Filomat 39:15 (2025), 5355–5376
https://doi.org/10.2298/FIL2515355M

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some new weighted Boole's type inequalities for differentiable generalized convex functions with their applications

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Abstract. This paper presents a rigorous proof of integral inequalities for first-time differentiable h-convex functions. The use of the h-convex function extends the results for convex functions and covers a large class of functions, which is the main motivation for using h-convexity. Initially, we derive a weighted Boole's formula type integral identity for differentiable functions. Utilizing this novel identity, we subsequently establish weighted Boole's formula type inequalities specifically developed for differentiable generalized convex function. We meticulously examine numerous special cases to provide comprehensive insights. These newly derived inequalities offer valuable tools for determining error bounds in various numerical integration techniques within classical calculus. To underscore the efficacy of our principal findings, we offer practical applications to weighted Boole's type quadrature formulas, continuous random variables, and special means for real numbers. These approximations highlight their potential impact on computational mathematics and related fields. Furthermore, we provide numerical examples of newly established inequalities to demonstrate that the results presented in this paper are numerically valid.

1. Introduction

Numerical analysis engaged the minds of many great mathematicians of ancient times, as evidenced by the titles of fundamental algorithms such as Euler's technique [1], Gaussian elimination, Lagrange interpolation polynomial, and Newton's method. Numerical analysis is a field that dates back many decades before modern computer systems were invented. The numerical quadrature, which related to the estimation of an integral $\int \phi(\omega)\, d\omega$ of function $\phi(\omega)$ by distinct summation $\sum g_i \phi(\omega_i)$ over points ω_i with weights g_i. Numerical quadrature techniques for distinct selections of points ω_i and weights g_i, with different degrees of precision for different types of functions $\phi(\omega)$ are shown in [2]. Boole's formula for numerical integration, derived from the fourth-order interpolating polynomial, is known for its superior accuracy with a global truncation error of $O\left(h^6\right)$. This is due to its ability to mitigate illogical fluctuations

2020 *Mathematics Subject Classification*. Primary 26D10; Secondary 34A08; 26A51; 26D15.
Keywords. weighted Boole's formula type inequality; Quadrature formulas; error bounds; h-convex function; convex function.
Received: 12 May 2024; Revised: 22 January 2025; Accepted: 22 April 2025
Communicated by Miodrag Spalević
This research was also partially supported by the National Natural Science Foundation of China 12571431.
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between data points, unlike higher-degree interpolating polynomials. Hence, Boole's formula is favored over others. To assess the numerical accuracy of various quadrature formulas, we utilized Boole's formula to compute the definite integral $\int_{-1}^{1}\phi\left(\omega^2+2\right)d\omega$ by dividing the interval $[-1,1]$ into 384 equal segments. The result obtained is as follows:

$$\begin{aligned}\int_{\omega_1}^{\omega_n}\phi(\omega)\,d\omega &= \sum_{j=1}^{\frac{n-1}{4}}\int_{\omega_{4j-3}}^{\omega_{4j+1}}\phi(\omega)\,d\omega\\ &= \sum_{j=1}^{\frac{n-1}{4}}\frac{h}{45}\left[14\left(\phi\left(\omega_{4j-3}\right)+\phi\left(\omega_{4j+1}\right)\right)+64\left(\phi\left(\omega_{4j-2}\right)+\phi\left(\omega_{4j}\right)\right)+24\phi\left(\omega_{4j-1}\right)\right],\end{aligned}$$

where $h=\omega_{4j-3}-\omega_{4j-2}=\ldots=\omega_{4j+2}-\omega_{4j+1}$, $\frac{n-1}{4}$ are positive integers.

As Boole's formula type inequality, the following inequality is widely recognized in the literature. This formula offers a fifth-order approximation, surpassing the accuracy of Simpson's formula, by approximating the integral over five points with a polynomial of degree four.

Theorem 1.1. *Let $\phi:[\xi_1,\xi_2]\to\mathbb{R}$, be a 6^{th} times continuously differentiable mappings on (ξ_1,ξ_2) and $\left\|\phi^6\right\|:=\sup_{\omega\in(\xi_1,\xi_2)}\left|\phi^6(\omega)\right|<\infty$. Then, the subsequent inequality holds:*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\left(\phi(\xi_1)+\phi(\xi_2)\right)+12\phi\left(\frac{\xi_1+\xi_2}{2}\right)+32\left(\phi\left(\frac{3\xi_1+\xi_2}{4}\right)+\phi\left(\frac{\xi_1+3\xi_2}{4}\right)\right)\right]\right.\\ &\left.-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)\,d\omega\right|\le\frac{(\xi_2-\xi_1)^6}{1935360}\left\|\phi^6\right\|_\infty.\end{aligned}$$

Krukowski [3] has investigated a logical evolution of Simpson's rule, which is Boole's formula. By leveraging computer software, novel error bounds for Boole's formula have been demonstrated. It is widely recognized that inequalities have been instrumental in advancing nearly all fields of sciences see [4–7].

The well-known Simpson's type inequality in literature is defined as follows:

Theorem 1.2. *Let $\phi:[\xi_1,\xi_2]\subset\mathbb{R}\to\mathbb{R}$ be 4^{th} times continuously differentiable mappings on (ξ_1,ξ_2), then we have*

$$\left|\frac{1}{6}\left(\phi(\xi_1)+4\phi\left(\frac{\xi_1+\xi_2}{2}\right)+\phi(\xi_2)\right)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)\,d\omega\right|\le\frac{(\xi_2-\xi_1)^4}{2880}\left\|\phi^{(4)}\right\|_\infty, \tag{1}$$

where $\left\|\phi^4\right\|_\infty:=\sup_{\omega\in[\xi_1,\xi_2]}\left|\phi^{(4)}(\omega)\right|<\infty$.

Some results regarding with inequality (1) and related inequalities, one can consult to [8–11].

Convexity is a fundamental concept in mathematics, particularly in the fields of calculus, optimization, and mathematical analysis. Mathematically, a function $\phi(\eta)$ defined on $[\xi_1,\xi_2]$ is convex, if for any pair of points $(\xi_1,\phi(\xi_1))$ and $(\xi_2,\phi(\xi_2))$ lying within the interval, the function satisfies the inequality:

$$\phi(\eta\xi_1+(1-\eta)\xi_2)\le\eta\phi(\xi_1)+(1-\eta)\phi(\xi_2), \tag{2}$$

for each values of η between 0 and 1. Hermite–Hadamard type inequalities are significant mathematical inequalities that involve convex mappings. They are stated as follows:

Let $\phi:[\xi_1,\xi_2]\subseteq\mathbb{R}\to\mathbb{R}^+$ be a convex function defined on the interval $[\xi_1,\xi_2]$ of real numbers. Then the following inequalities hold:

$$\phi\left(\frac{\xi_1+\xi_2}{2}\right)\le\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)\,d\omega\le\frac{\phi(\xi_1)+\phi(\xi_2)}{2}. \tag{3}$$

For concave function, then the inequalities mentioned above also hold, but in the opposite direction. In the past two decades, researchers have explored several new upper bounds for both the left and right sides of the inequality (3). In [12], Dragomir and Agarwal introduced the following trapezoidal type inequality along with its error bounds for differentiable convex mappings, one of which is as follows:

Theorem 1.3. *Let $\phi : [\xi_1, \xi_2] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mappings on (ξ_1, ξ_2), then we have*

$$\left| \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \phi(\omega)\, d\omega - \frac{\phi(\xi_1) + \phi(\xi_2)}{2} \right| \leq \frac{\xi_2 - \xi_1}{8} \left[\left|\phi'(\xi_1)\right| + \left|\phi'(\xi_2)\right| \right]. \tag{4}$$

Kirmaci [13], inspired by the work of Dragomir and Agarwal, proved some error bounds for the midpoint formula for differentiable convex functions. After these two publications, many researchers started their work to establish the error bounds for numerical quadrature formulas with different approaches. In their respective works [14] and [15], Alomari et al. and Dragomir et al. provided error bounds for Simpson's formula for both convex and general convex functions, along with their applications. On the other hand, authors established error bounds for Simpson's formula for two-variable functions in [16] and [17], respectively. Additionally, Du [18] utilized the general form of convexity to establish error bounds for Simpson's formula.

In [19], Breckner was the first mathematician who introduced a generalized convex function in 1979. The number of associations with s-convexity in the 1^{st} sense is negotiated in [20]. Direct proof of Breckner's result was esteemed in 2001, by Pycia [21]. Due to the importance of convexity and s-convexity in the investigation of optimality to resolve mathematical programming, many researchers focused seriously on s-convex functions. For example, H. Hudzik et al. presented two kinds of s-convexity $\{s \in (0,1)\}$ in [20]. They demonstrated that the 2^{nd} sense is fundamentally stronger than the s-convexity in the 1^{st} sense. We use s-convexity of a function in the 2^{nd} sense generally known as the s-convex function. Since $s \in (0,1)$, therefore this class of functions is more important than the convex. We also observe in the main section that the results obtained by s-convexity are much better than the convexity. Secondly, s-convexity is the generalization of a convex function, so we can obtain the results for convex functions by using $s = 1$ in the results of s-convex functions. This class is defined in the following as:

Definition 1.4. *A function $\phi : [0, \infty) \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be s-convex in the second sense, if*

$$\phi(\eta\xi_1 + (1-\eta)\xi_2) \leq \eta^s \phi(\xi_1) + (1-\eta)^s \phi(\xi_2), \ \forall\, \xi_1, \xi_2 \in [0, \infty),\ \eta \in [0,1], \tag{5}$$

for some fixed $s \in (0,1]$. The s-convexity reduce to ordinary convex when $s = 1$ in above inequality (5) which is defined on $[0, \infty)$.

