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On the joint spectra of operators and antiunitaries

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Abstract. Camara and Krejčiřík have studied properties of operators concerning with an antiunitary operator C (see [1]). In this paper we show that if $\mathbf{T} = (T_1, ..., T_n)$ is a commuting n-tuple of Hilbert space operators and C is an antinuitary, then $\sigma(C\mathbf{T}C^{-1}) = \sigma(\mathbf{T})^*$, where $\sigma(\mathbf{T})$ is the Taylor spectrum of $\mathbf{T}$, $\sigma(\mathbf{T})^* = \{\overline{z} = (\overline{z_1}, ..., \overline{z_n}) : z = (z_1, ..., z_n) \in \sigma(\mathbf{T})\}$ and $C\mathbf{T}C^{-1} = (CT_1C^{-1}, ..., CT_nC^{-1})$. Also we will show $\sigma_X(C\mathbf{T}C^{-1}) = \{\overline{z} : z \in \sigma_X(\mathbf{T})\}$, where $\sigma_X(\mathbf{T})$ is the Xia spectrum of $\mathbf{T}$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\ ,\ \rangle$ and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator C on $\mathcal{H}$ is said to be *antilinear* if $C(\alpha x + \beta y) = \overline{\alpha}Cx + \overline{\beta}Cy$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in \mathcal{H}$. An antilinear operator C is said to be *conjugation* if $\langle Cx\,,\, Cy\rangle = \langle y\,,\, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$, where I is the identity operator on $\mathcal{H}$. In [9], S. Jung, E. Ko and Ji Eun Lee showed that if C is a conjugation and $T \in B(\mathcal{H})$, then $\sigma(CTC) = \sigma(T)^*$, $\sigma_a(CTC) = \sigma_a(T)^*$ and $\sigma_p(CTC) = \sigma_p(T)^*$, where $\sigma(T), \sigma_a(T)$ and $\sigma_p(T)$ are the spectrum, the approximate point spectrum and the point spectrum of T, respectively.

For the study of the Pauli or Dirac operators, M. Cristina Câmara and D. Krejcirik have studied properties of operators concerning with an antilinear operator C satisfying $C^2 = -I$. They showed the following example.

Example 1.1. (see [1]) Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For the Hilbert space $L^2 := L^2(\mathbb{T})$, an operator C be defined by

$$(Cf)(z) = \frac{z - \overline{z}}{2}\,\overline{f(\overline{z})} + \frac{z + \overline{z}}{2}\,\overline{f(-\overline{z})}.$$

Then C is antilinear and satisfies $\langle Cf, Cg\rangle = \langle g, f\rangle$ for any $f, g \in L^2$ and $C^2 = -I$.

Definition 1.2. An antilinear operator C on $\mathcal{H}$ is said to be *antiunitary* if C satisfies $\langle Cx\,,\, Cy\rangle = \langle y\,,\, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = -I$.

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By the definition of an antiunitary operator C, C is onto, isometric and $C^{-1} = -C$.

Remark 1.3. It is easy to see that if an antilinear isometric operator C on $\mathcal{H}$ satisfies $C^2 = zI$, then $z = \pm 1$.

In [2] Chō and Ji Eun Lee showed that if C is antiunitary on $\mathcal{H}$, then $\sigma(CTC^{-1}) = \sigma(T)^*, \sigma_a(CTC^{-1}) = \sigma_a(T)^*$ and $\sigma_p(CTC^{-1}) = \sigma_p(T)^*$.

In this paper we show that if C is antiunitary and $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is a commuting n-tuple, then $\sigma(C\mathbf{T}C^{-1}) = \sigma(\mathbf{T})^*, \sigma_{ja}(C\mathbf{T}C^{-1}) = \sigma_{ja}(\mathbf{T})^*$ and $\sigma_{jp}(C\mathbf{T}C^{-1}) = \sigma_{jp}(\mathbf{T})^*$, where $C\mathbf{T}C^{-1} = (CT_1C^{-1}, ..., CT_nC^{-1})$, and $\sigma(\mathbf{T}), \sigma_{ja}(\mathbf{T})$ and $\sigma_{jp}(\mathbf{T})$ are the Taylor spectrum, the joint approximate point spectrum and the joint point spectrum of $\mathbf{T}$, respectively.

