(\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0} - \mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]\right)\right| \\
&= O_P\left(\frac{n^2p_n^3}{\delta_n^{k_0+2}\left(np_n\right)^{\frac{k_0}{2}+2}} + n^2p_n^3e^{-np_n\beta(1+o(1))}\right) \\
&= o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \qquad (45)
\end{aligned}
$$

Hence the last two terms of Y_{3n} are $o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right)$.

Now we show the second term of Y_{3n} is $o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right)$. In the expression of $(d_i-\mu_i)^s$ in (1), the sum over $\lambda_{t1},\ldots,\lambda_{tt}$ can be decomposed as two cases: (a) there exist distinct $r_1,r_2,\ldots,r_{t_1}\in\{1,2,3,\ldots,t\}$ such that $\lambda_{tr_1}=\lambda_{tr_2}=\cdots=\lambda_{tr_{t_1}}=1$ for $l\in\{1,2,\ldots,t_1\}$ with $t_1\geq 1$ and $\lambda_{tr}\geq 2$ for $r\notin\{r_1,r_2,\ldots,r_{t_1}\}$, (b) $\lambda_{tl}\geq 2$ for all $l=1,2,\ldots,t$. Then

$$
\begin{aligned}
(d_i-\mu_i)^s &= \sum_{t=1}^{s}\sum_{\lambda_{t1},\ldots,\lambda_{tt}}\sum_{j_1\neq\cdots\neq j_t\neq i}\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}} \\
&= \sum_{t=1}^{s}\sum_{(a)}\sum_{j_1\neq\cdots\neq j_t\neq i}\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}} \\
&\quad+\sum_{t=1}^{s}\sum_{\lambda_{t1}\geq 2,\ldots,\lambda_{tt}\geq 2}\sum_{j_1\neq\cdots\neq j_t\neq i}\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}}. \qquad (46)
\end{aligned}
$$

where $\sum_{(a)}$ represents sum over $\lambda_{t1},\ldots,\lambda_{tt}$ in case (a). In this case, we get

$$
\begin{aligned}
&(d_i-\mu_i)^s - \mathbb{E}\left[(d_i-\mu_i)^s\right] \\
&= \sum_{t=1}^{s}\sum_{(a)}\sum_{j_1\neq\cdots\neq j_t\neq i}\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}} \\
&\quad+\sum_{t=1}^{s}\sum_{\lambda_{t1}\geq 2,\ldots,\lambda_{tt}\geq 2}\sum_{j_1\neq\cdots\neq j_t\neq i}\left(\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}} - \mathbb{E}\left[\bar{A}_{ij_1}^{\lambda_{t1}}\bar{A}_{ij_2}^{\lambda_{t2}}\ldots\bar{A}_{ij_t}^{\lambda_{tt}}\right]\right). \qquad (47)
\end{aligned}
$$

For case (a), without loss of generality, we assume $\lambda_{t1} = \cdots = \lambda_{tt_1} = 1$ and $\lambda_{t,t_1+1} \geq 2, \ldots, \lambda_{tt} \geq 2$ for $t_1 \geq 1$. If $s = t$, then $t_1 = s \geq 2$. If $s > t$, then $2(s-t) \geq 2$. Hence, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq j_2\neq\ldots,j_t\neq i}\mathbb{E}[t_i]h^{(s)}(\mu_i)\prod_{l=1}^{t_1}\bar{A}_{ij_l}\prod_{l=t_1+1}^{t}\bar{A}_{ij_l}^{\lambda_{tl}}\right)^2\right]\\
\leq\ & \frac{1}{n^2}\sum_{\substack{j_1\neq j_2\neq\ldots,j_t\neq i\\ k_{t_1+1}\neq\cdots\neq k_s\neq i}}\left(\mathbb{E}[t_i]h^{(s)}(\mu_i)\right)^2\mathbb{E}\left[\prod_{l=1}^{t_1}\bar{A}_{ij_l}^2\prod_{l=t_1+1}^{t}\bar{A}_{ij_l}^{\lambda_{tl}}\prod_{l=t_1+1}^{s}\bar{A}_{ik_l}^{\lambda_{tl}}\right]\\
=\ & O\left(\frac{n^{2t-t_1+1}n^4p_n^6p_n^{2t-t_1}}{n^2(np_n)^{2s+4}}\right)\\
=\ & O\left(\frac{p_n^3}{(np_n)^{2(s-t)+t_1+1}}\right)\\
=\ & o\left(\frac{p_n}{n^2}+\frac{1}{n^2(np_n)}\right). \qquad (48)
\end{aligned}$$

By (48) and Markov's inequality, one has

$$\frac{1}{n}\sum_{j_1\neq j_2\neq\ldots,j_t\neq i}\mathbb{E}[t_i]h^{(s)}(\mu_i)\prod_{l=1}^{t_1}\bar{A}_{ij_l}\prod_{l=t_1+1}^{t}\bar{A}_{ij_l}^{\lambda_{tl}} = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \qquad (49)$$

Now suppose $\lambda_{tl} \geq 2$ for all $l = 1, 2, \ldots, t$ in (46). In this case, $t \leq \frac{s}{2}$. Then $s - t \geq 1$. Note that

$$\prod_{l=1}^{t}\bar{A}_{ij_l}^{\lambda_{tl}} - \prod_{l=1}^{t}\mathbb{E}[\bar{A}_{ij_l}^{\lambda_{tl}}] = \sum_{t_1=1}^{t}C_{t_1}\prod_{l=1}^{t_1}\left(\bar{A}_{ij_l}^{\lambda_{tl}} - \mathbb{E}[\bar{A}_{ij_l}^{\lambda_{tl}}]\right)\prod_{l=t_1+1}^{t}\mathbb{E}[\bar{A}_{ij_l}^{\lambda_{tl}}], \qquad (50)$$

for some constants C_{t_1}. Fix t_1 ($1 \leq t_1 \leq t$). Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq j_2\neq\ldots,j_t\neq i}\mathbb{E}[t_i]h^{(s)}(\mu_i)\prod_{l=1}^{t_1}\left(\bar{A}_{ij_l}^{\lambda_l} - \mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}]\right)\prod_{l=t_1+1}^{t}\mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}]\right)^2\right]\\
\leq\ & \frac{1}{n^2}\sum_{\substack{j_1\neq j_2\neq\ldots,j_t\neq i\\ k_{t_1+1},k_t}}\left(\mathbb{E}[t_i]h^{(s)}(\mu_i)\right)^2\prod_{l=1}^{t_1}\left(\bar{A}_{ij_l}^{\lambda_l} - \mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}]\right)^2\prod_{l=t_1+1}^{t}\mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}]\\
&\times\prod_{l=t_1+1}^{t}\mathbb{E}[\bar{A}_{ik_l}^{\lambda_l}]\\
=\ & O\left(\frac{n^{2t-t_1+1}n^4p_n^6p_n^{2t-t_1}}{n^2(np_n)^{2s+4}}\right)\\
=\ & O\left(\frac{p_n^3}{(np_n)^{2(s-t)+t_1+1}}\right)\\
=\ & o\left(\frac{p_n}{n^2}+\frac{1}{n^2(np_n)}\right). \qquad (51)
\end{aligned}$$

By (51) and Markov's inequality, we get

$$\frac{1}{n}\sum_{j_1\neq j_2\neq\ldots,j_t\neq i}\mathbb{E}[t_i]h^{(s)}(\mu_i)\prod_{l=1}^{t_1}\left(\bar{A}_{ij_l}^{\lambda_l} - \mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}]\right)\prod_{l=t_1+1}^{t}\mathbb{E}[\bar{A}_{ij_l}^{\lambda_l}] = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \qquad (52)$$

Since k_0 is a fixed constant, then by (47), (49) and (52), the second term of Y_{3n} is $o_P\left(\frac{\sqrt{p_n}}{n} + \frac{1}{n\sqrt{np_n}}\right)$. That is,

$$\sum_{s=2}^{k_0-1} \frac{1}{n} \sum_{i=1}^{n} \frac{h^{(s)}(\mu_i)\mathbb{E}[t_i]}{s!}\Big[(d_i-\mu_i)^s - \mathbb{E}[(d_i-\mu_i)^s]\Big] = o_P\left(\frac{\sqrt{p_n}}{n} + \frac{1}{n\sqrt{np_n}}\right). \tag{53}$$

Combining (40), (44), (45) and (53), we get

$$Y_{3n} = -\frac{1}{n}\sum_{i=1}^{n} \frac{\mathbb{E}[t_i](2\mu_i-1)}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i) + o_P\left(\frac{\sqrt{p_n}}{n} + \frac{1}{n\sqrt{np_n}}\right). \tag{54}$$

