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New construction methods for uninorms on bounded lattices

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Abstract. After Karaçal and Mesiar introduced uninorms on a bounded lattice in [25] and showed their existence on an arbitrary bounded lattice, construction methods of uninorms on bounded lattices have been widely studied in which the existence of t-norms and t-conorms on sublattices of the bounded lattice L was generally exploited. In this paper, we introduce two new construction methods for uninorms on a bounded lattice L by exploiting the existence of a triangular norm T and triangular conorm S on a sublattice of L, where $L \setminus \{0, 1\}$ has the bottom and the top elements. Then, we demonstrate that our new construction methods are also different from the existing construction methods in the literature. Additionally, some illustrative examples are provided. Finally, we generalize by induction our construction methods to a more general form.

1. Introduction

Aggregation functions satisfy the monotonicity and boundary conditions. The purpose of aggregation functions is simply to combine several inputs to a single output. The most well-known aggregation functions are triangular norms (conorms) (shortly t-norms (t-conorms)), uninorms and nullnorms, which also satisfy associativity.

Uninorms were introduced by Yager and Rybalov on the unit interval $[0, 1]$ in [37], as generalizations of t-norms and t-conorms since the neutral element of a uninorm is an arbitrary element of $[0, 1]$. They have been widely recognized as important aggregation functions in fuzzy logic, expert systems, neural networks.

In recent years, after Karaçal and Mesiar introduced and showed the existence of uninorms with the neutral element $e \in L \setminus \{0, 1\}$ on an arbitrary bounded lattice L [25], studying uninorms on bounded lattices has become a popular field due to the fact that bounded lattices are more general than unit interval ([1, 5, 8, 9, 11–14, 16, 18, 20, 26, 28, 37–39]). In this paper, we propose two construction methods for uninorms on a bounded lattice such that $L \setminus \{0, 1\}$ has the bottom and top elements. Also we give extensions of our construction methods by induction so that it can be practically implemented on proper lattices.

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The paper is structured as follows. In Section 2, we recall notions of a bounded lattice, t-norms, t-conorms and uninorms on bounded lattices. Then, we remind several construction methods for uninorms known in the literature. Section 3 contains the main results: new construction methods for uninorms on a bounded lattice L exploiting t-norms and t-conorms on subintervals of the bounded lattice L. Firstly, considering the existence of a t-norm on $[a,e]$ and a t-conorm on $[e,b]$, we introduce two new construction methods for uninorms on a bounded lattice L, where $L \setminus \{0,1\}$ has the bottom and the top elements. Furthermore, some examples are provided to illustrate that our new construction methods are different from the existing methods in the literature.

2. Notations, definitions and a review of previous results

In this section, we recall some basic notions and results.

Definition 2.1. *[4] A partially ordered set $(P,\leq)$ is called lattice if any two elements x, y in P have the greatest lower bound denoted by inf $\{x,y\}$ or $x \wedge y$ and the least upper bound denoted by sup $\{x,y\}$ or $x \vee y$.*

Definition 2.2. *[4] A lattice $(L,\leq)$ is a bounded lattice if L has the top element 1 and the bottom element 0, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.*

Definition 2.3. *[4] Let $(L,\leq,0,1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. The sublattice $[a,b]$ is defined as*

$$[a,b] = \{x \in L| \quad a \leq x \leq b\}.$$

Similarly, $(a,b] = \{x \in L| \quad a < x \leq b\}$, $[a,b) = \{x \in L| \quad a \leq x < b\}$ and $(a,b) = \{x \in L| \quad a < x < b\}$ can be defined.

Definition 2.4. *[4] Let $(L,\leq,0,1)$ be a bounded lattice. The elements x and y are called comparable and denoted by $x \nparallel y$ if $x \leq y$ or $y \leq x$. Otherwise, x and y are called incomparable and the notation $x\|y$ is used for such elements. In the following, I_a denotes the family of all elements incomparable with a, i.e., $I_a = \{x \in L : x\|a\}$. $A(e)$ is defined as $[0,e] \times [e,1] \cup [e,1] \times [0,e]$.*

We denote by $I_e^{a,b}$ for the set of elements which are incomparable with e but comparable with a and b, i.e., $I_e^{a,b} = \{x \in L : x\|e \text{ and } x \nparallel a \text{ and } x \nparallel b\}$. Similarly, we denote $I_a^{e,b} = \{x \in L : x\|a \text{ and } x \nparallel e \text{ and } x \nparallel b\}$, $I_b^{a,e} = \{x \in L : x\|b \text{ and } x \nparallel a \text{ and } x \nparallel e\}$, $I_{a,e}^{b} = \{x \in L : x\|a \text{ and } x\|e \text{ and } x \nparallel b\}$, $I_{e,b}^{a} = \{x \in L : x\|e \text{ and } x\|b \text{ and } x \nparallel a\}$ and $I_{a,e,b} = \{x \in L : x\|a \text{ and } x\|e \text{ and } x\|b\}$.

Definition 2.5. *[27] Let $(L,\leq,0,1)$ be a bounded lattice. An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).*

Example 2.6. *Let $(L,\leq,0,1)$ be a bounded lattice. The weakest t-norm T_W and the strongest t-norm $T_\wedge$ on bounded lattice L are given respectively as:*

$$T_W(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$T_\wedge(x,y) = x \wedge y.$$

The weakest t-conorm $S_\vee$ and the strongest t-conorm S_W on bounded lattice L are given respectively as:

$$S_\vee(x,y) = x \vee y$$

$$S_W(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.7. *[32] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $F : L^2 \to L$ is called a t-subnorm on L if it is commutative, associative, increasing with respect to both variables, and $F(x, y) \leq x \wedge y$ for all $x, y \in L$.*

Definition 2.8. *[32] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $R : L^2 \to L$ is called a t-subconorm on L if it is commutative, associative, increasing with respect to both variables, and $R(x, y) \geq x \vee y$ for all $x, y \in L$.*

Definition 2.9. *[25] Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation $U : L^2 \to L$ is called a uninorm if it has a neutral element $e \in L$ such that $U(e, x) = x$ for all $x \in L$ and satisfies the commutativity, associativity and it is increasing with respect to both variables.*

We denote by $U(e)$ the set of all uninorms on L with the neutral element $e \in L$.

A uninorm U is a t-norm T (t-conorm S) in the case $e = 1$ ($e = 0$).

Proposition 2.10. *[25] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then the following properties hold:*
i) $x \wedge y \leq U(x, y) \leq x \vee y$ for $(x, y) \in A(e)$.
ii) $U(x, y) \leq x$ for $(x, y) \in L \times [0, e]$.
iii) $U(x, y) \leq y$ for $(x, y) \in [0, e] \times L$.
iv) $x \leq U(x, y)$ for $(x, y) \in L \times [e, 1]$.
v) $y \leq U(x, y)$ for $(x, y) \in [e, 1] \times L$.

Proposition 2.11. *[25] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L$ and U a uninorm with the neutral element e on L. Then*
(i) $T_* = U|_{[0,e]^2} : [0, e]^2 \to [0, e]$ is a t-norm on $[0, e]$.
(ii) $S_* = U|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ is a t-conorm on $[e, 1]$.

Definition 2.12. *[3] Let L be a bounded lattice and A and B be two aggregation functions on L. A is called smaller than B if for any elements $x, y \in L$, $A(x, y) \leq B(x, y)$.*

Definition 2.13. *[8] A uninorm U on L is called conjunctive (resp. disjunctive) if $U(0, 1) = 0$ (resp. $U(0, 1) = 1$).*

Now, let us recall some construction methods for uninorms on a bounded lattice presented in [8, 11, 18, 25].

Theorem 2.14. *[8] Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0, 1\}$. If T_e is a t-norm on $[0, e]^2$ and S_e is a t-conorm on $[e, 1]^2$, then the functions $U_{t_1} : L^2 \to L$ and $U_{s_1} : L^2 \to L$ defined as follows*

$$U_{t_1}(x, y) = \begin{cases} T_e(x, y) & (x, y) \in [0, e]^2, \\ y & (x, y) \in [0, e] \times I_e, \\ x & (x, y) \in I_e \times [0, e], \\ x \vee y & \text{otherwise} \end{cases} \tag{1}$$

and

$$U_{s_1}(x, y) = \begin{cases} S_e(x, y) & (x, y) \in [e, 1]^2, \\ y & (x, y) \in [e, 1] \times I_e, \\ x & (x, y) \in I_e \times [e, 1], \\ x \wedge y & \text{otherwise} \end{cases} \tag{2}$$

are uninorms on L.