Definition 1.5. *[22] A function $\phi : [0, \infty) \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be h-convex, if*

$$\phi(\eta\xi_1 + (1-\eta)\xi_2) \leq h(\eta)\phi(\xi_1) + h(1-\eta)\phi(\xi_2), \ \forall\, \xi_1, \xi_2 \in [0, \infty),\ \eta \in (0,1). \tag{6}$$

Remark 1.6. *If $h(\omega) = \omega$ in (6), the Definition 1.5 bring to inequality (2).*

Remark 1.7. *If $h(\omega) = \omega^s$ in (5), the Definition 1.5 becomes the Definition 1.4.*

Remark 1.8. *If $h(\omega) = 1$ in (6), the Definition 1.5 become the Definition for P-convex function.*

In [23], the following Hadamard's inequality in term of s-convex functions in the second sense holds:

Theorem 1.9. *If $\phi : [0, \infty) \to [0, \infty)$ is s-convex function in the second sense by $0 < s < 1$ and $\xi_1, \xi_2 \in [0, \infty)$, $\xi_1 < \xi_2$. If $\phi \in L^1[\xi_1, \xi_2]$, then the following inequalities hold:*

$$2^{s-1}\phi\left(\frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \phi(\omega)\, d\omega \leq \frac{\phi(\xi_1) + \phi(\xi_2)}{s+1}. \tag{7}$$

Matloka introduced weighted bounds for Simpson's type inequalities applicable to h-convex functions in [24]. Kashuri et al. [25], in their work, established error bounds for Simpson's type inequalities in the form of weighted through differentiable s-convex functions. Authors in [26] presented bounds for weighted midpoint type inequalities for s-convex functions utilizing fractional integrals. Sarikaya demonstrated weighted midpoint-type inequalities specifically for convex functions in [27]. Wajeeha et al. contributed weighted integral inequalities suitable for differentiable h-preinvex functions in [28]. For further exploration of weighted type inequalities and related references, interested readers can refer to [29–31].

Inspired by the research mentioned earlier, we derive several integral inequalities by utilizing the differentiable generalized convexity characteristic of the function. The h-convexity provides a wide range of bounds variation as compared to convex functions. Therefore, these inequalities for h-convex functions offer a more versatile and refined framework compared to classical convex functions. Due to the unique nature of the weighted functions, setting the kernel was a challenging problem. We provide applications to quadrature formulas, r-moment, and special means for real numbers of these established inequalities. Furthermore, we provide numerical examples to demonstrate the effectiveness and validity of the newly derived inequalities. These contributions offer a significant advancement in the field of numerical integration, providing both theoretical foundations and practical tools for enhancing the reliability and efficiency of integration techniques.

The motivation behind this work is to establish weighted Boole's formula type inequalities for a class of functions whose derivatives are h-convex functions. These inequalities allow us to approximate the error bounds in Boole's formula without needing to compute its higher derivatives, which may not exist or may be difficult to find. This approach utilizes results within the domain of classical calculus.

The paper organization is as follows: Section 2 presents our main results of weighted Boole's type formula for differentiable h-convex functions. In Section 3, we provide numerical examples to demonstrate the newly established results. Real-life applications to the newly proved results are given in Section 4. Concluding remarks on this work and potential future directions are provided in Section 5.

2. Main Results

For the sake of simplicity, we define the following notations which will be used throughout the paper.

$$\begin{aligned}
&\varphi_{\phi,g}\left(\xi_1,\xi_2,\omega\right)\\
&:= \frac{1}{90\left(\xi_2-\xi_1\right)}\left[7\phi(\xi_1)+32\phi\left(\frac{3\xi_1+\xi_2}{4}\right)+12\phi\left(\frac{\xi_1+\xi_2}{2}\right)+32\phi\left(\frac{\xi_1+3\xi_2}{4}\right)+7\phi(\xi_2)\right]\\
&\times\int_{\xi_1}^{\xi_2} g\left(\omega\right)d\omega-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} g\left(\omega\right)\phi(\omega)d\omega.
\end{aligned}$$

Now, to establish our main theorems for h-convex functions, we rely on the following lemma.

Lemma 2.1. *Let $\phi : I \subset [\xi_1,\xi_2] \to \mathbb{R}$ be a differentiable mapping on I°, where $\xi_1,\xi_2 \in I^\circ$ with $\xi_1 < \xi_2$ and $g : [\xi_1,\xi_2] \to \mathbb{R}$ be continuous and symmetric function with respect to $\frac{\xi_1+\xi_2}{2}$, $\frac{3\xi_1+\xi_2}{4}$ and $\frac{\xi_1+3\xi_2}{4}$. If $\phi', g \in L[\xi_1,\xi_2]$, then the subsequent equality is satisfied:*

$$\begin{aligned}
&\varphi_{\phi,g}\left(\xi_1,\xi_2,\omega\right)\\
&=\frac{\xi_2-\xi_1}{16}\left[\int_0^1 I_1\left(\eta\right)\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)d\eta+\int_0^1 I_2\left(\eta\right)\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)d\eta\right.\\
&\left.+\int_0^1 I_3\left(\eta\right)\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)d\eta+\int_0^1 I_4\left(\eta\right)\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)d\eta\right],
\end{aligned}$$

where

$$I_1\left(\eta\right):=\int_0^\eta g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du-\frac{28}{90}\int_0^1 g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du,$$

$$I_2(\eta) := \int_0^{\eta} g\left(\frac{3-u}{4}\xi_1 + \frac{1+u}{4}\xi_2\right)du - \frac{44}{60}\int_0^1 g\left(\frac{3-u}{4}\xi_1 + \frac{1+u}{4}\xi_2\right)du,$$
$$I_3(\eta) := \int_0^{\eta} g\left(\frac{2-u}{4}\xi_1 + \frac{2+u}{4}\xi_2\right)du - \frac{24}{90}\int_0^1 g\left(\frac{2-u}{4}\xi_1 + \frac{2+u}{4}\xi_2\right)du,$$

and

$$I_4(\eta) := \int_0^{\eta} g\left(\frac{1-u}{4}\xi_1 + \frac{3+u}{4}\xi_2\right)du - \frac{62}{90}\int_0^1 g\left(\frac{1-u}{4}\xi_1 + \frac{3+u}{4}\xi_2\right)du.$$

Proof. By considering the fundamental rules of integration by parts and changing the variables $x = \frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2$, it is suffices to note that

$$\begin{aligned}
J_1 &= \int_0^1 I_1(\eta)\phi'\left(\frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2\right)d\eta \\
&= \int_0^1 \left[\int_0^{\eta} g\left(\frac{4-u}{4}\xi_1 + \frac{u}{4}\xi_2\right)du - \frac{28}{90}\int_0^1 g\left(\frac{4-u}{4}\xi_1 + \frac{u}{4}\xi_2\right)du\right]\phi'\left(\frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2\right)d\eta \\
&= \frac{4}{\xi_2-\xi_1}\left[\int_0^{\eta} g\left(\frac{4-u}{4}\xi_1 + \frac{u}{4}\xi_2\right)du - \frac{28}{90}\int_0^1 g\left(\frac{4-u}{4}\xi_1 + \frac{u}{4}\xi_2\right)du\right]\phi\left(\frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2\right)\Bigg|_0^1 \\
&- \frac{4}{\xi_2-\xi_1}\int_0^1 \phi\left(\frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2\right)g\left(\frac{4-\eta}{4}\xi_1 + \frac{\eta}{4}\xi_2\right)d\eta \\
&= \frac{4}{(\xi_2-\xi_1)^2}\left[\frac{62}{90}\phi\left(\frac{3\xi_1+\xi_2}{4}\right) + \frac{28}{90}\phi(\xi_1)\right]\int_{\xi_1}^{\xi_2} g(\omega)\,d\omega - \frac{4}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2} g(\omega)\phi(\omega)\,d\omega.
\end{aligned} \tag{8}$$

Similarly, we obtain

$$\begin{aligned}
J_2 &= \int_0^1 I_2(\eta)\phi'\left(\frac{3-\eta}{4}\xi_1 + \frac{1+\eta}{4}\xi_2\right)d\eta \\
&= \frac{4}{(\xi_2-\xi_1)^2}\left[\frac{44}{60}\phi\left(\frac{3\xi_1+\xi_2}{4}\right) + \frac{16}{90}\phi\left(\frac{\xi_1+\xi_2}{2}\right)\right]\int_{\xi_1}^{\xi_2} g(\omega)\,d\omega - \frac{4}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2} g(\omega)\phi(\omega)\,d\omega,
\end{aligned} \tag{9}$$

$$\begin{aligned}
J_3 &= \int_0^1 I_3(\eta)\phi'\left(\frac{2-\eta}{4}\xi_1 + \frac{2+\eta}{4}\xi_2\right)d\eta \\
&= \frac{4}{(\xi_2-\xi_1)^2}\left[\frac{66}{90}\phi\left(\frac{\xi_1+3\xi_2}{4}\right) + \frac{24}{90}\phi\left(\frac{\xi_1+\xi_2}{2}\right)\right]\int_{\xi_1}^{\xi_2} g(\omega)\,d\omega - \frac{4}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2} g(\omega)\phi(\omega)\,d\omega,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
J_4 &= \int_0^1 I_4(\eta)\phi'\left(\frac{1-\eta}{4}\xi_1 + \frac{3+\eta}{4}\xi_2\right)d\eta \\
&= \frac{4}{(\xi_2-\xi_1)^2}\left[\frac{28}{90}\phi(\xi_2) + \frac{62}{90}\phi\left(\frac{\xi_1+3\xi_2}{4}\right)\right]\int_{\xi_1}^{\xi_2} g(\omega)\,d\omega - \frac{4}{(\xi_2-\xi_1)^2}\int_{\xi_1}^{\xi_2} g(\omega)\phi(\omega)\,d\omega.
\end{aligned} \tag{11}$$

Since $g(\omega)$ is symmetric with relate to $\frac{\xi_1+\xi_2}{2}$, $\frac{3\xi_1+\xi_2}{4}$ and $\frac{\xi_1+3\xi_2}{4}$, one has

$$\int_{\xi_1}^{\frac{3\xi_1+\xi_2}{4}} g(\omega)\,d\omega = \int_{\frac{3\xi_1+\xi_2}{4}}^{\frac{\xi_1+\xi_2}{2}} g(\omega)\,d\omega = \int_{\frac{\xi_1+\xi_2}{2}}^{\frac{\xi_1+3\xi_2}{4}} g(\omega)\,d\omega = \int_{\frac{\xi_1+3\xi_2}{4}}^{\xi_2} g(\omega)\,d\omega = \frac{1}{4}\int_{\xi_1}^{\xi_2} g(\omega)\,d\omega.$$

Thus, we obtain the following equality by adding (8)-(11) and then multiplying by $\frac{\xi_2-\xi_1}{16}$.

$$\frac{\xi_2-\xi_1}{16}\left[J_1+J_2+J_3+J_4\right]=\varphi_{\phi,g}\left(\xi_1,\xi_2,\omega\right).$$

The desired equality is obtained. □

Corollary 2.2. *If we take $g(\omega)=1$ in Lemma 2.1, then we obtain the following classical Boole's formula type equality:*

$$\begin{aligned}
&\frac{1}{90}\left[7\phi(\xi_1)+32\phi\left(\frac{3\xi_1+\xi_2}{4}\right)+12\phi\left(\frac{\xi_1+\xi_2}{2}\right)+32\phi\left(\frac{\xi_1+3\xi_2}{4}\right)+7\phi(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)d\omega\\
&=\frac{\xi_2-\xi_1}{16}\left[\int_0^1\left(\eta-\frac{28}{90}\right)\left[\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right]d\eta+\int_0^1\left(\eta-\frac{44}{60}\right)\left[\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right]d\eta\right.\\
&\left.+\int_0^1\left(\eta-\frac{24}{90}\right)\left[\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right]d\eta+\int_0^1\left(\eta-\frac{62}{90}\right)\left[\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right]d\eta\right].
\end{aligned}$$

The next theorem gives a new refinement of Boole's formula type inequality for h-convex functions.