(b) Order of Y_{1n}.Straightforward calculation yields

$$\begin{aligned} A_{ij}A_{jk}A_{ki} - \mu_{ij}\mu_{jk}\mu_{ki} &= \bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} + \bar{A}_{ij}\bar{A}_{ik}\mu_{jk} + \bar{A}_{ij}\bar{A}_{jk}\mu_{ik} + \bar{A}_{ik}\bar{A}_{jk}\mu_{ij} \\ &\quad + \bar{A}_{jk}\mu_{ij}\mu_{ik} + \bar{A}_{ij}\mu_{jk}\mu_{ik} + \bar{A}_{ik}\mu_{jk}\mu_{ij}. \end{aligned} \tag{55}$$

Then Y_{1n} can be expressed as follows.

$$\begin{aligned} Y_{1n} &= \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\bar{A}_{ik}\mu_{jk} \\ &\quad + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\bar{A}_{jk}\mu_{ik} + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ik}\bar{A}_{jk}\mu_{ij} \\ &\quad + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{jk}\mu_{ij}\mu_{ik} + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\mu_{jk}\mu_{ik} \\ &\quad + \frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ik}\mu_{jk}\mu_{ij} \end{aligned} \tag{56}$$

We find the order of each term of Y_{1n} in (56). To this end, we will find the second moment of each term and then use Markov's inequality to get an upper bound of each term. Recall that we denote $a_i = \mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]$. The second moment of the first term of (56) is equal to

$$\begin{aligned} &\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k} a_i\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\right)^2\right] \\ &= \frac{1}{n^2}\sum_{i\neq j\neq k}(a_i^2 + a_ia_j + a_ia_k)\mathbb{E}[\bar{A}_{ij}^2]\mathbb{E}[\bar{A}_{ik}^2]\mathbb{E}[\bar{A}_{jk}^2] \\ &= \frac{1}{n^2}\sum_{i\neq j\neq k}(a_i^2 + a_ia_j + a_ia_k)\mu_{ij}(1-\mu_{ij})\mu_{ik}(1-\mu_{ik})\mu_{jk}(1-\mu_{jk}) \\ &= \Theta\left(\frac{1}{n^2(np_n)}\right). \end{aligned} \tag{57}$$

By (57) and Markov's inequality, one gets

$$\frac{1}{n}\sum_{i\neq j\neq k} a_i\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} = O_P\left(\frac{1}{n\sqrt{np_n}}\right). \tag{58}$$

Similarly, the order of the second moment of the second term of (56) is equal to

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\right)^2\right] = \Theta\left(\frac{p_n}{n^2(np_n)}\right). \tag{59}$$

By (59) and Markov's inequality, one gets

$$\frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}=O_P\left(\frac{\sqrt{p_n}}{n\sqrt{np_n}}\right). \tag{60}$$

The order of the 3rd term and the 4th term in (56) are the same as (60).

The order of the second moment of the 5th term of (56) is

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)^2\right]=\Theta\left(\frac{p_n}{n^2}\right). \tag{61}$$

By (61) and Markov's inequality, one gets

$$\frac{1}{n}\sum_{i\neq j\neq k}\mathbb{E}\left[\frac{1}{d_i(d_i-1)}\right]\bar{A}_{jk}\mu_{ij}\mu_{ik}=O_P\left(\frac{\sqrt{p_n}}{n}\right). \tag{62}$$

The order of the last two terms in (56) are the same as (62).

Recall that $p_n=n^{-\alpha}$ and $\alpha\in(0,1)$. Note that

$$\frac{1}{n^2(np_n)}\frac{n^2}{p_n}=\frac{1}{n^{1-2\alpha}}.$$

For $\alpha<\frac{1}{2}$, $\frac{1}{n^2(np_n)}=o\left(\frac{p_n}{n^2}\right)$. For $\alpha>\frac{1}{2}$, $\frac{p_n}{n^2}=o\left(\frac{1}{n^2(np_n)}\right)$. For $\alpha=\frac{1}{2}$, $\frac{1}{n^2(np_n)}=\frac{p_n}{n^2}$. The leading terms of Y_{1n} can be expressed as follows.

If $\alpha>\frac{1}{2}$, then by (58), (60) and (62), we have

$$Y_{1n} = \frac{1}{n}\sum_{i\neq j\neq k}a_i\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}+o_P\left(\frac{1}{n\sqrt{np_n}}\right). \tag{63}$$

If $\alpha<\frac{1}{2}$, then by (58), (60) and (62), we have

$$Y_{1n}=\frac{1}{n}\sum_{i\neq j\neq k}a_i\left(\bar{A}_{jk}\mu_{ij}\mu_{ik}+\bar{A}_{ij}\mu_{jk}\mu_{ik}+\bar{A}_{ik}\mu_{jk}\mu_{ij}\right)+o_P\left(\frac{\sqrt{p_n}}{n}\right). \tag{64}$$

If $\alpha=\frac{1}{2}$, then by (58), (60) and (62), we have

$$Y_{1n} = \frac{1}{n}\sum_{i\neq j\neq k}a_i\left(\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}+\bar{A}_{jk}\mu_{ij}\mu_{ik}+\bar{A}_{ij}\mu_{jk}\mu_{ik}+\bar{A}_{ik}\mu_{jk}\mu_{ij}\right)+o_P\left(\frac{\sqrt{p_n}}{n}\right). \tag{65}$$

(c) Order of Y_{2n}. Now we show that

$$Y_{2n}=o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \tag{66}$$

By (37) and (39), one has

$$\begin{aligned}Y_{2n} &= -\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)+\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[\bar{A}_{ij}^2]\mu_{ik}\mu_{jk}\\ &\quad+\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}-\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]\end{aligned}$$

$$+\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\sum_{s=2}^{k_0-1}\frac{h^{(s)}(\mu_i)}{s!}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big]. \tag{67}$$

Next we bound each term in (67).

Firstly, we consider the 3rd term and the 4-th term in (67). Note that

$$\begin{aligned}
& \mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|\right] \\
= & \mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[X_i\le\delta_n np_n]\right] \\
& +\mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[X_i>\delta_n np_n]\right].
\end{aligned} \tag{68}$$

By (55), it is easy to verify that

$$\mathbb{E}[(t_i-\mathbb{E}[t_i])^2]=O\left(n^3p_n^5+n^2p_n^3\right). \tag{69}$$

By (11) and (69), we get

$$\begin{aligned}
& \mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[X_i>\delta_n np_n]\right] \\
= & O\left(\frac{\sqrt{\mathbb{E}\left[(d_i-\mu_i)^{2k_0}\right]\mathbb{E}[(t_i-\mathbb{E}[t_i])^2]}}{\left(\delta_n np_n\right)^{k_0+2}}\right) \\
= & O\left(\frac{p_n}{\delta_n^{k_0+2}\left(np_n\right)^{\frac{k_0+1}{2}}}+\frac{1}{\sqrt{n}\delta_n^{k_0+2}\left(np_n\right)^{\frac{k_0+1}{2}}}\right).
\end{aligned} \tag{70}$$

Since $k_0=\lceil 1+\frac{1}{1-\alpha}\rceil+2>\max\{3,\frac{1}{1-\alpha}\}$, then

$$\frac{p_n n\sqrt{np_n}}{\left(np_n\right)^{\frac{k_0+1}{2}}}=\frac{1}{\left(np_n\right)^{\frac{k_0}{2}-1}}=o(1),$$

$$\frac{n\sqrt{np_n}}{\sqrt{n}\left(np_n\right)^{\frac{k_0+1}{2}}}=o(1).$$

Hence, by (70) we get

$$\mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[X_i>\delta_n np_n]\right]=o\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \tag{71}$$

If $X_i\le\delta_n np_n$, then $d_i\le\delta_n np_n$. Otherwise, X_i can not be between d_i and μ_i. In this case, $|h^{(k)}(x)|=O(1)$. Then by Lemma 3.1, we have

$$\begin{aligned}
& \mathbb{E}\left[\left|h^{(k_0)}(X_i)(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[X_i\le\delta_n np_n]\right] \\
\le & O(1)\mathbb{E}\left[\left|(t_i-\mathbb{E}[t_i])(d_i-\mu_i)^{k_0}\right|I[d_i\le\delta_n np_n]\right] \\
\le & O(n^2(np_n)^{k_0}))\sum_{t=2}^{\delta_n np_n}\mathbb{P}(d_i=t) \\
= & e^{-np_n\beta(1+o(1))}.
\end{aligned} \tag{72}$$

Hence, by (68), (71), (72) and Markov's inequality, one gets

$$\left|\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right| = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \tag{73}$$

