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The existence, uniqueness and asymptotic stability of solutions to fractional stochastic impulsive neutral systems

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Abstract. The paper presents novel results concerning mild solutions to fractional stochastic impulsive neutral systems, focusing on their existence, uniqueness, and asymptotic stability. A general existence and uniqueness theorem of mild solutions is established by utilizing the method of Picard successive approximation. Moreover, sufficient conditions regarding the impulse intensity and frequency is derived to achieve asymptotic stability for fractional stochastic impulsive neutral systems in mean-square, assuming a Lipschitz condition. Ultimately, the theoretical results are confirmed through numerical examples.

1. Introduction

Recently, fractional systems have attracted a great attention in many fields. Fractional systems are built upon fractional-order calculus, which generalizes the concepts of integer-order calculus. Actually, fractional-order calculus involves differentiation and integration of any arbitrary order. Unlike integer-order calculus, fractional calculus is characterized by memory and heredity, which offer a precise overview of past information. Since then, there has been an increasing trend in modeling realistic systems using fractional dynamic systems, including applications in viscoelasticity, chemistry, anomalous diffusion processes, automatic control, complex networks, and others [8, 12, 23, 24, 28, 33].

In actual scenario, many uncertain factors would influence the dynamic behaviers of systems. Uncertain factors can be characterized by stochastic processes including Brownian motion, fractional Brownian motion (fBm), and others. Building upon this foundation, fractional stochastic systems (FSSs) are developed. The existence and uniqueness results of mild solutions to various FSSs were guaranteed by using methods of Picard successive approximation [15, 19, 34] or Banach fixed point theorem [35]. For most cases, stability constitutes an essential structural feature of the system. Recently, some stability results of FSSs have been given. For instance, [33] presented mean-square asymptotic stability results for FSSs utilizing Mittag-Leffler functions. The asymptotic behavior of solutions to fractional stochastic evolution system was concerned

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in [19]. Fractional stochastic neutral systems (FSNSs) are a subclass of FSSs that depend on past or current values and encompass derivatives or the function itself. The papers [2, 3] extensively explored the existence and Ulam-Hyers stability of mild solutions to FSNSs.

Impulsive systems arise in multifarious processes where states change abruptly at specific instants in time. They are extensively researched and utilized in economy, biology, environment, and power electronics, due to their effective modeling of both continuous and discrete behaviors. The impulsive effects are categorized into stabilizing impulses and perturbing impulses. Stabilizing impulses can activate dynamic behaviors to suppress unstable continuous behavior [4, 17, 18, 20, 30]. In the field of control, stabilizing impulses will be regarded as impulsive controller. On the other hand, stable systems can be disrupted or even compromised by sudden uncertainty phenomena [29]. Hence, there are many results about fractional stochastic impulsive systems (FSISs) and fractional stochastic impulsive neutral systems (FSINSs). The existence results of mild solutions to FSISs were given by fixed point theorem in [26]. Moreover, long time behaviour of FSISs driven by fBm were studied in [31]. In recent years, existence rusults of solutions for FSINSs were studied by fixed point theorem [6, 9, 32] or Carathéodory approximation approach [1]. The asymptotic stability of FSINSs with fractional integral operator was explored in [32]. While [9, 21] investigated the exponential stability of FSINSs with fractional integral operator, the fractional order $\alpha \in (1, 2)$.

Building upon the aforementioned discussions, our the primary objective is to investigate the existence, uniqueness and asymptotic stability of mild solutions to the following system:

$$\begin{cases} {}^{C}_{t_{k-1}}\mathcal{D}^{\alpha}_{t}[x(t) - h(t, x(t))] = Ax(t) + f(t, x(t)) + g(t, x(t))\dfrac{\mathrm{d}B(t)}{\mathrm{d}t}, \ t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \\ x(t_0) = x_0, \end{cases}$$

where $k \in \mathbb{N}^+$, ${}^{C}_{t_{k-1}}\mathcal{D}^{\alpha}_{t}$ is the Caputo-type fractional derivative and α is the fractional-order of system. Linear operator A is a closed densely defined. $\{t_k : k \in \mathbb{N}^+\}$ is fixed time series, on which the impulse take place. Let $\{B(t) : t \geq 0\}$ be a Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, which is a filtered complete probability space. The previous results such as [21, 31] utilize the conditions $\|T_\alpha(t)\| \leq Me^{-pt}$ and $\|S_\alpha(t)\| \leq Mt^{\alpha-1}e^{-pt}$ to ensure the stability. Nevertheless, as stated by [33], the conditions above are impossible to achieve. Hence Mittag-Leffler-type conditions are employed. The key contributions of this work are as outlined below:

1) Existence and uniqueness theorem for the mild solutions to FSNISs is demonstrated by applying the method of Picard successive approximation, which necessitates fewer conditions compared to the Banach fixed point theorem approach.
2) Feasible sufficient conditions about inpulsive intensity and frequency are provided to ensure the asymptotic stability of FSNISs in mean-square based on characteristics of Mittag-Leffler function.
3) In comparison with [2], the system investigated in this work incorporates impulsive effects. Unlike the integro-differential systems analyzed in [32], [9], and [21], the mild solution to system in this work is constracted from operators that present greater analytical challenges.

The remaining parts of this paper are structured as outlined below: Sec. 2 briefly covers preliminaries and delineates the investigated system. Sec. 3 establishes the main results. Sec. 4 validates these results by two numerical examples. Sec. 5 presents the conclusion.

Notations: Denote $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ as a complete filtered probability space, where $\mathcal{F}_0$ contains every $\mathbf{P}$-null sets of $\mathcal{F}$. $\mathbf{E}$ represents the mathematical expectation. Denote $\mathcal{B}(\cdot, \cdot)$ as the Eular Beta function and $\Gamma(\cdot)$ as Eular Gamma function. Let $\mathbb{H}, \mathbb{K}$ denote two Hilbert spaces with the property of separability. Denote $\mathcal{L}(\mathbb{K}, \mathbb{H})$ as the space of consisting bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. Specially, let $\mathcal{L}(\mathbb{K}) := \mathcal{L}(\mathbb{K}, \mathbb{K})$. Denote $\|\cdot\|$ as the norm in Hilbert space $\mathbb{H}, \mathbb{K}$, and $\mathcal{L}(\mathbb{K}, \mathbb{H})$.

2. Preliminaries

Given that Hilbert space $\mathbb{K}$ has a complete orthonormal basis $\{e_m : m \geq 1\}$. The stochastic process $\{B(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Brownian motion [10] defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with a finite trace nuclear covariance operator $Q \geq 0$. Denote $\text{Tr}(Q) = \sum_{m=1}^{\infty} \lambda_m < \infty$, which is the trace of operator Q and staisfies $Qe_m = \lambda_m e_m$, $m \in \mathbb{N}$. Let $\{\beta_m(t) : m \geq 1\}$ denote a sequence of mutually independent one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, satisfying

$$B(t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m}\beta_m(t)e_m,\ t \geq 0.$$

Let operators $\xi_1, \xi_2 \in \mathcal{L}(\mathbb{K}, \mathbb{H})$, and ξ_1^* be the adjoint of the operator ξ_1. Denote $(\xi_1, \xi_2) = \text{Tr}[\xi_1 Q \xi_2^*]$. Forthermore, if ξ_1 is a bounded operator, we can denote

$$\|\xi_1\|_Q^2 = \text{Tr}[\xi_1 Q \xi_1^*] = \sum_{k=1}^{\infty} \left\| \sqrt{\lambda_k}\xi_1 e_k \right\|^2.$$

If $\|\xi_1\|_Q^2 < \infty$, ξ_1 is referred to as a Q-Hilbert–Schmidt operator.

Lemma 2.1. *[10] For positive scalar $b > 0$, Assume $g(t)$ is an $\mathcal{F}_t$-measurable, stochastic process and take values in $\mathcal{L}(\mathbb{K}, \mathbb{H})$. And $\forall\ t \in [0, b]$, $\int_0^b \mathbf{E}\|g(t)\|^2 \mathrm{d}s < \infty$, then*

$$\mathbf{E}\left\| \int_0^t g(s)\mathrm{d}B(s)\right\|^2 \leq \text{Tr}(Q)\int_0^t \mathbf{E}\|g(s)\|^2 \mathrm{d}s.$$

We investigate the fractional stochastic impulsive neutral system presented as follows:

$$\begin{cases} {}_{t_{k-1}}^{\ \ C}\mathcal{D}_t^{\alpha}[x(t) - h(t, x(t))] = Ax(t) + f(t, x(t)) + g(t, x(t))\dfrac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $k \in \mathbb{N}^+$, system fractional-order $\alpha \in (0, 1)$, and $x(\cdot)$ resides in $\mathbb{H}$. Initial value x_0 is a $\mathbb{H}$-valued random variable, satisfying $\mathbf{E}\|x_0\|^2 < \infty$. Operator $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is closed and densely defined, which is the infinitesimal generator of operators $S_\alpha(t)$ and $T_\alpha(t)$. $f(\cdot, \cdot), h(\cdot, \cdot) \in \mathcal{L}^1([0, \infty) \times \mathbb{H}; \mathbb{H})$, $g(\cdot, \cdot) \in \mathcal{L}^2([0, \infty) \times \mathbb{H}; \mathcal{L}(\mathbb{K}, \mathbb{H}))$ are continuous nonlinearities. $h(\cdot, \cdot)$ is the neutral function and $g(\cdot, \cdot)$ is the noise intensity function. Denote $\{t_k : k \in \mathbb{N}^+\}$ as fixed time series, on which the impulse effects take place, satisfying $0 = t_0 < t_1 < \cdots < t_k \to \infty$ as $k \to \infty$. $I_k(\cdot) \in \mathcal{L}\{\mathbb{H}\}$ is inpulsive intensity function at t_k,which is a continuous function. Assuming for simplicity that the solutions systisfying $x(t) = x(t^+) := \lim_{\delta\to 0^+} x(t + \delta)$.

Denote $\mathcal{L}^2(\Omega; \mathbb{H})$ as the space of all $\mathcal{F}_t$-measurable, square-integrable and $\mathbb{H}$-valued random variables X defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$. $\mathcal{L}^2(\Omega; \mathbb{H})$ forms a Banach space with the norm $\mathbf{E}\|X\|^2 < \infty$. Let $\chi := [\tau_1, \tau_2] \subset [0, \infty)$ and define $PC(\chi; \mathcal{L}^2(\Omega; \mathbb{H}))$ as the Banach space of all $\mathbb{H}$-valued functions ψ defined on χ. $PC(\chi; \mathcal{L}^2(\Omega; \mathbb{H}))$, furnished with norm

$$\|\psi(t)\|_{PC} = \mathbf{E}\left(\sup_{t\in\chi} \|\psi(t)\|^2\right)^{\frac{1}{2}} < \infty.$$

Subsequent to this, basic knowledge concerning Caputo fractional derivatives and Mittag-Leffler functions are succinctly presented.

Definition 2.2. *[25] The Caputo fractional derivative of α for a function $f(t)$ is defined as*

$${}_{a}^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)}\int_a^t \frac{f^{(n)}(s)}{(t - s)^{\alpha - n + 1}}\mathrm{d}s,$$

where $\alpha > 0, t \geq t_0$, $n = [\alpha] + 1$. $f(t)$ should possess absolutely continuous derivatives up to order $n - 1$.