Theorem 2.3. *Suppose that all conditions made in Lemma 2.1. If $\left|\phi'\right|^q$ is h-convex on $[\xi_1,\xi_2]$ and $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\varphi_{\phi,g}\left(\xi_1,\xi_2,\omega\right)\right|\leq\frac{\xi_2-\xi_1}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\left[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{28}{90}\right|h\left(\frac{4-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q\right.\right.\\
&\left.+\int_0^1\left|\eta-\frac{28}{90}\right|h\left(\frac{\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right)^{\frac{1}{q}}+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{44}{60}\right|h\left(\frac{3-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q\right.\\
&\left.+\int_0^1\left|\eta-\frac{44}{60}\right|h\left(\frac{1+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right)^{\frac{1}{q}}+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{24}{90}\right|h\left(\frac{2-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q\right.\\
&\left.+\int_0^1\left|\eta-\frac{24}{90}\right|h\left(\frac{2+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right)^{\frac{1}{q}}+\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1\left|\eta-\frac{62}{90}\right|h\left(\frac{1-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q+\int_0^1\left|\eta-\frac{62}{90}\right|h\left(\frac{3+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right)^{\frac{1}{q}}\right]. \qquad (12)
\end{aligned}$$

Proof. Using Lemma 2.1 and noticing that $\left\|g\right\|_{\left[\xi_1,\frac{3\xi_1+\xi_2}{4}\right],\infty}, \left\|g\right\|_{\left[\frac{3\xi_1+\xi_2}{4},\frac{\xi_1+\xi_2}{2}\right],\infty}, \left\|g\right\|_{\left[\frac{\xi_1+\xi_2}{2},\frac{\xi_1+3\xi_2}{4}\right],\infty}, \left\|g\right\|_{\left[\frac{\xi_1+3\xi_2}{4},\xi_2\right],\infty}\leq\left\|g\right\|_{[\xi_1,\xi_2],\infty}$, we have

$$\begin{aligned}
&\left|\varphi_{\phi,g}\left(\xi_1,\xi_2,\omega\right)\right|\\
&\leq\frac{\xi_2-\xi_1}{16}\left[\int_0^1\left(\left|\int_0^\eta g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du-\frac{28}{90}\int_0^1 g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du\right|\times\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|\right)d\eta\right.\\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{3-u}{4}\xi_1+\frac{1+u}{4}\xi_2\right)du-\frac{44}{60}\int_0^1 g\left(\frac{3-u}{4}\xi_1+\frac{1+u}{4}\xi_2\right)du\right|\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|\right)d\eta\\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{2-u}{4}\xi_1+\frac{2+u}{4}\xi_2\right)du-\frac{24}{90}\int_0^1 g\left(\frac{2-u}{4}\xi_1+\frac{2+u}{4}\xi_2\right)du\right|\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|\right)d\eta\\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{1-u}{4}\xi_1+\frac{3+u}{4}\xi_2\right)du-\frac{62}{90}\int_0^1 g\left(\frac{1-u}{4}\xi_1+\frac{3+u}{4}\xi_2\right)du\right|\right.\\
&\left.\left.\times\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|\right)d\eta\right]
\end{aligned}$$

$$\leq \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\left[\int_0^1\left|\int_0^\eta du-\frac{28}{90}\int_0^1 du\right|\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|d\eta\right.$$
$$+\int_0^1\left|\int_0^\eta du-\frac{44}{60}\int_0^1 du\right|\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|d\eta$$
$$+\int_0^1\left|\int_0^\eta du-\frac{24}{90}\int_0^1 du\right|\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|d\eta$$
$$\left.+\int_0^1\left|\int_0^\eta du-\frac{62}{90}\int_0^1 du\right|\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|d\eta\right]$$
$$=\frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\left[\int_0^1\left|\eta-\frac{28}{90}\right|\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|d\eta\right.$$
$$+\int_0^1\left|\eta-\frac{44}{60}\right|\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|d\eta+\int_0^1\left|\eta-\frac{24}{90}\right|\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|d\eta$$
$$\left.+\int_0^1\left|\eta-\frac{62}{90}\right|\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|d\eta\right]. \tag{13}$$

Using the power-mean inequality, we have

$$\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right|$$
$$\leq \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\left[\left(\int_0^1\left|\eta-\frac{28}{90}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{28}{90}\right|\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^1\left|\eta-\frac{44}{60}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{44}{60}\right|\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|^q d\eta\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1\left|\eta-\frac{24}{90}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{24}{90}\right|\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|^q d\eta\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_0^1\left|\eta-\frac{62}{90}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\eta-\frac{62}{90}\right|\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|^q d\eta\right)^{\frac{1}{q}}\right]. \tag{14}$$

Using h-convexity of $|\phi'|^q$ on the interval $[\xi_1,\xi_2]$, we have

$$\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|^q\leq h\left(\frac{4-\eta}{4}\right)\left|\phi'(\xi_1)\right|^q+h\left(1-\frac{4-\eta}{4}\right)\left|\phi'(\xi_2)\right|^q, \tag{15}$$

$$\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|^q\leq h\left(\frac{3-\eta}{4}\right)\left|\phi'(\xi_1)\right|^q+h\left(1-\frac{3-\eta}{4}\right)\left|\phi'(\xi_2)\right|^q, \tag{16}$$

$$\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|^q\leq h\left(\frac{2-\eta}{4}\right)\left|\phi'(\xi_1)\right|^q+h\left(1-\frac{2-\eta}{4}\right)\left|\phi'(\xi_2)\right|^q, \tag{17}$$

and

$$\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|^q\leq h\left(\frac{1-\eta}{4}\right)\left|\phi'(\xi_1)\right|^q+h\left(1-\frac{1-\eta}{4}\right)\left|\phi'(\xi_2)\right|^q. \tag{18}$$

Here, we used the equalities:

$$\int_0^1 \left|\eta - \frac{28}{90}\right| d\eta = \int_0^1 \left|\eta - \frac{62}{90}\right| d\eta = \frac{1157}{4050}, \tag{19}$$

and

$$\int_0^1 \left|\eta - \frac{44}{60}\right| d\eta = \int_0^1 \left|\eta - \frac{24}{90}\right| d\eta = \frac{137}{450}. \tag{20}$$

Using (15)-(18) in (14), we obtained (12). This completes the proof of Theorem 2.3. □

Corollary 2.4. *By taking $h(\eta) = \eta$ in Theorem 2.3, the we have the following inequality:*

$$\begin{aligned} \left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty} \Bigg[& \left(\frac{1157}{4050}\right)^{1-\frac{1}{q}} \left(\frac{130523}{546750}|\phi'(\xi_1)|^q + \frac{12836}{273375}|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \\ & + \left(\frac{137}{450}\right)^{1-\frac{1}{q}} \left(\frac{4127}{20250}|\phi'(\xi_1)|^q + \frac{1019}{10125}|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \\ & + \left(\frac{137}{450}\right)^{1-\frac{1}{q}} \left(\frac{1019}{10125}|\phi'(\xi_1)|^q + \frac{4127}{20250}|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \\ & + \left(\frac{1157}{4050}\right)^{1-\frac{1}{q}} \left(\frac{12836}{273375}|\phi'(\xi_1)|^q + \frac{130523}{546750}|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \Bigg]. \end{aligned}$$

Corollary 2.5. *By taking $h(\eta) = \eta^s$ with $s \in (0,1)$ in Theorem 2.3, then we have the following inequality:*

$$\begin{aligned} \left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty} \Bigg[& \left(\frac{1157}{4050}\right)^{1-\frac{1}{q}} \left(E_1(s)|\phi'(\xi_1)|^q + E_8(s)|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \\ & + \left(\frac{137}{450}\right)^{1-\frac{1}{q}} \left(E_3(s)|\phi'(\xi_1)|^q + E_6(s)|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} + \left(\frac{137}{450}\right)^{1-\frac{1}{q}} \left(E_5(s)|\phi'(\xi_1)|^q + E_4(s)|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \\ & + \left(\frac{1157}{4050}\right)^{1-\frac{1}{q}} \left(E_7(s)|\phi'(\xi_1)|^q + E_2(s)|\phi'(\xi_2)|^q\right)^{\frac{1}{q}} \Bigg], \end{aligned}$$

where

$$\begin{aligned} E_1(s) &:= \int_0^1 \left|\eta - \frac{28}{90}\right| \left(\frac{4-\eta}{4}\right)^s d\eta = \int_0^{\frac{28}{90}} \left(\frac{28}{90} - \eta\right)\left(\frac{4-\eta}{4}\right)^s d\eta + \int_{\frac{28}{90}}^1 \left(\eta - \frac{28}{90}\right)\left(\frac{4-\eta}{4}\right)^s d\eta \\ &= \frac{4^{-s}45^{-s-2}\left(45^{s+1}\left(2^{2s+3}(7s-76) - 3^{s+1}(31s+197)\right) + 2^{s+3}83^{s+2}\right)}{(s+1)(s+2)}, \\ E_2(s) &:= \int_0^1 \left|\eta - \frac{62}{90}\right| \left(1 - \frac{1-\eta}{4}\right)^s d\eta \\ &= \frac{4^{-s}45^{-s-2}\left(45^{s+1}\left(2^{2s+3}(7s-76) - 3^{s+1}(31s+197)\right) + 2^{s+3}83^{s+2}\right)}{(s+1)(s+2)}, \\ E_3(s) &:= \int_0^1 \left|\eta - \frac{44}{60}\right| \left(\frac{3-\eta}{4}\right)^s d\eta \\ &= \frac{4^{-s}15^{-s-2}\left(2^{s+3}17^{s+2} - 15^{s+1}\left(2^{s+2}(2s+19) + 3^{s+1}(23-11s)\right)\right)}{(s+1)(s+2)}, \end{aligned}$$