By a similar argument, it is easy to get

$$\left|\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\mathbb{E}\left[\frac{h^{(k_0)}(X_i)}{k_0!}(d_i-\mu_i)^{k_0}\right]\right| = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \tag{74}$$

Next we show the sum of the first two terms of Y_{2n} in (67) is equal to $o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right)$. By (55), we only need to show the following equations.

$$\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \tag{75}$$

$$\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk} = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \tag{76}$$

$$\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{ij}\bar{A}_{jk}\mu_{ik} = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \tag{77}$$

$$\begin{aligned}&\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\left(\bar{A}_{ij}\mu_{ik}\mu_{jk}+\bar{A}_{ik}\mu_{ij}\mu_{jk}+\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)\\ &-\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[\bar{A}_{ij}^2]\mu_{ik}\mu_{jk} = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right).\end{aligned} \tag{78}$$

The left-hand side of (75) can be expressed as

$$\begin{aligned}\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} &= \frac{1}{n}\sum_{i\neq j\neq k,l\neq i}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\\ &= \frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\\ &\quad+\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}^2\bar{A}_{ik}\bar{A}_{jk}.\end{aligned} \tag{79}$$

The second moment of the first term in (79) is bounded by

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\right)^2\right] = O\left(\frac{n^4p_n^4}{n^2(np_n)^6}\right) = O\left(\frac{1}{n^2(np_n)^2}\right). \tag{80}$$

The second moment of the second term in (79) is bounded by

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}^2\bar{A}_{ik}\bar{A}_{jk}\right)^2\right] = O\left(\frac{n^3p_n^3}{n^2(np_n)^6}\right) = O\left(\frac{1}{n^2(np_n)^3}\right). \tag{81}$$

By (79), (80), (81) and Markov's inequality, (75) holds.

Now we prove (76). The left-hand side of (76) can be written as

$$\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk} = \frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\bar{A}_{ik}\mu_{jk} + \frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}^2\bar{A}_{ik}\mu_{jk}. \quad (82)$$

It is easy to verify that

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\right)^2\right] = O\left(\frac{p_n}{n^2(np_n)^2}\right), \quad (83)$$

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}^2\bar{A}_{ik}\mu_{jk}\right)^2\right] = O\left(\frac{p_n}{n^2(np_n)^2}\right). \quad (84)$$

By (82), (83), (84) and Markov inequality, (76) holds. Similarly, (77) holds.

Now we prove (78). The left-hand side of (78) can be written as

$$\begin{aligned}
&\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\left(\bar{A}_{ij}\mu_{ik}\mu_{jk}+\bar{A}_{ik}\mu_{ij}\mu_{jk}+\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)\\
&-\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[\bar{A}_{ij}^2]\mu_{ik}\mu_{jk}\\
=\ &\frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\mu_{ik}\mu_{jk}+\frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ik}\mu_{ij}\mu_{jk}\\
&+\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\left(\bar{A}_{ij}^2-\mathbb{E}[\bar{A}_{ij}^2]\right)\mu_{ik}\mu_{jk}\\
&+\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{jk}\mu_{ij}\mu_{ik}. \qquad (85)
\end{aligned}$$

It is easy to verify that

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k\neq l}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{il}\bar{A}_{ij}\mu_{ik}\mu_{jk}\right)^2\right] = O\left(\frac{p_n}{n^2(np_n)}\right), \quad (86)$$

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\left(\bar{A}_{ij}^2-\mathbb{E}[\bar{A}_{ij}^2]\right)\mu_{ik}\mu_{jk}\right)^2\right] = O\left(\frac{p_n}{n^2(np_n)^2}\right). \quad (87)$$

Moreover, we have

$$\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)\bar{A}_{jk}\mu_{ij}\mu_{ik} = \frac{1}{n}\sum_{i\neq j\neq k\neq s}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{is}\bar{A}_{jk}\mu_{ij}\mu_{ik} + \frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}\bar{A}_{jk}\mu_{ij}\mu_{ik}, \quad (88)$$

and

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k\neq s}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{is}\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)^2\right] = O\left(\frac{p_n^2}{n^2(np_n)^2}\right), \quad (89)$$

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\bar{A}_{ij}\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)^2\right] = O\left(\frac{p_n^3}{n^2(np_n)^3}\right). \quad (90)$$

By (85)-(90) and Markov's inequality, (78) holds.

Combining (75)- (78) yields

$$-\frac{1}{n}\sum_{i=1}^{n}(t_i-\mathbb{E}[t_i])\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}(d_i-\mu_i)+\frac{2}{n}\sum_{i\neq j\neq k}\frac{2\mu_i-1}{\mu_i^2(\mu_i-1)^2}\mathbb{E}[\bar{A}_{ij}^2]\mu_{ik}\mu_{jk}$$
$$= o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \quad (91)$$

Finally, we prove the last term in (67) is equal to $o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right)$. By (55), we only need to show the following equations.

$$\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big] = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \quad (92)$$

$$\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big] = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \quad (93)$$

$$\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\bar{A}_{ij}\bar{A}_{jk}\mu_{ik}\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big] = o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right), \quad (94)$$

$$\sum_{s=2}^{k_0-1}\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\left(\bar{A}_{ij}\mu_{ik}\mu_{jk}+\bar{A}_{ik}\mu_{ij}\mu_{jk}+\bar{A}_{jk}\mu_{ij}\mu_{ik}\right)\Big[(d_i-\mu_i)^s-\mathbb{E}[(d_i-\mu_i)^s]\Big]$$
$$= o_P\left(\frac{\sqrt{p_n}}{n}+\frac{1}{n\sqrt{np_n}}\right). \quad (95)$$

Firstly we prove (92). It is easy to verify that

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\mathbb{E}[(d_i-\mu_i)^s]\right)^2\right] = O\left(\frac{1}{n^2(np_n)^{s+1}}\right). \quad (96)$$

Recall the expression of $(d_i-\mu_i)^s$ in (1). Given $\lambda_{t1},\lambda_{t2},\dots,\lambda_{tt}$ such that $\lambda_{t1}+\lambda_{t2}+\cdots+\lambda_{tt}=s$. Suppose $\lambda_l\geq 2$ for all $l=1,2,\dots,t$. If $s\geq 3$, then $s-t\geq 2$. In this case,

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq j_2\neq\dots,j_t\neq i\neq j\neq k}h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\prod_{l=1}^{t}\bar{A}_{ij_l}^{\lambda_l}\right|\right] &= O\left(\frac{(np_n)^{t+3}}{n(np_n)^{s+2}}\right)\\ &= O\left(\frac{1}{n(np_n)^{s-t-1}}\right)\\ &= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \quad (97)\end{aligned}$$

When $j\in\{j_1,j_2,\dots,j_t\}$ or $k\in\{j_1,j_2,\dots,j_t\}$ in the summation of (97), (97) still holds.

If $s = 2$, then $t = 1$. In this case,

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\bar{A}_{ij_1}^2\right)^2\right] &= O\left(\frac{(np_n)^5}{n^2(np_n)^8}\right)\\ &= o\left(\frac{1}{n^2(np_n)}+\frac{p_n}{n^2}\right), \quad (98)\\ \mathbb{E}\left[\left|\frac{1}{n}\sum_{i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}^3\bar{A}_{ik}\bar{A}_{jk}\right|\right] &= O\left(\frac{(np_n)^3}{n(np_n)^4}\right)\\ &= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \quad (99)\end{aligned}$$

Suppose $\lambda_{t1} = \cdots = \lambda_{tt_1} = 1$ and $\lambda_{t,t_1+1} \geq 2, \ldots, \lambda_{tt} \geq 2$ for $t_1 \geq 1$. If $s - t \geq 2$, then (97) holds. Assume $s = t$. In this case, we have

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_s}\right)^2\right] &= O\left(\frac{(np_n)^{s+3}}{n^2(np_n)^{2s+4}}\right)\\ &= o\left(\frac{p_n}{n^2}+\frac{p_n}{n^2(np_n)}\right), \quad (100)\\ \mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq i\neq k} h^{(s)}(\mu_i)\bar{A}_{ij_1}^2\bar{A}_{ik}\bar{A}_{j_1k}\bar{A}_{ij_2}\ldots\bar{A}_{ij_s}\right)^2\right] &= O\left(\frac{(np_n)^{s+2}}{n^2(np_n)^{2s+4}}\right)\\ &= o\left(\frac{p_n}{n^2}+\frac{p_n}{n^2(np_n)}\right), \quad (101)\\ \mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq i} h^{(s)}(\mu_i)\bar{A}_{ij_1}^2\bar{A}_{ij_2}^2\bar{A}_{j_1j_2}\bar{A}_{ij_3}\ldots\bar{A}_{ij_s}\right|\right] &= O\left(\frac{(np_n)^{s+1}}{n(np_n)^{s+2}}\right)\\ &= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \quad (102)\end{aligned}$$