For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we explain the Taylor spectrum $\sigma(\mathbf{T})$ of $\mathbf{T}$ shortly. Let E^n be the exterior algebra on n generators, that is, E^n is the complex algebra with identity e generated by indeterminates $e_1, ..., e_n$. Let $E^n_k(\mathcal{H}) = \mathcal{H} \otimes E^n_k$. Define $d^n_k : E^n_k(\mathcal{H}) \longrightarrow E^n_{k-1}(\mathcal{H})$ by

$$d^n_k(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_k}) := \sum_{i=1}^{k} (-1)^{i-1} T_{j_i} x \otimes e_{j_1} \wedge \cdots \wedge \check{e}_{j_i} \wedge \cdots \wedge e_{j_k},$$

where $\check{e}_{j_i}$ means deletion. We denote d^n_k by d_k simply. We think Koszul complex $E(\mathbf{T})$ of $\mathbf{T}$ as follows:

$$(*) \quad E(\mathbf{T}) \ : \ 0 \longrightarrow E^n_n(\mathcal{H}) \xrightarrow{d_n} E^n_{n-1}(\mathcal{H}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} E^n_1(\mathcal{H}) \xrightarrow{d_1} E^n_0(\mathcal{H}) \longrightarrow 0.$$

It is easy to see that $E^n_k(\mathcal{H}) \cong \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{\frac{n!}{(n-k)!k!}}$ $(k = 1, ..., n)$ and $\mathrm{Im}\, d_j \subset \ker d_{j-1}$ $(j = 2, ..., n)$. Koszul complex $E(\mathbf{T})$ is said to be *exact* if $\ker d_n = \{0\}$, $\mathrm{Im}\, d_j = \ker d_{j-1}$ for all j $(j = 2, ..., n)$ and $\mathrm{Im}\, d_1 = \mathcal{H}$.

Definition 1.4. *A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is said to be nonsingular if and only if the Koszul complex $E(\mathbf{T})$ is exact.*

Definition 1.5. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $z = (z_1, ..., z_n) \notin \sigma(\mathbf{T})$ (Taylor spectrum) if $\mathbf{T} - z = (T_1 - z_1, ..., T_n - z_n)$ is nonsingular.*

About the definition of the Taylor spectrum, see details J. L. Taylor [10] and [11].

The joint approximate point spectrum of $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is denoted by $\sigma_{ja}(\mathbf{T})$, i.e., $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$ if and only if there exists a sequence $\{x_k\}$ of unit vectors such that

$$(T_j - z_j)x_k \ \longrightarrow \ 0 \ \text{ as } k \to \infty \text{ for all } \ j = 1, ..., n.$$

The joint point spectrum $\sigma_{jp}(\mathbf{T})$ of $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ is the set of all $(z_1, ..., z_n) \in \mathbb{C}^n$ which there exists a nonzero vector x such that $(T_j - z_j)x = 0$ for all $j = 1, ..., n$.

2. Taylor spectrum

First we need the following result by R. Curto [7] .

Proposition 2.1. (pp.131-132, [7]) *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, $0 = (0, ..., 0) \notin \sigma(\mathbf{T})$ if and only if*

$$\alpha(\mathbf{T}) := \begin{pmatrix} d_1 & 0 & \cdots & \cdots \\ d_2^* & d_3 & \cdots & \cdots \\ 0 & d_4^* & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \ \textit{is invertible on} \ \overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}},$$

where d_k is the mapping of $(*)$ $(k = 1, 2, ..., n)$.

For an antiunitary C on $\mathcal{H}$, let $C\mathbf{T}C^{-1} = (CT_1C^{-1}, ..., CT_nC^{-1})$. If $\mathbf{T} = (T_1, ..., T_n)$ is a commuting n-tuple, then $C\mathbf{T}C^{-1}$ is also commuting n-tuple.

First we start the following lemma. Since a proof is easy, we omit it.