$$E_4(s) := \int_0^1 \left|\eta - \frac{24}{90}\right| \left(1 - \frac{2-\eta}{4}\right)^s d\eta$$
$$= \frac{4^{-s}15^{-s-2}\left(2^{s+3}17^{s+2} - 15^{s+1}\left(2^{s+2}(2s+19) + 3^{s+1}(23-11s)\right)\right)}{(s+1)(s+2)},$$

$$E_5(s) := \int_0^1 \left|\eta - \frac{24}{90}\right| \left(\frac{2-\eta}{4}\right)^s d\eta$$
$$= \frac{4^{-s}15^{-s-2}\left(15^{s+1}\left(-11s + 2^{s+2}(2s-11) - 37\right) + 2^{s+3}13^{s+2}\right)}{(s+1)(s+2)},$$

$$E_6(s) := \int_0^1 \left|\eta - \frac{44}{60}\right| \left(1 - \frac{3-\eta}{4}\right)^s d\eta$$
$$= \frac{4^{-s}15^{-s-2}\left(15^{s+1}\left(-11s + 2^{s+2}(2s-11) - 37\right) + 2^{s+3}13^{s+2}\right)}{(s+1)(s+2)},$$

$$E_7(s) := \int_0^1 \left|\eta - \frac{62}{90}\right| \left(\frac{1-\eta}{4}\right)^s d\eta = \frac{4^{-s}45^{-s-2}\left(45^{s+1}(31s+17) + 2^{s+3}7^{s+2}\right)}{(s+1)(s+2)},$$

and

$$E_8(s) := \int_0^1 \left|\eta - \frac{28}{90}\right| \left(1 - \frac{4-\eta}{4}\right)^s d\eta = \frac{4^{-s}45^{-s-2}\left(45^{s+1}(31s+17) + 2^{s+3}7^{s+2}\right)}{(s+1)(s+2)}.$$

Corollary 2.6. *By taking $h(\eta) = 1$ in Theorem 2.3, then we have the following inequality:*

$$\left|\varphi_{\phi,g}(\xi_1, \xi_2, \omega)\right| \leq \frac{239(\xi_2 - \xi_1)}{3240} \|g\|_{[\xi_1,\xi_2],\infty} \left[\left|\phi'(\xi_1)\right|^q + \left|\phi'(\xi_2)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 2.7. *By taking $h(\eta) = \eta(1-\eta)$ in Theorem 2.3, then we have the following inequality:*

$$\left|\varphi_{\phi,g}(\xi_1, \xi_2, \omega)\right| \leq \frac{\xi_2 - \xi_1}{8} \|g\|_{[\xi_1,\xi_2],\infty} \left(\frac{1157\left(\cosh\left(\frac{\log\left(\frac{224920800}{29691973}\right)}{q}\right) - \sinh\left(\frac{\log\left(\frac{224920800}{29691973}\right)}{q}\right)\right)}{4050}\right.$$
$$\left. + \frac{137^{1-\frac{1}{q}}641413^{1/q}}{450\left(\sinh\left(\frac{\log(21600)}{q}\right) + \cosh\left(\frac{\log(21600)}{q}\right)\right)}\right)\left[\left|\phi'(\xi_1)\right|^q + \left|\phi'(\xi_2)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 2.8. *If we take $g(\omega) = 1$ in Corollary 2.4, then we have the following inequality:*

$$\left|\frac{1}{90}\left[7\phi(\xi_1) + 32\phi\left(\frac{3\xi_1 + \xi_2}{4}\right) + 12\phi\left(\frac{\xi_1 + \xi_2}{2}\right) + 32\phi\left(\frac{\xi_1 + 3\xi_2}{4}\right) + 7\phi(\xi_2)\right]\right.$$
$$\left. - \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \phi(\omega) d\omega\right|$$
$$\leq \frac{\xi_2 - \xi_1}{16}\left[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left(\frac{130523}{546750}|\left|\phi'(\xi_1)|\right|^q + \frac{12836}{273375}\left|\phi'(\xi_2)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+ \left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left(\frac{4127}{20250}\left|\phi'(\xi_1)\right|^q + \frac{1019}{10125}\left|\phi'(\xi_2)\right|^q\right)^{\frac{1}{q}}$$
$$+ \left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left(\frac{1019}{10125}\left|\phi'(\xi_1)\right|^q + \frac{4127}{20250}|\phi'(\xi_2)|^q\right)^{\frac{1}{q}}$$

$$+\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left(\frac{12836}{273375}\left|\phi'(\xi_1)\right|^q+\frac{130523}{546750}\left|\phi'(\xi_2)\right|^q\right)^{\frac{1}{q}}\Bigg].$$

Corollary 2.9. *If we take $q = 1$ in Theorem 2.3, then we have obtained the following Boole's type inequality for h-convex:*

$$\begin{aligned}\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \le \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}&\Bigg[\left(\int_0^1\left|\eta-\frac{28}{90}\right|h\left(\frac{4-\eta}{4}\right)d\eta|\phi'(\xi_1)|+\int_0^1\left|\eta-\frac{28}{90}\right|h\left(\frac{\eta}{4}\right)d\eta|\phi'(\xi_2)|\right)\\&+\left(\int_0^1\left|\eta-\frac{44}{60}\right|h\left(\frac{3-\eta}{4}\right)d\eta|\phi'(\xi_1)|+\int_0^1\left|\eta-\frac{44}{60}\right|h\left(\frac{1+\eta}{4}\right)d\eta|\phi'(\xi_2)|\right)\\&+\left(\int_0^1\left|\eta-\frac{24}{90}\right|h\left(\frac{2-\eta}{4}\right)d\eta|\phi'(\xi_1)|+\int_0^1\left|\eta-\frac{24}{90}\right|h\left(\frac{2+\eta}{4}\right)d\eta|\phi'(\xi_2)|\right)\\&+\left(\int_0^1\left|\eta-\frac{62}{90}\right|h\left(\frac{1-\eta}{4}\right)d\eta|\phi'(\xi_1)|+\int_0^1\left|\eta-\frac{62}{90}\right|h\left(\frac{3+\eta}{4}\right)d\eta|\phi'(\xi_2)|\right)\Bigg].\end{aligned}$$

Corollary 2.10. *If we take $g(\omega) = 1$ and $h(\eta) = \eta$ in Corollary 2.9, then we have following Boole's type inequality:*

$$\begin{aligned}&\left|\frac{1}{90}\left[7\phi(\xi_1)+32\phi\left(\frac{3\xi_1+\xi_2}{4}\right)+12\phi\left(\frac{\xi_1+\xi_2}{2}\right)+32\phi\left(\frac{\xi_1+3\xi_2}{4}\right)+7\phi(\xi_2)\right]-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)d\omega\right|\\&\le\frac{239(\xi_2-\xi_1)}{6480}\left[|\phi'(\xi_1)|+|\phi'(\xi_2)|\right].\end{aligned}$$

Corollary 2.11. *In Corollary 2.10, if $\phi(\xi_1) = \phi\left(\frac{\xi_1+\xi_2}{2}\right) = \phi\left(\frac{3\xi_1+\xi_2}{4}\right) = \phi\left(\frac{\xi_1+3\xi_2}{4}\right) = \phi(\xi_2)$, one can obtain new inequality of midpoint type, which is as follows:*

$$\left|\phi\left(\frac{\xi_1+\xi_2}{2}\right)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\phi(\omega)d\omega\right|\le\frac{239(\xi_2-\xi_1)}{6480}\left[|\phi'(\xi_1)|+|\phi'(\xi_2)|\right]. \tag{21}$$

Corollary 2.12. *If we take $h(\eta) = \eta^s$ in Corollary 2.9, then we have the following Boole's type inequality for s-convex:*

$$\begin{aligned}\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \le \frac{(\xi_2-\xi_1)}{16}\|g\|_{[\xi_1,\xi_2],\infty}\times\frac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}&\left(45^{s+1}\left(3^{s+1}+7\,4^{s+1}-1\right)s-19\,4^{s+2}45^{s+1}\right.\\&\left.-47\,45^{s+1}-133\,135^{s+1}+14^{s+2}+78^{s+2}-90^{s+2}+102^{s+2}+166^{s+2}\right)\left[\left|\phi'(\xi_1)\right|+\left|\phi'(\xi_2)\right|\right].\end{aligned} \tag{22}$$

Remark 2.13. *By taking mapping for $h = 1$ and $h = \eta(1-\eta)$ in Corollary 2.9, one can establish more Boole's type inequalities similar to Corollary 2.6 and Corollary 2.7.*

Theorem 2.14. *Suppose that all conditions made in Lemma 2.1. If $\left|\phi'\right|^q$ is h-convex on $[\xi_1,\xi_2]$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \le \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}&\Bigg[(E_9(q))^{1-\frac{1}{q}}\left[\int_0^1 h\left(\frac{4-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q+\int_0^1 h\left(\frac{\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\\&+(E_{10}(q))^{1-\frac{1}{q}}\left[\int_0^1 h\left(\frac{3-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q+\int_0^1 h\left(\frac{1+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\\&+(E_{11}(q))^{1-\frac{1}{q}}\left[\int_0^1 h\left(\frac{2-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q+\int_0^1 h\left(\frac{2+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\end{aligned}$$

$$+\left(E_{12}(q)\right)^{1-\frac{1}{q}}\left[\int_0^1 h\left(\frac{1-\eta}{4}\right)d\eta|\phi'(\xi_1)|^q+\int_0^1 h\left(\frac{3+\eta}{4}\right)d\eta|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\Bigg], \tag{23}$$

where

$$E_9(q):=\frac{434^{\frac{1}{q-1}}2025^{-\frac{q}{q-1}}\left(961\left(\frac{45}{14}\right)^{\frac{1}{q-1}}+196\left(\frac{45}{31}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

$$E_{10}(q):=\frac{44^{\frac{1}{q-1}}225^{-\frac{1}{q-1}-1}\left(121\left(\frac{15}{4}\right)^{\frac{1}{q-1}}+16\left(\frac{15}{11}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

$$E_{11}(q):=\frac{44^{\frac{1}{q-1}}225^{-\frac{1}{q-1}-1}\left(121\left(\frac{15}{4}\right)^{\frac{1}{q-1}}+16\left(\frac{15}{11}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

and

$$E_{12}(q):=\frac{434^{\frac{1}{q-1}}2025^{-\frac{1}{q-1}-1}\left(961\left(\frac{45}{14}\right)^{\frac{1}{q-1}}+196\left(\frac{45}{31}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1}.$$