Assume $t = s - 1$. In this case, we have

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_{s-1}\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right)^2\right]\\ =\ & O\left(\frac{(np_n)^{s+3}}{n^2(np_n)^{2s+4}}\right)\\ =\ & o\left(\frac{p_n}{n^2}+\frac{p_n}{n^2(np_n)}\right), \quad (103)\\ &\mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq\cdots\neq j_{s-1}\neq i\neq k} h^{(s)}(\mu_i)\bar{A}_{ij_1}^2\bar{A}_{ik}\bar{A}_{j_1k}\bar{A}_{ij_2}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right|\right]\\ =\ & O\left(\frac{(np_n)^{s+1}}{n(np_n)^{s+2}}\right)\\ =\ & o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right), \quad (104)\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq\cdots\neq j_{s-1}\neq i} h^{(s)}(\mu_i)\bar{A}_{ij_1}^2\bar{A}_{ij_2}^2\bar{A}_{j_1j_2}\bar{A}_{ij_3}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right|\right] \\
= \; & O\left(\frac{(np_n)^s}{n(np_n)^{s+2}}\right) \\
= \; & o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \qquad (105)
\end{aligned}$$

Combining (96)-(105), Markov's inequality and the fact that k_0 is a fixed constant, (92) holds.

Now we prove (93). The proof is similar to the proof of (92). It is easy to verify that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\mathbb{E}[(d_i-\mu_i)^s]\right)^2\right] &= O\left(\frac{p_n}{n^2(np_n)^{s+1}}\right) \\
&= o\left(\frac{1}{n^2np_n}+\frac{p_n}{n^2}\right). \qquad (106)
\end{aligned}$$

Recall $(d_i-\mu_i)^s$ in (1). Suppose $\lambda_{tl}\geq 2$ for all $l=1,2,\ldots,t$. If $s-t\geq 2$, that is, $s\geq 3$, then

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq j_2\neq\cdots\neq j_t\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\prod_{l=1}^{t}\bar{A}_{ij_l}^{\lambda_{tl}}\right|\right] &= O\left(\frac{n^{t+3}p_n^{t+3}}{n(np_n)^{s+2}}\right) \\
&= O\left(\frac{1}{n(np_n)^{s-t-1}}\right) \\
&= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right) \qquad (107)
\end{aligned}$$

If $s=2$, then $t=1$ and

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq i\neq j\neq k} h^{(2)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\bar{A}_{ij_1}^2\right)^2\right] = O\left(\frac{n^5p_n^6}{n^2(np_n)^8}\right)=o\left(\frac{1}{n^2np_n}+\frac{p_n}{n^2}\right), \qquad (108)$$

and

$$\mathbb{E}\left[\left|\frac{1}{n}\sum_{i\neq j\neq k} h^{(2)}(\mu_i)\bar{A}_{ik}\mu_{jk}\bar{A}_{ij}^3\right|\right] = O\left(\frac{n^3p_n^3}{n(np_n)^4}\right)=o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \qquad (109)$$

Suppose $\lambda_{t1}=\cdots=\lambda_{tt_1}=1$ and $\lambda_{t,t_1+1}\geq 2$, $\lambda_{tt}\geq 2$ for $t_1\geq 1$. If $s-t\geq 2$, then (107) holds. Assume $s=t$. Then

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_s}\right)^2\right] &= O\left(\frac{n^{s+3}p_n^{s+4}}{n^2(np_n)^{2s+4}}\right) \\
&= o\left(\frac{1}{n^2np_n}+\frac{p_n}{n^2}\right). \qquad (110)
\end{aligned}$$

For $j\in\{j_1,j_2,\ldots,j_s\}$ or $k\in\{j_1,j_2,\ldots,j_s\}$ in the summation of (110), (110) still hods.

Assume $t=s-1$. Then

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_{s-1}\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right)^2\right]$$

$$
\begin{aligned}
&= O\left(\frac{n^{s+3}p_n^{s+3}}{n^2(np_n)^{2s+4}}\right)\\
&= o\left(\frac{1}{n^2np_n}+\frac{p_n}{n^2}\right), \qquad (111)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{n}\sum_{\substack{j_1\neq\cdots\neq j_{s-1}\neq i\\ \{j,k\}\cap\{j_1,\ldots,j_{s-1}\}\neq\emptyset}} h^{(s)}(\mu_i)\bar{A}_{ij}\bar{A}_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right|\right]\\
&= O\left(\frac{n^{s+1}p_n^{s+1}}{n(np_n)^{s+2}}\right)\\
&= o\left(\frac{1}{n^2np_n}+\frac{p_n}{n^2}\right). \qquad (112)
\end{aligned}
$$

By (106)-(112), Markov's inequality and the fact that k_0 is a fixed constant, (93) holds. Similarly (94) holds.

Now we prove (95). It is easy to verify that

$$
\mathbb{E}\left[\left(\frac{1}{n}\sum_{i\neq j\neq k}\frac{h^{(s)}(\mu_i)}{s!}\bar{A}_{ij}\mu_{ik}\mu_{jk}\mathbb{E}[(d_i-\mu_i)^s]\right)^2\right]=O\left(\frac{p_n}{n^2(np_n)^s}\right). \qquad (113)
$$

Recall $(d_i-\mu_i)^s$ in (1). Suppose $\lambda_l\geq 2$ for all $l=1,2,\ldots,t$. If $s\geq 3$, then

$$
\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{n}\sum_{j_1\neq j_2\neq\ldots,j_t\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\prod_{l=1}^{t}\bar{A}_{ij_l}^{\lambda_l}\right|\right] &= O\left(\frac{n^{t+3}p_n^{t+3}}{n(np_n)^{s+2}}\right)\\
&= O\left(\frac{1}{n(np_n)^{s-t-1}}\right)\\
&= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right). \qquad (114)
\end{aligned}
$$

If $s=2$, then $t=1$ and

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\bar{A}_{ij_1}^2\right)^2\right] &= O\left(\frac{n^6p_n^7}{n^2(np_n)^8}\right)\\
&= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right), \qquad (115)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{n}\sum_{i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\bar{A}_{ik}^2\right|\right] &= O\left(\frac{n^3p_n^4}{n(np_n)^4}\right)\\
&= o\left(\frac{1}{n\sqrt{np_n}}+\frac{\sqrt{p_n}}{n}\right), \qquad (116)
\end{aligned}
$$

$$
\mathbb{E}\left[\left|\frac{1}{n}\sum_{i\neq j\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}^3\mu_{ik}\mu_{jk}\right|\right] = O\left(\frac{n^3p_n^3}{n(np_n)^4}\right)
$$

$$= o\left(\frac{1}{n\sqrt{np_n}} + \frac{\sqrt{p_n}}{n}\right). \quad (117)$$

Suppose $\lambda_{t1} = \cdots = \lambda_{tt_1} = 1$ and $\lambda_{t,t_1+1} \geq 2, \ldots, \lambda_{tt} \geq 2$ for $t_1 \geq 1$. If $s - t \geq 2$, then (114) holds. Assume $s = t$. Then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq j\neq i\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_s}\right)^2\right] &= O\left(\frac{n^{s+4}p_n^{s+5}}{n^2(np_n)^{2s+4}}\right)\\ &= o\left(\frac{1}{n^2np_n} + \frac{p_n}{n^2}\right), \quad (118)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_s\neq i\neq k} h^{(s)}(\mu_i)\bar{A}_{ij_1}^2\mu_{ik}\mu_{j_1k}\bar{A}_{ij_2}\ldots\bar{A}_{ij_s}\right)^2\right] &= O\left(\frac{n^{s+4}p_n^{s+4}}{n^2(np_n)^{2s+4}}\right)\\ &= o\left(\frac{1}{n^2np_n} + \frac{p_n}{n^2}\right), \quad (119)\end{aligned}$$

Assume $t = s - 1$. Then

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{n}\sum_{j_1\neq\cdots\neq j_{s-1}\neq j\neq i\neq k} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right)^2\right]\\ &= O\left(\frac{n^{s+4}p_n^{s+5}}{n^2(np_n)^{2s+4}}\right)\\ &= o\left(\frac{1}{n^2np_n} + \frac{p_n}{n^2}\right), \quad (120)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{n}\sum_{\substack{j_1\neq\cdots\neq j_{s-1}\neq i\neq k\\ j\in\{j_1,\ldots,j_{s-1}\}}} h^{(s)}(\mu_i)\bar{A}_{ij}\mu_{ik}\mu_{jk}\bar{A}_{ij_1}\ldots\bar{A}_{ij_{s-2}}\bar{A}_{ij_{s-1}}^2\right|\right] &= O\left(\frac{n^{s+1}p_n^{s+1}}{n(np_n)^{s+2}}\right)\\ &= o\left(\frac{1}{n^2np_n} + \frac{p_n}{n^2}\right). \quad (121)\end{aligned}$$