Definition 2.3. *[25] The definition of the two-parameter Mittag–Leffler function is as follows:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \ \ z, \beta \in \mathbb{C}, \ \Re\mathrm{e}(\alpha) > 0.$$

$\Re\mathrm{e}(\alpha)$ is the real part of α. Particularly, we denote $E_\alpha(z) = E_{\alpha,1}(z)$.

Lemma 2.4. *[25] If $\alpha \in (0,1)$, $\beta \in (0,1)$, $E_\alpha(-t)$ and $E_{\alpha,\alpha}(-t)$ are monotonically decreasing. Besides, $E_\alpha(-t) \leq E_\alpha(0) = 1$ and $E_{\alpha,\beta}(-\lambda t^\alpha) \leq E_{\alpha,\beta}(0) = 1/\Gamma(\beta)$.*

Lemma 2.5. *[25] Assume $\alpha \in (0,2)$ and $\beta > 0$, $\upsilon \in (\alpha\pi/2, \ \pi \wedge \alpha\pi)$. Whenever $|\mathrm{Arg}(z)| \in [\upsilon, \pi]$, we have $|E_{\alpha,\beta}(z)| < \mathcal{M}_0(1 + |z|)^{-1}$, where $\mathcal{M}_0 \geq \Gamma(\beta - \alpha)^{-1}$.*

Lemma 2.6. *[16] Let $\Re\mathrm{e}(\alpha) \geq 0$, $\Re\mathrm{e}(\beta) > 0$, $\lambda \in \mathbb{C}$,*

$$\int_a^t (s-a)^{\beta-1} E_{\alpha,\beta}(\lambda(s-a)^\alpha)\mathrm{d}s = (t-a)^\beta E_{\alpha,\beta+1}(\lambda(t-a)^\alpha), \ 0 \leq a \leq t.$$

Notably, for $\alpha = \beta$, and $a = 0$, thus

$$\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(\lambda s^\alpha)\mathrm{d}s = t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha).$$

Definition 2.7. *[13] Let A denote a linear closed operator with domain $D(A)$ defined on $\mathbb{H}$. Let $\varrho(A)$ denote the resolvent set of A, $\mathcal{R}(\lambda, A) = (\lambda I - A)^{-1}$ denote the resolvent operator of A. A is deemed sectorial if it fulfills the subsequent properties: (i) $\varrho(A) \subset \Sigma_\varpi(\vartheta) = \{\lambda \in \mathbb{C} \setminus \{0\} : |\mathrm{Arg}(\lambda - \varpi)| < \vartheta\}$, where $\vartheta \in [\pi/2, \pi]$, $\varpi \in \mathbb{R}$. (ii) $\|\mathcal{R}(\lambda, A)\| \leq \mathcal{N}|\lambda - \varpi|^{-1}$ for $\mathcal{N} > 0$, $\lambda \in \Sigma_\varpi(\vartheta)$.*

Definition 2.8. *[5] Let A denote a linear closed operator with domain $D(A)$ defined on $\mathbb{H}$. Strongly continuous function $T_\alpha(t) : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$ is designated as the α-order fractional solution operator generated by A, if $\exists\, \varpi \geq 0$ s.t. $\{\lambda^\alpha \mid \Re\mathrm{e}(\lambda) > \varpi\} \subset \varrho(A)$, it holds that:*

$$\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} y = \int_0^\infty e^{-\lambda s} T_\alpha(s) y \mathrm{d}s, \ \forall\, y \in \mathbb{H}, \ \Re\mathrm{e}(\lambda) > \varpi.$$

Definition 2.9. *[5] Let A denote a linear closed operator with domain $D(A)$ defined on $\mathbb{H}$. Strongly continuous function $S_\alpha(t) : \mathbb{R}^+ \to \mathcal{L}(\mathbb{H})$ is designated as the α-resolvent family generated by A, if $\exists\, \varpi \geq 0$ s.t. $\{\lambda^\alpha : \Re\mathrm{e}(\lambda) > \varpi\} \subset \varrho(A)$, it holds that:*

$$(\lambda^\alpha I - A)^{-1} y = \int_0^\infty e^{-\lambda s} S_\alpha(s) y \mathrm{d}s, \ \forall\, y \in \mathbb{H}, \ \Re\mathrm{e}(\lambda) > \varpi.$$

Remark 2.10. *[26, 33] From Definitions 2, 4, and 5, one can obtain $T_\alpha(t) = E_\alpha(At^\alpha)$ and $S_\alpha(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$.*

Definition 2.11. *[7, 27] Operator $T_\alpha(t)$ is termed analytic if $T_\alpha(t)$ admits an analytic extension to a sector $\Sigma_{\vartheta_0} := \{z \in \mathbb{C}/\{0\} : |\mathrm{Arg}(z)| < \vartheta_0\}$ for $\vartheta_0 \in (0, \pi/2]$. An analytic solution operator is said to be of analyticity type (ϖ_0, ϑ_0) if for each $\vartheta < \vartheta_0$ and $\varpi > \varpi_0$, $\exists\, M_0 = M_0(\varpi, \vartheta)$ s.t. $\|T_\alpha(t)\| \leq M_0 e^{\varpi \Re\mathrm{e}(z)}$, $\forall\, z \in \Sigma_\vartheta$. Denote $\mathcal{A}^\alpha(\varpi_0, \vartheta_0)$ is a set of operator generating analytic T_α of type (ϖ_0, ϑ_0).*

Definition 2.12. *[7] If $T_\alpha(t)$ is compact for all $t > 0$, α-order fractional solution operator $\{T_\alpha(t) : t \geq 0\}$ is termed compact.*

Arguing as in the proof of Lemma 10 in [11], we can obtain the continuity of the α-order fractional solution operator $T_\alpha(t)$ and α-resolvent family $S_\alpha(t)$ in the uniform operator topology.

Lemma 2.13. *[7] Assume that* $A \in \mathcal{A}^{\alpha}(\varpi, \vartheta_0)$*, then* $\forall\ t > 0$, $\|T_\alpha(t)\| \le M_1 e^{\varpi t}$, $\|S_\alpha(t)\| \le M_2 e^{\varpi t}(t^{\alpha-1} + 1)$*, where* $\varpi > \varpi_0$*. Assume* $M_T = \sup_{t\in[0,T]} \|T_\alpha(t)\|$, $M_S = \sup_{t\in[0,T]} M_2 e^{\varpi t}(1 + t^{1-\alpha})$*, one has* $\|T_\alpha(t)\| \le M_T$, $\|S_\alpha(t)\| \le M_S t^{\alpha-1}$.

Lemma 2.14. *[11] Assume that* $A \in \mathcal{A}^{\alpha}(\varpi_0, \vartheta_0)$*, if* $\{T_\alpha(t) : t \ge 0\}$, $\{S_\alpha(t) : t \ge 0\}$ *are compact, it follows that:*

$$\lim_{\Delta t\to 0} \|T_\alpha(t + \Delta t) - T_\alpha(t)\| = 0 \text{ and } \lim_{\Delta t\to 0} \|S_\alpha(t + \Delta t) - S_\alpha(t)\| = 0,\ \forall\ t > 0.$$

Lemma 2.15. *[22] Given* $a_1 \ge 0$, $a_2 \ge 0$*, and* $\upsilon \in (0, 1)$*, the following holds:* $(a_1 + a_2)^2 \le a_1^2/\upsilon + a_2^2/1 - \upsilon$.

Lemma 2.16. *[14] (Gronwall–Bellman inequality) For* $t \ge 0$*, consider* $\psi_1(\cdot)$ *and* $\psi_2(\cdot)$ *as real continuous functions. Assume* $\psi_3(\cdot)$ *is real function that is integrable on* $\forall\ [\nu_1, \nu_2] \subset [0, \infty)$*. If*

$$\psi_1(t) \le \psi_3(t) + \int_a^t \psi_1(s)\psi_2(s)\mathrm{d}s,\ \forall\ t \ge a,$$

then

$$\psi_1(t) \le \psi_3(t) + \int_a^t \psi_2(s)\psi_3(s) \exp\left(\int_s^t \psi_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s,\ \forall\ t \ge a.$$

Lemma 2.17. *[14] Consider* $w_1(\cdot)$ *as a real function and* $w_2(\cdot)$ *as a locally integrable and nonnegative function on* $[0, b]$*. Then* $\exists\ c > 0$, $\exists\ \alpha \in (0, 1)$ *s.t.*

$$w_1(t) \le w_2(t) + c\int_0^t (t - s)^{-\alpha} w_1(s)\mathrm{d}s.$$

As a result, there exists a constant $\mathcal{K}_\alpha$*, dependent solely on* α*,*

$$w_1(t) \le w_2(t) + c\mathcal{K}_\alpha \int_0^t (t - s)^{-\alpha} w_2(s)\mathrm{d}s,\ 0 \le t \le b$$

3. Main results

3.1. Existance and uniqueness of the mild solutions

Next, we shall establish the existence and uniqueness theorem of mild solutions to system (1).

Definition 3.1. *An* $\mathcal{F}_t$*-adapted stochastic process* $x : [0, \infty) \to \mathbb{H}$ *is called a mild solution to system (1), if* $x \in PC([0, \infty); \mathcal{L}^2(\Omega; \mathbb{H}))$ *satisfying the piecewise integral equation*

$$x(t) = \begin{cases} T_\alpha(t)[x_0 - h(t_0, x_0)] + h(t, x(t)) + \int_0^t AS_\alpha(t - s)h(s, x(s))\mathrm{d}s + \int_0^t S_\alpha(t - s)f(s, x(s))\mathrm{d}s \\ \quad + \int_0^t S_\alpha(t - s)g(s, x(t))\mathrm{d}B(s), \quad t \in [0,\ t_1) \\ T_\alpha(t - t_1)[x(t_1) - h(t_1, x(t_1)))] + h(t, x(t)) + \int_{t_1}^t AS_\alpha(t - s)h(s, x(s))\mathrm{d}s + \int_{t_1}^t S_\alpha(t - s)f(s, x(s))\mathrm{d}s \\ \quad + \int_{t_1}^t S_\alpha(t - s)g(s, x(t))\mathrm{d}B(s), \quad t \in [t_1,\ t_2) \\ \quad \cdots, \\ T_\alpha(t - t_n)[x(t_n) - h(t_n, x(t_n)))] + h(t, x(t)) + \int_{t_n}^t AS_\alpha(t - s)h(s, x(s))\mathrm{d} + \int_{t_n}^t S_\alpha(t - s)f(s, x(s))\mathrm{d}s \\ \quad + \int_{t_n}^t S_\alpha(t - s)g(s, x(t))\mathrm{d}B(s),\ \ t \in [t_n,\ t_{n+1}) \\ \quad \cdots \end{cases} \tag{2}$$

Assumption 3.2. *(Lipschitz condition)[22]* $\forall\ u,\ v \in \mathbb{H}$ *and* $t \in [0, \infty)$, $\exists\ \bar{K} > 0$ *s.t.*

$$\|f(t,u) - f(t,v)\|^2 \vee \|g(t,u) - g(t,v)\|^2 \leq \bar{K}\|u - v\|^2.$$

Assumption 3.3. *[22]* $\forall\ u,\ v \in \mathbb{H}$ *and* $t \in [0, \infty)$, $\exists\ l \in (0, 1)$ *s.t.*

$$\|h(t,u) - h(t,v)\| \leq l\|u - v\|.$$

Assumption 3.4. *Assume* $f(t, 0) = g(t, 0) = h(t, 0) \equiv 0$ *if* $t \in [0, \infty)$, *ensuring that* $x(t) \equiv 0$ *is the trivial solution.*