Proof. Continuing from (13) in the proof of Theorem 2.3 and using Hölder integral inequality, we have

$$\begin{aligned}\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \le \frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\Bigg[&\left(\int_0^1\left|\eta-\frac{28}{90}\right|^{\frac{q}{q-1}}d\eta\right)^{1-\frac{1}{q}}\left[\int_0^1\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|^q d\eta\right]^{\frac{1}{q}}\\ &+\left(\int_0^1\left|\eta-\frac{44}{60}\right|^{\frac{q}{q-1}}d\eta\right)^{1-\frac{1}{q}}\left[\int_0^1\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|^q d\eta\right]^{\frac{1}{q}}\\ &+\left(\int_0^1\left|\eta-\frac{24}{90}\right|^{\frac{q}{q-1}}d\eta\right)^{1-\frac{1}{q}}\left[\int_0^1\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|^q d\eta\right]^{\frac{1}{q}}\\ &+\left(\int_0^1\left|\eta-\frac{62}{90}\right|^{\frac{q}{q-1}}d\eta\right)^{1-\frac{1}{q}}\left[\int_0^1\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|^q d\eta\right]^{\frac{1}{q}}\Bigg].\end{aligned} \tag{24}$$

Utilizing (15), (16), (17) and (18) in (24), we obtain the desired inequality (23). Hence,

$$\int_0^1\left|\eta-\frac{28}{90}\right|^{\frac{q}{q-1}}d\eta=\frac{434^{\frac{1}{q-1}}2025^{-\frac{q}{q-1}}\left(961\left(\frac{45}{14}\right)^{\frac{1}{q-1}}+196\left(\frac{45}{31}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

$$\int_0^1\left|\eta-\frac{44}{60}\right|^{\frac{q}{q-1}}d\eta=\frac{44^{\frac{1}{q-1}}225^{-\frac{1}{q-1}-1}\left(121\left(\frac{15}{4}\right)^{\frac{1}{q-1}}+16\left(\frac{15}{11}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

$$\int_0^1\left|\eta-\frac{24}{90}\right|^{\frac{q}{q-1}}d\eta=\frac{44^{\frac{1}{q-1}}225^{-\frac{1}{q-1}-1}\left(121\left(\frac{15}{4}\right)^{\frac{1}{q-1}}+16\left(\frac{15}{11}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1},$$

and

$$\int_0^1\left|\eta-\frac{62}{90}\right|^{\frac{q}{q-1}}d\eta=\frac{434^{\frac{1}{q-1}}2025^{-\frac{q}{q-1}}\left(961\left(\frac{45}{14}\right)^{\frac{1}{q-1}}+196\left(\frac{45}{31}\right)^{\frac{1}{q-1}}\right)(q-1)}{2q-1}.$$

Thus, the proof is completed. □

Corollary 2.15. *By taking $h(\eta)=\eta$ in Theorem 2.14, then we have the following inequality:*

$$\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\frac{1}{(8)^{\frac{1}{q}}}\Big[(E_9(q))^{1-\frac{1}{q}}\left[7|\phi'(\xi_1)|^q+|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}$$
$$+(E_{10}(q))^{1-\frac{1}{q}}\left[5|\phi'(\xi_1)|^q+3|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}+(E_{11}(q))^{1-\frac{1}{q}}\left[3|\phi'(\xi_1)|^q+5|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}$$
$$+(E_{12}(q))^{1-\frac{1}{q}}\left[|\phi'(\xi_1)|^q+7|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\Big]. \tag{25}$$

Corollary 2.16. *By taking $h(\eta)=\eta^s$ in Theorem 2.14, then we have the following inequality:*

$$\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\Bigg[(E_9(q))^{1-\frac{1}{q}}\left[\frac{4-3^{s+1}4^{-s}}{s+1}|\phi'(\xi_1)|^q+\frac{4^{-s}}{s+1}|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}$$
$$+(E_{10}(q))^{1-\frac{1}{q}}\left[\frac{4^{-s}\left(3^{s+1}-2^{s+1}\right)}{s+1}|\phi'(\xi_1)|^q+\frac{4^{-s}\left(2^{s+1}-1\right)}{s+1}|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}$$
$$+(E_{11}(q))^{1-\frac{1}{q}}\left[\frac{4^{-s}\left(2^{s+1}-1\right)}{s+1}|\phi'(\xi_1)|^q+\frac{4^{-s}\left(3^{s+1}-2^{s+1}\right)}{s+1}||\phi'(\xi_2)|^q\right]^{\frac{1}{q}}$$
$$+(E_{12}(q))^{1-\frac{1}{q}}\left[\frac{4^{-s}}{s+1}|\phi'(\xi_1)|^q+\frac{4-3^{s+1}4^{-s}}{s+1}|\phi'(\xi_2)|^q\right]^{\frac{1}{q}}\Bigg]. \tag{26}$$

Remark 2.17. *By taking mapping for $h=1$ and $h=\eta(1-\eta)$ in Theorem 2.14, one can establish more Boole's type inequalities similar to Corollary 2.6 and Corollary 2.7.*

Theorem 2.18. *Suppose that all conditions made in Lemma 2.1. If $\left|\phi'\right|$ is h-convex on $[\xi_1,\xi_2]$ and $q>1$, then the following inequality holds:*

$$\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\Bigg[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\Bigg[\left(\int_0^1\left|\eta-\frac{28}{90}\right|h^q\left(\frac{4-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|$$
$$+\left(\int_0^1\left|\eta-\frac{28}{90}\right|h^q\left(\frac{\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\Bigg]+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1\left|\eta-\frac{44}{60}\right|h^q\left(\frac{3-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{44}{60}\right|h^q\left(\frac{1+\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right]$$
$$+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\Bigg[\left(\int_0^1\left|\eta-\frac{24}{90}\right|h^q\left(\frac{2-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|$$
$$+\left(\int_0^1\left|\eta-\frac{24}{90}\right|h^q\left(\frac{2+\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\Bigg]+\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\Bigg[\left(\int_0^1\left|\eta-\frac{62}{90}\right|h^q\left(\frac{1-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|$$
$$+\left(\int_0^1\left|\eta-\frac{62}{90}\right|h^q\left(\frac{3+\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\Bigg]\Bigg]. \tag{27}$$

Proof. From Lemma 2.1 and h-convexity of $\left|\phi'\right|^q$ on $[\xi_1,\xi_2]$, we have

$$\left|\varphi_{\phi,g}(\xi_1,\xi_2,\omega)\right| \leq \frac{\xi_2-\xi_1}{16}\left[\int_0^1\left(\left|\int_0^\eta g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du-\frac{28}{90}\int_0^1 g\left(\frac{4-u}{4}\xi_1+\frac{u}{4}\xi_2\right)du\right|\right.\right.$$

$$
\begin{aligned}
&\times\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|\Bigg)d\eta \\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{3-u}{4}\xi_1+\frac{1+u}{4}\xi_2\right)du-\frac{44}{60}\int_0^1 g\left(\frac{3-u}{4}\xi_1+\frac{1+u}{4}\xi_2\right)du\right|\right. \\
&\times\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|\Bigg)d\eta \\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{2-u}{4}\xi_1+\frac{2+u}{4}\xi_2\right)du-\frac{24}{90}\int_0^1 g\left(\frac{2-u}{4}\xi_1+\frac{2+u}{4}\xi_2\right)du\right|\right. \\
&\times\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|\Bigg)d\eta \\
&+\int_0^1\left(\left|\int_0^\eta g\left(\frac{1-u}{4}\xi_1+\frac{3+u}{4}\xi_2\right)du-\frac{62}{90}\int_0^1 g\left(\frac{1-u}{4}\xi_1+\frac{3+u}{4}\xi_2\right)du\right|\right. \\
&\times\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|\Bigg)d\eta\Bigg] \\
&\le\frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\left[\int_0^1\left|\eta-\frac{28}{90}\right|\left|\phi'\left(\frac{4-\eta}{4}\xi_1+\frac{\eta}{4}\xi_2\right)\right|d\eta+\int_0^1\left|\eta-\frac{44}{60}\right|\right. \\
&\times\left|\phi'\left(\frac{3-\eta}{4}\xi_1+\frac{1+\eta}{4}\xi_2\right)\right|d\eta+\int_0^1\left|\eta-\frac{24}{90}\right|\left|\phi'\left(\frac{2-\eta}{4}\xi_1+\frac{2+\eta}{4}\xi_2\right)\right|d\eta \\
&+\int_0^1\left|\eta-\frac{62}{90}\right|\left|\phi'\left(\frac{1-\eta}{4}\xi_1+\frac{3+\eta}{4}\xi_2\right)\right|d\eta\Bigg] \\
&\le\frac{\xi_2-\xi_1}{16}\|g\|_{[\xi_1,\xi_2],\infty}\left[\int_0^1\left|\eta-\frac{28}{90}\right|\left[h\left(\frac{4-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta\right. \\
&+\int_0^1\left|\eta-\frac{44}{60}\right|\left[h\left(\frac{3-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{1+\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta \\
&+\int_0^1\left|\eta-\frac{24}{90}\right|\left[h\left(\frac{2-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{2+\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta \\
&+\int_0^1\left|\eta-\frac{62}{90}\right|\left[h\left(\frac{1-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{3+\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta\Bigg].
\end{aligned}
\tag{28}
$$