Theorem 2.15. *[11] Let $e \in L \setminus \{0, 1\}$. If T_e is a t-norm on $[0, e]^2$ and S_e is a t-conorm on $[e, 1]^2$, then the functions $U_e^T : L^2 \to L$ and $U_e^S : L^2 \to L$ are uninorms on L with the neutral element e, where*

$$U_e^T(x, y) = \begin{cases} T_e(x, y) & (x, y) \in [0, e]^2, \\ x \vee y & (x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e], \\ y & (x, y) \in [0, e] \times I_e, \\ x & (x, y) \in I_e \times [0, e], \\ x \vee y \vee e & (x, y) \in I_e \times I_e, \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

and

$$U_e^S(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ x \wedge y & (x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e], \\ y & (x,y) \in [e,1] \times I_e, \\ x & (x,y) \in I_e \times [e,1], \\ x \wedge y \wedge e & (x,y) \in I_e \times I_e, \\ 0 & \textit{otherwise.} \end{cases} \tag{4}$$

Theorem 2.16. *[18] Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. If T_e is a t-norm on $[0,e]^2$ and S_e is a t-conorm on $[e,1]^2$, then the functions $U_{(T,e)} : L^2 \to L$ and $U_{(S,e)} : L^2 \to L$ defined as follows*

$$U_{(T,e)}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ y & (x,y) \in [0,e] \times I_e, \\ x & (x,y) \in I_e \times [0,e], \\ x \vee y \vee e & \textit{otherwise} \end{cases} \tag{5}$$

and

$$U_{(S,e)}(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ y & (x,y) \in [e,1] \times I_e, \\ x & (x,y) \in I_e \times [e,1], \\ x \wedge y \wedge e & \textit{otherwise} \end{cases} \tag{6}$$

are uninorms on L.

Theorem 2.17. *[25] Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. If T_e is a t-norm on $[0,e]^2$ and S_e is a t-conorm on $[e,1]^2$, then the functions $U_t : L^2 \to L$ and $U_s : L^2 \to L$ defined as follows*

$$U_t(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ x \vee y & (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e], \\ y & (x,y) \in [0,e] \times I_e, \\ x & (x,y) \in I_e \times [0,e], \\ 1 & \textit{otherwise} \end{cases} \tag{7}$$

and

$$U_s(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ x \wedge y & (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e), \\ y & (x,y) \in [e,1] \times I_e, \\ x & (x,y) \in I_e \times [e,1], \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

are uninorms on L.

In addition to those methods in Theorems 2.11, 2.12, 2.13 and 2.14, there exist some construction methods for uninorms under some constraints [1, 9, 13, 16].

Theorem 2.18. *[1] Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$ such that for all $x \in I_e$ and $y \in (0,e]$ it holds $x \| y$. Given t-norm T_e on $[0,e]$, then the function $U_{T_e} : L^2 \to L$ defined as follows is a uninorm on L with the neutral element e, $U_{T_e} \in U(e)$ where*

$$U_{T_e}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ y & (x,y) \in [e,1] \times I_e, \\ x & (x,y) \in I_e \times [e,1], \\ 0 & (x,y) \in [0,e) \times I_e \cup I_e \times [0,e), \\ x \wedge y & (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \cup I_e \times I_e, \\ x \vee y & \textit{otherwise.} \end{cases} \tag{9}$$

Theorem 2.19. *[1] Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0,1\}$ such that for all $x \in I_e$ and $y \in [e,1)$ it holds $x \| y$. Given t-conorm S_e on $[e,1]$, then the function $U_{S_e} : L^2 \to L$ defined as follows is a uninorm on L with the neutral element e, $U_{S_e} \in \mathcal{U}(e)$ where*

$$U_{S_e}(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ y & (x,y) \in [0,e] \times I_e, \\ x & (x,y) \in I_e \times [0,e], \\ 1 & (x,y) \in (e,1] \times I_e \cup I_e \times (e,1], \\ x \vee y & (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e] \cup I_e \times I_e, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{10}$$

Theorem 2.20. *[9] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, T_e be a t-norm on $[0,e]$. If $x \vee y > e$ for all $x, y \in I_e$ or $x \vee y \in I_e$ for all $x, y \in I_e$, then the function $U_{t_e} : L^2 \to L$ defined as follows is a uninorm on L with the neutral element e:*

$$U_{t_e}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ x \vee y & (x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e] \cup I_e \times I_e, \\ y & (x,y) \in [0,e] \times I_e, \\ x & (x,y) \in I_e \times [0,e], \\ 1 & \text{otherwise.} \end{cases} \tag{11}$$

Theorem 2.21. *[9] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$, S_e be a t-norm on $[e,1]$. If $x \wedge y < e$ for all $x, y \in I_e$ or $x \wedge y \in I_e$ for all $x, y \in I_e$, then the function $U_{s_e} : L^2 \to L$ defined as follows is a uninorm on L with the neutral element e:*

$$U_{s_e}(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ x \wedge y & (x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e] \cup I_e \times I_e, \\ y & (x,y) \in [e,1] \times I_e, \\ x & (x,y) \in I_e \times [e,1], \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Theorem 2.22. *[13] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$ and T_e be a t-norm on $[0,e]^2$. $T_e(x,y) \in (0,e]$ for all $x, y \in (0,e]$ if and only if the function $U_{T_1} : L^2 \to L$ is a uninorm on L with the neutral element e, where*

$$U_{T_1}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ y & (x,y) \in (0,e] \times I_e, \\ x & (x,y) \in I_e \times (0,e], \\ x \wedge y & (x,y) \in [e,1] \times \{0\} \cup \{0\} \times [e,1] \cup I_e \times \{0\} \cup \\ & \{0\} \times I_e, \\ x \vee y & \text{otherwise.} \end{cases} \tag{13}$$

Theorem 2.23. *[13] Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$ and S_e be a t-conorm on $[e,1]^2$. $S_e(x,y) \in [e,1)$ for all $x, y \in [e,1)$ if and only if the function $U_{S_1} : L^2 \to L$ is a uninorm on L with the neutral element e, where*

$$U_{S_1}(x,y) = \begin{cases} S_e(x,y) & (x,y) \in [e,1]^2, \\ y & (x,y) \in [e,1) \times I_e, \\ x & (x,y) \in I_e \times [e,1), \\ x \vee y & (x,y) \in [0,e] \times \{1\} \cup \{1\} \times [0,e] \cup I_e \times \{1\} \cup \\ & \{1\} \times I_e, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{14}$$

Theorem 2.24. *[16] Suppose that $e \in L \setminus \{0,1\}$ and $a < c$ for all $a < e$ and $c \| e$. Given a t-norm $T_e : [0,e]^2 \to [0,e]$*

and a t-conorm $S_e : [e,1]^2 \to [e,1]$ *, then the function* $U^{'} : L^2 \to L$ *expressed by*

$$U^{'}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ S_e(x,y) & (x,y) \in [e,1]^2, \\ x \wedge y & (x,y) \in [0,e) \times I_e \cup I_e \times [0,e), \\ y & (x,y) \in \{e\} \times I_e, \\ x & (x,y) \in I_e \times \{e\}, \\ x \vee y & \text{otherwise} \end{cases} \tag{15}$$

is a uninorm on L that possesses the neutral element e if and only if $c < b$ *and* $c \vee d \| e$ *for all* $e < b$ *and* $c, d \| e$.

Theorem 2.25. *[16] Suppose that* $e \in L \setminus \{0,1\}$ *and* $a > c$ *for all* $a > e$ *and* $c \| e$. *Given a t-norm* $T_e : [0,e]^2 \to [0,e]$ *and a t-conorm* $S_e : [e,1]^2 \to [e,1]$ *, then the function* $U^{''} : L^2 \to L$ *expressed by*

$$U^{''}(x,y) = \begin{cases} T_e(x,y) & (x,y) \in [0,e]^2, \\ S_e(x,y) & (x,y) \in [e,1]^2, \\ x \vee y & (x,y) \in (e,1] \times I_e \cup I_e \times (e,1], \\ y & (x,y) \in \{e\} \times I_e, \\ x & (x,y) \in I_e \times \{e\}, \\ x \wedge y & \text{otherwise} \end{cases} \tag{16}$$

is a uninorm on L that possesses the neutral element e if and only if $b < c$ *and* $c \wedge d \| e$ *for all* $b < e$ *and* $c, d \| e$.

For more details about construction methods for uninorms from triangular norms (conorms), we recommend [1, 2, 5, 8, 9, 11–16, 18, 20, 25, 26, 35, 36]. It is worth noting that all construction methods recalled in this section fall within classes introduced in [41].