Lemma 2.2. *Let $T \in B(\mathcal{H})$ and C be antiunitary. Then $(CTC^{-1})^* = CT^*C^{-1}$.*

Lemma 2.3. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and any antiunitary C, $0 = (0, ..., 0) \notin \sigma(\mathbf{T})$ if and only if $0 = (0, ..., 0) \notin \sigma(C\mathbf{T}C^{-1})$, where $C\mathbf{T}C^{-1} = (CT_1C^{-1}, ..., CT_nC^{-1})$.*

Proof. It holds $CT_iC^{-1} \cdot CT_jC^{-1} = CT_iT_jC^{-1}$ and $(CT_iC^{-1})^* = CT_i^*C^{-1}$ by Lemma 2.2. Hence we have

$$\alpha(C\mathbf{T}C^{-1}) = \begin{pmatrix} C & 0 & \cdots & \cdots & 0 \\ 0 & C & \cdots & \cdots & 0 \\ 0 & 0 & C & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C \end{pmatrix} \cdot \alpha(\mathbf{T}) \cdot \begin{pmatrix} C^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & C^{-1} & \cdots & \cdots & 0 \\ 0 & 0 & C^{-1} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & C^{-1} \end{pmatrix}$$

on $\overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}}$. Since $\tilde{C} = \overbrace{C \oplus \cdots \oplus C}^{2^{n-1}}$ is an antiunitary on $\overbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}^{2^{n-1}}$, it holds that $\alpha(\mathbf{T})$ is invertible if and only if $\alpha(C\mathbf{T}C^{-1})$ is invertible. □

Theorem 2.4. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and any antiunitary C, it holds $\sigma(C\mathbf{T}C^{-1}) = \sigma(\mathbf{T})^*$, $\sigma_{ja}(C\mathbf{T}C^{-1}) = \sigma_{ja}(\mathbf{T})^*$ and $\sigma_{jp}(C\mathbf{T}C^{-1}) = \sigma_{jp}(\mathbf{T})^*$, where $E^* = \{\bar{z} = (\overline{z_1}, ..., \overline{z_n}) : z \in E\} \subset \mathbb{C}^n$.*

Proof. It holds that $(C(T_1 - z_1)C^{-1}, ..., C(T_n - z_n)C^{-1}) = (CT_1C^{-1} - \overline{z_1}, ..., CT_nC^{-1} - \overline{z_n}) = C\mathbf{T}C^{-1} - \bar{z}$, where $z = (z_1, ..., z_n) \in \mathbb{C}^n$. Hence proof follows from Lemma 2.3. □

Remark 2.5. It does not need the commutativity for the joint approximate point spectrum and the joint point spectrum. Hence, for any n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and any antiunitary C, it holds $\sigma_{ja}(C\mathbf{T}C^{-1}) = \sigma_{ja}(\mathbf{T})^*$ and $\sigma_{jp}(C\mathbf{T}C^{-1}) = \sigma_{jp}(\mathbf{T})^*$.

3. Properties of joint approximate point spectra of commuting tuples

For a multi-index $j = (j_1, ..., j_n)$ and $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, we define $|j| = j_1 + \cdots + j_n$, $j! = j_1! \cdots j_n!$ and $\mathbf{T}^j = T_1^{j_1} \cdots T_n^{j_n}$.

Definition 3.1. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$, let $\mathbf{T}^* = (T_1^*, ..., T_n^*)$ and we define $\mathcal{P}_m(\mathbf{T})$ by*

$$\mathcal{P}_m(\mathbf{T}) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \Big(\sum_{|j|=k} \frac{k!}{j!} \mathbf{T}^{*j} \cdot \mathbf{T}^j \Big).$$

$\mathbf{T} = (T_1, ..., T_n)$ is said to be an m-isometric tuple if $\mathcal{P}_m(\mathbf{T}) = 0$.

Then in [8] J. Gleason and S. Richter proved the following result.

Proposition 3.2. (Lemma 3.2, [8])
Let $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ be an m-isometric tuple. If $z = (z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$, then $|z|^2 = |z_1|^2 + \cdots + |z_n|^2 = 1$.

We introduce m-symmetric tuples as follows.