Using power-mean inequality, we have

$$
\begin{aligned}
&\int_0^1\left|\eta-\frac{28}{90}\right|\left[h\left(\frac{4-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta \\
&\le\left(\int_0^1\left|\eta-\frac{28}{90}\right|d\eta\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{28}{90}\right|h^q\left(\frac{4-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{28}{90}\right|h^q\left(\frac{\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right],
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\int_0^1\left|\eta-\frac{44}{60}\right|\left[h\left(\frac{3-\eta}{4}\right)|\phi'(\xi_1)|+h\left(\frac{1+\eta}{4}\right)|\phi'(\xi_2)|\right]d\eta \\
&\le\left(\int_0^1\left|\eta-\frac{44}{60}\right|d\eta\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{44}{60}\right|h^q\left(\frac{3-\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{44}{60}\right|h^q\left(\frac{1+\eta}{4}\right)d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right],
\end{aligned}
\tag{30}
$$

$$\int_0^1 \left|\eta - \frac{24}{90}\right| \left[h\left(\frac{2-\eta}{4}\right)\left|\phi'(\xi_1)\right| + h\left(\frac{2+\eta}{4}\right)\left|\phi'(\xi_2)\right| \right] d\eta$$
$$\leq \left(\int_0^1 \left|\eta - \frac{24}{90}\right| d\eta\right)^{1-\frac{1}{q}} \left[\left(\int_0^1 \left|\eta - \frac{24}{90}\right| h^q\left(\frac{2-\eta}{4}\right) d\eta\right)^{\frac{1}{q}} \left|\phi'(\xi_1)\right| + \left(\int_0^1 \left|\eta - \frac{24}{90}\right| h^q\left(\frac{2+\eta}{4}\right) d\eta\right)^{\frac{1}{q}} \left|\phi'(\xi_2)\right| \right], \tag{31}$$

and

$$\int_0^1 \left|\eta - \frac{62}{90}\right| \left[h\left(\frac{1-\eta}{4}\right)\left|\phi'(\xi_1)\right| + h\left(\frac{3+\eta}{4}\right)\left|\phi'(\xi_2)\right| \right] d\eta$$
$$\leq \left(\int_0^1 \left|\eta - \frac{62}{90}\right| d\eta\right)^{1-\frac{1}{q}} \left[\left(\int_0^1 \left|\eta - \frac{62}{90}\right| h^q\left(\frac{1-\eta}{4}\right) d\eta\right)^{\frac{1}{q}} \left|\phi'(\xi_1)\right| + \left(\int_0^1 \left|\eta - \frac{62}{90}\right| h^q\left(\frac{3+\eta}{4}\right) d\eta\right)^{\frac{1}{q}} \left|\phi'(\xi_2)\right| \right]. \tag{32}$$

Using equalities (19), (20) and inequalities (29), (30), (31), (32) in (28), we obtained the desired result inequality (27). This completes the proof. □

Remark 2.19. *One can obtain numbers of weighted Boole's formula type inequalities similar to Corollaries 2.4, 2.5, 2.6 and 2.7, by taking different mapping of h in Theorem 2.18.*

3. Numerical Examples

In this section, we provide numerical examples for newly established weighted Boole's formula type inequalities for h-convex functions. These computations show that the inequalities presented in this work are numerically valid.

Example 3.1. *Let $\phi : [\xi_1, \xi_2] = [1, 2] \to \mathbb{R}$ be a function defined by $\phi(\omega) = \omega^{s+1}$. By taking $g(\eta) = \xi_2 - \xi_1$, $q = 1$ and $h(\eta) = \eta^s$. Then by applying inequality (12) to the function $\phi(\omega) = \omega^{s+1}$, we have*

$$\int_1^2 \omega^{s+1} d\omega = \frac{2^{2+s} - 1}{s+2},$$

$$\frac{1}{90}\left[7\phi(\xi_1) + 32\phi\left(\frac{3\xi_1 + \xi_2}{4}\right) + 12\phi\left(\frac{\xi_1 + \xi_2}{2}\right) + 32\phi\left(\frac{\xi_1 + 3\xi_2}{4}\right) + 7\phi(\xi_2)\right]$$
$$= \frac{1}{45\,(2)^{1+2s}}\left(7 \times 2^{1+3s} + 2^{1+s} \times 3^{2+s} + 7 \times 4^s + 8 \times 5^{1+s} + 8 \times 7^{1+s}\right),$$

for $s = \frac{1}{2}$, the left-hand side of (12) is

$$\left| \frac{1}{90\,(\xi_2 - \xi_1)}\left[7\phi(\xi_1) + 32\phi\left(\frac{3\xi_1 + \xi_2}{4}\right) + 12\phi\left(\frac{\xi_1 + \xi_2}{2}\right) + 32\phi\left(\frac{\xi_1 + 3\xi_2}{4}\right) + 7\phi(\xi_2)\right] \right.$$
$$\left. \times \int_{\xi_1}^{\xi_2} g(x)dx - \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \phi(\omega)g(\omega)d\omega \right| = 4.5373 \times 10^{-7}, \tag{33}$$

and the right-hand side of (12), for $s = \frac{1}{2}$ is

$$\frac{(\xi_2 - \xi_1)\left\|g\right\|_{[\xi_1,\xi_2],\infty}}{16} \frac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)} \left(45^{s+1}\left(3^{s+1} + 7\,4^{s+1} - 1\right)s - 19\,4^{s+2}45^{s+1}\right.$$

$$-47\,45^{s+1} - 133\,135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2}\Big)\Big[|\phi'(\xi_1)| + |\phi'(\xi_2)|\Big]$$
$$= 1.798 \times 10^{-1}. \tag{34}$$

From (33) and (34), it is clear that the left-hand side is less than the right-hand side

$$4.5373 \times 10^{-7} < 1.798 \times 10^{-1}.$$

This demonstrate that the inequality (12) is numerically valid.

Example 3.2. *Let $\phi : [\xi_1, \xi_2] = [1, 2] \to \mathbb{R}$ be a function defined by $\phi(\omega) = (\omega - 1)^2$. By taking $g(\eta) = \xi_2 - \xi_1$, $q = 2$ and $h(\eta) = \eta(1 - \eta)$. Then by applying inequality (12) to the function $\phi(\omega) = (\omega - 1)^2$. Then, we have the left-hand side of inequality (12) is*

$$\left|\varphi_{\phi,g}(\xi_1, \xi_2, \omega)\right| = 3.3333e^{-11} \tag{35}$$

the right-hand side of the inequality is (12) is

$$\frac{\xi_2 - \xi_1}{8}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\left(\frac{1157\left(\cosh\left(\frac{\log\left(\frac{224920800}{29691973}\right)}{q}\right) - \sinh\left(\frac{\log\left(\frac{224920800}{29691973}\right)}{q}\right)\right)}{4050}\right.$$
$$\left.+\frac{(137)^{1-\frac{1}{q}}\,(641413)^{1/q}}{450\left(\sinh\left(\frac{\log(21600)}{q}\right) + \cosh\left(\frac{\log(21600)}{q}\right)\right)}\right)\left[|\phi'(\xi_1)|^q + |\phi'(\xi_2)|^q\right]^{\frac{1}{q}} = 1.0612. \tag{36}$$

From (35) and (36), it is clear that the left-hand side is less than the right-hand side

$$3.3333e^{-11} < 1.0612.$$

This demonstrates that the inequality (12) is also numerically valid for this case as well.

Remark 3.3. *From these above examples, one can easily observe that the errors obtained here are better than the existing inequalities in the literature.*

Remark 3.4. *We discuss two different cases for Theorem 2.3 in the above examples, other theorems can be verified similarly, left for the readers.*

4. Real-Life Applications of Boole's Formula

Using the results of section 2, we now provide some applications for numerical quadrature formulas.

4.1. Application to Numerical Quadrature Formulas

Let Υ be the division of the interval $[\xi_1, \xi_2]$, $\Upsilon : \xi_1 = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_n = \xi_2$, and let $\eta_j := \omega_{j+1} - \omega_j$ for $j = 0, 1, \ldots, n - 1$. Let us consider the following quadrature formula:

$$I := \int_{\xi_1}^{\xi_2} \phi(\eta) g(\eta)\, d\eta = S_B\left(\phi, g, \Upsilon\right) + \mathcal{R}_B\left(\phi, g, \Upsilon\right), \tag{37}$$

where

$$S_B\left(\phi, g, \Upsilon\right) = \sum_{j=0}^{n-1} \frac{1}{90}\left[7\phi(\omega_j) + 12\phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right) + 32\phi\left(\frac{3\omega_j + \omega_{j+1}}{4}\right)\right.$$

$$+ 32\phi\left(\frac{\omega_j + 3\omega_{j+1}}{4}\right) + 7\phi(\omega_{j+1})\Big]\int_{\omega_j}^{\omega_{j+1}} g(\omega)\, d\omega,$$

for the weighted Boole's type formula and $\mathcal{R}_B\left(\phi, g, \Upsilon\right)$ denotes the related approximation error of the integral $\int_{\xi_1}^{\xi_2} \phi(\eta)g(\eta)\,d\eta$.

Now, we obtain an error estimate related to the weighted Boole's type formula. In the following, we give estimations for the remainder term $\mathcal{R}_B\left(\phi, g, \Upsilon\right)$ in terms of the first derivative.

Proposition 4.1. *Suppose that all assumptions made in Theorem 2.3 are satisfied and $q = 1$, then the following weighted Boole's type error estimate satisfies:*

$$\left|\mathcal{R}_B\left(\phi, g, \Upsilon\right)\right| \leq \frac{239}{6480}\sum_{j=0}^{n-1}\eta_j^2\left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right]\|g\|_{[\xi_1,\xi_2],\infty}. \tag{38}$$

Proof. Applying the Corollary 2.4 with $q = 1$ on the subinterval $\left[\omega_j, \omega_{j+1}\right]$ $(j = 0, 1, 2, \ldots, n-1)$, we have

$$\begin{aligned}&\left|\frac{1}{90\eta_j}\left[7\phi(\omega_j) + 32\phi\left(\frac{3\omega_j + \omega_{j+1}}{4}\right) + 12\phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right)\right.\right.\\&\left.\left.+32\phi\left(\frac{\omega_j + 3\omega_{j+1}}{4}\right) + 7\phi(\omega_{j+1})\right]\int_{\omega_j}^{\omega_{j+1}} g(\omega)d\omega - \frac{1}{\eta_j}\int_{\omega_j}^{\omega_{j+1}} g(\omega)\,\phi(\omega)d\omega\right|\\&\leq \frac{239\eta_j}{6480}\|g\|_{[\xi_1,\xi_2],\infty}\left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right].\end{aligned} \tag{39}$$

Summing over j from 0 to $n-1$ on both sides of (39) and using the triangle inequality, we have that

$$\begin{aligned}\left|\mathcal{R}_B\left(\phi, g, \Upsilon\right)\right| &= \left|S_B\left(\phi, g, \Upsilon\right) - \int_{\xi_1}^{\xi_2}\phi(\eta)g(\eta)\,d\eta\right|\\&= \sum_{j=0}^{n-1}\left|\frac{1}{90}\left[7\phi(\omega_j) + 32\phi\left(\frac{3\omega_j + \omega_{j+1}}{4}\right) + 12\phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right)\right.\right.\\&\left.\left.+32\phi\left(\frac{\omega_j + 3\omega_{j+1}}{4}\right) + 7\phi(\omega_{j+1})\right]\int_{\omega_j}^{\omega_{j+1}} g(\omega)d\omega - \int_{\omega_j}^{\omega_{j+1}} g(\omega)\,\phi(\omega)d\omega\right|\\&\leq \frac{239}{6480}\sum_{j=0}^{n-1}\eta_j^2\left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right]\|g\|_{[\xi_1,\xi_2],\infty}.\end{aligned}$$