Assumption 3.5. *Suppose the inpulsive intensity function* $I_k(\cdot) \in \mathcal{L}\{\mathbb{H}\}$ *are continuous, satisfying* $I_k(0) \equiv 0$, $k \in \mathbb{N}^+$. *Then* $\exists\ \mu > 0$ *s.t. ,*

$$\|I_k(u) - I_k(v)\|^2 \leq \mu\|u - v\|^2, \forall\ k \in \mathbb{N}^+,\ \forall\ u, v \in \mathbb{H}$$

Assumption 3.6. $\exists\ \gamma > 0, \varsigma > 0$ *s.t.*

$$\gamma = \sup_{k \in \mathbb{N}^+}\{t_k - t_{k-1}\} < \infty, \text{ and } \varsigma = \inf_{k \in \mathbb{N}^+}\{t_k - t_{k-1}\} > 0.$$

Assumption 3.7. $\exists\ \lambda > 0, \mathcal{M}_1 > 0, \mathcal{M}_2 > 0$ *s.t.*

$$\|T_\alpha(t)\| \leq \mathcal{M}_1 E_\alpha(-\lambda t^\alpha) \text{ and } \|S_\alpha(t)\| \leq \mathcal{M}_2 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),\ \forall\ t \geq 0.$$

The value of λ *determines the decay rate of* $T_\alpha(\lambda t^\alpha)$ *and* $S_\alpha(\lambda t^\alpha)$.

Remark 3.8. *Assumption 3.6 ensures that the impulsive frequency ranges between* γ^{-1} *and* ς^{-1}.

Remark 3.9. *[33] Based on Remark 2.10 and the property of the Mittag-Leffler functions,* $\forall\ \lambda > 0$, *we have*

$$\lim_{t \to \infty} \frac{E_{\alpha,\beta}(-\lambda t^\alpha)}{e^{-\lambda t}} = \infty.$$

Hence, finding a positive constant $M_0 > 0$ *is unattainable. Therefore it is impossible to find positive constant* $M > 0$ *satisfying* $\|T_\alpha(t)\| \leq M_0 e^{-\lambda t}$ *and* $\|S_\alpha(t)\| \leq M_0 t^{\alpha-1} e^{-\lambda t}$, *which are considered in previous references. Hence, Assumption 3.7 is utilized in this paper based on Mittag-Leffler function.*

Next, the main theorems are presented and proved.

Theorem 3.10. *Assume that Assumptions 3.2-3.6 hold. Let* $A \in \mathcal{A}^\alpha(\varpi_0, \vartheta_0)$ *with* $\varpi_0 \in \mathbb{R}$ *and* $\vartheta_0 \in (0, \pi/2]$. *For* $t \geq 0$, *let* $T_\alpha(t)$ *and* $S_\alpha(t)$ *are compact. Then for every positive number* T, *it can be shown that a unique mild solution to system (1) exists on the interval* $[0, T)$.

Proof. Define the Picard iterations sequence on interval $[0, \eta_1 \wedge t_1)$ as

$$\begin{cases} \varphi_n(t) = T_\alpha(t)[x_0 - h(0, x_0)] + h(t, \varphi_{n-1}(t)) + \int_0^t AS_\alpha(t-s)h(s, \varphi_{n-1}(s))\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s, \varphi_{n-1}(s))\mathrm{d}s \\ \qquad + \int_0^t S_\alpha(t-s)g(s, \varphi_{n-1}(s))\mathrm{d}B(t), \\ \varphi_0(t) = x_0. \end{cases} \tag{3}$$

where η_1 is a small enough positive constant we chosen satisfying

$$\delta_1 = l^2 + \frac{6M_S^2(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{2\alpha - 1} < 1 \text{ and } \delta_2 = l + \frac{3M_S^2(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{(1-l)(2\alpha - 1)} < 1.$$

The above serves as additional conditions. Remaining steps are structured into eight parts.

Step 1: Let us verify $\varphi_n(t)$ is uniformly bounded on the interval $[0, \eta_1 \wedge t_1)$. For $n = 1$, using elementary inequality, the Hölder inequality, Lemma 2.1, Lemma 2.13 and Assumption 3.2-3.4, one can derive that

$$\begin{aligned}
\mathbf{E}\|\varphi_1(t)\|^2 \leq & 6\mathbf{E}\,\|T_\alpha(t)x_0\|^2 + 6\mathbf{E}\|h(t,x_0)\|^2 + 6\mathbf{E}\|T_\alpha(t)h(0,x_0)\|^2 + 6\eta_1\int_0^t \|AS_\alpha(t-s)\|^2\mathbf{E}\|h(s,x_0)\|^2 \mathrm{d}s \\
& + 6\eta_1\int_0^t \|S_\alpha(t-s)\|^2\mathbf{E}\|f(s,x_0)\|^2\mathrm{d}s + 6\mathrm{Tr}(Q)\int_0^t \|S_\alpha(t-s)\|^2\mathbf{E}\|g(s,x_0)\|^2\mathrm{d}B(s). \\
\leq & 6(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\mathbf{E}\|x_0\|^2\int_0^t \|S_\alpha(t-s)\|^2\mathrm{d}s + 6[M_T^2(1+l^2)+l^2]\mathbf{E}\|x_0\|^2 \\
\leq & 6\mathbf{E}\|x_0\|^2\left[M_T^2(1+l^2) + l^2 + \frac{6M_S^2(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{2\alpha-1}\right] \\
= & \mathcal{U}_1
\end{aligned} \tag{4}$$

where $\|A\|$ is the norm of For $n \geq 2$, we use mathematical induction. Suppose that when $n = p$, $\mathbf{E}\|\varphi_p(t)\|^2 \leq \mathcal{U}_p$ is bounded. then for $n = p + 1$,

$$\begin{aligned}
\mathbf{E}\|\varphi_{p+1}(t)\|^2 \leq & 6[M_T^2(1+l^2)+l^2]\mathbf{E}\|x_0\|^2 + \left[l^2 + \frac{6M_S^2(\eta_1 l^2\|A\|^2\eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{2\alpha-1}\right]\mathcal{U}_p \\
= & 6[M_T^2(1+l^2)+l^2]\mathbf{E}\|x_0\|^2 + \delta_1\mathcal{U}_p \\
= & \mathcal{U}_{p+1}
\end{aligned}$$

Through induction on n, we consequently derive that

$$\begin{aligned}
\mathbf{E}\|\varphi_n(t)\|^2 \leq & \frac{6(\delta_1^{n-1}-1)[M_T^2(1+l^2)+l^2]\mathbf{E}\|x_0\|^2}{\delta_1 - 1} + \delta_1^{n-1}\mathcal{U}_1 \\
\leq & \mathcal{U}_1 + 6[M_T^2(1+l^2)+l^2]\mathbf{E}\|x_0\|^2 = \mathcal{U}.
\end{aligned}$$

Because of the additional condition of η_1, the iterative sequence $\mathbf{E}\|\varphi_n(t)\|^2$ is uniformly bounded on interval $[0, t_1)$.

Step 2: Let us verify that $\varphi_n(t) \in C([0, t_1 \wedge \eta_1), \mathbb{H})$. Considering $\zeta > 0$ sufficiently small, for $n = 1$,

$$\begin{aligned}
\mathbf{E}\|\varphi_1(t+\zeta) - \varphi_1(t)\|^2 \leq & 8\mathbf{E}\|T_\alpha(t+\zeta) - T_\alpha(t)\|^2[x_0 - h(0,x_0)]^2 + 8\mathbf{E}\|h(t+\zeta,x_0) - h(t,x_0)\|^2 \\
& + 8\mathbf{E}\left\|\int_0^t A(S_\alpha(t+\zeta-s) - S_\alpha(t-s))h(s,x_0)\mathrm{d}s\right\|^2 + 8\mathbf{E}\left\|\int_t^{t+\zeta} AS_\alpha(t+\zeta-s)h(s,x_0)\mathrm{d}s\right\|^2 \\
& + 8\mathbf{E}\left\|\int_0^t (S_\alpha(t+\zeta-s) - S_\alpha(t-s))f(s,x_0)\mathrm{d}s\right\|^2 + 8\mathbf{E}\left\|\int_t^{t+\zeta} S_\alpha(t+\zeta-s)f(s,x_0)\mathrm{d}s\right\|^2 \\
& + 8\mathbf{E}\left\|\int_0^t (S_\alpha(t+\zeta-s) - S_\alpha(t-s))g(s,x_0)\mathrm{d}B(s)\right\|^2 + 8\mathbf{E}\left\|\int_t^{t+\zeta} S_\alpha(t+\zeta-s)g(s,x_0)\mathrm{d}B(s)\right\|^2 \\
= & 8\sum_{i=1}^{8} H_i.
\end{aligned} \tag{5}$$

Utilizing the continuity of $h(t, x(t))$ and Lemma 2.14, it follows $H_1 \to 0$ and $H_2 \to 0$ as $\zeta \to 0$.

For H_3, let $\iota \in (0, t)$ is a arbitrary constant, by virtue of Lemma 2.13, Assumptions 3.3,3.4 and C-S inequality , we can ascertain

$$\begin{aligned}
H_3 \leq & \int_0^t \|A\|^2 \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\| \mathbf{E}\|h(s,x_0)\|^2 \mathrm{d}s \int_0^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\| \mathrm{d}s \\
\leq & \eta_1 l^2 \|A\|^2 \mathbf{E}\|x_0\|^2 \left(\int_{t_k}^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\| \mathrm{d}s \right)^2 \\
\leq & \eta_1 l^2 \|A\|^2 \mathbf{E}\|x_0\|^2 \int_0^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s \\
= & \eta_1 l^2 \|A\|^2 \mathbf{E}\|x_0\|^2 \left[\int_0^{t-\iota} \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s + \int_{t-\iota}^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s \right] \\
\leq & \eta_1 l^2 \|A\|^2 \mathbf{E}\|x_0\|^2 \int_{t-\iota}^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s + \frac{2\eta_1 l^2 \|A\|^2 M_S^2 \mathbf{E}\|x_0\|^2}{2\alpha-1} \left((\zeta+\iota)^{2\alpha-1} + \iota^{2\alpha-1} \right).
\end{aligned} \tag{6}$$

Hence, because the arbitrary of ι, we can obtain than $H_3 \to 0$ as $\zeta \to 0$.