Definition 3.3. *Let, for commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *and* $A \in B(\mathcal{H})$,

$$\mathcal{S}_{\mathbf{T}}(A) := (T_1 + \cdots + T_n)^* A - A(T_1 + \cdots + T_n).$$

An n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *is said to be an m-symmetric tuple if*

$$\mathcal{S}^m_{\mathbf{T}}(I) = 0.$$

Then it holds

$$\mathcal{S}^m_{\mathbf{T}}(I) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (T_1 + \cdots + T_n)^j.$$

We define a m-complex symmetric tuple and skew m-complex symmetric tuple as follows:

Definition 3.4. *For a commuting n-tuple* $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ *and antiunitary C, we define* $r_m(\mathbf{T}; C)$ *and* $\mathcal{R}_m(\mathbf{T}; C)$ *by*

$$r_m(\mathbf{T}; C) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^j$$

and

$$\mathcal{R}_m(\mathbf{T}; C) = \sum_{j=0}^{m} \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^j.$$

A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ *is said to be a m-complex symmetric tuple and a skew m-complex symmetric tuple with an antiunitary C if* $r_m(\mathbf{T}; C) = 0$ *and* $\mathcal{R}_m(\mathbf{T}; C) = 0$*, respectively.*

Theorem 3.5. *Let* $\mathbf{T} = (T_1, ..., T_n)$ *be a commuting n-tuple.*
(1) If $\mathbf{T}$ *is an m-complex symmetric tuple with antiunitary C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1^* + \cdots + T_n^*$*. Hence if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1^* + \cdots + T_n^*)$.
(2) If $\mathbf{T}$ *is a skew m-complex symmetric tuple with antiunitary C and* $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n})$ *belongs to the approximate point spectrum of* $T_1^* + \cdots + T_n^*$*. Hence if* $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$*, then* $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1^* + \cdots + T_n^*)$.

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

Then it holds (**) $CT_j^{k_j} C^{-1} Cx_k = C(T_j^{k_j} - z_j^{k_j})x_k + \overline{z}_j^{k_j} C_j x_k$ for any j and k_j.

(1) If $\mathbf{T}$ is an m-complex symmetric tuple with antiunitary C, by (**) we have

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^j \Big) Cx_k\|$$

$$= \lim_{k\to\infty} \|\Big((T_1^* + \cdots + T_n^*) - (\overline{z_1} + \cdots + \overline{z_n}) \Big)^m Cx_k\|.$$

Since $\{Cx_k\}$ is a sequence of unit vectors, $\overline{z_1} + \cdots + \overline{z_n}$ belongs to the approximate point spectrum of $T_1^* + \cdots + T_n^*$. In the case of the joint point spectrum, it is clear.
(2) If $\mathbf{T}$ is a skew m-complex symmetric tuple with antiunitary C, it holds

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} \binom{m}{j} (T_1^* + \cdots + T_n^*)^{m-j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^j \Big) Cx_k\|$$

$$= \lim_{k\to\infty} \|\Big((T_1^* + \cdots + T_n^*) + (\overline{z_1} + \cdots + \overline{z_n})\Big)^m Cx_k\|.$$

Similarly, we have $-(\overline{z_1} + \cdots + \overline{z_n})$ belongs to the approximate point spectrum of $T_1^* + \cdots + T_n^*$. It is clear for eigenvalue case. □

Next we define an $[m, C]$-symmetric tuple and a skew $[m, C]$-symmetric tuple as follows:

Definition 3.6. *For a commuting n-tuple $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ and an antiunitary C, we define $w_m(\mathbf{T}; C)$ and $\mathcal{W}_m(\mathbf{T}; C)$ by*

$$w_m(\mathbf{T}; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^{m-j}(T_1 + \cdots + T_n)^j$$

and

$$\mathcal{W}_m(\mathbf{T}; C) = \sum_{j=0}^{m} \binom{m}{j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^{m-j}(T_1 + \cdots + T_n)^j.$$

A commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ is said to be an $[m, C]$-symmetric tuple and a skew $[m, C]$-symmetric tuple with antiunitary C if $w_m(\mathbf{T}; C) = 0$ and $\mathcal{W}_m(\mathbf{T}; C) = 0$, respectively.