Definition 2.26. *[21] Let* $(L, \leq, \wedge, \vee)$ *be a lattice. A mapping* $cl : L \to L$ *is said to be a closure operator if, for any* $x, y \in L$*, it satisfies the following three conditions:*
(i) $x \leq cl(x)$ *(expansion);*
(ii) $cl(x \vee y) = cl(x) \vee cl(y)$ *(preservation of join);*
(iii) $cl(cl(x)) = cl(x)$ *(idempotence).*

Definition 2.27. *[21] Let* $(L, \leq, \wedge, \vee)$ *be a lattice. A mapping* $int : L \to L$ *is said to be an interior operator if, for any* $x, y \in L$*, it satisfies the following three conditions:*
(i) $int(x) \leq x$ *(contraction);*
(ii) $int(x \wedge y) = int(x) \wedge int(y)$ *(preservation of meet);*
(iii) $int(int(x)) = int(x)$ *(idempotence).*

Theorem 2.28. *[19] Let L be a meet semilattice and let* $h : L \to L$ *be an interior operator on L. Let M denote the image of L under h, i.e.,* $h(L) = M$*. Then,*
(i) *M is a meet sub-semilattice of L with the bottom element* 0 *and the top element* 1,
(ii) *if V is a t-norm on M, then there exists its extension to a t-norm T on L as follows:*

$$T(x,y) = \begin{cases} V(h(x), h(y)) & x, y \in L \setminus \{1\}, \\ x \wedge y & \text{otherwise}. \end{cases} \tag{17}$$

Theorem 2.29. *[10] Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $a \in L \setminus \{0,1\}$. *If V is a t-norm on* $[a,1]$ *and W is a t-conorm on* $[0,a]$*, then the functions* $T : L^2 \to L$ *and* $S : L^2 \to L$ *are, respectively, a t-norm and a t-conorm on L, where*

$$T(x,y) = \begin{cases} V(x,y) & (x,y) \in [a,1[^2, \\ x \wedge y & 1 \in \{x,y\}, \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

and

$$S(x,y)=\begin{cases} W(x,y) & (x,y)\in]0,a]^2,\\ x\vee y & 0\in\{x,y\},\\ 1 & \textit{otherwise}.\end{cases} \tag{19}$$

In the following result, we remind the ordinal sum construction of Clifford [7] as it was formulated in [27].

Theorem 2.30. *Let $A\neq\emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha=(X_\alpha,*_\alpha)$ be a family of semigroups. Assume that for all $\alpha,\beta\in A$ with $\alpha<\beta$ the sets X_α and X_β are either disjoint or that $X_\alpha\cap X_\beta=\{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma\in A$ with $\alpha<\gamma<\beta$ we have $X_\gamma=\{x_{\alpha,\beta}\}$. Put $X=\bigcup\limits_{\alpha\in A}X_\alpha$ and define the binary operation $*$ on X by*

$$x*y=\begin{cases} x *_\alpha y & \text{if } (x,y)\in X_\alpha\times X_\alpha,\\ x & \text{if } (x,y)\in X_\alpha\times X_\beta \text{ and } \alpha<\beta,\\ y & \text{if } (x,y)\in X_\alpha\times X_\beta \text{ and } \alpha>\beta.\end{cases} \tag{20}$$

Then $G=(X,*)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha\in A$ the semigroup G_α is commutative.

In [29] the authors generalize ordinal sum construction into the z-ordinal sum construction we give this result in the following.

Theorem 2.31. *Let A and B be two index sets such that $A\cap B=\emptyset$ and $C=A\cup B\neq\emptyset$. Let $(G_\alpha)_{\alpha\in C}$ with $G_\alpha=(X_\alpha,*_\alpha)$ be a family of semigroups and let set C be partially ordered by the binary relation $\preceq$ such that $(C,\preceq)$ is a meet semilattice. Further suppose that each semigroup G_α for $\alpha\in A$ possesses an annihilator z_α, and for all $\alpha,\beta\in C$ such that α and β are incomparable there is $\alpha\wedge\beta\in A$. Assume that for all $\alpha,\beta\in C$, $\alpha\neq\beta$, the sets X_α and X_β are either disjoint or that $X_\alpha\cap X_\beta=\{x_{\alpha,\beta}\}$. In the second case suppose that for all $\gamma\in C$ which is incomparable with $\alpha\wedge\beta$ there is $\alpha\wedge\gamma=\beta\wedge\gamma$ and for each $\gamma\in C$ with $\alpha\wedge\beta\prec\gamma\prec\alpha$ or $\alpha\wedge\beta\prec\gamma\prec\beta$ we have $X_\gamma=\{x_{\alpha,\beta}\}$. Further,*

(i) in the case that $\alpha\wedge\beta\in A$ then $x_{\alpha,\beta}=z_{\alpha\wedge\beta}$ is the annihilator of both G_β and G_α;
(ii) in the case that $\alpha\wedge\beta=\alpha\in B$ then $x_{\alpha,\beta}$ is both the annihilator of G_β and the neutral element of G_α.
Put $X=\bigcup\limits_{\alpha\in C}X_\alpha$ and define the binary operation $*$ on X by

$$x*y=\begin{cases} x *_\alpha y & \text{if } (x,y)\in X_\alpha\times X_\alpha,\\ x & \text{if } (x,y)\in X_\alpha\times X_\beta,\ \alpha\neq\beta, \text{ and } \alpha\wedge\beta=\alpha\in B,\\ y & \text{if } (x,y)\in X_\alpha\times X_\beta,\ \alpha\neq\beta, \text{ and } \alpha\wedge\beta=\beta\in B,\\ z_\gamma & \text{if } (x,y)\in X_\alpha\times X_\beta,\ \alpha\neq\beta, \text{ and } \alpha\wedge\beta=\gamma\in A.\end{cases} \tag{21}$$

Then $G=(X,*)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha\in C$ the semigroup G_α is commutative.

Note that if $A=\emptyset$ then the z-ordinal sum reduces to the standard ordinal sum. For more details we recommend [29–31, 34].

3. Construction of uninorms on bounded lattices

In this section, we propose two construction methods for uninorms on a bounded lattice L by exploiting the existence of a t-norm T on $[a,e]$ and a t-conorm S on $[e,b]$ in Theorems 3.1 and 3.8, where $L\setminus\{0,1\}$ has the bottom and the top elements. We compare our construction methods with those construction methods in the literature by choosing the same underlying t-norm and t-conorm properly. And, we emphasize the differences between our construction methods and those in the literature. Some illustrative examples for our construction methods on a bounded lattice are also provided. Finally, we extend our methods recursively to proper bounded lattices as in Theorems 3.14 and 3.16.

In the following theorem, we introduce a new construction method for uninorms on a bounded lattice. Our construction method is obtained exploiting the existence of a t-norm T on the subinterval $[a,e]$ of the bounded lattice L and a t-conorm S on the subinterval $[e,b]$ of the bounded lattice L, where we have the bottom element k and the top element l of $L \setminus \{0,1\}$.

Theorem 3.1. *Let $(L,\leq,0,1)$ be a bounded lattice, and let $a,b,e,k,l \in L\setminus\{0,1\}$, where $k \leq a < e < b \leq l$ and k be the bottom element and l be the top element of $L\setminus\{0,1\}$. If T is a t-norm on $[a,e]^2$ and S is a t-conorm on $[e,b]^2$, then the function $U^*: L^2 \to L$ defined by*

$$U^*(x,y) = \begin{cases} T(x\wedge e, y\wedge e) & (x,y) \in [a,e]^2 \cup [a,e) \times I_e^a \cup I_e^a \times ([a,e) \cup I_e^a), \\ S(x,y) & (x,y) \in [e,b]^2, \\ l & (x,y) \in (e,l)^2 \setminus (e,b]^2, \\ x & (x,y) \in ([0,e) \cup I_e^a \cup I_{a,e}) \times [e,1] \cup (e,1] \times \{e\} \cup \{0\} \times (L \setminus (e,1)) \\ & \cup \{1\} \times (e,1], \\ y & (x,y) \in [e,1] \times ([0,e) \cup I_e^a \cup I_{a,e}) \cup \{e\} \times (e,1] \cup (L\setminus(e,1)) \times \{0\} \\ & \cup (e,1] \times \{1\}, \\ k & \text{otherwise}, \end{cases} \tag{22}$$

is a uninorm on L with the neutral element e.

Proof. (i) *Monotonicity*: Let us show that for every elements $x,y \in L$ with $x \leq y$, $U^*(x,z) \leq U^*(y,z)$ for all $z \in L$. If x and y are both elements of $[k,l]\setminus[e,1]$ or $[e,l]$, $U^*(x,z) \leq U^*(y,z)$ is always satisfied for all $z \in L$ since $x \leq y$ and $k \leq T(x,y)$ for all $x,y \in [a,e]$. If $0,1 \in \{x,y,z\}$, the inequality is satisfied, therefore, they are omitted. The proof is then split into all the remain possible cases as follows.