By (113)-(121), Markov's inequality and the fact that k_0 is a fixed constant, (95) holds. By (55) and (92)-(95), we get

$$\frac{1}{n}\sum_{i=1}^{n}(t_i - \mathbb{E}[t_i])\sum_{s=2}^{k_0-1}\frac{h^{(s)}(\mu_i)}{s!}\left[(d_i-\mu_i)^s - \mathbb{E}[(d_i-\mu_i)^s]\right] = o_P\left(\frac{\sqrt{p_n}}{n} + \frac{1}{n\sqrt{np_n}}\right). \quad (122)$$

By (35), (54),(63), (64), (65) and (66), $\overline{C}_n - \mathbb{E}[\overline{C}_n]$ has the following asymptotic expression.

$$\overline{C}_n - \mathbb{E}[\overline{C}_n] = \begin{cases} \frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} + o_P\left(\frac{1}{n\sqrt{np_n}}\right), & \text{if } \alpha > \frac{1}{2},\\ \frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{ik}\bar{A}_{jk} + \frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij} + o_P\left(\frac{\sqrt{p_n}}{n} + \frac{1}{n\sqrt{np_n}}\right), & \text{if } \alpha = \frac{1}{2},\\ \frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij} + o_P\left(\frac{\sqrt{p_n}}{n}\right), & \text{if } \alpha < \frac{1}{2}.\end{cases} \quad (123)$$

By (28),(29), (57) and (61), we have

$$\begin{aligned}\sigma_{1n}^2 &= Var\left(\frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\right) = \Theta\left(\frac{1}{n^2(np_n)}\right),\\ \sigma_{2n}^2 &= Var\left(\frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij}\right) = \Theta\left(\frac{p_n}{n^2}\right).\end{aligned}$$

For $\alpha = \frac{1}{2}$, σ_{1n}^2 and σ_{2n}^2 have the same order. Recall that $\sigma_n^2 = \sigma_{1n}^2 + \sigma_{2n}^2$. Let $\lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$ in Lemma 3.4, we get

$$\frac{\frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki} + \frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij}}{\sqrt{\sigma_{1n}^2 + \sigma_{2n}^2}} \Rightarrow \mathcal{N}(0,1). \tag{124}$$

For $\alpha > \frac{1}{2}$, then $\sigma_{2n}^2 = o(\sigma_{1n}^2)$. In this case, $\sigma_n^2 = \sigma_{1n}^2 + \sigma_{2n}^2 = \sigma_{1n}^2(1 + o(1))$ and Lemma 3.4 still holds. Taking $\lambda_1 = 1$ and $\lambda_2 = 0$ in Lemma 3.4 yields

$$\frac{\frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\sqrt{\sigma_{1n}^2 + \sigma_{2n}^2}} = \frac{1}{1+o(1)}\frac{\frac{2}{n}\sum_{i<j<k} a_{ijk}\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\sqrt{\sigma_{1n}^2}} \Rightarrow \mathcal{N}(0,1). \tag{125}$$

For $\alpha < \frac{1}{2}$, then $\sigma_{1n}^2 = o(\sigma_{2n}^2)$. In this case, $\sigma_n^2 = \sigma_{1n}^2 + \sigma_{2n}^2 = \sigma_{2n}^2(1 + o(1))$ and Lemma 3.4 still holds. Taking $\lambda_1 = 0$ and $\lambda_2 = 1$ in Lemma 3.4 yields

$$\frac{\frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij}}{\sqrt{\sigma_{1n}^2 + \sigma_{2n}^2}} = \frac{1}{1+o(1)}\frac{\frac{2}{n}\sum_{i<j} e_{ij}\bar{A}_{ij}}{\sqrt{\sigma_{2n}^2}} \Rightarrow \mathcal{N}(0,1). \tag{126}$$

Combining (124)-(126), the proof is complete. □

3.2. Proof of Theorem 2.4

Recall $\Delta_{ijk} = A_{ij}A_{jk}A_{ki}$. For distinct indices i, j, k, define $d_{i(jk)} = 2 + \sum_{l\notin\{i,j,k\}} A_{il}$. Then

$$\mathcal{T}_n = \sum_{i<j<k} \frac{\Delta_{ijk}}{d_{i(jk)}d_{j(ki)}d_{k(ij)}}.$$

Note that $d_{i(jk)}$, $d_{j(ki)}$, $d_{k(ij)}$ are independent. With a little abuse of notation, we still write $d_{i(jk)}$ as d_i for convenience. Let $g(x, y, z) = \frac{1}{xyz}$. Then the k-th derivative of g is

$$\frac{\partial^k g(x,y,z)}{\partial x^r \partial y^m \partial z^l} = \frac{r!m!l!(-1)^{r+m+l}}{x^{r+1}y^{m+1}x^{l+1}},$$

where r, m, l are non-negative integers such that $r + m + l = k$. Let $k_0 = 3 + \lceil \frac{1}{1-\alpha} \rceil$. By **the** Taylor expansion, we have

$$\begin{aligned}\sum_{i<j<k}\frac{\Delta_{ijk}}{d_id_jd_k} &= \sum_{i<j<k}\frac{\Delta_{ijk}}{\mu_i\mu_j\mu_k} - \sum_{i<j<k}\left(\frac{\Delta_{ijk}(d_i-\mu_i)}{\mu_i^2\mu_j\mu_k} + \frac{\Delta_{ijk}(d_j-\mu_j)}{\mu_i\mu_j^2\mu_k} + \frac{\Delta_{ijk}(d_k-\mu_k)}{\mu_i\mu_j\mu_k^2}\right)\\ &\quad + \sum_{t=2}^{k_0-1}\sum_{r+m+l=t}(-1)^t\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}}\end{aligned}$$

$$+(-1)^{k_0}\sum_{r+m+l=k_0}\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}, \tag{127}$$

where X_s is between μ_s and d_s for $s\in\{i,j,k\}$. Then

$$\begin{aligned}
&\sum_{i<j<k}\left(\frac{\Delta_{ijk}}{d_id_jd_k}-\mathbb{E}\left[\frac{\Delta_{ijk}}{d_id_jd_k}\right]\right)\\
=&\sum_{i<j<k}\frac{\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}]}{\mu_i\mu_j\mu_k}-\sum_{i<j<k}\left(\frac{\Delta_{ijk}(d_i-\mu_i)}{\mu_i^2\mu_j\mu_k}+\frac{\Delta_{ijk}(d_j-\mu_j)}{\mu_i\mu_j^2\mu_k}+\frac{\Delta_{ijk}(d_k-\mu_k)}{\mu_i\mu_j\mu_k^2}\right)\\
&+\sum_{t=2}^{k_0-1}\sum_{r+m+l=t}(-1)^t\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}}\\
&-\sum_{t=2}^{k_0-1}\sum_{r+m+l=t}(-1)^t\sum_{i<j<k}\frac{\mathbb{E}[\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l]}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}}\\
&+(-1)^{k_0}\sum_{r+m+l=k_0}\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\\
&-(-1)^{k_0}\sum_{r+m+l=k_0}\sum_{i<j<k}\mathbb{E}\left[\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right].
\end{aligned} \tag{128}$$

Next we find the order of each term in (128).
By (55), we have

$$\begin{aligned}
\sum_{i<j<k}\frac{\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}]}{\mu_i\mu_j\mu_k}=&\sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}+\sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\mu_{ki}+\bar{A}_{ij}\mu_{jk}\bar{A}_{ki}+\mu_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}\\
&+\sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}.
\end{aligned} \tag{129}$$

Now we find the order of each term in (129) by calculating the second moment of each term. Note that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}\right)^2\right]&=\sum_{i<j<k}\frac{\mathbb{E}[\bar{A}_{ij}^2]\mathbb{E}[\bar{A}_{jk}^2]\mathbb{E}[\bar{A}_{ki}^2]}{\mu_i^2\mu_j^2\mu_k^2}\\
&=\sum_{i<j<k}\frac{\mu_{ij}(1-\mu_{ij})\mu_{jk}(1-\mu_{jk})\mu_{ki}(1-\mu_{ki})}{\mu_i^2\mu_j^2\mu_k^2}\\
&=\Theta\left(\frac{1}{(np_n)^3}\right),
\end{aligned} \tag{130}$$

