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On a class of harmonic functions associated with the μth order differential subordination

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Abstract. In this paper, a class $\mathcal{P}_H^{\mu}(\alpha,\beta)$ of functions $f = h + \overline{g}$ which are the harmonic shears of the analytic functions $h + g$ is defined and studied. A sufficient coefficient condition for functions $f = h + \overline{g} \in \mathcal{P}_H^{\mu}(\alpha,\beta)$ is obtained. It is proved that this coefficient condition is necessary and sufficient for functions belonging to its subclass $\mathcal{NP}_H^{\mu}(\alpha,\beta)$. Results on bounds, extreme points, convolution, convex combinations and integral operator for functions belonging to the subclass $\mathcal{NP}_H^{\mu}(\alpha,\beta)$ are obtained. Inequalities for certain hypergeometric harmonic functions belonging to these classes are also given and the results based on one special case when $\mu = 4$ are included.

1. Introduction and preliminaries

Let $\mathcal{H}$ denotes a class of complex-valued functions $f = u + iv$ which are harmonic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in $\mathbb{U}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{U}$, called the analytic and co-analytic parts of f, respectively. The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$.

According to the Lewy's [12], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{U}$ if and only if $J_f(z) > 0$ in $\mathbb{U}$ which is equivalent to the existence of an analytic function $\omega_f(z) = g'(z)/h'(z)$ in $\mathbb{U}$ such that $|\omega_f(z)| < 1$ for all $z \in \mathbb{U}$ [7]. The function $\omega_f(z)$ is called the dilatation of f. A class of all univalent, sense preserving harmonic functions $f = h+\overline{g} \in \mathcal{H}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $S_{\mathcal{H}}$.

Harmonic mappings techniques have been used to study and solve fluid flow problems (see[4]). A subclass of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $S_{\mathcal{H}}^0$. If $f = h + \overline{g} \in S_{\mathcal{H}}^0$, then h and g are of the form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

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Further, if $g(z) \equiv 0$, the class $S_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of normalized univalent functions.

Ahuja and Jahangiri [1] introduced the class $\mathcal{P}_H(\gamma)$ of harmonic functions $f \in S_{\mathcal{H}}$ satisfying the condition

$$\Re e\left(\frac{\frac{\partial}{\partial\theta} f(z)}{\frac{\partial z}{\partial\theta}}\right) > \gamma \quad \left(0 \le \gamma < 1; z = re^{i\theta} \in \mathbb{U}\right)$$

and they proved a sufficient coefficient condition for $f \in \mathcal{P}_H(\gamma)$ which is given by

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) \le 2 - \gamma.$$

The class $\mathcal{P}_H(\gamma)$ of functions $f = h + \overline{g}$ reduces to a well known class $\mathcal{P}(\gamma)$ if $g(z) \equiv 0$ and then the analytic univalent functions $h \in \mathcal{P}(\gamma)$ satisfy the condition $\Re e\left(h'(z)\right) > \gamma$. Functions in the class $\mathcal{P}(\gamma)$ are called the functions of bounded turning [9].

A harmonic convolution "$*$" of functions $f = h + \bar{g} \in \mathcal{H}$ and $F = H + \overline{G} \in \mathcal{H}$, where h and g are of the form (1) and

$$H(z) = z + \sum_{n=2}^{\infty} A_n z^n \qquad \text{and} \qquad G(z) = \sum_{n=1}^{\infty} B_n z^n, \tag{2}$$

is defined by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n z^n}. \tag{3}$$

We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{U}$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \tag{4}$$

Clunie and Sheil-Small in [6] studied univalent harmonic functions through some geometric properties of related analytic functions and gives a fundamental theorem called *Shearing Theorem*, which is as follows:

Theorem 1.1. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{U}$ is a univalent harmonic mapping of $\mathbb{U}$ onto a domain convex in the horizontal direction if and only if $h + \epsilon g$ ($|\epsilon| = 1$) is univalent analytic mapping of $\mathbb{U}$ onto a domain convex in the horizontal direction (CHD).*