1. Let $x \in [k,l]\setminus[e,1]$.

1.1. $y \in [e,l]$,

1.1.1. If $z \in [k,l]\setminus[e,1]$, since $U^*(x,z) \in \{k, T(y\wedge e, z\wedge e)\}$, $U^*(x,z) \leq z = U^*(y,z)$.

1.1.2. If $z \in [e,l]$, since $U^*(y,z) \in \{l, S(y,z)\}$, $U^*(x,z) = x \leq U^*(y,z)$.

(ii) *Associativity*: To show associativity of U^*, we consider ordinal sum/ z-ordinal sum construction as we remind in Theorems 2.30 and 2.31. The operator U^* in the formula (22) can be obtained as an ordinal sum of the semigroups $G_1 = (\{0\}, Id)$, $G_2 = ([e,1), U_1)$, $G_3 = (\{1\}, Id)$ and $G_4 = (L\setminus([e,1]\cup\{0\}), U_2)$, the operators U_1, U_2 are obtained as follows: U_1 is extension of U_1^* on $[e,l]\setminus I_b$ via closure operator $cl : L \to L$,

$$cl\,(x) = \begin{cases} x \vee e \vee b, & \text{if } x \in I_b \\ x \vee e, & \text{if } x \notin I_b \end{cases}$$

where U_1^* is an z-ordinal sum of semigroups $H_1 = (\{l\}, Id)$, $H_2 = ((e,b], S)$, $H_3 = ((b,l], *_3 = l)$ and $H_4 = (\{e\}, Id)$ with respect to a partially ordered an index set $(C_1, \preceq_1)$, where $1 = 2 \wedge_1 3$, $2 \prec_1 4$, $3 \prec_1 4$ and $2 \|_1 3$. U_2 is extension of U_2^* on $L\setminus([e,1]\cup I_a \cup I_e \cup \{0\})$ via interior operator $int : L \to L$,

$$int\,(x) = \begin{cases} x \wedge e \wedge a, & \text{if } x \in I_a \\ x \wedge e, & \text{if } x \notin I_a \end{cases}$$

where U_2^* is an z-ordinal sum of semigroups $K_1 = (\{k\}, Id)$, $K_2 = ([k,a), *_2 = k)$, and $K_3 = ([a,e), T)$ with respect to a partially ordered an index set $(C_2, \preceq_2)$, where $1 = 2 \wedge_2 3$ and $2 \|_2 3$.

It is easy to observe the commutativity of U^* and the fact that e is a neutral element of U^*.

Therefore, U^* is a uninorm on L with the neutral element e. □

The structure of the uninorm U^* given in formula (22) can be summarized Figure 1, in which $T(x_e, y_e)$ denotes $T(x \wedge e, y \wedge e)$. In order to avoid any confusion, we would like to specify the values of U^* on the borders seperately: $U^*(x, y) = 0$ when $(x, y) \in \{0\} \times L$, $U^*(x, y) = 1$ when $(x, y) \in [e, 1] \times \{1\}$, $U^*(x, y) = x$ when $(x, y) \in ([0, e] \cup I_{a,e}) \times \{1\}$, $U^*(x, y) = a$ when $(x, y) \in L^2 \setminus ([e, 1] \times \{a\})$, $U^*(x, y) = b$ when $(x, y) \in [e, b] \times \{b\}$, $U^*(x, y) = k$ when $(x, y) \in ([0, a) \cup I_{a,e}) \times \{a\}$, $U^*(x, y) = a$ when $(x, y) \in ([e, 1] \times \{a\}$, $U^*(x, y) = x$ when $(x, y) \in ([0, e) \cup I_{a,e}) \times \{b\}$, $U^*(x, y) = l$ when $(x, y) \in (b, 1] \times \{b\}$.

Let us define the sets $L_1 = \{x \in L \mid x \ngtr a\}$, $L_2 = [a, e] \cup I_e^a$, $L_3 = [e, b]$ and $L_4 = [b, 1] \cup I_b^{a,e}$.

L_4	x	x	l	l
L_3	x	x	$S(x, y)$	l
L_2	k	$T(x_e, y_e)$	y	y
L_1	k	k	y	y
	L_1	L_2	L_3	L_4

Figure 1: The structure of the uninorm U^*

In the following example, we exemplify how to apply Theorem 3.1.

Example 3.2. *Consider the bounded lattice* $(L_1 = \{0, a, b, c, d, e, f, k, l, m, n, t, 1\}, \leq, 0, 1)$ *characterized by the Hasse diagram in Figure* 2. *It is easy to check that* L_1 *satisfies the constraints of Theorem* 3.1.

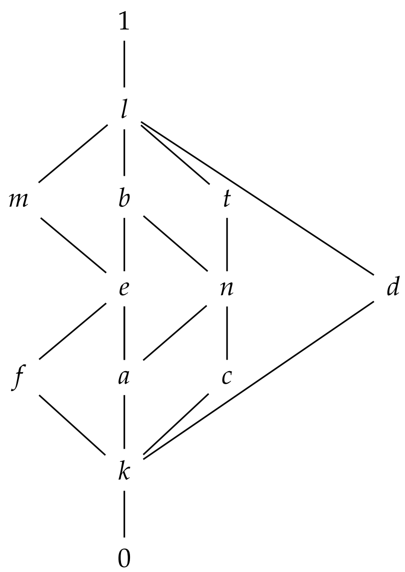

Figure 2: Lattice diagram of L_1.

It is easy to see that $I_a^{e,b} = \{f\}, I_{a,e}^b = \{c\}, I_{a,e,b} = \{d\}, I_b^{a,e} = \{m\}$ *and* $I_{e,b}^a = \{t\}$. *If we apply the formula* (22) *in Theorem* 3.1, *when* $T = T_\wedge$ *on* $[a, e]$ *and* $S = S_\vee$ *on* $[e, b]$, *the uninorm* U^* *on* L_1 *is obtained as in Table* 1.

U^*	0	k	a	f	c	n	d	t	e	b	m	l	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
k	0	k	k	k	k	k	k	k	k	k	k	k	k
a	0	k	a	k	k	a	k	a	a	a	a	a	a
f	0	k	k	k	k	k	k	k	f	f	f	f	f
c	0	k	k	k	k	k	k	k	c	c	c	c	c
n	0	k	a	k	k	a	k	a	n	n	n	n	n
d	0	k	k	k	k	k	k	k	d	d	d	d	d
t	0	k	a	k	k	a	k	a	t	t	t	t	t
e	0	k	a	f	c	n	d	t	e	b	m	l	1
b	0	k	a	f	c	n	d	t	b	b	l	l	1
m	0	k	a	f	c	n	d	t	m	l	l	l	1
l	0	k	a	f	c	n	d	t	l	l	l	l	1
1	0	k	a	f	c	n	d	t	1	1	1	1	1

Table 1: The uninorm U^* induced by the formula (22) in Theorem 3.1.

Remark 3.3.
Note that the constraint of Theorem 3.1 *cannot be omitted in general. The following example illustrates this fact.*

The lattice L_2 is a negative example that does not satisfy the constraint of Theorem 3.1. It is easily seen that $L_2 \setminus \{0, 1\}$ has not the bottom and top elements.

Example 3.4. *Consider the bounded lattice* $(L_2 = \{0, a, b, c, d, e, f, k, l, 1\}, \leq, 0, 1)$ *characterized by the Hasse diagram in Figure* 3 *.*

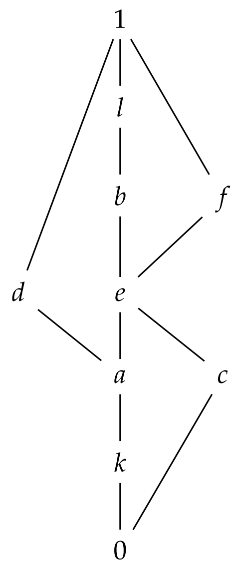

Figure 3: Lattice diagram of L_2.

U^* does not satisfy the monotonicity on L_2 since $U^*(k, c) = k \nleq c = U^*(l, c)$, while $k < l$ for the elements $k, l \in L_2$. Therefore, U^* is not a uninorm on L_2.