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k}\right)^2\right]&=\sum_{i<j<k}\frac{\mathbb{E}[\bar{A}_{ij}^2]\mathbb{E}[\bar{A}_{jk}^2]\mu_{ki}^2}{\mu_i^2\mu_j^2\mu_k^2}\\
&=\sum_{i<j<k}\frac{\mu_{ij}(1-\mu_{ij})\mu_{jk}(1-\mu_{jk})\mu_{ki}^2}{\mu_i^2\mu_j^2\mu_k^2}\\
&=\Theta\left(\frac{p_n}{(np_n)^3}\right),
\end{aligned} \tag{131}$$

and

$$\mathbb{E}\left[\left(\sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k}\right)^2\right] = \sum_{i<j<k,i<j<k_1}\frac{\mathbb{E}[\bar{A}_{ij}^2]\mu_{jk}\mu_{ki}\mu_{jk_1}\mu_{k_1i}}{\mu_i^2\mu_j^2\mu_k\mu_{k_1}}$$
$$= \sum_{i<j<k,k_1\neq k}\frac{\mu_{ij}(1-\mu_{ij})\mu_{jk}\mu_{ki}\mu_{jk_1}\mu_{k_1i}}{\mu_i^2\mu_j^2\mu_k\mu_{k_1i}} + \sum_{i<j<k}\frac{\mu_{ij}(1-\mu_{ij})\mu_{jk}^2\mu_{ki}^2}{\mu_i^2\mu_j^2\mu_k^2}$$
$$= \Theta\left(\frac{p_n}{(np_n)^2}\right). \tag{132}$$

Combining (130), (131) and (132), we get the leading terms of (129) as follows.

If $\alpha > \frac{1}{2}$, then

$$\sum_{i<j<k}\frac{\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}]}{\mu_i\mu_j\mu_k} = \sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} + o_P\left(\frac{1}{np_n\sqrt{np_n}}\right). \tag{133}$$

If $\alpha < \frac{1}{2}$, then

$$\sum_{i<j<k}\frac{\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}]}{\mu_i\mu_j\mu_k} = \sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} + o_P\left(\frac{1}{n\sqrt{p_n}}\right). \tag{134}$$

If $\alpha = \frac{1}{2}$, then

$$\sum_{i<j<k}\frac{\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}]}{\mu_i\mu_j\mu_k} = \sum_{i<j<k}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} + \sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}$$
$$+o_P\left(\frac{1}{n\sqrt{p_n}}+\frac{1}{np_n\sqrt{np_n}}\right). \tag{135}$$

Next, we consider the second term of (128). By (55), one gets

$$\sum_{i\neq j\neq k}\frac{\Delta_{ijk}(d_i-\mu_i)}{\mu_i^2\mu_j\mu_k} = \sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} + 3\sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\bar{A}_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}$$
$$+3\sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} + \sum_{i\neq j\neq k\neq l}\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}.$$

Similar to (130), (131) and (132), it is easy to verify that

$$\sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} = O_P\left(\frac{1}{(np_n)^2}\right),$$
$$\sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\bar{A}_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} = O_P\left(\frac{\sqrt{p_n}}{(np_n)^2}\right),$$
$$\sum_{i\neq j\neq k\neq l}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} = O_P\left(\frac{1}{np_n\sqrt{n}}\right),$$
$$\sum_{i\neq j\neq k\neq l}\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k} = O_P\left(\frac{\sqrt{p_n}}{np_n}\right).$$

Hence the second term of (128) is equal to

$$\begin{aligned}
&\sum_{i<j<k}\left(\frac{\Delta_{ijk}(d_i-\mu_i)}{\mu_i^2\mu_j\mu_k}+\frac{\Delta_{ijk}(d_j-\mu_j)}{\mu_i\mu_j^2\mu_k}+\frac{\Delta_{ijk}(d_k-\mu_k)}{\mu_i\mu_j\mu_k^2}\right)\\
=&\sum_{i<j<k,l\notin\{i,j,k\}}\left(\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{jl}}{\mu_i\mu_j^2\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{kl}}{\mu_i\mu_j\mu_k^2}\right)\\
&+o_P\left(\frac{1}{n\sqrt{p_n}}+\frac{1}{np_n\sqrt{np_n}}\right). \qquad (136)
\end{aligned}$$

Now we consider the last two terms of (128). Given r,m,l such that $r+m+l=k_0$, we have

$$\begin{aligned}
&\sum_{i<j<k}\mathbb{E}\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right|\right]\\
=&\sum_{i<j<k}\left(\mathbb{E}\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right|I[X_i\geq\delta_nnp_n,X_j\geq\delta_nnp_n,X_k\geq\delta_nnp_n]\right]\right.\\
&\left.+\mathbb{E}\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right|I[X_i<\delta_nnp_n\ \cup\ X_j<\delta_nnp_n\cup X_k<\delta_nnp_n]\right]\right). \qquad (137)
\end{aligned}$$

Recall that $k_0=3+\lceil\frac{1}{1-\alpha}\rceil$. Then $k_0>\max\{3,1+\frac{1}{1-\alpha}\}$. Moreover, d_i,d_j,d_k are independent. Hence,

$$\begin{aligned}
&\sum_{i<j<k}\mathbb{E}\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right|I[X_i\geq\delta_nnp_n,X_j\geq\delta_nnp_n,X_k\geq\delta_nnp_n]\right]\\
\leq&\frac{1}{\delta_n^{k_0+3}(np_n)^{k_0+3}}\sum_{i<j<k}\mathbb{E}[\Delta_{ijk}]\mathbb{E}[|(d_i-\mu_i)^r|]\mathbb{E}[|(d_j-\mu_j)^m|]\mathbb{E}[|(d_k-\mu_k)^l|]\\
=&O\left(\frac{(np_n)^{\frac{k_0}{2}+3}}{\delta_n^{k_0+3}(np_n)^{k_0+3}}\right)\\
=&o\left(\frac{1}{np_n\sqrt{np_n}}+\frac{\sqrt{p_n}}{np_n}\right). \qquad (138)
\end{aligned}$$

Since $d_i\geq1$ and X_i is between d_i and μ_i, then $X_i\geq1$. Hence

$$\begin{aligned}
\sum_{i<j<k}\mathbb{E}&\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{X_i^{r+1}X_j^{m+1}X_k^{l+1}}\right|I[X_i<\delta_nnp_n\ \cup\ X_j<\delta_nnp_n\cup X_k<\delta_nnp_n]\right]\\
\leq&\sum_{i<j<k}\mathbb{E}\left[\left|\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l\right|I[X_i<\delta_nnp_n]\right]\\
&+\sum_{i<j<k}\mathbb{E}\left[\left|\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l\right|I[X_j<\delta_nnp_n]\right]\\
&+\sum_{i<j<k}\mathbb{E}\left[\left|\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l\right|I[X_k<\delta_nnp_n]\right]. \qquad (139)
\end{aligned}$$

By Lemma 3.1, we have

$$\sum_{i<j<k}\mathbb{E}\left[\left|\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l\right|I[X_i<\delta_nnp_n]\right]$$

$$\leq \quad n^{k_0+3}\mathbb{P}(d_i < \delta_n np_n) = \exp\left(-\beta np_n(1+o(1))\right). \tag{140}$$

Combining (136)-(140), the last two terms of (128) are $o_P\left(\frac{1}{np_n\sqrt{np_n}} + \frac{\sqrt{p_n}}{np_n}\right)$.

Now we consider the third term and the forth term of (128). Given $t \geq 2$ and r, m, l such that $r+m+l = t$, we have

$$\begin{aligned}
& \sum_{i<j<k} \mathbb{E}\left[\left|\frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}}\right|\right] \\
\leq & \sum_{i<j<k} \frac{\mathbb{E}[\Delta_{ijk}]\mathbb{E}[|(d_i-\mu_i)^r|]\mathbb{E}[|(d_j-\mu_j)^m|]\mathbb{E}[|(d_k-\mu_k)^l|]}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}} \\
= & O\left(\frac{1}{(np_n)^{\frac{t}{2}}}\right). \qquad (141)
\end{aligned}$$

If $t \geq 4$, then

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i-\mu_i)^r(d_j-\mu_j)^m(d_k-\mu_k)^l}{\mu_i^{r+1}\mu_j^{m+1}\mu_k^{l+1}} = o_P\left(\frac{1}{np_n\sqrt{np_n}} + \frac{\sqrt{p_n}}{np_n}\right).$$

Next we consider the case $t = 2, 3$.