This completes the proof. □

Remark 4.2. *In a singular way, one can obtain an approximation for Theorem 2.14 and Theorem 2.18. However, the details are left to the interested reader.*

Remark 4.3. *The error approximations in classical are based on Taylor expansion for the Boole's formula involve the 6^{th} derivative $\|\phi^{(6)}\|_\infty$. If the $\phi^{(6)}$ derivative does not exists or is very large at some points in $[\xi_1, \xi_2]$, the classical approximation cannot be applied and thus (38) provide alternative approximation for Boole's formula.*

4.2. Application to Midpoint Formula

Let Υ be the division of the interval $[\xi_1, \xi_2]$, $\Upsilon : \xi_1 = \omega_0 < \omega_1 < \omega_2 < \ldots < \omega_n = \xi_2$, and consider the following midpoint quadrature formula:

$$M\left(\phi, \Upsilon\right) = \sum_{j=0}^{n-1}\left(\omega_{j+1} - \omega_j\right)\phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right). \tag{40}$$

It is well-known that if the mapping $\phi : [\xi_1, \xi_2] \to \mathbb{R}$, is differentiable such that $\phi'(\omega)$ exists on (ξ_1, ξ_2), then

$$I := \int_{\xi_1}^{\xi_2} \phi(\eta) d\eta = M\left(\phi, \Upsilon\right) + \mathcal{R}_M\left(\phi, \Upsilon\right), \tag{41}$$

where the approximation error $\mathcal{R}_M\left(\phi, \Upsilon\right)$ of the integral I by midpoint formula $M\left(\phi, \Upsilon\right)$ satisfies

$$\left|\mathcal{R}_M\left(\phi, \Upsilon\right)\right| \leq \sum_{j=0}^{n-1} \left(\omega_{j+1} - \omega_j\right) \phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right). \tag{42}$$

Here, we drive some error bounds for new midpoint formula type inequality. In the following, we propose some new approximations for the remainder term $\mathcal{R}_M\left(\phi, \Upsilon\right)$ in terms of the 1^{st} derivative.

Proposition 4.4. *Consider $\phi : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $\phi' \in L[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in I$ with $\xi_1 < \xi_2$. If $\left|\phi'\right|$ is convex on $[\xi_1, \xi_2]$, then in (41) for every division Υ of $[\xi_1, \xi_2]$, the following inequality holds:*

$$\left|\mathcal{R}_M\left(\phi, \Upsilon\right)\right| \leq \frac{239}{6480} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right]. \tag{43}$$

Proof. Applying the Corollary 2.11 on the subinterval $\left[\omega_j, \omega_{j+1}\right]$ $(j = 0, 1, 2, \ldots, n-1)$, we have

$$\begin{aligned} &\left|\left(\omega_{j+1} - \omega_j\right) \phi\left(\frac{\omega_j + \omega_{j+1}}{2}\right) - \int_{\omega_j}^{\omega_{j+1}} \phi(\omega) d\omega\right| \\ &\leq \frac{239(\omega_{j+1} - \omega_j)^2}{6480} \left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right]. \end{aligned} \tag{44}$$

Summing over j from 0 to $n-1$ on both sides of (44) and taking into account that $\left|\phi'\right|$ is convex, we deduce by the triangle inequality, that

$$\left|\mathcal{R}_M\left(\phi, \Upsilon\right)\right| \leq \frac{239}{6480} \sum_{j=0}^{n-1} (\omega_{j+1} - \omega_j)^2 \left[|\phi'(\omega_j)| + |\phi'(\omega_{j+1})|\right].$$

This completes the proof. □

Remark 4.5. *The estimations for midpoint type inequality given in this work are better than the estimation proved in [13].*

4.3. Application to Random variable

Let $0 < \xi_1 < \xi_2$, $r \in \mathbb{R}$, and let Υ be a continuous random variable having the continuous probability density mapping $g : [\xi_1, \xi_2] \to [0, 1]$, which is symmetric with respect to $\frac{3\xi_1+\xi_2}{4}$, $\frac{\xi_1+\xi_2}{2}$ and $\frac{\xi_1+3\xi_2}{4}$. Also, the r-moment is defined by

$$\Xi_r(\Upsilon) := \int_{\xi_1}^{\xi_2} \varkappa^r g(\varkappa) d\varkappa,$$

which is assumed to be finite. Now, we obtain the following results.

Proposition 4.6. *Suppos that all assumptions made in Theorem 2.18 are satisfied, then we have the following inequality:*

$$\left|\frac{1}{90}\left[7\xi_1^r+32\left(\frac{3\xi_1+\xi_2}{4}\right)^r+12\left(\frac{\xi_1+\xi_2}{2}\right)^r+32\left(\frac{\xi_1+3\xi_2}{4}\right)^r+7\left(\xi_2\right)^r\right]-\Xi_r\left(\Upsilon\right)\right|$$
$$\leq\frac{|r|\left(\xi_2-\xi_1\right)^2}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\left[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left[\left((E_{13})^{\frac{1}{q}}+(E_{19})^{\frac{1}{q}}\right)\xi_1^{r-1}+\left((E_{14})^{\frac{1}{q}}+(E_{20})^{\frac{1}{q}}\right)\right.\right.$$
$$\left.\left.\times\xi_2^{r-1}\right]+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left[\left((E_{15})^{\frac{1}{q}}+(E_{17})^{\frac{1}{q}}\right)\xi_1^{r-1}+\left((E_{16})^{\frac{1}{q}}+(E_{18})^{\frac{1}{q}}\right)\xi_2^{r-1}\right]\right],$$

where

$$E_{13}:=-\frac{45q\tau\left(31\times\ 3^{q\tau+1}4^{-q\tau}-56\right)+985\ \times 3^{q\tau+3}4^{-q\tau}-2^{3-q\tau}45^{-q\tau}83^{q\tau+2}+27360}{2025\left(q^2\tau^2+3q\tau+2\right)},$$
$$E_{14}:=\frac{2^{-2q\tau}45^{-q\tau-2}\left(31q\tau 45^{q\tau+1}+17\ 45^{q\tau+1}+2^{q\tau+3}7^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$
$$E_{15}:=\frac{1}{(q\tau+1)(q\tau+2)}4^{-q\tau}15^{-q\tau-2}\left(q\tau\left(-15^{q\tau+1}\right)\left(2^{q\tau+3}-11\ 3^{q\tau+1}\right)\right.$$
$$\left.-19\ 2^{q\tau+2}15^{q\tau+1}-23\ 45^{q\tau+1}+2^{q\tau+3}17^{q\tau+2}\right),$$
$$E_{16}:=\frac{4^{-q\tau}15^{-q\tau-2}\left(q\tau 15^{q\tau+1}\left(2^{q\tau+3}-11\right)-11\ 2^{q\tau+2}15^{q\tau+1}-37\ 15^{q\tau+1}+2^{q\tau+3}13^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$
$$E_{17}:=\frac{4^{-q\tau}15^{-q\tau-2}\left(q\tau 15^{q\tau+1}\left(2^{q\tau+3}-11\right)-11\ 2^{q\tau+2}15^{q\tau+1}-37\ 15^{q\tau+1}+2^{q\tau+3}13^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$
$$E_{18}:=\frac{1}{(q\tau+1)(q\tau+2)}4^{-q\tau}15^{-q\tau-2}\left(q\tau\left(-15^{q\tau+1}\right)\left(2^{q\tau+3}-11\ 3^{q\tau+1}\right)\right.$$
$$\left.-19\ 2^{q\tau+2}15^{q\tau+1}-23\ 45^{q\tau+1}+2^{q\tau+3}17^{q\tau+2}\right),$$
$$E_{19}:=\frac{2^{-2q\tau}45^{-q\tau-2}\left(31q\tau 45^{q\tau+1}+17\ 45^{q\tau+1}+2^{q\tau+3}7^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$

and

$$E_{20}:=-\frac{45q\tau\left(31\ 3^{q\tau+1}4^{-q\tau}-56\right)+985\ 3^{q\tau+3}4^{-q\tau}-2^{3-q\tau}45^{-q\tau}83^{q\tau+2}+27360}{2025\left(q^2\tau^2+3q\tau+2\right)}.$$

Proof. Let $\phi(\mu)=\frac{\mu^r}{r}$, $\mu\in[\xi_1,\xi_2]$ with $r\in(-\infty,0)\cup(0,1]\cup[2,+\infty)$, and let $h(\eta)=\eta^\tau$ for $\tau\in(-\infty,-1)\cup(-1,1]$, then $|\phi'|$ is h-convex see Example 7 in [22]. Now applying $\phi(\mu)=\frac{\mu^r}{r}$, $\mu\in[\xi_1,\xi_2]$ to the inequality (27), we have

$$\left|\frac{1}{90}\left[7\xi_1^r+32\left(\frac{3\xi_1+\xi_2}{4}\right)^r+12\left(\frac{\xi_1+\xi_2}{2}\right)^r+32\left(\frac{\xi_1+3\xi_2}{4}\right)^r+7\left(\xi_2\right)^r\right]-\Xi_r\left(\Upsilon\right)\right|$$
$$\leq\frac{|r|\left(\xi_2-\xi_1\right)^2}{16}\left\|g\right\|_{[\xi_1,\xi_2],\infty}\left[\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{28}{90}\right|\left(\frac{4-\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|\right.\right.$$
$$\left.+\left(\int_0^1\left|\eta-\frac{28}{90}\right|\left(\frac{\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right]$$

$$+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{44}{60}\right|\left(\frac{3-\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{44}{60}\right|\left(\frac{1+\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right]$$

$$+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{24}{90}\right|\left(\frac{2-\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{24}{90}\right|\left(\frac{2+\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right]$$

$$+\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\eta-\frac{62}{90}\right|\left(\frac{1-\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_1)|+\left(\int_0^1\left|\eta-\frac{62}{90}\right|\left(\frac{3+\eta}{4}\right)^{q\tau}d\eta\right)^{\frac{1}{q}}|\phi'(\xi_2)|\right]\Bigg].$$