Remark 3.5.
(i) In general, it should be pointed out that the uninorm U^* defined in Theorem 3.1 is different from the uninorms $U_{t_1}, U_{s_1}, U_e^T, U_e^S, U_{(T,e)}, U_{(S,e)}, U_t$ and U_s defined in Theorems 2.14, 2.15, 2.16 and 2.17 on any bounded lattice L such that $L \setminus \{0,1\}$ has the bottom element k and the top element l regardless of the choices of t-norm T and/or t-conorm S.
- $U_{t_1}(k,l) = k \vee l = l \neq k = U^*(k,l)$;
- $U_{s_1}(x,y) = x \neq k = U^*(x,y)$ *when* $(x,y) \in (k,a) \times (a,e)$;
- $U_e^T(k,l) = l \neq k = U^*(k,l)$;
- $U_e^S(k,k) = 0 \neq k = U^*(k,k)$;
- $U_{(T,e)}(k,l) = l \neq k = U^*(k,l)$;
- $U_{(S,e)}(x,y) = x \neq k = U^*(x,y)$ *when* $(x,y) \in (k,a) \times (a,e)$;
- $U_t(k,l) = l \neq k = U^*(k,l)$;
- $U_s(k,k) = 0 \neq k = U^*(k,k)$.
(ii) Note that, on the lattices that simultaneously satisfy the conditions of the relevant theorems, the uninorm U^* defined in Theorem 3.1 is different from the uninorms $U_{t_1}, U_{s_1}, U_e^T, U_e^S, U_{(T,e)}, U_{(S,e)}, U_t$ and U_s defined in Theorems 2.14, 2.15, 2.16 and 2.17 regardless of the choices of t-norm T and/or t-conorm S.
- $U_{T_e}(x,y) = 0 \neq k = U^*(k,1)$ *when* $(x,y) \in (0,a) \times I_e^a$;
- $U_{S_e}(k,l) = l \neq k = U^*(k,l)$;
- $U_{t_e}(k,l) = l \neq k = U^*(k,l)$;
- $U_{s_e}(k,k) = 0 \neq k = U^*(k,k)$;
- $U_{T_1}(k,l) = l \neq k = U^*(k,l)$;
- $U_{S_1}(k,1) = 1 \neq k = U^*(k,1)$;
- $U'(k,l) = l \neq k = U^*(k,l)$;
- $U''(x,y) = x \neq y = U^*(x,y)$ *when* $(x,y) \in (b,1) \times I_e^a$.

Theorem 3.1 can be generalized as stated in Theorem 3.6 as follows.

Theorem 3.6. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $a, b, e, t, s \in L \setminus \{0,1\}$ such that $t \leq a < e < b \leq s$, T_1 be t-norm on $[a,e]^2$, T_2 on $[0,t]^2$ be t-subnorm, S_1 be t-conorm on $[e,b]^2$, S_2 be a t-superconorm $[s,1]^2$. Then the function $U_\alpha : L^2 \to L$ defined by*

$$U_\alpha(x,y) = \begin{cases} T_1(x \wedge e, y \wedge e) & (x,y) \in [a,e]^2 \cup [a,e) \times I_e^a \cup I_e^a \times [a,e) \cup I_e^a \times I_e^a, \\ T_2(x \wedge t, y \wedge t) & (x,y) \in [0,a)^2 \cup [0,a] \times I_a^* \cup I_a^* \times ([0,a] \cup I_a^*) \cup \\ & [0,a) \times \{a\} \cup \{a\} \times [0,a), \\ S_1(x \vee e, y \vee e) & (x,y) \in [e,b]^2, \\ S_2(x \vee s, y \vee s) & (x,y) \in (b,1]^2 \cup [b,1] \times I_b^{a,e} \cup I_b^{a,e} \times ([b,1] \cup I_b^{a,e}) \\ & (b,1] \times \{b\} \cup \{b\} \times (b,1], \\ x \wedge t & (x,y) \in ([0,a) \cup I_a^*) \times ((a,e) \cup I_e^a), \\ y \wedge t & (x,y) \in ((a,e) \cup I_e^a) \times ([0,a) \cup I_a^*), \\ x \vee s & (x,y) \in ((b,1] \cup I_b^{a,e}) \times (e,b), \\ y \vee s & (x,y) \in (e,b) \times ((b,1] \cup I_b^{a,e}), \\ x & (x,y) \in ([0,e) \cup I_e^a) \times [e,1] \cup ((b,1] \cup I_b^{a,e}) \times \{e\}, \\ y & \text{otherwise}, \end{cases} \tag{23}$$

is a uninorm on L with the neutral element e.

Proof. Let us define the sets $L_1 = \{x \in L \mid x \not> a\}$, $L_2 = [a,e] \cup I_e^a$, $L_3 = [e,b]$ and $L_4 = [b,1] \cup I_b^{a,e}$.
(i) *Monotonicity*: Let us show that for every elements $x, y \in L$ with $x \leq y$, $U^*(x,z) \leq U^*(y,z)$ for all $z \in L$. If x and y are both elements of L_1 or L_2, L_3 and L_4, $U^*(x,z) \leq U^*(y,z)$ is always satisfied for all $z \in L$ since $x \leq y$. The proof is then split into all the remain possible cases as follows.
1. Let $x \in L_1$ and $y \in L \setminus L_1$.
    1.1. If $z \in L_1$, then $U_\alpha(x,z) \leq z \wedge t \leq U_\alpha(y,z)$.
    1.1. If $z \in L \setminus L_1$, then $U_\alpha(x,z) \leq a \leq U_\alpha(y,z)$.

2. Let $x \in L_2$ and $y \in L \setminus (L_1 \cup L_2)$.
 2.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x,z) \le z = U_\alpha(y,z)$.
 2.2. If $z \in L \setminus (L_1 \cup L_2)$, then $U_\alpha(x,z) \le e \le U_\alpha(y,z)$.
3. Let $x \in L_3$ and $y \in L_4$.
 3.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x,z) = z = U_\alpha(y,z)$.
 3.1. If $z \in L_3$, then $U_\alpha(x,z) \le b \le U_\alpha(y,z)$.
 3.1. If $z \in L_4$, then $U_\alpha(x,z) = z \vee s \le S_2(y \vee s, z \vee s) = U_\alpha(y,z)$.

(ii) *Associativity*: We demonstrate that $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(U_\alpha(x,y),z)$ for all $x,y,z \in L$. remain possible cases by considering the relationships between the elements x, y and z as follows.

1. Let $x \in L_1$.
1.1. $y \in L_1$,
 1.1.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_2(y \wedge t, z \wedge t)) = T_2(x \wedge t, T_2(y \wedge t, z \wedge t)) = T_2(T_2(x \wedge t, y \wedge t), z) = U_\alpha(T_2(x \wedge t, y \wedge t), z) = U_\alpha(U_\alpha(x,y),z)$.
 1.1.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \wedge t) = T_2(x \wedge t, y \wedge t) = T_2(x \wedge t, y \wedge t) \wedge t = U_\alpha(T_2(x \wedge t, y \wedge t), z) = U_\alpha(U_\alpha(x,y),z)$.
 1.1.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = T_2(x \wedge t, y \wedge t) = U_\alpha(T_2(x \wedge t, y \wedge t), z) = U_\alpha(U_\alpha(x,y),z)$.
1.2. $y \in L_2$,
 1.2.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \wedge t) = T_2(x \wedge t, z \wedge t) = U_\alpha(x \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
 1.2.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_1(y \wedge e, z \wedge e)) = x \wedge t = U_\alpha(x \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
 1.2.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = x \wedge t = U_\alpha(x \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
1.3. $y \in L_3$,
 1.3.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = T_2(x \wedge t, z \wedge t) = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.3.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = x \wedge t = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.3.3. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_1(y \vee e, z \vee e)) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.3.4. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \vee s) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
1.4. $y \in L_4$,
 1.4.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = T_2(x \wedge t, z \wedge t) = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.4.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = x \wedge t = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.4.3. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \vee s) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 1.4.4. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_2(y \vee s, z \vee s)) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
2. Let $x \in L_2$.
2.1. $y \in L_1$,
 2.1.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_2(y \wedge t, z \wedge t)) = T_2(x \wedge t, y \wedge t) \wedge t = T_2(x \wedge t, y \wedge t) = U_\alpha(y \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
 2.1.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \wedge t) = y \wedge t = U_\alpha(y \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
 2.1.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = y \wedge t = U_\alpha(y \wedge t, z) = U_\alpha(U_\alpha(x,y),z)$.
2.2. $y \in L_2$,
 2.2.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \wedge t) = z \wedge t = U_\alpha(T_1(x \wedge e, y \wedge e), z) = U_\alpha(U_\alpha(x,y),z)$.
 2.2.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_1(y \wedge e, z \wedge e)) = T_1(x \wedge e, T_1(y \wedge e, z \wedge e)) = T_1(T_1(x \wedge e, y \wedge e), z \wedge e) = U_\alpha(T_1(x \wedge e, y \wedge e), z) = U_\alpha(U_\alpha(x,y),z)$.
 2.2.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = T_1(x \wedge e, y \wedge e) = U_\alpha(T_1(x \wedge e, y \wedge e), z) = U_\alpha(U_\alpha(x,y),z)$.
2.3. $y \in L_3$,
 2.3.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z \wedge t = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.3.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = T_1(x \wedge e, z \wedge e) = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.3.3. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_1(y \vee e, z \vee e)) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.3.4. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \vee s) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
2.4. $y \in L_4$,
 2.4.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z \wedge t = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.4.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = T_1(x \wedge e, z \wedge e) = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.4.3. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \vee s) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
 2.4.4. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_2(y \vee s, z \vee s)) = x = U_\alpha(x,z) = U_\alpha(U_\alpha(x,y),z)$.
3. Let $x \in L_3$.