For H_4, by Hölder inequality and Assumption 3.3-3.4, we determine that

$$\begin{aligned}
H_4 \leq & \int_t^{t+\zeta} \|A\|^2 \|S_\alpha(t+\zeta-s)\| \mathbf{E}\|h(s,x_0)\|^2 \mathrm{d}s \int_t^{t+\zeta} \|S_\alpha(t+\zeta-s)\| \mathrm{d}s \\
\leq & \, l^2 \|A\|^2 \mathbf{E}\|x_0\|^2 \left(\int_t^{t+\zeta} (t+\zeta-s)^{\alpha-1} \mathrm{d}s \right)^2 \\
= & \frac{l^2 \|A\|^2 M_S^2 \zeta^{2\alpha}}{\alpha^2} \mathbf{E}\|x_0\|^2 \to 0 \text{ as } \zeta \to 0.
\end{aligned} \tag{7}$$

For H_5, H_6, using the same method as above by Assumption 3.2 and Assumption 3.4, it have

$$H_5 \leq \eta_1 \bar{K} \mathbf{E}\|x_0\|^2 \int_{t-\iota}^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s + \frac{2\eta_1 \bar{K} M_S^2 \mathbf{E}\|x_0\|^2}{2\alpha-1} \left((\zeta+\iota)^{2\alpha-1} + \iota^{2\alpha-1} \right) \to 0 \text{ as } \zeta \to 0 \tag{8}$$

and

$$H_6 \leq \frac{\bar{K} M_S^2 \zeta^{2\alpha}}{\alpha^2} \mathbf{E}\|x_0\|^2 \to 0 \text{ as } \zeta \to 0. \tag{9}$$

For H_7, in the similar way to deal with H_3, H_5, by Lemma 2.1 Assumption 3.2 and Assumption 3.4, we can obtain

$$\begin{aligned}
H_7 \leq & \mathrm{Tr}(Q) \bar{K} \mathbf{E}\|x_0\|^2 \int_0^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s \\
\leq & \mathrm{Tr}(Q) \bar{K} \mathbf{E}\|x_0\|^2 \int_{t-\iota}^t \|S_\alpha(t+\zeta-s) - S_\alpha(t-s)\|^2 \mathrm{d}s + \frac{2\mathrm{Tr}(Q) \bar{K} M_S^2 \mathbf{E}\|x_0\|^2}{2\alpha-1} \left((\zeta+\iota)^{2\alpha-1} + \iota^{2\alpha-1} \right) \to 0 \text{ as } \zeta \to 0.
\end{aligned} \tag{10}$$

For H_8, similar as H_4, H_6, it follows from Lemma 2.1, Assumption 3.2 and Assumption 3.4

$$\begin{aligned}
H_8 \leq & \frac{\bar{K} \mathbf{E}\|x_0\|^2 \mathrm{Tr}(Q)}{\Gamma(\alpha)^2} \int_t^{t+\zeta} (t+\zeta-s)^{2\alpha-2} \mathrm{d}s \\
= & \frac{\mathrm{Tr}(Q) \bar{K} M_S^2 \mathbf{E}\|x_0\|^2 \zeta^{2\alpha-1}}{2\alpha-1} \to \text{ as } \zeta \to 0.
\end{aligned} \tag{11}$$

Therefore, combining Eqs. (5)-(11), as $\zeta \to 0$, $\mathbf{E}\|\varphi_1(t+\zeta) - \varphi_1(t)\|^2$ converges to zero, implying $\varphi_1(t) \in C([0, t_1 \wedge \eta_1), \mathcal{L}^2(\Omega; \mathbb{H}))$.

Now, let us verify that $\varphi_n(t) \in C([0, t_1 \wedge \eta_1), \mathbb{H})$ when $n \geq 2$. Assume that $\mathbf{E}\|\varphi_q(t+\zeta) - \varphi_q(t)\|^2 \to 0$ as $\zeta \to 0$. For $n = q+1$,

$$\begin{aligned}
&\mathbf{E}\|\varphi_{q+1}(t+\zeta) - \varphi_{q+1}(t)\|^2 \leq 8\mathbf{E}\|T_\alpha(t+\zeta) - T_\alpha(t)\|^2[x_0 - h(0,x_0)]^2 \\
&+ 8\mathbf{E}\|h(t+\zeta, \varphi_q(t+\zeta)) - h(t, \varphi_q(t))\|^2 + 8\mathbf{E}\left\|\int_0^t (S_\alpha(t+\zeta-s) - S_\alpha(t-s))h(s,\varphi_q(t))\mathrm{d}s\right\|^2 \\
&+ 8\mathbf{E}\left\|\int_t^{t+\zeta} S_\alpha(t+\zeta-s)h(s,\varphi_q(t))\mathrm{d}s\right\|^2 + 8\mathbf{E}\left\|\int_0^t (S_\alpha(t+\zeta-s) - S_\alpha(t-s))f(s,\varphi_q(t))\mathrm{d}s\right\|^2 \\
&+ 8\mathbf{E}\left\|\int_t^{t+\zeta} S_\alpha(t+\zeta-s)f(s,\varphi_q(t))\mathrm{d}s\right\|^2 + 8\mathbf{E}\left\|\int_0^t (S_\alpha(t+\zeta-s) - S_\alpha(t-s))g(s,\varphi_q(t))\mathrm{d}B(s)\right\|^2 \\
&+ 8\mathbf{E}\left\|\int_t^{t+\zeta} S_\alpha(t+\zeta-s)g(s,\varphi_q(t))\mathrm{d}s\right\|^2.
\end{aligned} \tag{12}$$

In Step 1, we've proven that the iterations sequence $\varphi_n(t)$ is bounded. By the similar way, under Assumption 3.3, we can acquire that

$$\begin{aligned}
\mathbf{E}\|\varphi_{q+1}(t+\zeta) - \varphi_{q+1}(t)\|^2 \leq & 7H_1 + 7\frac{\mathbf{E}\|\varphi_q(t)\|^2}{\mathbf{E}\|x_0\|^2}\sum_{i=3}^{8} H_i + l^2\mathbf{E}\|\varphi_q(t+\zeta) - \varphi_q(t)\|^2 \\
& + \mathbf{E}\|h(t+\zeta, \varphi_q(t)) - h(t, \varphi_q(t))\|^2 \to 0 \text{ as } \zeta \to 0.
\end{aligned}$$

According to the principle of induction, we can obtain $\varphi_n(t) \in C([0, t_1 \wedge \eta_1), \mathbb{H})$, for all $n \in \mathbb{N}^+$.

Step 3: Now, let's demonstrate that $\{\varphi_n(t)\}$ forms a Cauchy sequence in $C([0, t_1 \wedge \eta_1), \mathbb{H})$. Especially, when $n = 0$, by Hölder inequality, Lemma 2.1, Lemma 2.13, Assumption 3.2 and Assumption 3.4,

$$\begin{aligned}
\mathbf{E}\|\varphi_1 - \varphi_0\|^2 =& \mathbf{E}\|\varphi_1 - x_0\|^2 + 6\mathbf{E}\|x_0\|^2(\|T_\alpha(t) - 1\|^2) + 6\mathbf{E}\|x_0\|^2(l^2\|T_\alpha(t)\|^2 + l^2) \\
&+ 6\mathbf{E}\left\|\int_0^t AS_\alpha(t-s)h(s,x_0)\mathrm{d}s\right\|^2 + 6\mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s,x_0)\mathrm{d}s\right\|^2 + 6\mathbf{E}\left\|\int_0^t S_\alpha(t-s)g(s,x_0)\mathrm{d}B(s)\right\|^2 \\
\leq& 6\mathbf{E}\|x_0\|^2\left[(M_T - 1)^2 + l^2(M_T^2 + 1) + \frac{M_S^2(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{2\alpha - 1}\right] = \mathcal{V}
\end{aligned} \tag{13}$$

Applying Lemma 3.6 letting $v = l$ one derives that

$$\mathbf{E}\|\varphi_{n+1}(t) - \varphi_n(t)\|^2 \leq \frac{1}{l}\mathbf{E}\|h(t,\varphi_n(t)) - h(t,\varphi_{n-1}(t))\|^2 + \frac{1}{1-l}\mathbf{E}\|J(t)\|^2 \leq l\mathbf{E}\|\varphi_n(t) - \varphi_{n-1}(t)\|^2 + \frac{1}{1-l}\mathbf{E}\|J(t)\|^2, \tag{14}$$

where

$$\begin{aligned}
J(t) =& \int_0^t AS_\alpha(t-s)[h(s,\varphi_n(s)) - h(s,\varphi_{n-1}(s))]\mathrm{d}s + \int_0^t S_\alpha(t-s)[f(s,\varphi_n(s)) - f(s,\varphi_{n-1}(s))]\mathrm{d}s \\
&+ \int_0^t S_\alpha(t-s)[g(s,\varphi_n(s)) - g(s,\varphi_{n-1}(s))]\mathrm{d}B(s).
\end{aligned}$$

Same method used above,

$$\mathbf{E}\|J(t)\|^2 \leq 3M_S^2(\eta_1 l^2\|A\|^2 + \eta_1\bar{K} + \mathrm{Tr}(Q)\bar{K})\int_0^t (t-s)^{2\alpha-2}\mathbf{E}\|\varphi_n(s) - \varphi_{n-1}(s)\|^2\mathrm{d}s. \tag{15}$$

Hence, combined Eqs.(14)(15)

$$\begin{aligned}\sup_{0\le s\le t_1\wedge\eta_1}(\mathbf{E}\|\varphi_{n+1}(t)-\varphi_n(t)\|^2) \le&\left[l+\frac{3M_S^2(\eta_1 l^2\|A\|^2+\eta_1\bar{K}+\mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{(1-l)(2\alpha-1)}\right]\sup_{0\le s\le t_1\wedge\eta_1}(\mathbf{E}\|\varphi_n(t)-\varphi_{n-1}(t)\|^2)\\ \le&\left[l+\frac{3M_S^2(\eta_1 l^2\|A\|^2+\eta_1\bar{K}+\mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{\Gamma(\alpha)^2(1-l)(2\alpha-1)}\right]^n\sup_{0\le s\le t_1\wedge\eta_1}(\mathbf{E}\|\varphi_1(t)-\varphi_0(t)\|^2)\\ \le&\delta_2^n\mathcal{V}.\end{aligned}\tag{16}$$

where $\delta_2<1$. By virtue of Weierstrass discriminant method

$$\lim_{n\to\infty}\sup_{0\le s\le t_1\wedge\eta_1}(\mathbf{E}\|\varphi_{n+1}(t)-\varphi_n(t)\|^2)\le\lim_{n\to\infty}\delta_2^n\mathcal{V}=0,$$

where Eq. (13) has been utilized, hence $\{\varphi_n(t)\}$ forms a uniformly Cauchy sequence in $C([0,t_1\wedge\eta_1),\mathbb{H})$ in mean-square sense. Therefore, there exist a continuous function $\varphi(\cdot)$ s.t.

$$\lim_{n\to\infty}\sup_{0\le s\le t_1\wedge\eta_1}(\mathbf{E}\|\varphi_n(t)-\varphi(t)\|^2)=0.$$

And one can show from Eq.(16). Obviously, from the previous argument, Picard iterations sequence $\{\varphi_n(t)\}$ is uniformly bounded, continous and $\mathcal{F}_t$-adapted on the interval $[0,t_1\wedge\eta_1)$.

Step 4: Furthermore, we need to verify that the limit of the sequence $\{\varphi_n(t)\}$ is a mild solution to system (1).