Theorem 3.7. *Let $\mathbf{T} = (T_1, ..., T_n) \in B(\mathcal{H})^n$ be a commuting n-tuple.*
(1) If $\mathbf{T}$ is an $[m, C]$-symmetric tuple with antiunitary C and $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$, then $(\overline{z_1} + \cdots + \overline{z_n})$ belongs to the approximate point spectrum of $T_1 + \cdots + T_n$. Hence, if $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$, then $(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1 + \cdots + T_n)$.
(2) If $\mathbf{T}$ is a skew $[m, C]$-symmetric tuple with antiunitary C and $(z_1, ..., z_n) \in \sigma_{ja}(\mathbf{T})$, then $-(\overline{z_1} + \cdots + \overline{z_n})$ belongs to the approximate point spectrum of $T_1 + \cdots + T_n$. Hence, if $(z_1, ..., z_n) \in \sigma_{jp}(\mathbf{T})$, then $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_p(T_1 + \cdots + T_n)$.

Proof. Let $\{x_k\}$ be a sequence of unit vectors such that

$$(T_j - z_j)x_k \longrightarrow 0 \text{ as } k \to \infty \text{ for all } j = 1, ..., n.$$

(1) If $\mathbf{T}$ is an $[m, C]$-symmetric tuple with antiunitary C, we have

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^{m-j}(T_1 + \cdots + T_n)^j\Big)x_k\|$$

$$= \lim_{k\to\infty} \|\Big((CT_1C^{-1} + \cdots + CT_nC^{-1}) - (z_1 + \cdots + z_n)\Big)^m x_k\|.$$

Hence $z_1 + \cdots + z_n$ belongs to the approximate point spectrum of $CT_1C^{-1} + \cdots + CT_nC^{-1} = C(T_1 + \cdots + T_n)C^{-1}$ and therefore, by Lemma 3.21 of [9], we have $\overline{z_1} + \cdots + \overline{z_n} \in \sigma_a(T_1 + \cdots + T_n)$. In the case of the joint point spectrum, it is clear.
(2) If $\mathbf{T}$ is skew $[m, C]$-symmetric with antiunitary C, it holds

$$0 = \lim_{k\to\infty} \|\Big(\sum_{j=0}^{m} \binom{m}{j} (CT_1C^{-1} + \cdots + CT_nC^{-1})^{m-j}(T_1 + \cdots + T_n)^j\Big)x_k\|$$

$$= \lim_{k\to\infty} \|\Big((CT_1C^{-1} + \cdots + CT_nC^{-1}) + (z_1 + \cdots + z_n)\Big)^m x_k\|.$$

Therefore we have $-(z_1 + \cdots + z_n) \in \sigma_a(CT_1C^{-1} + \cdots + CT_nC^{-1}) = \sigma(C(T_1 + \cdots + T_n)C^{-1})$. By Lemma 3.21 of [9], we have $-(\overline{z_1} + \cdots + \overline{z_n}) \in \sigma_a(T_1 + \cdots + T_n)$. It is clear in the eigenvalue case. □

4. Xia spectra of doubly commuting tuples

In this section, for a doubly commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ and antiunitary C we will study relation between Xia spctra of $\mathbf{T} = (T_1, ..., T_n)$ and $C\mathbf{T}C^{-1} = (CT_1C^{-1}, ..., CT_nC^{-1})$. We assume that the polar decomposition of every operator T_j has the form $T_j = U_j|T_j|$ such that U_j is unitary ($j = 1, ..., n$). We start the following lemma.

Lemma 4.1. *Let U be unitary and C be antiuniary. Then CUC^{-1} is unitary.*

Proof. By Lemma 2.2, it holds $(CUC^{-1})^* = CU^*C^{-1}$. Hence we have

$$(CUC^{-1})^*CUC^{-1} = CU^*C^{-1}CUC^{-1} = CU^*UC^{-1} = I.$$

and $CUC^{-1}(CU^{-1})^* = CUC^{-1}CU^*C^{-1} = I$. Hence CUC^{-1} is unitary. □

Therefore, if $T = U|T|$ is the polar decomposition of T, then $CTC^{-1} = CUC^{-1}|CTC^{-1}| = CUC^{-1} \cdot C|T|C^{-1}$ is the polar decomposition of CTC^{-1}, because $(CTC^{-1})^*CTC^{-1} = CT^*TC^{-1} = C|T|^2C^{-1} = (C|T|C^{-1})^2$.