Let $F = h + g$, where h and g are of the form (1), and let the operator D^μ be defined by

$$D^0F(z) = F(z),\ D^1F(z) = DF(z) = zF'(z) \text{ and}$$
$$D^\mu F(z) = D\left(D^{\mu-1}F(z)\right)\ (\mu \in \mathbb{N}).$$

We define a class $\mathcal{P}_H^\mu(\alpha, \beta)$ of harmonic functions $f = h + \overline{g} \in S_{\mathcal{H}}^0$ which are the harmonic shears of analytic functions $F = h + g$ satisfy the condition

$$(1-\alpha)F'(z) + \alpha\frac{D^\mu F(z)}{z} \prec \frac{1 + (1-2\beta)z}{1-z} \tag{5}$$
$$\left(\mu \in \mathbb{N}, \alpha \ge 0, 0 \le \beta < 1; z \in \mathbb{U}\right).$$

The operator D^μ was defined by Sălăgean [17]. Classes of harmonic functions associated with subordination have been studied in [3, 13, 16], also see [20, 25].

In view of the identity [14, 15]:

$$n^{\mu} = n + \lambda_1 n(n-1) + ... + \lambda_{\mu-2} n(n-1)...(n-\mu+2) + n(n-1)...(n-\mu+1), \tag{6}$$

where $\lambda_1, ..., \lambda_{\mu-2}$ are positive integers depend only on $\mu \in \mathbb{N}$ (in case $\mu > 2$) not on n, we may observe that for these $\lambda_1, ..., \lambda_{\mu-2}$,

$$\frac{D^{\mu}F(z)}{z} = F'(z) + \lambda_1 zF''(z) + ... + \lambda_{\mu-2} z^{\mu-2} F^{(\mu-1)}(z) + z^{\mu-1} F^{(\mu)}(z).$$

Thus, the class condition (5) of the class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ is associated with μth order differential subordination and is also given by

$$F'(z) + \alpha\lambda_1 zF''(z) + ... + \alpha\lambda_{\mu-2} z^{\mu-2} F^{(\mu-1)}(z) + \alpha z^{\mu-1} F^{(\mu)}(z) \prec \frac{1+(1-2\beta)z}{1-z} \tag{7}$$

$$(\mu \in \mathbb{N}, \alpha \geq 0, 0 \leq \beta < 1, z \in \mathbb{U}).$$

The solution of this μth order differential subordination (7) is given by

$$F'(z) \prec \frac{1+(1-2\beta)z}{1-z} \ (0 \leq \beta < 1; z \in \mathbb{U})$$

which implies that $F \in \mathcal{P}(\beta)$. Thus $\mathcal{P}_H^{\mu}(\alpha, \beta)$ a class of function $f = h + \bar{g}$, which are the harmonic shear of F, of bounded turning in $\mathbb{U}$. Further, functions in the class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ satisfy

$$\Re e\left\{(1-\alpha)F'(z) + \alpha\frac{D^{\mu}F(z)}{z}\right\} > \beta \ (z \in \mathbb{U}). \tag{8}$$

Note that the class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ with the class condition (8) was earlier studied for some special values of μ as follows:

(i) $\mathcal{P}_H^{1}(\alpha, \beta) = \mathcal{HP}(\beta)$ studied by Yalcin et al. [27] ([2]).

(ii) $\mathcal{P}_H^{2}(\alpha, 0)) = HP(\alpha)$ studied by Yalcin and Oztork [28].

(iii) $\mathcal{P}_H^{2}(\alpha, \beta) = \mathcal{HP}(\alpha, \beta)$ investigated by Chandrashekhar et al. [5].

(iv) $\mathcal{P}_H^{3}(\alpha, \beta) = \mathcal{H}(\alpha, \beta)$ studied by Sokół et al. [19].