Let $t = 2$. Without loss of generality, assume $r = 2, m = l = 0$ or $r = m = 1, l = 0$. Consider $r = 2, m = l = 0$ first. In this case, we have

$$\begin{aligned}
& \sum_{i<j<k} \frac{\Delta_{ijk}(d_i-\mu_i)^2 - \mathbb{E}[\Delta_{ijk}(d_i-\mu_i)^2]}{\mu_i^3\mu_j\mu_k} \\
= & \sum_{i<j<k, l\neq s} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{is}}{\mu_i^3\mu_j\mu_k} + \sum_{i<j<k, l} \frac{\Delta_{ijk}\bar{A}_{il}^2 - \mathbb{E}[\Delta_{ijk}\bar{A}_{il}^2]}{\mu_i^3\mu_j\mu_k} \\
= & \sum_{i<j<k, l\neq s} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{is}}{\mu_i^3\mu_j\mu_k} + \sum_{i<j<k, l} \frac{\Delta_{ijk}(\bar{A}_{il}^2 - \mathbb{E}[\bar{A}_{il}^2])}{\mu_i^3\mu_j\mu_k} + \sum_{i<j<k, l} \frac{(\Delta_{ijk} - \mathbb{E}[\Delta_{ijk}])\mathbb{E}[\bar{A}_{il}^2]}{\mu_i^3\mu_j\mu_k}. \qquad (142)
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i<j<k, l\neq s} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{is}}{\mu_i^3\mu_j\mu_k}\right)^2\right] &= O\left(\frac{p_n}{(np_n)^3}\right), \\
\mathbb{E}\left[\left(\sum_{i<j<k, l} \frac{\Delta_{ijk}(\bar{A}_{il}^2 - \mathbb{E}[\bar{A}_{il}^2])}{\mu_i^3\mu_j\mu_k}\right)^2\right] &= O\left(\frac{p_n}{(np_n)^4}\right), \qquad (143) \\
\mathbb{E}\left[\left(\sum_{i<j<k, l} \frac{(\Delta_{ijk} - \mathbb{E}[\Delta_{ijk}])\mathbb{E}[\bar{A}_{il}^2]}{\mu_i^3\mu_j\mu_k}\right)^2\right] &= O\left(\frac{p_n}{(np_n)^4}\right), \qquad (144)
\end{aligned}$$

Hence

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i-\mu_i)^2 - \mathbb{E}[\Delta_{ijk}(d_i-\mu_i)^2]}{\mu_i^3\mu_j\mu_k} = o_P\left(\frac{1}{np_n\sqrt{np_n}} + \frac{\sqrt{p_n}}{np_n}\right). \tag{145}$$

Consider $t = 2$ and $r = m = 1, l = 0$. In this case,

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i - \mu_i)(d_j - \mu_j)}{\mu_i^2 \mu_j^2 \mu_k} = \sum_{i<j<k, l\neq s} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{js}}{\mu_i^2 \mu_j^2 \mu_k} + \sum_{i<j<k, l} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}}{\mu_i^2 \mu_j^2 \mu_k}.$$

It is easy to verify that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i<j<k, l\neq s} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{js}}{\mu_i^2 \mu_j^2 \mu_k}\right)^2\right] &= O\left(\frac{p_n^2}{(np_n)^4}\right), \\
\mathbb{E}\left[\left(\sum_{i<j<k, l} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}}{\mu_i^2 \mu_j^2 \mu_k}\right)^2\right] &= O\left(\frac{p_n^2}{(np_n)^5}\right),
\end{aligned}$$

Hence

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i - \mu_i)(d_j - \mu_j)}{\mu_i^2 \mu_j^2 \mu_k} = o_P\left(\frac{1}{np_n\sqrt{np_n}} + \frac{\sqrt{p_n}}{np_n}\right). \tag{146}$$

Let $t = 3$. Without loss of generality, assume $r = 3, m = l = 0$ or $r = 2, m = 1, l = 0$ or $r = m = l = 1$. Consider $r = m = l = 1$. In this case, one has

$$\begin{aligned}
&\sum_{i<j<k} \frac{\Delta_{ijk}(d_i - \mu_i)(d_j - \mu_j)(d_k - \mu_k)}{\mu_i^2 \mu_j^2 \mu_k^2} \\
= &\sum_{\substack{i<j<k \\ l\neq s\neq m}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jm}\bar{A}_{ks}}{\mu_i^2 \mu_j^2 \mu_k^2} + 3\sum_{\substack{i<j<k \\ l\neq s}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}\bar{A}_{ks}}{\mu_i^2 \mu_j^2 \mu_k^2} + \sum_{\substack{i<j<k \\ l}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}\bar{A}_{kl}}{\mu_i^2 \mu_j^2 \mu_k^2}
\end{aligned}$$

It is easy to verify that

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{\substack{i<j<k \\ l\neq s\neq m}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jm}\bar{A}_{ks}}{\mu_i^2 \mu_j^2 \mu_k^2}\right)^2\right] &= O\left(\frac{p_n^3}{(np_n)^6}\right), \\
\mathbb{E}\left[\left(\sum_{\substack{i<j<k \\ l\neq s}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}\bar{A}_{ks}}{\mu_i^2 \mu_j^2 \mu_k^2}\right)^2\right] &= O\left(\frac{p_n}{(np_n)^7}\right), \\
\mathbb{E}\left[\left(\sum_{\substack{i<j<k \\ l}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{jl}\bar{A}_{kl}}{\mu_i^2 \mu_j^2 \mu_k^2}\right)^2\right] &= O\left(\frac{p_n^2}{(np_n)^8}\right).
\end{aligned}$$

Hence,

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i - \mu_i)(d_j - \mu_j)(d_k - \mu_k)}{\mu_i^2 \mu_j^2 \mu_k^2} = o_P\left(\frac{1}{np_n\sqrt{np_n}} + \frac{\sqrt{p_n}}{np_n}\right). \tag{147}$$

Consider $r = 2, m = 1, l = 0$. In this case, one has

$$\sum_{i<j<k} \frac{\Delta_{ijk}(d_i - \mu_i)^2(d_j - \mu_j)}{\mu_i^3 \mu_j^2 \mu_k} = \sum_{\substack{i<j<k \\ l\neq s\neq m}} \frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{im}\bar{A}_{js}}{\mu_i^3 \mu_j^2 \mu_k} + \sum_{\substack{i<j<k \\ l\neq s}} \frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{js}}{\mu_i^3 \mu_j^2 \mu_k}$$

$$+2\sum_{\substack{i<j<k\\ l\neq s}}\frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{is}\bar{A}_{js}}{\mu_i^3\mu_j^2\mu_k}+\sum_{\substack{i<j<k\\ l}}\frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{jl}}{\mu_i^3\mu_j^2\mu_k}.$$

It is easy to verify that

$$\mathbb{E}\left[\left(\sum_{\substack{i<j<k\\ l\neq s\neq m}}\frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{im}\bar{A}_{js}}{\mu_i^3\mu_j^2\mu_k}\right)^2\right] = O\left(\frac{p_n}{(np_n)^5}\right),$$

$$\mathbb{E}\left[\left(\sum_{\substack{i<j<k\\ l\neq s}}\frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{js}}{\mu_i^3\mu_j^2\mu_k}\right)^2\right] = O\left(\frac{p_n}{(np_n)^4}\right),$$

$$\mathbb{E}\left[\left(\sum_{\substack{i<j<k\\ l\neq s}}\frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{is}\bar{A}_{js}}{\mu_i^3\mu_j^2\mu_k}\right)^2\right] = O\left(\frac{p_n^2}{(np_n)^6}\right),$$

$$\mathbb{E}\left[\left|\sum_{\substack{i<j<k\\ l}}\frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{jl}}{\mu_i^3\mu_j^2\mu_k}\right|\right] = O\left(\frac{p_n}{(np_n)^2}\right).$$

Hence

$$\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^2(d_j-\mu_j)}{\mu_i^3\mu_j^2\mu_k}=o_P\left(\frac{1}{np_n\sqrt{np_n}}+\frac{\sqrt{p_n}}{np_n}\right). \quad (148)$$

Consider $r=3, m=l=0$. In this case, one has

$$\begin{aligned}&\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^3-\mathbb{E}[\Delta_{ijk}(d_i-\mu_i)^3]}{\mu_i^4\mu_j\mu_k}\\ =&\sum_{\substack{i<j<k\\ l\neq s\neq m}}\frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{im}\bar{A}_{is}}{\mu_i^4\mu_j\mu_k}+3\sum_{\substack{i<j<k\\ l\neq s}}\frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{is}}{\mu_i^4\mu_j\mu_k}\\ &+\sum_{\substack{i<j<k\\ l}}\frac{\Delta_{ijk}(\bar{A}_{il}^3-\mathbb{E}[\bar{A}_{il}^3])}{\mu_i^4\mu_j\mu_k}+\sum_{\substack{i<j<k\\ l}}\frac{(\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}])\mathbb{E}[\bar{A}_{il}^3]}{\mu_i^4\mu_j\mu_k}\end{aligned}$$