The required inequality obtained from the above inequality by simple computations that

$$\int_0^1\left|\eta-\frac{28}{90}\right|\left(\frac{4-\eta}{4}\right)^{q\tau}d\eta=-\frac{45q\tau\left(31\times\ 3^{q\tau+1}4^{-q\tau}-56\right)+985\ \times 3^{q\tau+3}4^{-q\tau}-2^{3-q\tau}45^{-q\tau}83^{q\tau+2}+27360}{2025\,(q^2\tau^2+3q\tau+2)},$$

$$\int_0^1\left|\eta-\frac{28}{90}\right|\left(\frac{\eta}{4}\right)^{q\tau}d\eta=\frac{2^{-2q\tau}45^{-q\tau-2}\left(31q\tau45^{q\tau+1}+17\ 45^{q\tau+1}+2^{q\tau+3}7^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$

$$\int_0^1\left|\eta-\frac{44}{60}\right|\left(\frac{3-\eta}{4}\right)^{q\tau}d\eta=\frac{1}{(q\tau+1)(q\tau+2)}4^{-q\tau}15^{-q\tau-2}\left(q\tau\left(-15^{q\tau+1}\right)\left(2^{q\tau+3}-11\ 3^{q\tau+1}\right)\right.$$
$$\left.-19\ 2^{q\tau+2}15^{q\tau+1}-23\ 45^{q\tau+1}+2^{q\tau+3}17^{q\tau+2}\right),$$

$$\int_0^1\left|\eta-\frac{44}{60}\right|\left(\frac{1+\eta}{4}\right)^{q\tau}d\eta=\frac{4^{-q\tau}15^{-q\tau-2}\left(q\tau15^{q\tau+1}\left(2^{q\tau+3}-11\right)-11\ 2^{q\tau+2}15^{q\tau+1}-37\ 15^{q\tau+1}+2^{q\tau+3}13^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$

$$\int_0^1\left|\eta-\frac{24}{90}\right|\left(\frac{2-\eta}{4}\right)^{q\tau}d\eta=\frac{1}{(q\tau+1)(q\tau+2)}4^{-q\tau}15^{-q\tau-2}\left(q\tau\left(-15^{q\tau+1}\right)\left(2^{q\tau+3}-11\ 3^{q\tau+1}\right)\right.$$
$$\left.-19\ 2^{q\tau+2}15^{q\tau+1}-23\ 45^{q\tau+1}+2^{q\tau+3}17^{q\tau+2}\right),$$

$$\int_0^1\left|\eta-\frac{62}{90}\right|\left(\frac{1-\eta}{4}\right)^{q\tau}d\eta=\frac{2^{-2q\tau}45^{-q\tau-2}\left(31q\tau45^{q\tau+1}+17\ 45^{q\tau+1}+2^{q\tau+3}7^{q\tau+2}\right)}{(q\tau+1)(q\tau+2)},$$

and

$$\int_0^1\left|\eta-\frac{62}{90}\right|\left(\frac{3+\eta}{4}\right)^{q\tau}d\eta=-\frac{45q\tau\left(31\ 3^{q\tau+1}4^{-q\tau}-56\right)+985\ 3^{q\tau+3}4^{-q\tau}-2^{3-q\tau}45^{-q\tau}83^{q\tau+2}+27360}{2025\,(q^2\tau^2+3q\tau+2)}.$$

This completes the proof. □

4.4. Application to Special Means

Let $s\in(0,1]$ and $\eta_1,\eta_2,\eta_3\in\mathbb{R}$. We define a function $\phi,g:[0,\infty)\to\mathbb{R}$ as

$$\phi(\omega)\,g(\omega)=\begin{cases}\eta_1, & \omega=0,\\ \eta_2\tau^s+\eta_3, & \omega>0\,.\end{cases}\ .$$

If $\eta_2\geq0$ and $0\leq\eta_3\leq\eta_1$, then $\phi\in\mathcal{K}_s^2$ (see [20]). Hence $\eta_1=\eta_3=0$, $\eta_2=1$, we have $g,\phi:[\xi_1,\xi_2]\to\mathbb{R}$, $\phi(\omega)=\omega^s$, $\phi\in\mathcal{K}_s^2$.

A simple consequence of the previous result may be stated as follows:

Corollary 4.7. *Let $\mathcal{G} : I \to I_1 \subset [0,\infty)$ be a non-negative convex function on I, then $\mathcal{G}^s(\eta)$ is s-convex on $[0,\infty)$, $0 < s < 1$.*

We shall consider the following special means. For arbitrary real numbers $\xi_1, \xi_{11}, \xi_{12}, \ldots, \xi_{1n}, \xi_2$, we have

The arithmetic mean: $\mathcal{A} := \mathcal{A}(\xi_{11}, \xi_{12}, \ldots, \xi_{1n}) = \frac{\xi_{11}+\xi_{12}+\ldots+\xi_{1n}}{n}$.

The harmonic mean: $\mathcal{H} := \mathcal{H}(\xi_1, \xi_2) = \frac{2}{\frac{1}{\xi_1}+\frac{1}{\xi_2}}, \quad \xi_1, \xi_2 > 0.$

The logarithm mean: $\mathcal{L} := \mathcal{L}(\xi_1, \xi_2) = \frac{\xi_2-\xi_1}{\ln \xi_2 - \ln \xi_1}, \qquad \xi_1 \neq \xi_2, \quad \xi_1, \xi_2 \in \mathbb{R}^+.$

The p-logarithmic mean: $\mathcal{L}_p := \mathcal{L}_p(\xi_1, \xi_2) = \left(\frac{\xi_2^{p+1}-\xi_1^{p+1}}{(p+1)(\xi_2-\xi_1)}\right)^{\frac{1}{p}}, \quad \xi_1, \xi_2 > 0, \quad \xi_1 \neq \xi_2$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

It is well-known that $\mathcal{L}_p$ is monotonic non-decreasing over $p \subset \mathbb{R}$, with $\mathcal{L}_{-1} := \mathcal{L}$ and $\mathcal{L}_0 := \mathcal{I}$. We have the following inequality $\mathcal{L} \leq \mathcal{A}$.

Based on the outcomes presented in Section 2, we establish a new inequalities for the means mentioned above.

Proposition 4.8. *Consider $\phi, g : [\xi_1, \xi_2] \to \mathbb{R}$, $(0 < \xi_1 < \xi_2)$, $g(\omega) = \phi(\omega) = \omega^s$, $s\,(0,1]$. Then, we have*

$$\begin{aligned}
\frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} g(\tau)\,\phi(\tau)\,d\tau &= \left(\mathcal{L}_s^s(\xi_1, \xi_2)\right)^2, \\
\frac{\phi(\xi_1) + \phi(\xi_2)}{2} &= \mathcal{A}(\xi_1^s, \xi_2^s), \\
\phi\left(\frac{\xi_1 + \xi_2}{2}\right) &= \mathcal{A}^s(\xi_1, \xi_2), \\
\phi\left(\frac{3\xi_1 + \xi_2}{4}\right) &= \mathcal{A}^s(\xi_1, \xi_1, \xi_1, \xi_2), \\
\phi\left(\frac{\xi_1 + 3\xi_2}{4}\right) &= \mathcal{A}^s(\xi_1, \xi_2, \xi_2, \xi_2).
\end{aligned}$$

(1) *Using the inequality (22), we obtain*

$$\begin{aligned}
&\left|\frac{1}{90}\left[14\mathcal{A}(\xi_1^s, \xi_2^s) + 32\mathcal{A}^s(\xi_1, \xi_1, \xi_1, \xi_2) + 12\mathcal{A}^s(\xi_1, \xi_2) + 32\mathcal{A}^s(\xi_1, \xi_2, \xi_2, \xi_2)\right]\mathcal{L}_s^s(\xi_1, \xi_2) - \left(\mathcal{L}_s^s(\xi_1, \xi_2)\right)^2\right| \\
&\leq \frac{s(\xi_2 - \xi_1)\left\|g\right\|_{[\xi_1,\xi_2],\infty}}{16}\,\frac{2^{1-2s}}{(45)^{s+2}(s+1)(s+2)}\Big(45^{s+1}\left(3^{s+1} + 7\,4^{s+1} - 1\right)s - 19\,4^{s+2}45^{s+1} - 47\,45^{s+1} \\
&-133\,135^{s+1} + 14^{s+2} + 78^{s+2} - 90^{s+2} + 102^{s+2} + 166^{s+2}\Big)\left[|\xi_1|^{s-1} + |\xi_2|^{s-1}\right]. \qquad (45)
\end{aligned}$$

For instance, if $s = 1$ in (45), then we have

$$\left|\frac{1}{90}\left[26\mathcal{A}(\xi_1, \xi_2) + 32\mathcal{A}(\xi_1, \xi_1, \xi_1, \xi_2) + 32\mathcal{A}(\xi_1, \xi_2, \xi_2, \xi_2)\right] - \mathcal{L}(\xi_1, \xi_2)\right| \leq \frac{239(\xi_2 - \xi_1)\left\|g\right\|_{[\xi_1,\xi_2],\infty}}{6480}.$$

(2) *Using the inequality (21), we obtain*

$$\left|\mathcal{A}^s(\xi_1, \xi_2) - \mathcal{L}_s^s(\xi_1, \xi_2)\right| \leq \frac{239s(\xi_2 - \xi_1)}{6480}\left[\xi_1^{s-1} + \xi_2^{s-1}\right]. \qquad (46)$$

For instance, if $s = 1$ in (46), then we have

$$|\mathcal{A}(\xi_1, \xi_2) - \mathcal{L}(\xi_1, \xi_2)| \leq \frac{239(\xi_2 - \xi_1)}{6480}.$$

Remark 4.9. *We gave applications for different cases of Theorem 2.3. Similarly, one can easily obtain more results for different cases of other theorems, details are left behind for interested researchers.*

5. Conclusion

The primary objective of this study is to introduce novel weighted Boole's formula type inequalities applicable to single-time differentiable for general convex functions. To achieve this goal, we used an integral identity and subsequently demonstrated new inequalities of Boole's formula type for differentiable h-convex functions. The inequalities established in this paper can be helpful in finding the bounds for Boole's and midpoint formulas. Moreover, the error bounds given in this work are better than those of some existing results. Applications to quadrature formulas, r-moment, and special means for real numbers are given. Furthermore, numerical examples show that the obtained results in this work are numerically valid. This research extensively delved into various classes of functions through specific methodologies. It is an interesting and new problem that the new researcher can explore further extensions and delve deeper into the implications of these findings across other mathematical domains.

Competing Interests

The authors declare no competing interests.

Data availability

Data sharing is not relevant to this paper, as there was no generation or analysis of new data during this study.

Authors Contributions

Conceptualization; A.M., Methodology; A.M., Software; A.M., Validation; A.M., M.T., A.K., and Z.Z., Writing-original draft; A.M., Writing-review & editing; A.M., M.T., A.K., and Z.Z., Project administration; A.M., and A.K., Supervision; Z.Z. All authors have read and agreed to the final version of the manuscript.

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