3.1. $y \in L_1$,

3.1.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_2(y \wedge t, z \wedge t)) = T_2(y \wedge t, z \wedge t) = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.1.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \wedge t) = y \wedge t = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.1.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = y = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.2. $y \in L_2$,

3.2.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \wedge t) = z \wedge t = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.2.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_1(y \wedge e, z \wedge e)) = T_1(y \wedge e, z \wedge e) = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.2.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = y = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

3.3. $y \in L_3$,

3.3.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z = U_\alpha(S_1(x \vee e, y \vee e), z) = U_\alpha(U_\alpha(x,y),z)$.

3.3.2. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_1(y \vee e, z \vee e)) = S_1(x \vee e, S_1(y \vee e, z \vee e)) = S_1(S_1(x \vee e, y \vee e), z \vee e) = U_\alpha(S_1(x \vee e, y \vee e), z) = U_\alpha(U_\alpha(x,y),z)$.

3.3.3. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \vee e) = z \vee e = U_\alpha(S_1(x \vee e, y \vee e), z) = U_\alpha(U_\alpha(x,y),z)$.

3.4. $y \in L_4$,

3.4.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z = U_\alpha(y \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

3.4.2. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \vee s) = y \vee s = U_\alpha(y \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

3.4.3. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_2(y \vee s, z \vee s)) = S_2(y \vee s, z \vee s) \vee s = S_2(y \vee s, z \vee s) = U_\alpha(y \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

4. Let $x \in L_4$.

4.1. $y \in L_1$,

4.1.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_2(y \wedge t, z \wedge t)) = T_2(y \wedge t, z \wedge t) = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.1.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \wedge t) = y \wedge t = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.1.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = y = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.2. $y \in L_2$,

4.2.1. If $z \in L_1$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \wedge t) = z \wedge t = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.2.2. If $z \in L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, T_1(y \wedge e, z \wedge e)) = T_1(y \wedge e, z \wedge e) = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.2.3. If $z \in L_3 \cup L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,y) = y = U_\alpha(y,z) = U_\alpha(U_\alpha(x,y),z)$.

4.3. $y \in L_3$,

4.3.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z = U_\alpha(x \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

4.3.2. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_1(y \vee e, z \vee e)) = x \vee s = U_\alpha(x \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

4.3.3. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, z \vee s)) = S_2(x \vee s, z \vee s) = U_\alpha(x \vee s, z) = U_\alpha(U_\alpha(x,y),z)$.

4.4. $y \in L_4$,

4.4.1. If $z \in L_1 \cup L_2$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x,z) = z = U_\alpha(S_2(x \vee s, y \vee s), z) = U_\alpha(U_\alpha(x,y),z)$.

4.4.2. If $z \in L_3$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, y \vee s) = S_2(x \vee s, y \vee s) = U_\alpha(S_2(x \vee s, y \vee s), z) = U_\alpha(U_\alpha(x,y),z)$.

4.4.3. If $z \in L_4$, then $U_\alpha(x, U_\alpha(y,z)) = U_\alpha(x, S_2(y \vee s, z \vee s)) = S_2(x \vee s, S_2(y \vee s, z \vee s)) = S_2(S_2(x \vee s, y \vee s), z \vee s)) = U_\alpha(S_2(x \vee s, y \vee s), z) = U_\alpha(U_\alpha(x,y),z)$.

It is easy to observe the commutativity of U_α and the fact that e is a neutral element of U^*.

Therefore, U_α is a uninorm on L with the neutral element e. □

The structure of the uninorm U_α given in formula (23) can be summarized Figure 4, in which $T_1(x_e, y_e)$, $T_2(x_t, y_t)$, $S_1(x^e, y^e)$, $S_2(x^s, y^s)$ mean that $T_1(x \wedge e, y \wedge e)$, $T_2(x \wedge t, y \wedge t)$, $S_1(x \vee e, y \vee e)$, $S_2(x \vee s, y \vee s)$, $I_a^* = I_a^{e,b} \cup I_{a,e,b} \cup I_{a,e}^b$, respectively.

L_4	x	x	$y \vee s$	$S_2(x^s, y^s)$
L_3	x	x	$S_1(x^e, y^e)$	$x \vee s$
L_2	$x \wedge t$	$T_1(x_e, y_e)$	y	y
L_1	$T_2(x_t, y_t)$	$y \wedge t$	y	y
	L_1	L_2	L_3	L_4

Figure 4: The structure of the uninorm U_α

Remark 3.7.
If L is a bounded lattice such that k be the bottom element and l be the top element of $L \setminus \{0,1\}$, $t = k$, $T = T_1$, $T_2(x,y) = \begin{cases} y & x = a, \\ x & y = a, \\ 0 & 0 \in \{x,y\}, \\ k & \text{otherwise}, \end{cases}$, $S_1 = S$, $S_2(x,y) = \begin{cases} y & x = b, \\ x & y = b, \\ 1 & 1 \in \{x,y\}, \\ l & \text{otherwise}, \end{cases}$ *and $s = l$ in the formula* (21)*, the formulas* (22) *and* (23) *are the same. Therefore, it is obvious that Theorem* 3.6 *is a generalization of Theorem* 3.1*. Moreover, the formula* (23) *in Theorem* 3.6 *produces different uninorm construction methods depending on $t \in (0,a]$ and $s \in [b,1)$.*

We introduce another construction method for uninorms on bounded lattice L as follows.

Theorem 3.8. *Let $(L, \leq, 0, 1)$ be a bounded lattice, and let $a,b,e,k,l \in L \setminus \{0,1\}$, where $k \leq a < e < b \leq l$ and k be the bottom element and l be the top element of $L \setminus \{0,1\}$. If T is a t-norm on $[a,e]^2$ and S is a t-conorm on $[e,b]^2$, then the function $U_* : L^2 \to L$ defined by*

$$U_*(x,y) = \begin{cases} T(x,y) & (x,y) \in [a,e]^2, \\ S(x \vee e, y \vee e) & (x,y) \in [e,b]^2 \cup (e,b] \times I_e^b \cup I_e^b \times ((e,b] \cup I_e^b), \\ k & (x,y) \in (k,e)^2 \setminus [a,e)^2, \\ x & (x,y) \in ([e,1] \cup I_e^b \cup I_{e,b}) \times [0,e] \cup [0,e) \times \{e\} \cup \{1\} \times (L \setminus (0,e)) \\ & \cup \{0\} \times [0,e), \\ y & (x,y) \in [0,e] \times ([e,1] \cup I_e^b \cup I_{e,b}) \cup \{e\} \times [0,e) \cup (L \setminus (0,e)) \times \{1\} \\ & \cup [0,e) \times \{0\}, \\ l & \text{otherwise}, \end{cases} \tag{24}$$

is a uninorm on L with the neutral element e.

Proof. It can be proved in an analogous way to the proof of Theorem 3.1. Therefore, we omit it. □

The structure of U_* on L given in the Theorem 3.8 is designed as shown in Figure 5, in which $S(x^e, y^e)$ means that $S(x \vee e, y \vee e)$, $I_b^* = I_b^{a,e} \cup I_{e,b}$. $U^*(x,y) = 1$ when $(x,y) \in \{1\} \times L$, $U^*(x,y) = 0$ when $(x,y) \in ((0,e) \cup I_{a,e}) \times \{0\}$, $U^*(x,y) = x$ when $(x,y) \in L \setminus ((0,e) \cup I_{a,e}) \times \{0\}$, $U^*(x,y) = a$ when $(x,y) \in [a,e] \times \{a\}$, $U^*(x,y) = k$ when $(x,y) \in ((0,e) \cup I_{a,e}) \times \{a\}$, $U^*(x,y) = x$ when $(x,y) \in L \setminus ((0,e) \cup I_{a,e}) \times \{a\}$, $U^*(x,y) = b$ when $(x,y) \in L^2 \setminus ((0,e) \times \{b\})$, $U^*(x,y) = y$ when $(x,y) \in ((0,e) \cup I_{a,e}) \times \{b\}$, $U^*(x,y) = l$ when $(x,y) \in ((b,1) \cup I_b^*) \times \{b\}$.

Let us define the sets $L_1 = \{x \in L \mid x \not> a\}$, $L_2 = [a,e]$, $L_3 = [e,b] \cup I_e^b$ and $L_4 = [b,1] \cup I_b^*$.