Under Lemma 2.1, Lemma 2.13, and Assumption 3.2-3.4,

$$\begin{aligned}\mathbf{E}\|\varphi_n(t)-\varphi(t)\|^2\le&3\mathbf{E}\left\|\int_0^t S_\alpha(t-s)[f(s,\varphi_{n-1})-f(s,\varphi(s))]\mathrm{d}s\right\|^2+3\mathbf{E}\left\|\int_0^t S_\alpha(t-s)[f(s,\varphi_{n-1})-f(s,\varphi(s))]\mathrm{d}s\right\|^2\\ &+3\mathbf{E}\left\|\int_0^t S_\alpha(t-s)[g(s,\varphi_{n-1})-g(s,\varphi(s))]\mathrm{d}B(s)\right\|^2\\ \le&\frac{3M_S^2(\eta_1 l^2\|A\|^2+\eta_1\bar{K}+\mathrm{Tr}(Q)\bar{K})\eta_1^{2\alpha-1}}{2\alpha-1}\sup_{0\le s\le t_1+\wedge\eta_1}(\mathbf{E}\|\varphi_n(t)-\varphi(t)\|^2)\to0\text{ as }n\to\infty.\end{aligned}$$

Therefore, $\varphi(t)$ is the mild solution to systems in interval $[0,t_1\wedge\eta_1)$ as the sense of Definition 3.1.

Step 5: Let $\phi(t)$ be another continuous mild solution on the interval $[0,t_1\wedge\eta_1)$.

$$\begin{aligned}\mathbf{E}\|\varphi(t)-\phi(t)\|^2\le&\frac{3}{1-l}\mathbf{E}\left\|\int_0^t AS_\alpha(t-s)[h(s,\varphi(s))-h(s,\phi(s))]\mathrm{d}t\right\|^2+\frac{3}{1-l}\mathbf{E}\left\|\int_0^t S_\alpha(t-s)[f(s,\varphi(s))-f(s,\phi(s))]\mathrm{d}t\right\|^2\\ &+\frac{3}{1-l}\mathbf{E}\left\|\int_0^t S_\alpha(t-s)[g(s,\varphi(s))-g(s,\phi(s))]\mathrm{d}B(t)\right\|^2\end{aligned}$$

which implies

$$\mathbf{E}\|\varphi(t)-\phi(t)\|^2\le\frac{3M_S^2(\eta_1 l^2\|A\|^2+\eta_1\bar{K}+\mathrm{Tr}(Q)\bar{K})}{(1-l)^2}\int_0^t(t-s)^{2\alpha-2}\mathbf{E}\|\varphi(s)-\phi(s)\|^2\mathrm{d}s\tag{17}$$

Using Lemma 2.17, $\mathbf{E}\|\varphi(t)-\phi(t)\|^2=0$ is derived. Hence, $\forall\, t\in[0,t_1\wedge\eta_1)$, $\varphi(t)=\phi(t)$ almost surely. The uniqueness has been demonstrated.

Step 6: We need to remove the additional condition of η_1, and verify the mild solution on interval $[0,t_1)$. If $\eta_1<t_1$,we can rewrite the mild solution on interval $\eta_1\le t<(\eta_1+\eta_2)\wedge t_1$ as

$$x(t)=x_{01}(t)+h(t,x(t))+\int_{\eta_1}^t AS_\alpha(t-s)h(s,x(s))\mathrm{d}s+\int_{\eta_1}^t S_\alpha(t-s)f(s,x(s))\mathrm{d}s+\int_{\eta_1}^t S_\alpha(t-s)g(s,x(t))\mathrm{d}B(s)\tag{18}$$

where $x_{01}(t)$ as defined by

$$x_{01}(t) = T_\alpha(t)[x_0 - h(0,x_0)] + \int_0^{\eta_1} AS_\alpha(t-s)h(s,x(s))\mathrm{d}s + \int_0^{\eta_1} S_\alpha(t-s)f(s,x(s))\mathrm{d}s + \int_0^{\eta_1} S_\alpha(t-s)g(s,x(t))\mathrm{d}B(s) \tag{19}$$

is known function. Applying the same rationale as previously, we deduce the existence of a unique mild solution over $\eta_1 \le t < (\eta_1+\eta_2)\wedge t_1$. Repeatedly applying this process leads to the conclusion that a unique solution $x(t)$ to system (1) exists on the interval $[0,t_1)$.

Step 7: Denoted $x_{(1)}(t)$ as the unique mild solution on the interval $[0,t_1)$, .

$$x_{(1)}(t) = T_\alpha(t)[x_0 - h(0,x_0)] + h(t,x_{(1)}(t)) + \int_0^t AS_\alpha(t-s)h(s,x_{(1)}(s))\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s,x_{(1)}(s))\mathrm{d}s$$
$$+ \int_0^t S_\alpha(t-s)g(s,x_{(1)}(s))\mathrm{d}B(s),\ t\in[0,t_1),$$

Applying Lemma 2.15 twice, one derives that

$$\|x_{(1)}(t)\|^2 \le \frac{1}{l}\|h(t_1,x_{(1)}(t)) - T_\alpha(t)h(0,x_0)\|^2 + \frac{1}{1-l}\|\mathcal{J}(t)\|^2 \le \sqrt{l}\|x_{(1)}(t)\|^2 + \frac{l}{1-\sqrt{l}}\|T_\alpha(t)x_0\|^2 + \frac{1}{1-l}\|\mathcal{J}(t)\|^2,$$

where

$$\mathcal{J}(t) = T_\alpha(t)x_0 + \int_0^{t_1} AS_\alpha(t_1-s)h(s,x(s))\mathrm{d}s + \int_0^t S_\alpha(t-s)f(s,x(s))\mathrm{d}s + \int_0^t S_\alpha(t-s)g(s,x(s))\mathrm{d}B(t)$$

Combine the above derivation and Gronwall's inequality

$$\begin{aligned}
\mathbf{E}\|x_{(1)}(t)\|^2 \le& \frac{l(1-l)+4(1-\sqrt{l})^2}{(1-\sqrt{l})^2(1-l)}\|T_\alpha(t)\|^2\mathbf{E}\|x_0\|^2 + \frac{4}{1-l}\mathbf{E}\left\|\int_0^t AS_\alpha(t-s)h(s,x_{(1)}(t))\mathrm{d}s\right\|^2 \\
&+ \frac{4}{1-l}\mathbf{E}\left\|\int_0^t S_\alpha(t-s)f(s,x_{(1)}(t))\mathrm{d}s\right\|^2 + \frac{4}{1-l}\mathbf{E}\left\|\int_0^t S_\alpha(t-s)g(s,x_{(1)}(t))\mathrm{d}B(s)\right\|^2 \\
\le& C_1 + C_2\int_0^t (t-s)^{2\alpha-2}\mathbf{E}\|x_{(1)}(s)\|^2\mathrm{d}s \\
\le& C_1 e^{C_2\int_0^t (t-s)^{2\alpha-2}\mathrm{d}s} = C_1 e^{\frac{C_2 t^{2\alpha-1}}{2\alpha-1}} < \infty,
\end{aligned}$$

where

$$C_1 = \frac{l(1-l)+4(1-\sqrt{l})^2}{(1-\sqrt{l})^2(1-l)}\sup_{0\le t\le t_1}\|T_\alpha(t)\|^2\mathbf{E}\|x_0\|^2 \text{ and } C_2 = \frac{4}{1-l}M_S(\Gamma l^2\|A\|^2 + \Gamma\bar{K} + \mathrm{Tr}(Q)\bar{K}).$$

Define the Picard iterations sequence once again on interval $[t_1,(t_1+\eta_1)\wedge t_2)$ as

$$\begin{cases}
\varphi_n(t) = T_\alpha(t)[x(t_1) - h(t_1,x(t_1))] + h(t,\varphi_{n-1}(t)) + \int_{t_1}^t AS_\alpha(t-s)h(s,\varphi_{n-1}(s))\mathrm{d}s + \int_{t_1}^t S_\alpha(t-s)f(s,\varphi_{n-1}(s))\mathrm{d}s \\
\qquad + \int_{t_1}^t S_\alpha(t-s)g(s,\varphi_{n-1}(s))\mathrm{d}B(t), \\
\varphi_0(t) = x(t_1),
\end{cases}$$

By Assumption 3.5, we observe $\mathbf{E}\|x(t_1)\|^2 \leq \mu \mathbf{E}\|x_{(1)}(t_1^-)\|^2 < \infty$, similarly to the steps above, a unique solution exists on $[t_1, t_2)$. We denote the solution on the interval $[0, t_2)$ as $x_{(2)}(t)$, which satisfies

$$x_{(2)}(t) = \begin{cases} x_{(1)}(t), \quad t \in [0, \infty), \\ T_\alpha(t-t_0)[x_0 - h(t_0, x_0)] + h(t, x_{(1)}(t)) + \int_{t_0}^{t} AS_\alpha(t-s)h(s, x_{(2)}(s))\mathrm{d}s + \int_{t_0}^{t} S_\alpha(t-s)f(s, x_{(2)}(s))\mathrm{d}s \\ \quad + \int_{t_0}^{t} S_\alpha(t-s)g(s, x_{(2)}(s))\mathrm{d}B(s), \quad t \in [t_1, t_2), \end{cases}$$

Continuing this procedure, we can conclude that on the interval $[0, T)$, there exists a unique local mild solution $x_{(k)}(t)$, $\forall\, T > 0$.

Combining all the aforementioned steps, the proof is complete. □

Remark 3.11. *Throughout the above procedure, Assumption 3.4 was utilized for the sake of simplicity. In fact, if is not necessary. By Lipschitz condition and elementary inequality it can be observed*

$$\|h(t, x(t))\|^2 \leq 2l^2\|A\|^2\|x(t)\|^2 + 2\sup_{s\in[0,T)}\|h(s,0)\|^2 \ , \ \|f(t, x(t))\|^2 \leq 2\bar{K}\|x(t)\|^2 + 2\sup_{s\in[0,T)}\|f(s,0)\|^2,$$

and $g(\cdot)$ is similar to $f(\cdot)$. Utilizing the above inequalities, we can substitute the corresponding procedure in the proof.

Remark 3.12. *Compared to the conditions obtained by the Banach fixed point theorem such as (3.2) in [26] and (3.1) in [6], which is an inequality where right-hand side is* 1*. Our conditions derived by Picard successive approximation is more relaxable*

Theorem 3.13. *Under conditions of Theorem 3.10 and Assumption 3.7, there is a unique global mild solution to system (1) defined on* $[0, \infty)$ *as Definition 3.1.*

Proof. Theorem 3.10 have been deduced based on the results of Lemma 2.13. In this context, M_T is denoted as $\sup_{0\leq t\leq T}\|T_\alpha(t)\|$, and M_S as $\sup_{0\leq t\leq T} Ce^{\varpi t}(1 + t^{1-\alpha})$. Moreover, M_T and M_S depend on the finite number T. As $T \to \infty$, $M_T \to \infty$ and $M_S \to \infty$. In order to remove the dependence of T, Assumption 3.7 is introduced. Using the property of Mittag-Leffler monotonically decreasing property, one can find $M_T = \mathcal{M}_1$ and $M_S = \mathcal{M}_2\Gamma(\alpha)^{-1}$. Replacing M_T with $\mathcal{M}_1$ and M_S with $\mathcal{M}_2/\Gamma(\alpha)$ in the proof of Theorem 3.10 is also consistent.

According to Theorem 3.10, a unique solution $x_{(1)}(t)$ to system (1) exists on the interval $[0, t_1)$, and a solution $x_{(2)}(t)$ exists on the interval $[0, t_2)$. Following this procedure enables us to obtain a unique global mild solution to system (1), as defined in Definition 3.1. □

Remark 3.14. *In the proof of theorem 3.13, we have employed Assumption 3 for simplicity. If Assumption 3.4 is not considered, by Remark 3.9 and Theorem 4.2 of [19]. For any fixed $T > 0$, we should let $\|h(t,0)\|^2$, $\|f(t,0)\|^2$ and $\|g(t,0)\|^2$ is bounded on $t \in [0, T)$, .*

3.2. Stability analysis of FSINSs

Next, the stability conclusion will be presented in detail.