D. Xia introduced the Xia spectrum of a commuting $(n + 1)$-tuple of operators as follows (see Xia [13]).

Definition 4.2. *For an operator $T \in B(\mathcal{H})$, T is said to be semi-hyponormal if $|T| \geq |T^*|$.*

Lemma 4.3. *Let $T = U|T|$ be semi-hyponormal with unitary U. Then CTC^{-1} is semi-hyponormal with unitary CUC^{-1}.*

Proof. It is easy to see that $|CTC^{-1}| = C|T|C^{-1}$ and $|(CTC^{-1})^*| = C|T^*|C^{-1}$. Hence we have

$$|CTC^{-1}| - |(CTC^{-1})^*| = C(|T| - |T^*|)C^{-1} \geq 0.$$

Hence CTC^{-1} is semi-hyponormal with unitary CUC^{-1}. □

Let $\mathbf{U} = (U_1, ..., U_n)$ be an n-tuple of unitary operators. For $T \in B(\mathcal{H})$, an operator $\mathbf{Q}_j$ ($j = 1, ..., n$) on $B(\mathcal{H})$ is defined by

$$\mathbf{Q}_jT := T - U_jTU_j^*.$$

Definition 4.4. *Let $\mathbf{U} = (U_1, \cdots, U_n)$ be a commuting n-tuple of unitary operators and $A \geq 0$. An $(n + 1)$-tuple $(\mathbf{U}, A)$ is said to be a semi-hyponormal tuple if*

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m}A \geq 0 \text{ for all } 1 \leq j_1 < \cdots < j_m \leq n.$$

Let $\mathbf{U} = (U_1, \cdots, U_n)$ be an n-tupleof unitary operators and $T \in B(\mathcal{H})$. If

$$\mathcal{S}_j^{\pm}(T) := \text{s-}\lim_{n\to\pm\infty}(U_j^{-n}TU_j^n)$$

exist, then the operator $\mathcal{S}_j^{\pm}(T)$ are called the polar symbols of T. If $U_j|A|$ is semi-hyponormal, then the polar symbols $\mathcal{S}_j^{\pm}(T)$ exist.

For $k \in [0, 1]$ and $A \geq 0$, we denote

$$(k\mathcal{S}_j^+ + (1 - k)\mathcal{S}_j^-)A := k\mathcal{S}_j^+(A) + (1 - k)\mathcal{S}_j^-(A).$$

By the definition of $\mathcal{S}_j^{\pm}(A)$, it is clear that $(k\mathcal{S}_j^+ + (1 - k)\mathcal{S}_j^-)A \geq 0$ for all $k \in [0, 1]$.

Let $\mathbf{k} = (k_1, ..., k_n) \in [0, 1]^n$ and $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then the generalized polar symbols $A_{\mathbf{k}}$ of A are defined by

$$A_{\mathbf{k}} := \Pi_{j=1}^n\Big(k_j\mathcal{S}_j^+ + (1 - k_j)\mathcal{S}_j^-\Big)A.$$

Since $A \geq 0$, it holds $A_{\mathbf{k}} \geq 0$ for all $\mathbf{k} \in [0, 1]^n$. Hence since $(\mathbf{U}, A_{\mathbf{k}})$ is a commuting $(n + 1)$-tuple of normal operators for every $\mathbf{k} \in [0, 1]^n$, we have $\sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}) \neq \emptyset$.

Definition 4.5. *Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple. The the Xia spectrum $\sigma_X(\mathbf{U}, A)$ is defined by*

$$\sigma_X(\mathbf{U}, A) := \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}).$$

Proposition 4.6. (Theorem 5, Xia [13]) *Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple. Then*

$$\|\mathbf{Q}_1 \cdots \mathbf{Q}_n A\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U},A)} d\theta_1 \cdots d\theta_n \, dr.$$

Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of semi-hyponormal operators. When every U_j is unitary of the polar decomposition $T_j = U_j|T_j|$ of T_j $(j = 1, ..., n)$, let $\mathbf{U} = (U_1, ..., U_n)$ and $A = \Pi_{j=1}^n |T_j|$. It is easy to see that

$$\mathbf{Q}_{j_1} \cdots \mathbf{Q}_{j_m} A = (\Pi_{j\neq j_k}|T_j|) \cdot \Pi_{k=1}^m (|T_{j_k}| - |T^*_{j_k}|).$$

Hence, since $(\Pi_{j\neq j_k}|T_j|) \cdot \Pi_{k=1}^m (|T_{j_k}| - |T^*_{j_k}|)$ is a positive operator, $(\mathbf{U}, A)$ is a semi-hyponormal tuple. See Xia [13]. Hence, we have the following corollary.