L_4	y	y	l	l
L_3	y	y	$S(x^e, y^e)$	l
L_2	k	$T(x, y)$	x	x
L_1	k	k	x	x
	L_1	L_2	L_3	L_4

Figure 5: The structure of the uninorm U_*

In the following example, we exemplify how to apply Theorem 3.8.

Example 3.9. *Consider the lattice* $(L_1 = \{0, a, b, c, d, e, f, k, l, m, n, t, 1\}, \leq, 0, 1)$ *described in Figure* 2. *If we apply the formula* (24) *in Theorem* 3.8*, when* $T = T_\wedge$ *on* $[a, e]$ *and* $S = S_\vee$ *on* $[e, b]$*, the uninorm* U_* *on* L_1 *is obtained as in Table* 2*.*

U_*	0	k	a	f	c	n	d	t	e	b	m	l	1
0	0	0	0	0	c	n	d	t	0	b	m	l	1
k	0	k	k	k	c	n	d	t	k	b	m	l	1
a	0	k	a	k	c	n	d	t	a	b	m	l	1
f	0	k	k	k	c	n	d	t	f	b	m	l	1
c	c	c	c	c	b	b	l	l	c	b	l	l	1
n	n	n	n	n	b	b	l	l	n	b	l	l	1
d	d	d	d	d	l	l	l	l	d	l	l	l	1
t	t	t	t	t	l	l	l	l	t	l	l	l	1
e	0	k	a	f	c	n	d	t	e	b	m	l	1
b	b	b	b	b	b	b	l	l	b	b	l	l	1
m	m	m	m	m	l	l	l	l	m	l	l	l	1
l	l	l	l	l	l	l	l	l	l	l	l	l	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2: The uninorm U_* induced by the formula (24) in Theorem 3.8.

It is also worth noting that if we permut the elements $0, a, b, c, d, e, f, k, l, m, n, t$ and 1 with the element $1, b, a, t, d, e, m, l, k, f, n, c$ and 0, respectively, we obtain U^*.

Remark 3.10.
(i) In general, it should be pointed out that the uninorm U_* *defined in Theorem* 3.8 *is different from the uninorms* $U_{t_1}, U_{s_1}, U_e^T, U_e^S, U_{(T,e)}, U_{(S,e)}, U_t$ *and* U_s *defined in Theorems* 2.14, 2.15, 2.16 *and* 2.17 *on any bounded lattice L such that* $L \setminus \{0, 1\}$ *has the bottom element k and the top element l regardless of the choices of t-norm T and/or t-conorm S.*
- $U_{t_1}(x, y) = y \neq l = U_*(x, y)$ *when* $(x, y) \in (e, b) \times (b, l)$;
- $U_{s_1}(k, l) = k \neq l = U_*(k, l)$;
- $U_e^T(l, l) = 1 \neq l = U_*(l, l)$;
- $U_e^S(k, l) = k \neq l = U_*(k, l)$;
- $U_{(T,e)}(x, y) = y \neq l = U_*(x, y)$ *when* $(x, y) \in (e, b) \times (b, l)$;
- $U_{(S,e)}(k, l) = k \neq l = U_*(k, l)$;

- $U_t(l,l) = 1 \neq l = U_*(l,l)$ *and* $U_s(k,l) = k \neq l = U_*(k,l)$.
(ii) Note that, on the lattices that simultaneously satisfy the conditions of the relevant theorems, the uninorm U_* *defined in Theorem* 3.8 *is different from the uninorms* $U_{t_1}, U_{s_1}, U_e^T, U_e^S, U_{(T,e)}, U_{(S,e)}, U_t$ *and* U_s *defined in Theorems* 2.14, 2.15, 2.16 *and* 2.17 *regardless of the choices of t-norm T and/or t-conorm S.*
- $U_{T_e}(k,l) = k \neq l = U_*(k,l)$;
- $U_{S_e}(l,1) = l \neq 1 = U_*(l,1)$;
- $U_{t_e}(l,l) = 1 \neq l = U_*(l,l)$;
- $U_{s_e}(k,l) = k \neq l = U_*(k,l)$;
- $U_{T_1}(x,y) = x \neq l = U_*(x,y)$ *when* $(x,y) \in (b,1) \times I_e^b$;
- $U_{S_1}(k,l) = k \neq l = U_*(k,l)$;
- $U^{'}(x,y) = x \neq l = U_*(x,y)$ *when* $(x,y) \in (b,1) \times I_e^b$;
- $U^{''}(k,l) = k \neq l = U_*(k,l)$.

In the following Theorem 3.11, we propose a generalization of Theorem 3.8.

Theorem 3.11. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $a, b, e, t, s \in L \setminus \{0,1\}$ *such that* $t \leq a < e < b \leq s$, T_1 *be t-norm on* $[a,e]^2$, T_2 *on* $[0,t]^2$ *be t-subnorm,* S_1 *be t-conorm on* $[e,b]^2$, S_2 *be a t-superconorm* $[s,1]^2$*. Then the function* $U_\beta : L^2 \to L$ *defined by*

$$U_\beta(x,y) = \begin{cases} T_1(x \wedge e, y \wedge e) & (x,y) \in [a,e]^2, \\ T_2(x \wedge t, y \wedge t) & (x,y) \in [0,a)^2 \cup [0,a) \times \{a\} \cup \{a\} \times [0,a) \cup [0,a] \times I_a^{e,b} \\ & \cup I_a^{e,b} \times ([0,a] \cup I_a^{e,b}), \\ S_1(x \vee e, y \vee e) & (x,y) \in [e,b]^2 \cup (e,b] \times I_e \cup I_e \times ((e,b] \cup I_e), \\ S_2(x \vee s, y \vee s) & (x,y) \in [b,1]^2 \cup [b,1] \times I_b^* \cup I_b^* \times ([b,1] \cup I_b^*), \\ x \wedge t & (x,y) \in ([0,a) \cup I_a^{e,b}) \times (a,e), \\ y \wedge t & (x,y) \in (a,e) \times ([0,a) \cup I_a^{e,b}), \\ x \vee s & (x,y) \in ((b,1] \cup I_b^*) \times ((e,b) \cup I_e), \\ y \vee s & (x,y) \in ((e,b) \cup I_e) \times ((b,1] \cup I_b^*), \\ y & (x,y) \in ([0,e)) \times ([e,1] \cup I_b^* \cup I_e) \cup \{e\} \times [0,a], \\ x & \text{otherwise}, \end{cases} \tag{25}$$

is a uninorm on L with the neutral element e.

Remark 3.12.
If L is a bounded lattice such that k be the bottom element and l be the top element of $L \setminus \{0,1\}$, $t = k$, $T = T_1$, $T_2(x,y) = \begin{cases} y & x = a, \\ x & y = a, \\ 0 & 0 \in \{x,y\}, \\ k & \text{otherwise}, \end{cases}$, $S_1 = S$, $S_2(x,y) = \begin{cases} y & x = b, \\ x & y = b, \\ 1 & 1 \in \{x,y\}, \\ l & \text{otherwise}, \end{cases}$ *and* $s = l$ *in the formula* (25)*, the formulas* (24) *and* (25) *are the same. Therefore, it is obvious that Theorem* 3.11 *is a generalization of Theorem* 3.8*. Moreover, the formula* (25) *in Theorem* 3.11 *produces different uninorm construction methods depending on* $t \in (0,a]$ *and* $s \in [b,1)$.

Remark 3.13. *It should be pointed out that* U^* *obtained from the Theorem* 3.1 *are conjunctive uninorms and* U_* *obtained from the Theorem* 3.8 *are disjunctive uninorms, i.e.,* $U^*(0,1) = 0$ *and* $U_*(0,1) = 1$.

Our construction methods can be generalized by induction on an appropriate bounded lattice L as follows.