Theorem 3.15. *Assume that Assumptions 3.2-3.7 hold. Let $A \in \mathscr{A}^\alpha(\varpi_0, \vartheta_0)$ where $\varpi_0 \in \mathbb{R}$ and $\vartheta_0 \in (0, \pi/2]$. For $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are compact. If the fractional order of system (1) α and decay rate λ satisfies the condition $\alpha\lambda > \Phi$, with*

$$\Phi = 2(\Phi_1 + \Phi_2)^2 K_1^2, \ \Phi_1 = 5(\bar{K} + l^2\|A\|^2)\mathcal{M}^{3/2}\mathcal{M}_2^2\Gamma^{\alpha 3/2}(1 + \lambda\gamma^\alpha)^{-3/2},$$

$$\Phi_2 = 5\mathrm{Tr}(Q)\bar{K}\sqrt{\mathcal{M}}\mathcal{M}_2^2\left[\frac{1}{\Gamma(\alpha)^3(3\alpha-2)} + \frac{\gamma^\alpha\mathcal{M}}{\Gamma(\alpha)^2(1+\lambda\Gamma^\alpha)}\right]^{\frac{1}{2}},$$

and there is a positive scalar Ψ *s.t.*

$$\sqrt{2}K_2\Big[(1+\mu)^2\|T_\alpha(\varsigma)\|^4+\Xi\Psi^{(\frac{\Phi}{\lambda}-\alpha)}\Big]^{\frac{1}{2}}<1 \tag{20}$$

holds, where

$$K_1=\frac{1}{(1-l)(1-\sqrt{l})}\ \text{and}\ K_2=\frac{2l(1-l)+5(1-\sqrt{l})}{(1-\sqrt{l})^2(1-l)},$$

then system (1) achieves asymptotic stability in the mean-square sense.

Proof. The proof unfolds through three steps:

Step 1: For $t\in[0,t_1)$, according to Definition 3.1,

$$x(t)=T_\alpha(t)[x_0-h(0,x_0)]+h(t,x(t))+\int_0^t AS_\alpha(t-s)f(s,x(s))\mathrm{d}s+\int_0^t S_\alpha(t-s)f(s,x(s))\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x(t))\mathrm{d}B(s).$$

Applying Lemma 2.15 twice, one derives that

$$\|x(t)\|^2\leq\sqrt{l}\|x(t)\|^2+\frac{l}{1-\sqrt{l}}\|T_\alpha(t)x_0\|^2+\frac{1}{1-l}\|\mathcal{J}_0(t)\|^2, \tag{21}$$

where

$$\mathcal{J}_0(t)=T_\alpha(t)x_0+\int_0^t AS_\alpha(t-s)h(s,x(s))\mathrm{d}s+\int_0^t S_\alpha(t-s)f(s,x(s))\mathrm{d}s+\int_0^t S_\alpha(t-s)g(s,x(s))\mathrm{d}B(t)$$

Utilizing Lemma 2.1, Young's inequality, and the Cauchy-Schwarz inequality on the mild solution over the interval $[0,t_1]$, one has

$$\mathbf{E}\|\mathcal{J}_0(t)\|^2\leq 5\|T_\alpha(t)\|^2\mathbf{E}\|x_0\|^2+\sum_{i=1}^{3}\tilde{H}_i(t), \tag{22}$$

with

$$\begin{aligned}
\tilde{H}_1(t)&=5\int_0^t\|S_\alpha(t-s)\|\,\mathrm{d}s\int_0^t\|A\|^2\,\|S_\alpha(t-s)\|\,\mathbf{E}\|h(s,x(s))\|^2\mathrm{d}s,\\
\tilde{H}_2(t)&=5\int_0^t\|S_\alpha(t-s)\|\,\mathrm{d}s\int_0^t\|S_\alpha(t-s)\|\,\mathbf{E}\|f(s,x(s))\|^2\mathrm{d}s,\\
\tilde{H}_3(t)&=5\mathrm{Tr}(Q)\int_0^t\|S_\alpha(t-s)\|^2\,\mathbf{E}\|g(s,x(s))\|^2\mathrm{d}s.
\end{aligned}$$

From the elementary inequality, the coefficient Eq. (22) is 4, but in order to match the form of the following steps, we increase the coefficient to 5.

From Assumption 3.4 and Lemma 3.4, we can obtain

$$\begin{aligned}
\int_{t_{k-1}}^t\|S_\alpha(t-s)\|\,\mathrm{d}s&\overset{(u=t-s)}{=}\int_0^{t-t_{k-1}}\|S_\alpha(u)\|\,\mathrm{d}u\leq\mathcal{M}_2\int_0^{t-t_{k-1}}u^{\alpha-1}E_{\alpha,\alpha}(-\lambda u^\alpha)\mathrm{d}u\leq\mathcal{M}_2(t-t_{k-1})^\alpha E_{\alpha,\alpha+1}(-\lambda(t-t_{k-1})^\alpha)\\
&\leq\frac{(t-t_{k-1})^\alpha\mathcal{M}\mathcal{M}_2}{1+|\lambda(t-t_{k-1})^\alpha|}\leq\frac{\gamma^\alpha\mathcal{M}\mathcal{M}_2}{1+\lambda\gamma^\alpha}
\end{aligned}$$

Then, utilizing Assumption 3.2, Assumption 3.3, Hölder inequality and Lemma 2.5, one has

$$\begin{aligned}\tilde{H}_1(t) &\leq 5\bar{K}l^2\|A\|^2 \int_0^t \|S_\alpha(x)\|\,\mathrm{d}x \int_0^t \left[\|S_\alpha(t-s)\|^{\frac{1}{2}}\,\|S_\alpha(t-s)\|^{\frac{1}{2}}\,\|x(s)\|_{PC}^2\right]\mathrm{d}s \\ &\leq 5l^2\|A\|^2\bar{K}\int_0^t \|S_\alpha(x)\|\,\mathrm{d}x \left[\int_0^t \|S_\alpha(x)\|\,\mathrm{d}x\right]^{\frac{1}{2}} \left[\int_0^t \|S_\alpha(t-s)\|\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}} \\ &\leq \Upsilon_1\left[\int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha}\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}},\end{aligned} \tag{23}$$

where $\Upsilon_1 = 5l^2\|A\|^2\mathcal{M}^{3/2}\mathcal{M}_2^2\gamma^{3\alpha/2}(1+\lambda\gamma^\alpha)^{-3/2}$. Analogously,

$$\begin{aligned}\tilde{H}_2(t) \leq& 5\bar{K}\int_0^t \|S_\alpha(x)\|\,\mathrm{d}x \int_0^t \left[\|S_\alpha(t-s)\|^{\frac{1}{2}}\,\|S_\alpha(t-s)\|^{\frac{1}{2}}\,\|x(s)\|_{PC}^2\right]\mathrm{d}s \\ \leq& 5\bar{K}\int_0^t \|S_\alpha(x)\|\,\mathrm{d}x\left[\int_0^t \|S_\alpha(x)\|\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_0^t \|S_\alpha(t-s)\|\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}} \\ \leq& \Upsilon_2\left[\int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha}\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}},\end{aligned} \tag{24}$$

where $\Upsilon_2 = 5\bar{K}\mathcal{M}^{3/2}\mathcal{M}_2^2\gamma^{3\alpha/2}(1+\lambda\gamma^\alpha)^{-3/2}$. Combine Eqs. (23)(24) and let $\Phi_1 = \Upsilon_1 + \Upsilon_2$, we ascertain

$$\tilde{H}_1(t) + \tilde{H}_2(t) \leq \Phi_1\left[\int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha}\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}} \tag{25}$$

For $\tilde{H}_3(t)$,

$$\begin{aligned}\tilde{H}_3(t) &\leq 5\mathrm{Tr}(Q)\bar{K}\int_0^t \left[\|S_\alpha(t-s)\|^{\frac{3}{2}+\frac{1}{2}}\,\|x(s)\|_{PC}^2\right]\mathrm{d}s \\ &\leq 5\mathrm{Tr}(Q)\bar{K}\left[\int_0^t \|S_\alpha(x)\|^3\,\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_0^t \|S_\alpha(t-s)\|\left(\|x(s)\|_{PC}^2\right)^2\mathrm{d}s\right]^{\frac{1}{2}}.\end{aligned}$$

By Assumption 3.4 and Lemma 2.4, one can derive

$$\int_{t_{k-1}}^t \|S_\alpha(t-s)\|^3\mathrm{d}s \leq \mathcal{M}_2^3\int_0^{t-t_{k-1}} x^{3\alpha-3}\left[E_{\alpha,\alpha}(-\lambda x^\alpha)\right]^3\mathrm{d}x \leq \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^2}\int_0^\gamma x^{3\alpha-3}E_{\alpha,\alpha}(-\lambda x^\alpha)\mathrm{d}x.$$

if $\gamma \leq 1$,

$$\frac{\mathcal{M}_2^3}{\Gamma(\alpha)^2}\int_0^\gamma x^{3\alpha-3}E_{\alpha,\alpha}(-\lambda x^\alpha)\mathrm{d}x \leq \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^3}\int_0^1 x^{3\alpha-3}\mathrm{d}x \leq \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^3(3\alpha-2)};$$

and if $\gamma > 1$, by Lemma 2.6

$$\begin{aligned}\frac{\mathcal{M}_2^3}{\Gamma(\alpha)^2}\int_0^\gamma x^{3\alpha-3}E_{\alpha,\alpha}(-\lambda x^\alpha)\mathrm{d}x \leq& \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^3}\int_0^1 x^{3\alpha-3}\mathrm{d}x + \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^2}\int_1^\gamma x^{3\alpha-3}E_{\alpha,\alpha}(-\lambda x^\alpha)\mathrm{d}x \\ \leq& \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^3(3\alpha-2)} + \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^2}\gamma^\alpha E_{\alpha,\alpha+1}(-\lambda\gamma^\alpha) \\ \leq& \frac{\mathcal{M}_2^3}{\Gamma(\alpha)^3(3\alpha-2)} + \frac{\gamma^\alpha\mathcal{M}\mathcal{M}_2^3}{\Gamma(\alpha)^2(1+\lambda\gamma^\alpha)}\end{aligned}$$

Combined with the above derivation, it can be concluded that

$$\tilde{H}_3(t) \leq \Phi_2 \left[\int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha} \left(\|x(s)\|_{PC}^2 \right)^2 \mathrm{d}s \right]^{\frac{1}{2}}, \tag{26}$$

where $\Phi_2 = 5\mathrm{Tr}(Q) l^2 \sqrt{\mathcal{M}} \mathcal{M}_2^2 \left[\frac{1}{\Gamma(\alpha)^3(3\alpha-2)} + \frac{\gamma^\alpha \mathcal{M}}{\Gamma(\alpha)^2(1+\lambda\Gamma^\alpha)} \right]^{\frac{1}{2}}$.