A subclass of $\mathcal{P}_H^{\mu}(\alpha, \beta)$ is denoted by $\mathcal{NP}_H^{\mu}(\alpha, \beta)$ when the functions $f = h + \overline{g} \in S_{\mathcal{H}}^0$ and h and g are of the form:

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \qquad \text{and} \qquad g(z) = -\sum_{n=2}^{\infty} |b_n| z^n. \tag{9}$$

In this paper, a class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ of functions $f = h + \overline{g}$ which are the harmonic shears of analytic functions $h + g$ is defined in terms of the μth order differential subordination. A sufficient coefficient condition for functions $f = h + \overline{g} \in \mathcal{P}_H^{\mu}(\alpha, \beta)$ is obtained. It is proved that this coefficient condition is necessary for functions belonging to its subclass $\mathcal{NP}_H^{\mu}(\alpha, \beta)$. Using the coefficient condition, results on bounds, extreme points, convolution, convex combinations and integral operator for functions belonging to the subclass $\mathcal{NP}_H^{\mu}(\alpha, \beta)$ are obtained. Inequalities for certain hypergeometric harmonic functions belonging to these classes are also obtained and one special case when $\mu = 4$ is included as an extension of the previous work proved by AL-Khal and AL-Kharsani [2], Chandrashekar et al. [5] and Sokół et al. [19], where $\mu = 1, 2$ and 3, respectively, is taken, for more detail see also ([8, 11, 18, 21–24, 26]).

2. Main Results

Theorem 2.1. *Let $f = h + \bar{g} \in \mathcal{H}$, where h and g are of the form (1). If*

$$\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_n| + |b_n|) \leq 1-\beta, \tag{10}$$

where $\mu \in \mathbb{N}, 0 \leq \alpha \leq 1, 0 \leq \beta < 1$, then $f \in \mathcal{P}_H^{\mu}(\alpha,\beta)$. The result is sharp.

Proof. To prove $f = h + \bar{g} \in \mathcal{P}_H^{\mu}(\alpha,\beta)$, we first show that the function f is univalent and sense-preserving in $\mathbb{U}$. For this, suppose that $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$, then on using the series expansions (1) and the condition (10), we get

$$\begin{aligned}
&\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{n=2}^{\infty} b_n(z_1^n - z_2^n)}{(z_1-z_2) - \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right| \\
&> 1 - \frac{\sum_{n=2}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|a_n|} \geq 1 - \frac{\sum_{n=2}^{\infty} \frac{\{(1-\alpha)n+\alpha n^{\mu}\}}{1-\beta}|b_n|}{1-\sum_{n=2}^{\infty} \frac{\{(1-\alpha)n+\alpha n^{\mu}\}}{1-\beta}|a_n|} \geq 0,
\end{aligned}$$

since,$\{(1-\alpha)n + \alpha n^{\mu}\} \geq n,\ n \geq 2$ and $0 < 1-\beta \leq 1$. Hence, $\left|f(z_1) - f(z_2)\right| > 0$ and so f is univalent in $\mathbb{U}$. Also, we have

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{n=2}^{\infty} n\,|a_n|\,|z|^{n-1} > 1 - \sum_{n=2}^{\infty} n\,|a_n| \\
&\geq 1 - \sum_{n=2}^{\infty} \frac{\{(1-\alpha)n + \alpha n^{\mu}\}}{1-\beta}\,|a_n| \\
&\geq \sum_{n=2}^{\infty} \frac{\{(1-\alpha)n + \alpha n^{\mu}\}}{1-\beta}\,|b_n| > \sum_{n=2}^{\infty} n|b_n|\,|z|^{n-1} = |g'(z)|
\end{aligned}$$

which proves that f is sense preserving in $\mathbb{U}$. Now, to prove $f \in \mathcal{P}_H^{\mu}(\alpha,\beta)$, from the subordination class condition (5), we need to show

$$S_1 := \left|\frac{P(z)-1}{P(z)+1-2\beta}\right| < 1, \tag{11}$$

where

$$\begin{aligned}
P(z) &= (1-\alpha)\,F'(z) + \alpha \frac{D^{\mu}F(z)}{z} \\
&= 1 + \sum_{n=2}^{\infty} \{n(1-\alpha) + n^{\mu}\alpha\}\,(a_n + b_n)\,z^{n-1}.
\end{aligned}$$

Observe that if $a_n + b_n = 0$ $(n \geq 2)$, then $P(z) = 1$ $(z \in \mathbb{U})$ which proves (11), and if there is some $a_n + b_n \neq 0$