Theorem 3.14. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice, and let* $\{a_1, b_1, ..., a_n, b_n\}$ *be a finite chain in L such that* $0 = a_n < ... < a_2 < a_1 < e < b_1 < b_2 < ... < b_n = 1$*, where there exists a bottom element* k_{i-1} *and a top element* l_{i-1} *of* $L \setminus]a_i, b_i[$ *for* $i \in \{2, ..., n\}$ *and* T_1 *be a t-norm on* $[a_1, e]$ *and* S_1 *be a t-conorm on* $[e, b_1]$*. Then, the function* $U_i : L^2 \to L$ *defined recursively as follows is a uninorm on L with the neutral element e, where* T_{i-1} *and* S_{i-1} *are underlying t-norm and t-conorm of* U_{i-1} *for* $i > 2$*, respectively, the function* $U_i : [a_i, b_i]^2 \to [a_i, b_i]$ *is given by*

$$U_i(x,y)=\begin{cases} S_{i-1}(x,y) & (x,y)\in[e,b_{i-1}]^2,\\ T_{i-1}(x\wedge e,y\wedge e) & (x,y)\in[a_{i-1},e]^2\cup[a_{i-1},e)\times I_e^{a_{i-1}}\cup I_e^{a_{i-1}}\times([a_{i-1},e)\cup I_e^{a_{i-1}}),\\ k_{i-1} & (x,y)\in(k_{i-1},e)^2\setminus(a_{i-1},e)^2,\\ l_{i-1} & (x,y)\in(e,l_{i-1})^2\setminus(e,b_{i-1})^2,\\ x & (x,y)\in([a_i,e)\cup I_e^{a_{i-1}}\cup I_{a_{i-1},e})\times[e,b_i]\cup(e,b_i]\times\{e\}\cup\{a_i\}\times\\ & (L\setminus(e,b_i))\cup\{b_i\}\times(e,b_i],\\ y & (x,y)\in[e,b_i]\times([a_i,e)\cup I_e^{a_{i-1}}\cup I_{a_{i-1},e})\cup\{e\}\times(e,b_i]\cup\\ & (L\setminus(e,b_i))\times\{a_i\}\cup(e,b_i]\times\{b_i\}.\end{cases}\tag{26}$$

Proof. The proof follows easily from Theorem 3.1 by induction and therefore it is omitted. □

In the following example, we exemplify how to apply Theorem 3.14.

Example 3.15. *Consider the bounded lattice ($L_4=\{0,a_1,a_2,b_1,b_2,e,l_1,l_2,w,x,y,z,1\},\leq,0,1$) characterized by the Hasse diagram in Figure 6 and put the uninorm $U_1:[a_1,b_1]^2\to[a_1,b_1]$ as in Table 3. By applying construction method in Theorem 3.14, we obtain the uninorms $U_2:[a_2,b_2]^2\to[a_2,b_2]$ and $U_3:L_4^2\to L_4$ as in Tables 4 and 5, respectively.*

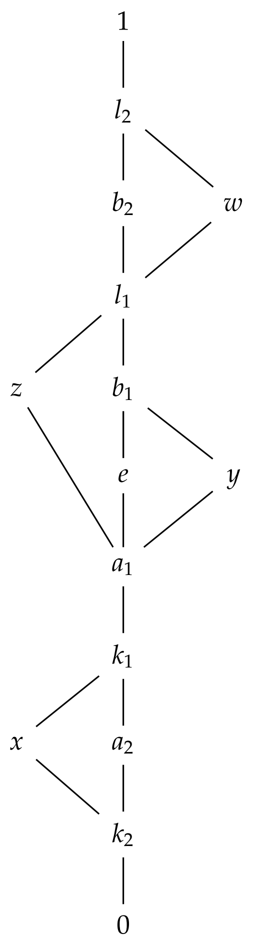

Figure 6: Lattice diagram of L_4.

If we apply the formula (26) in Theorem 3.14, when $T_i=\wedge$ on $[a_i,e]$ and $S_i=\vee$ on $[e,b_i]$, the uninorms U_1,U_2 and U_3 on L_4 is obtained as in Tables 3, 4 and 5.

U_1	a_1	e	b_1	y
a_1	a_1	a_1	a_1	a_1
e	a_1	e	b_1	y
b_1	a_1	b_1	b_1	y
y	a_1	y	y	a_1

Table 3: The uninorm U_1 on $[a_1, b_1]$.

U_2	a_2	k_1	a_1	e	y	b_1	z	b_2	l_1
a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
k_1	a_2	k_1	k_1	k_1	k_1	k_1	k_1	k_1	k_1
a_1	a_2	k_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
e	a_2	k_1	a_1	e	y	b_1	z	b_2	l_1
y	a_2	k_1	a_1	y	a_1	y	a_1	y	y
b_1	a_2	k_1	a_1	b_1	y	b_1	z	b_2	l_1
z	a_2	k_1	a_1	z	a_1	z	a_1	z	z
b_2	a_2	k_1	a_1	b_2	y	b_2	z	b_2	b_2
l_1	a_2	k_1	a_1	l_1	y	l_1	z	b_2	l_1

Table 4: The uninorm U_2 induced by the formula (26) in Theorem 3.14.

U_3	0	k_2	a_2	k_1	a_1	x	e	b_1	l_1	b_2	l_2	y	z	w	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
k_2	0	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2	k_2
a_2	0	k_2	a_2	a_2	a_2	k_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
k_1	0	k_2	a_2	k_1	k_1	k_2	k_1	k_1	k_1	k_1	k_1	k_1	k_1	k_1	k_1
a_1	0	k_2	a_2	k_1	a_1	k_2	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
x	0	k_2	k_2	k_2	k_2	k_2	x	x	x	x	x	k_2	k_2	x	x
e	0	k_2	a_2	k_1	a_1	x	e	b_1	l_1	b_2	l_2	y	z	w	1
b_1	0	k_2	a_2	k_1	a_1	x	b_1	b_1	l_1	b_2	l_2	y	z	l_2	1
l_1	0	k_2	a_2	k_1	a_1	x	l_1	l_1	l_1	b_2	l_2	y	z	l_2	1
b_2	0	k_2	a_2	k_1	a_1	x	b_2	b_2	b_2	b_2	l_2	y	z	l_2	1
l_2	0	k_2	a_2	k_1	a_1	x	l_2	l_2	l_2	l_2	l_2	y	z	l_2	1
y	0	k_2	a_2	k_1	a_1	k_2	y	y	y	y	y	a_1	a_1	y	y
z	0	k_2	a_2	k_1	a_1	k_2	z	z	z	z	z	a_1	a_1	z	z
w	0	k_2	a_2	k_1	a_1	x	w	l_2	l_2	l_2	l_2	y	z	l_2	1
1	0	k_2	a_2	k_1	a_1	x	1	1	1	1	1	y	z	1	1

Table 5: The uninorm U_3 induced by the formula (26) in Theorem 3.14.

Theorem 3.16. *Let $(L, \leq, 0, 1)$ be a bounded lattice, and let $\{a_1, b_1, ..., a_n, b_n\}$ be a finite chain in L such that $0 = a_n < ... < a_2 < a_1 < e < b_1 < b_2 < ... < b_n = 1$, where there exists a bottom element k_{j-1} and a top element l_{j-1} of $L \backslash]a_j, b_j[$ for $j \in \{2, ..., n\}$ and T_1 be a t-norm on $[a_1, e]$ and S_1 be a t-conorm on $[e, b_1]$. Then, the function $U_j : L^2 \to L$ defined recursively as follows is a uninorm on L with the neutral element e, where T_{j-1} and S_{j-1} are underlying t-norm and t-conorm of U_{j-1} for $j > 2$, respectively, the function $U_j : [a_j, b_j]^2 \to [a_j, b_j]$ is given by*

$$U_j(x,y)=\begin{cases} T_{j-1}(x,y) & (x,y)\in[a_{j-1},e]^2,\\ S_{j-1}(x\vee e,y\vee e) & (x,y)\in[e,b_{j-1}]^2\cup(e,b_{j-1}]\times I_e^{b_{j-1}}\cup I_e^{b_{j-1}}\times((e,b_{j-1}]\cup I_e^{b_{j-1}}),\\ k_{j-1} & (x,y)\in(k_{j-1},e)^2\setminus[a_{j-1},e)^2,\\ l_{j-1} & (x,y)\in(e,l_{j-1})^2\setminus[e,b_{j-1})^2,\\ x & (x,y)\in([e,b_j]\cup I_e^{b_{j-1}}\cup I_{e,b_{j-1}})\times[a_j,e]\cup[a_j,e)\times\{e\}\cup\{b_j\}\times\\ & (L\setminus(a_j,e))\cup\{a_j\}\times[a_j,e),\\ y & (x,y)\in[a_j,e]\times([e,b_j]\cup I_e^{b_{j-1}}\cup I_{e,b_{j-1}})\cup\{e\}\times[a_j,e)\cup\\ & (L\setminus(a_j,e))\times\{b_j\}\cup[a_j,e)\times\{a_j\}.\end{cases} \quad (27)$$

Proof. The proof follows easily from Theorem 3.8 by induction and therefore it is omitted. □

4. Concluding remarks

Uninorms on bounded lattices, particularly the construction of uninorms on related algebraic structures, are active research areas. In this paper, we proposed two construction methods for uninorms on a bounded lattice by exploiting the existence of t-norm T and t-conorm S on a sublattice of L, where $L\setminus\{0,1\}$ has the bottom and the top elements. We have also highlighted the differences of these construction methods from the existing methods. We have generalized by induction our construction methods to a more general form. We believe that our construction methods provide the inspiration for other aggregation functions.

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