It is easy to verify that

$$\mathbb{E}\left[\left(\sum_{\substack{i<j<k\\ l\neq s\neq m}}\frac{\Delta_{ijk}\bar{A}_{il}\bar{A}_{im}\bar{A}_{is}}{\mu_i^4\mu_j\mu_k}\right)^2\right] = O\left(\frac{p_n}{(np_n)^4}\right),$$

$$\mathbb{E}\left[\left(\sum_{\substack{i<j<k\\ l\neq s}}\frac{\Delta_{ijk}\bar{A}_{il}^2\bar{A}_{is}}{\mu_i^4\mu_j\mu_k}\right)^2\right] = O\left(\frac{p_n}{(np_n)^4}\right),$$

$$\mathbb{E}\left[\left|\sum_{\substack{i<j<k\\ l}}\frac{\Delta_{ijk}(\bar{A}_{il}^3-\mathbb{E}[\bar{A}_{il}^3])}{\mu_i^4\mu_j\mu_k}\right|\right] = O\left(\frac{1}{(np_n)^2}\right),$$

$$\mathbb{E}\left[\left|\sum_{\substack{i<j<k\\ l}}\frac{(\Delta_{ijk}-\mathbb{E}[\Delta_{ijk}])\mathbb{E}[\bar{A}_{il}^3]}{\mu_i^4\mu_j\mu_k}\right|\right] = O\left(\frac{1}{(np_n)^2}\right).$$

Hence,

$$\sum_{i<j<k}\frac{\Delta_{ijk}(d_i-\mu_i)^3-\mathbb{E}[\Delta_{ijk}(d_i-\mu_i)^3]}{\mu_i^4\mu_j\mu_k}=o_P\left(\frac{1}{np_n\sqrt{np_n}}+\frac{\sqrt{p_n}}{np_n}\right).$$

Note that

$$\sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} = \frac{1}{6}\sum_{i\neq j\neq k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k}$$

Let $\gamma_{ij}=\sum_{k\notin\{i,j\}}\frac{\mu_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k}$. Then

$$\sum_{i\neq j\neq k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k}=\sum_{i\neq j}\bar{A}_{ij}\sum_{k\notin\{i,j\}}\frac{\mu_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k}=2\sum_{i<j}\gamma_{ij}\bar{A}_{ij}.$$

Then

$$\sum_{i<j<k}\frac{\bar{A}_{ij}\mu_{jk}\mu_{ki}+\mu_{ij}\bar{A}_{jk}\mu_{ki}+\mu_{ij}\mu_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} = \sum_{i<j}\gamma_{ij}\bar{A}_{ij}.$$

$$\begin{aligned}&\sum_{i<j<k,l\notin\{i,j,k\}}\left(\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{jl}}{\mu_i\mu_j^2\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{kl}}{\mu_i\mu_j\mu_k^2}\right)\\ =&\ \frac{1}{6}\sum_{i\neq j\neq k\neq l}\left(\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{jl}}{\mu_i\mu_j^2\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{kl}}{\mu_i\mu_j\mu_k^2}\right)\end{aligned}$$

Let $\eta_i=\sum_{j\neq k}\frac{\mu_{ij}\mu_{jk}\mu_{ki}}{\mu_i^2\mu_j\mu_k}$. Then

$$\sum_{i\neq j\neq k\neq l}\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}=\sum_{i\neq l}\bar{A}_{il}\sum_{\substack{j\neq k\\ j,k\notin\{i,l\}}}\frac{\mu_{ij}\mu_{jk}\mu_{ki}}{\mu_i^2\mu_j\mu_k}=\sum_{i\neq l}\eta_i\bar{A}_{il}+\sum_{i\neq l}r_n\bar{A}_{il},$$

where $r_n=O\left(\frac{p_n^3}{(np_n)^4}\right)$. Hence

$$\begin{aligned}&\sum_{i<j<k,l\notin\{i,j,k\}}\left(\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{il}}{\mu_i^2\mu_j\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{jl}}{\mu_i\mu_j^2\mu_k}+\frac{\mu_{ij}\mu_{jk}\mu_{ki}\bar{A}_{kl}}{\mu_i\mu_j\mu_k^2}\right)\\ =&\ \frac{1}{2}\sum_{i\neq l}\eta_i\bar{A}_{il}+\frac{1}{2}\sum_{i\neq l}r_n\bar{A}_{il}\\ =&\ \frac{1}{2}\sum_{i<j}(\eta_i+\eta_j)\bar{A}_{ij}+O_P\left(\frac{p_n^2}{(np_n)^3}\right).\end{aligned}$$

Then we get

If $\alpha > \frac{1}{2}$, then

$$\mathcal{T}_n - \mathbb{E}[\mathcal{T}_n] = \sum_{i<j<k} \frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} + o_P\left(\frac{1}{np_n\sqrt{np_n}}\right).$$

If $\alpha < \frac{1}{2}$, then

$$\mathcal{T}_n - \mathbb{E}[\mathcal{T}_n] = \sum_{i<j}\left(\gamma_{ij} - \frac{\eta_i + \eta_j}{2}\right)\bar{A}_{ij} + o_P\left(\frac{1}{n\sqrt{p_n}}\right).$$

If $\alpha = \frac{1}{2}$, then

$$\mathcal{T}_n - \mathbb{E}[\mathcal{T}_n] = \sum_{i<j<k} \frac{\bar{A}_{ij}\bar{A}_{jk}\bar{A}_{ki}}{\mu_i\mu_j\mu_k} + \sum_{i<j}\left(\gamma_{ij} - \frac{\eta_i + \eta_j}{2}\right)\bar{A}_{ij} + o_P\left(\frac{1}{n\sqrt{p_n}} + \frac{1}{np_n\sqrt{np_n}}\right).$$

By Lemma 3.4, the proof is complete.
□

3.3. Proof of Corollary 2.5

We only need to find the orders of v_{1n}, v_{2n} and show $v_{2n} = o(v_{1n})$. Note that $w = \Theta(n)$. Given distinct indices i, j, k, we have

$$\mu_i = p_n w_i \sum_{l \neq i} w_l = w p_n w_i \left(1 + O\left(\frac{1}{n}\right)\right),$$
$$v_{1n}^2 = \frac{n^3}{6w^6 p_n^3}(1 + o(1)).$$

Moreover, direct calculation yields

$$\begin{aligned}
\eta_i &= \sum_{j\neq k} \frac{\mu_{ij}\mu_{jk}\mu_{ki}}{\mu_i^2\mu_j\mu_k} \\
&= \sum_{j\neq k} \frac{p_n^3 w_i^2 w_j^2 w_k^2}{w^4 p_n^4 w_i^2 w_j w_k \left(1 + O\left(\frac{1}{n}\right)\right)} \\
&= \frac{1}{w^4 p_n} \sum_{j\neq k} w_j w_k + O\left(\frac{1}{n^3 p_n}\right) \\
&= \frac{1}{w^2 p_n} + O\left(\frac{1}{n^3 p_n}\right),
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{ij} &= \sum_{k\notin\{i,j\}} \frac{\mu_{jk}\mu_{ki}}{\mu_i\mu_j\mu_k} \\
&= \sum_{k\notin\{i,j\}} \frac{p_n^2 w_i w_j w_k^2}{w^3 p_n^3 w_i w_j w_k \left(1 + O\left(\frac{1}{n}\right)\right)} \\
&= \frac{1}{w^3 p_n} \sum_{k\notin\{i,j\}} w_k + O\left(\frac{1}{n^3 p_n}\right) \\
&= \frac{1}{w^2 p_n} + O\left(\frac{1}{n^3 p_n}\right).
\end{aligned}$$

Hence we get

$$\begin{aligned} v_{2n}^2 &= \sum_{i<j}\left(\gamma_{ij} - \frac{\eta_i + \eta_j}{2}\right)^2 \mu_{ij}(1-\mu_{ij}) \\ &= O\left(\frac{1}{n^6 p_n^2}\right)\sum_{i<j}\mu_{ij}(1-\mu_{ij}) \\ &= O\left(\frac{1}{n^4 p_n}\right). \end{aligned}$$

Then $v_{2n} = o(v_{1n})$. Then the proof is complete. □

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