Corollary 4.7. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of semi-hyponormal operators with unitary U_j $(j = 1, ..., n)$. Then*

$$\|\Pi_{j=1}^n (|T_j| - |T_j^*|)\| \leq \frac{1}{(2\pi)^n} \int \cdots \int_{\sigma_X(\mathbf{U},A)} d\theta_1 \cdots d\theta_n \, dr,$$

where $\mathbf{U} = (U_1, ..., U_n)$ and $A = \Pi_{j=1}^n |T_j|$.

Definition 4.8. *Let $\mathbf{T} = (T_1, ..., T_n)$ be a doubly commuting n-tuple of semi-hyponormal operators with unitary U_j $(j = 1, ..., n)$. Then the Xia spectrum $\sigma_X(\mathbf{T})$ of $\mathbf{T}$ is defined by $\sigma_X(\mathbf{T}) := \sigma_X(\mathbf{U}, A)$, where $\mathbf{U} = (U_1, ..., U_n)$ and $A = \Pi_{j=1}^n |T_j|$.*

For antiunitary C and a doubly commuting n-tuple $\mathbf{T} = (T_1, ..., T_n)$ of semi-hyponormal operators with unitary U_j $(j = 1, ..., n)$, let $C\mathbf{T}C^{-1} := (CT_1C^{-1}, ..., CT_nC^{-1})$. Then by Lemma 4.3, it holds that $C\mathbf{T}C^{-1}$ is a doubly commuting n-tuple of semi-hyponormal operators.

Theorem 4.9. *Let $(\mathbf{U}, A)$ be a semi-hyponormal tuple and C be antiunitary. Then $\sigma_X(C\mathbf{T}C^{-1}) = \{\overline{z} : z \in \sigma_X(\mathbf{T})\}$.*

Proof. Let $\mathbf{U} = (U_1, ..., U_n)$, $A = |T_1| \cdots |T_n|$, $\mathbf{V} = (CU_1C^{-1}, ..., CU_nC^{-1})$ and $B = C|T_1|C^{-1} \cdots C|T_n|C^{-1} = CAC^{-1}$. Then by the definition of the Xia spectrum it holds

$$\sigma_X(\mathbf{T}) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}}) \text{ and } \sigma_X(C\mathbf{T}C^{-1}) = \bigcup_{\mathbf{k}\in[0,1]^n} \sigma_{ja}(\mathbf{V}, B_{\mathbf{k}}).$$

Hence, we will show that $\sigma_{ja}(\mathbf{V}, B_{\mathbf{k}}) = \{\overline{z} : z \in \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}})\}$ for all $\mathbf{k} = (k_1, ..., k_n) \in [0, 1]^n$. Since, for any $\mathbf{k} = (k_1, ..., k_n) \in [0, 1]^n$, it holds

$$(\mathbf{V}, B_{\mathbf{k}}) = C(\mathbf{U}, A_{\mathbf{k}})C^{-1},$$

by Remark 2.5, we have

$$\sigma_{ja}(\mathbf{V}, B_{\mathbf{k}}) = \{\overline{z} : z \in \sigma_{ja}(\mathbf{U}, A_{\mathbf{k}})\}.$$

It completes the proof. □

Remark 4.10. For a semi-hyponormal opertaor $T = U|T| \in B(\mathcal{H})$ with unitary U, since it holds $A = |T|$, the Xia spectrum $\sigma_X(T)$ of T is defined by $\sigma_X(T) := \bigcup_{0\leq k\leq 1} \sigma_{ja}(U, |T|_k)$ and it holds $\sigma(T) = \{ae^{i\theta} : (e^{i\theta}, a) \in \sigma_X(T)\}$. See [5].

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