$(n \geq 2)$, then from (10) it follows that

$$S_1 = \left|\frac{P(z)-1}{P(z)+1-2\beta}\right|$$
$$< \frac{\sum\limits_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}|a_n+b_n|}{2(1-\beta)-\sum\limits_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}|a_n+b_n|}$$
$$\leq \frac{\sum\limits_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)}{2(1-\beta)-\sum\limits_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)} \leq 1,$$

since the denominator is non-zero as

$$\sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}\frac{1}{2(1-\beta)}(|a_n|+|b_n|)$$
$$< \sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}\frac{1}{1-\beta}(|a_n|+|b_n|) \leq 1.$$

Sharpness can be verified for the harmonic function

$$f_1(z) = z + \sum_{n=2}^{\infty}\frac{1-\beta}{(1-\alpha)n+\alpha n^{\mu}}x_n z^n$$
$$+ \sum_{n=2}^{\infty}\frac{1-\beta}{(1-\alpha)n+\alpha n^{\mu}}\overline{y_n z}^n \quad (z \in \mathbb{U}),$$

where $\mu \in \mathbb{N}, 0 \leq \alpha \leq 1, 0 \leq \beta < 1$ and $\sum_{n=2}^{\infty}(|x_n|+|y_n|) = 1$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f = h + \bar{g}$ be such that h and g are given by (9). Then $f \in \mathcal{NP}_H^{\mu}(\alpha,\beta)$ if and only if the condition (10) holds. The result is sharp.*

Proof. Since $\mathcal{NP}_H^{\mu}(\alpha,\beta) \subseteq \mathcal{P}_H^{\mu}(\alpha,\beta)$, the *if* part is already proved in Theorem 2.1. We only need to prove the *only if* part of this theorem for functions $f = h + \bar{g} \in \mathcal{NP}_H^{\mu}(\alpha,\beta)$, where h and g are of the form (9). From the subordination class condition (5), we have

$$\left|\frac{P(z)-1}{P(z)+1-2\beta}\right| < 1 \quad (z \in \mathbb{U}),$$

where

$$P(z) = (1-\alpha)F'(z) + \alpha\frac{D^{\mu}F(z)}{z}$$
$$= 1 - \sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)z^{n-1}.$$

Since,

$$-\Re e\left(\frac{P(z)-1}{P(z)+1-2\beta}\right) \leq \left|\frac{P(z)-1}{P(z)+1-2\beta}\right| < 1,$$

we get

$$\frac{\sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)}{2(1-\beta)-\sum\limits_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)}\leq 1$$

as $z\to 1^-$ along real axis, which gives the condition (10). Sharpness can be seen for the function

$$\begin{aligned}f_2(z)&=z-\sum_{n=2}^{\infty}\frac{1-\beta}{(1-\alpha)n+\alpha n^{\mu}}|x_n|z^n\\&\quad-\sum_{n=2}^{\infty}\frac{1-\beta}{(1-\alpha)n+\alpha n^{\mu}}\left|y_n\right|\overline{z}^n\quad(z\in\mathbb{U}),\end{aligned}$$

where $\mu\in\mathbb{N}, 0\leq\alpha\leq 1, 0\leq\beta<1$ and $\sum_{n=2}^{\infty}(|x_n|+|y_n|)=1$. This proves Theorem 2.2. □

Theorem 2.3. *Let $f=h+\bar{g}\in\mathcal{NP}_H^{\mu}(\alpha,\beta)$, where h and g are of the form (9). Then for $|z|=r<1$,*

$$r-\frac{1-\beta}{2(1-\alpha)+2^{\mu}\alpha}r^2\leq\left|f(z)\right|\leq r+\frac{1-\beta}{2(1-\alpha)+2^{\mu}\alpha}r^2.$$

The result is sharp.