Combining Eqs. (21)(22)(25)(26), we have

$$\|x(t)\|_{PC}^2 \leq K_2 \|T_\alpha(t)\|^2 \mathbf{E}\|x_0\|^2 + K_1(\Phi_1 + \Phi_2) \left[\int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha} \left(\|x(s)\|_{PC}^2 \right)^2 \mathrm{d}s \right]^{\frac{1}{2}}. \tag{27}$$

Square the expression given in Eq. (27) , utilize elementary inequality and Gronwall-Bellman inequality (Lemma 2.16)

$$\begin{aligned} \left(\|x(s)\|_{PC}^2 \right)^2 &\leq 2K_2^2 \|T_\alpha(t)\|^4 \left(\mathbf{E}\|x_0\|^2 \right)^2 + \Phi \int_0^t \frac{(t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha} \left(\|x(s)\|_{PC}^2 \right)^2 \mathrm{d}s \\ &\leq 2K_2^2 \left(\mathbf{E}\|x_0\|^2 \right)^2 \mathcal{Y}(t), \end{aligned}$$

where

$$\mathcal{Y}(t) = \|T_\alpha(t)\|^4 + \int_0^t \frac{\Phi \|T_\alpha(s)\|^4 (t-s)^{\alpha-1}}{1+\lambda(t-s)^\alpha} \exp\left(\Phi \int_s^t \frac{(t-\tau)^{\alpha-1}}{1+\lambda(t-\tau)^\alpha} \mathrm{d}\tau \right) \mathrm{d}s.$$

Moreover,

$$\begin{aligned} \mathcal{Y}(t) &\leq \|T_\alpha(t)\|^4 + \int_0^t \frac{\Phi \mathcal{M}^4 (t-s)^{\alpha-1}}{(1+\lambda s^\alpha)^4 [1+\lambda(t-s)^\alpha]} \exp\left(-\frac{\Phi}{\alpha\lambda} \ln(\lambda(t-\tau)^\alpha + 1)|_{\tau=s}^t \right) \mathrm{d}s \\ &\leq \|T_\alpha(t)\|^4 + \int_0^t \frac{\Phi \mathcal{M}^4 (t-s)^{\alpha-1} \mathrm{d}s}{(1+\lambda s^\alpha) [1+\lambda(t-s)^\alpha]^{1-\frac{\Phi}{\alpha\lambda}}} \\ &\leq \|T_\alpha(t)\|^4 + \Phi \mathcal{M}^4 \lambda^{(\frac{\Phi}{\alpha\lambda}-2)} \int_0^t s^{-\alpha} (t-s)^{(\frac{\Phi}{\lambda}-1)} \mathrm{d}s. \\ &= \|T_\alpha(t)\|^4 + \Phi \mathcal{M}^4 \lambda^{(\frac{\Phi}{\alpha\lambda}-2)} \left[\int_0^1 \tau^{-\alpha} (1-\tau)^{(\frac{\Phi}{\lambda}-1)} \mathrm{d}\tau \right] t^{(\frac{\Phi}{\lambda}-\alpha)} \\ &= \|T_\alpha(t)\|^4 + \Phi \mathcal{M}^4 \lambda^{(\frac{\Phi}{\alpha\lambda}-2)} \mathcal{B}(\frac{\Phi}{\lambda}, 1-\alpha) t^{(\frac{\Phi}{\lambda}-\alpha)}. \end{aligned}$$

In the first equal sign of the above equation, we substituted $\tau = s/t$. Combining the above equation, it can be shown that

$$\|x(t)\|_{PC}^2 \leq \sqrt{2} K_2 \mathbf{E}\|x_0\|^2 \left[\|T_\alpha(t)\|^4 + \Xi_1 t^{(\frac{\Phi}{\lambda}-\alpha)} \right]^{\frac{1}{2}}.$$

where $\Xi = \Phi \mathcal{M}_1^4 \lambda^{(\Phi/\alpha\lambda-2)} \mathcal{B}(\Phi/\lambda, 1-\alpha)$. Hence, one has

$$\mathbf{E}\|x(t_1^-)\|^2 \leq \sqrt{2} K_2 \mathbf{E}\|x_0\|^2 \left[\|T_\alpha(t_1)\|^4 + \Xi_1 t_1^{(\frac{\Phi}{\lambda}-\alpha)} \right]^{\frac{1}{2}} = \Theta_1.$$

Step 2: According to Definition 3.1, for $t_k \leq t \leq t_{k+1}$, $\forall\, k \geq 1$, the mild solution is

$$\begin{aligned} x(t) =& T_\alpha(t-t_1)[x(t_k) - h(t_k, x(t_k))] + h(t, x(t)) + \int_{t_k}^t A S_\alpha(t-s) h(s, x(s)) \mathrm{d}s + \int_{t_k}^t S_\alpha(t-s) f(s, x(s)) \mathrm{d}s \\ &+ \int_{t_k}^t S_\alpha(t-s) g(s, x(t)) \mathrm{d}B(s). \end{aligned}$$

Similarly, applying Lemma 2.15 twice, using Assumption 3.6, one can show that

$$\begin{aligned}\|x(t)\|^2 \leq& \frac{1}{l}\|h(t,x(t)) - T_\alpha(t-t_k)h(t_k, x(t_k^+))\|^2 + \frac{1}{1-l}\|\mathcal{J}_k(t)\|^2\\ \leq& \sqrt{l}\|x(t)\|^2 + \frac{l}{1-\sqrt{l}}\|T_\alpha(t-t_k)x(t_k^+)\|^2 + \frac{1}{1-l}\|\mathcal{J}_k(t)\|^2\\ \leq& \sqrt{l}\|x(t)\|^2 + \frac{2l(1+\mu)}{1-\sqrt{l}}\|T_\alpha(t-t_k)x(t_k)\|^2 + \frac{1}{1-l}\|\mathcal{J}_k(t)\|^2,\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_k(t) =& T_\alpha(t-t_k)x(t_k^-) + T_\alpha(t-t_k)I_k(x(t_k^-)) + \int_{t_k}^t AS_\alpha(t-s)h(s,x(s))\mathrm{d}s + \int_{t_k}^t S_\alpha(t-s)f(s,x(s))\mathrm{d}t\\ &+ \int_{t_k}^t S_\alpha(t-s)g(s,x(s))\mathrm{d}B(t),\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\|\mathcal{J}_k(t)\|^2 \leq& 5\mathbf{E}\|x(t_k^-)\|^2\|T_\alpha(t-t_k)\|^2 + 5\mu\mathbf{E}\|x(t_k^-)\|^2\|T_\alpha(t-t_k)\|^2 + \check{H}_1(t)+\check{H}_2(t)+\check{H}_3(t)\\ \leq& 5\mathbf{E}\|x(t_k^-)\|^2(1+\mu)\|T_\alpha(t-t_k)\|^2 + \sum_{i=1}^{3}\check{H}_i(t),\end{aligned}$$

with

$$\begin{aligned}\check{H}_1(t) =& 5\int_{t_1}^t \|S_\alpha(t-s)\|\,\mathrm{d}s \int_{t_1}^t \|A\|^2\,\|S_\alpha(t-s)\|\,\mathbf{E}\|h(s,x(s))\|^2\mathrm{d}s,\\ \check{H}_2(t) =& \int_{t_1}^t \|S_\alpha(t-s)\|\,\mathrm{d}s 5\int_{t_1}^t \|S_\alpha(t-s)\|\,\mathbf{E}\|f(s,x(s))\|^2\mathrm{d}s,\\ \check{H}_3(t) =& 5\mathrm{Tr}(Q)\int_{t_1}^t \|S_\alpha(t-s)\|^2\,\mathbf{E}\|g(s,x(s))\|^2\mathrm{d}s.\end{aligned}$$

Similar to the derivation in Step 1, we have

$$\|x(t)\|_{PC}^2 \leq \sqrt{2}K_2\mathcal{B}_k\Big[(1+\mu)^2\|T_\alpha(t-t_1)\|^4 + \Xi t^{(\frac{\Phi}{\Lambda}-\alpha)}\Big]^{\frac{1}{2}}. \tag{28}$$

and

$$\mathbf{E}\|x(t_k^-)\|^2 \leq \sqrt{2}K_2\Theta_{k-1}\Big[(1+\mu)^2\|T_\alpha(t_2-t_1)\|^4 + \Xi t_2^{(\frac{\Phi}{\Lambda}-\alpha)}\Big]^{\frac{1}{2}} = \Theta_k. \tag{29}$$

Step 3: From Eq.(29), we can know that $\mathcal{B}_k$ has a iterative relation as the type of

$$\sqrt{2}K_2\Theta_k\Big[(1+\mu)^2\|T_\alpha(t_{k+1}-t_k)\|^4 + \Xi t_{k+1}^{(\frac{\Phi}{\Lambda}-\alpha)}\Big]^{\frac{1}{2}} = \Theta_{k+1}$$

According to the above discussion, Assumption 3.6 and the conditon (20), there exists a large enough number Ψ s.t. $\Theta_{k+1} \leq \Theta_k,\ t_k \geq \Psi$. So sequence $\{\Theta_k\}_{k\in\mathbb{N}^+}$ has an upper bound. Assume that $\Theta_{k^*} = \max_{k\in\mathbb{N}^+}\Theta_k$, where k^* is a finite scalar. Let

$$l_k = \sqrt{2}K_2\Big[(1+\mu)^2\|T_\alpha(t_k-t_{k-1})\|^4 + \Xi t_k^{(\frac{\Phi_k}{\Lambda}-\alpha)}\Big]^{\frac{1}{2}},$$

then $\Theta_{k^*} = \|x_0\|^2 \prod_{i=1}^{k^*} l_i$. Therefore, we conclude that

$$\|x(t)\|_{PC}^2 \leq \sqrt{2}K_2\|x_0\|^2 \left(\prod_{i=1}^{k^*} l_i\right)\Big[(1+\mu)^2\|T_\alpha(\varsigma)\|^4 + \Xi t^{(\frac{\Phi}{\lambda}-\alpha)}\Big]^{\frac{1}{2}}, \forall\, t \geq 0$$

By the definition of the stability, $\forall\ \varepsilon > 0$,

$$\exists\, \delta = \frac{\varepsilon \prod_{i=1}^{k^*} l_i^{-1}}{\sqrt{2}K_2}\left[(1+\mu)^2\|T_\alpha(\varsigma)\|^4 + \Xi t^{(\frac{\Phi}{\lambda}-\alpha)}\right]^{-\frac{1}{2}},$$

when $t > 0$ and $\mathbf{E}\|x_0\|^2 < \delta$, it implies $\|x(t)\|_{PC}^2 < \varepsilon$, e.g. system (1) is stability. Additionally, the condition $\alpha\lambda > \Phi$ results in $\lim_{t\to\infty} \Xi t^{(\frac{\Phi}{\lambda}-\alpha)} = 0$. If t_k is big enough, by Assumption 3.4, Assumption 3.6 and condition (20)

$$\sqrt{2}K_2(1+\mu)\|T_\alpha(t_{k+1}-t_k)\|^2\Theta_k = \Theta_{k+1}$$

and

$$\sqrt{2}K_2(1+\mu)\|T_\alpha(t_{k+1}-t_k)\|^2 \leq \sqrt{2}K_2(1+\mu)\|T_\alpha(\varsigma)\|^2 < 1,$$

implying $\Theta_k \to 0$ as $k \to \infty$. In fact, $k(t)$ behaves like a step function over t, with $k(t)$ being non-decreasing and tending to infinity as t increases. By Eq.(28), we can obtain $\mathbf{E}\|x(t)\|^2 \to 0$ as $t \to \infty$. Accordingly, system (1) achieves asymptotic stability in mean-square.

Combining all the aforementioned steps, the proof is complete. □

Remark 3.16. *The system studied in this paper is a semi-linear system. According to the Assumption 3.7 and Lemma 2.4, λ is determined by the linear part and determines the decay rate. When λ is large enough, the decay rate of the mild solution will be larger, so it is easier to approach stability, which is consistent with the conclusion.*

Remark 3.17. *It is worth noting that substantiating condition (20) is straightforward. If the remaining conditions of Theorem 3.15 satisfied,* $\Xi t^{(\frac{\Phi}{\lambda}-\alpha)} \to 0$ as $t \to \infty$. *By the $\varepsilon - N$ language of limits, condition (20) can be evaluated by verifying* $\sqrt{2}K_2(1+\mu)\|T_\alpha(\varsigma)\|^2 < 0$.

3.3. Stability analysis of fractional stochastic dynamic systems without neutral function

Contemplate the ensuing fractional stochastic impulsive system without neutral function

$$\begin{cases} {}^{C}_{t_{k-1}}\mathcal{D}_t^{\alpha} x(t) = Ax(t) + f(t,x(t)) + g(t,x(t))\dfrac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,... \\ x(t_0) = x_0, \end{cases} \tag{30}$$

where $t \geq 0$. Neutral function $h(t,x(t)) \equiv 0$, the remaining parts are analogous to system (1).

Corollary 3.18. *Assume that Assumptions 3.2-3.4 and 3.6-3.7 hold. Let $A \in \mathcal{A}^\alpha(\varpi_0, \vartheta_0)$ with $\varpi_0 \in \mathbb{R}$, $\vartheta_0 \in (0, \pi/2]$. For $t \geq 0$, $T_\alpha(t)$ and $S_\alpha(t)$ are compact, if the order of system (30) α satisfies the condition $\alpha\lambda > \Phi$, with*

$$\Phi = 2(\Phi_1+\Phi_2)^2, \Phi_1 = 4\bar{K}\mathcal{M}^{3/2}\mathcal{M}_2^2\gamma^{3\alpha/2}(1+\lambda\gamma^\alpha)^{-3/2},$$

$$\Phi_2 = 4\mathrm{Tr}(Q)\bar{K}\sqrt{\mathcal{M}}\mathcal{M}_2^2\left[\frac{1}{\Gamma(\alpha)^3(3\alpha-2)} + \frac{\gamma^\alpha\mathcal{M}}{\Gamma(\alpha)^2(1+\lambda\gamma^\alpha)}\right]^{\frac{1}{2}}.$$

and there is a positive scalar Ψ s.t.

$$4\sqrt{2}\Big[(1+\mu)^2\|T_\alpha(\varsigma)\|^4 + \Xi\Psi^{(\frac{\Phi}{\lambda}-\alpha)}\Big]^{\frac{1}{2}} < 1 \tag{31}$$

holds, then system (30) achieves asymptotic stability in the mean-square.

4. Numerical examples

Leveraging the fractional-order predictor-corrector scheme and Euler-Maruyama method, two examples are presented to validate the obtained results.

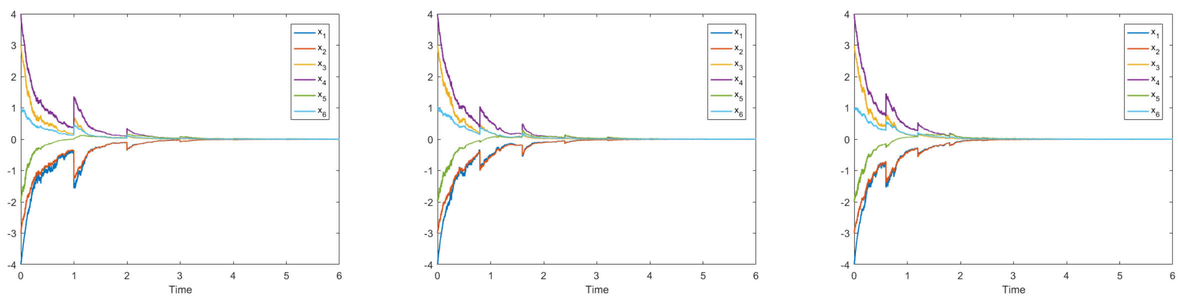
4.1. Example 1

First, we examine the FSINS as described below:

$$\begin{cases} {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_1(t)-\frac{1}{4}\sin(x_1(t))] = -3x_1(t)+15\sin(x_1(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_2(t)+\frac{1}{5}\tanh(x_2(t))] = -3x_2(t) + \frac{4}{5}\cos(x_2(t)) - \frac{4}{5} + 10\tanh(x_2(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_3(t) - \frac{2}{11}\tanh(x_3(t))] = -3x_3(t) - \sin(x_3(t)) + 15\sin(x_3(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_4(t)-\frac{1}{5}\cos(x_4(t))+\frac{1}{5}] = -3x_4(t) + 10\tanh(x_4(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_5(t)-\frac{1}{4}\tanh(x_5(t))] = -3x_5(t) - \sin(x_2(t))t + 15\sin(x_5(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ {}_{t_{k-1}}^{\ C}\mathcal{D}_t^{\alpha}[x_6(t)+\frac{2}{9}\sin(x_6(t))] = -3x_6(t) - \cos(x_6(t)) + 1 + 10\tanh(x_6(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k, \\ \Delta x_i|_{t=t_k} = \sqrt{\mu_1}x_i(t_k^-), \quad i = 1,...,6,\ k \in \mathbb{N}^+ \end{cases} \tag{32}$$

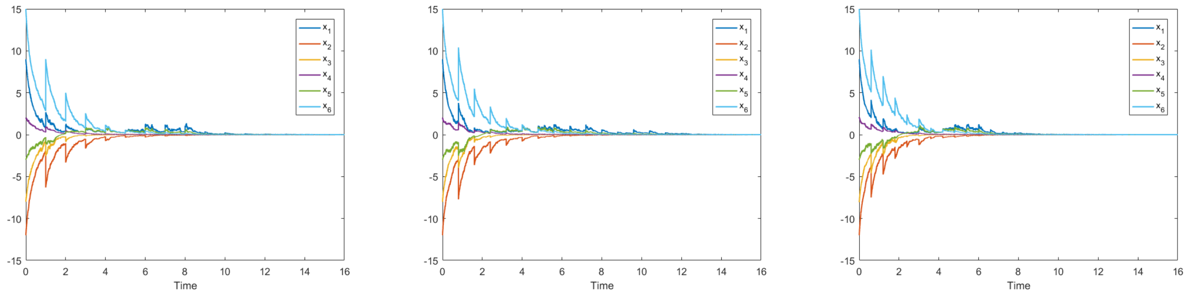
where $\alpha = 0.9$, $t_k - t_{k-1} = \varsigma_1 = \gamma_1$. The initial values are set as $x_1(0) = -4,\ x_2(0) = -3,\ x_3(0) = -3,\ x_4(0) = 4,\ x_5(0) = -2,\ x_6(0) = 1$ is not random variables. The parameters of the system are given by the theorem that, because of the condition $\alpha\lambda > \Phi$, we need only find λ large enough to make the system itself stable. When determining the parameters of the impulse, we can first determine its maximum impulse interval ς, and then use condition (20) to find the maximum impulse intensity μ that can guarantee stability. Firstly, we set ς_1 as $1, 0.8, 0.6$ respectively. Using a simplified version of the condition (20) in Remark 3.17, we can calculate the corresponding values of μ_1 as $5.55, 2.29, 0.46$, and all remaining conditions of Theorem 3.15 are fulfilled. Figure 1 plot the paths of $x_i(t)$ $(i = 1, 2, \ldots, 6)$, illustrating that $\lim_{t\to\infty} \mathbf{E}\|x_i(t)\|^2 = 0$ for $i = 1, 2, \ldots, 6$.

Remark 4.1. *Suppose one would like to let the initial value x_0 be a random variable, On the basis of Sec 4.2 in [22], we need only to discuss the case of constant initial values.*



(a) $\mu_1 = 5.55,\ \varsigma_1 = 1.0$ (b) $\mu_1 = 2.29,\ \varsigma_1 = 0.8$ (c) $\mu_1 = 0.46,\ \varsigma_1 = 0.6$

Figure 1: Solution trajectories of system (32)



(a) $\mu_2 = 4.64,\ \varsigma_2 = 1.0$ (b) $\mu_2 = 2.47,\ \varsigma_2 = 0.8$ (c) $\mu_2 = 0.99,\ \varsigma_2 = 0.6$

Figure 2: Solution trajectories of system (33)

4.2. Example 2

Examine the FSIS without neutral function $h(\cdot,\cdot)$ as described below:

$$\begin{aligned}
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_1(t) = -2x_1(t) + x_5(t) + 15\sin(x_1(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_2(t) = -2x_2(t) + \frac{4}{5}\cos(x_4(t)) - \frac{4}{5} + 10\tanh(x_2(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_3(t) = -2x_3(t) - \sin(x_3(t))15\sin(x_3(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_4(t) = -2x_4(t) + 10\tanh(x_4(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_5(t) = -2x_5(t) - \sin(x_2(t)) + 15\sin(x_5(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&{}_{t_{k-1}}^{C}\mathcal{D}_t^{\alpha} x_6(t) = -2x_6(t) - \cos(x_3(t)) + 1 + 10\tanh(x_6(t))\frac{\mathrm{d}B(t)}{\mathrm{d}t},\ t \neq t_k,\\
&\Delta x_i|_{t=t_k} = \sqrt{\mu_2} x_1(t_k^-),\ i = 1,...,6,\ k = 1,2,...
\end{aligned} \tag{33}$$

where $\alpha = 0.85$, $t_k - t_{k-1} = \varsigma_2 = \gamma_2$. The initial values are set as $x_1(0) = 9,\ x_2(0) = 12,\ x_3(0) = -8,\ x_4(0) = 2,\ x_5(0) = -3,\ x_6(0) = 15$. Use the same method as in the previous example In this example we set ς_2 as $1, 0.8, 0.6$ respectively. Given the conditon (31) ,we obtain corresponding values of μ_2 are $4.64, 2.47, 0.99$, and all remaining conditions of Corollary 3.18 are fulfilled. Figure 2 plot the paths of $x_i(t)$ $(i = 1, 2, \ldots, 6)$, illustrating that $\lim_{t\to\infty} \mathbf{E}\|x_i(t)\|^2 = 0$ for $i = 1, 2, \ldots, 6$.

5. Conclusions

We had proved the existance and uniqueness results of local and global mild solutions to FSINSs in Hilbert space. Next, the asymptotic stability of the system mean-square was investigated. Of note, the fractional solution operator and resolvent family exhibit Mittag–Leffler characteristics. Consequently, gentler sufficient conditions were derived to ensure the asymptotic stability of FSINSs in mean-square. Utilizing both a predictor-corrector scheme and the stochastic Euler-Maruyama method, numerical examples were implemented to substantiate the validity of the proposed theorems. In future studies, consideration will be given to the effect of time delay on the system. Moreover, random impulsive effect will also be examined.

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