Proof. Let $f=h+\bar{g}\in\mathcal{NP}_H^{\mu}(\alpha,\beta)$, where h and g are of the form (9). Then from Theorem 2.2, we have coefficient condition (10). Taking the absolute value of f, we obtain for $|z|=r<1$,

$$\begin{aligned}\left|f(z)\right|&\leq r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n\\&\leq r+r^2\frac{1}{2(1-\alpha)+2^{\mu}\alpha}\sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)\\&\leq r+\frac{1-\beta}{2(1-\alpha)+2^{\mu}\alpha}r^2\end{aligned}$$

and similarly,

$$\begin{aligned}\left|f(z)\right|&\geq r-\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n\\&\geq r-r^2\frac{1}{2(1-\alpha)+2^{\mu}\alpha}\sum_{n=2}^{\infty}\{(1-\alpha)n+\alpha n^{\mu}\}(|a_n|+|b_n|)\\&\geq r-\frac{1-\beta}{2(1-\alpha)+2^{\mu}\alpha}r^2.\end{aligned}$$

Sharpness can be verified for the harmonic function given by

$$f(z)=z-\frac{1-\beta}{2(1-\alpha)+2^{\mu}\alpha}\overline{z}^2\quad(z\in\mathbb{U}),$$

where $\mu\in\mathbb{N}, 0\leq\alpha\leq 1, 0\leq\beta<1$. This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let $f = h + \bar{g}$ be such that h and g are given by (9). Then the harmonic function $f \in co\overline{\mathcal{NP}_H^{\mu}(\alpha,\beta)}$ if and only if it is given by*

$$f(z) = \sum_{n=1}^{\infty}(x_n h_n(z) + y_n g_n(z)), \tag{12}$$

where

$$h_1(z) = z,\ h_n(z) = z - \frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} z^n \quad (n = 2,3,...),$$

$$g_1(z) = z,\ g_n(z) = z - \frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} \overline{z}^n \quad (n = 2,3,...),$$

$$\sum_{n=1}^{\infty}(x_n + y_n) = 1, \quad x_n, y_n \geq 0 \ \ (n = 1,2,3,...)\,. \tag{13}$$

In particular, $\{h_n\}$ and $\{g_n\}$ are the extreme points of closed convex hull of $\mathcal{NP}_H^{\mu}(\alpha,\beta)$ denoted by $co\overline{\mathcal{NP}_H^{\mu}(\alpha,\beta)}$.

Proof. Let a function f be given by (12). Then by (13) it is of the form

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} x_n z^n - \sum_{n=2}^{\infty} \frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} y_n \overline{z}^n \tag{14}$$

which satisfy

$$\sum_{n=2}^{\infty}\left(\frac{(1-\alpha)n + \alpha n^{\mu}}{1-\beta}\frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} x_n + \frac{(1-\alpha)n + \alpha n^{\mu}}{1-\beta}\frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} y_n\right) = \sum_{n=2}^{\infty}(x_n + y_n) = 1 - x_1 \leq 1,$$

the coefficient inequality for the function (14) and hence, by Theorem 2.2, $f \in co\overline{\mathcal{NP}_H^{\mu}(\alpha,\beta)}$. Conversely, let $f = h + \bar{g} \in co\overline{\mathcal{NP}_H^{\mu}(\alpha,\beta)}$ be such that h and g are given by (9). Define

$$x_n = \frac{(1-\alpha)n + \alpha n^{\mu}}{1-\beta}|a_n| \quad (n = 2,3,...),$$

$$y_n = \frac{(1-\alpha)n + \alpha n^{\mu}}{1-\beta}|b_n| \quad (n = 2,3,...)$$

and

$$x_1 + y_1 = 1 - \sum_{n=2}^{\infty}(x_n + y_n)\,.$$

Then the function

$$f(z) = z - \sum_{n=2}^{\infty}|a_n| z^n - \sum_{n=2}^{\infty}|b_n| \overline{z}^n$$

reduces to the form (14) and by (13) it is of the form of (12). This proves Theorem 2.4. □

Theorem 2.5. *The class $\mathcal{NP}^{\mu}_{H}(\alpha,\beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, ...,$ let $f_i \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$, where

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i.n}| z^n - \sum_{n=2}^{\infty} |b_{i.n}| \overline{z}^n.$$

Then by Theorem 2.2, we have

$$\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_{i.n}| + |b_{i.n}|) \leq 1 - \beta.$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, let f be the convex combination of f_i $(i = 1, 2, ...)$. Then

$$f(z) = \sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i.n}| \right) z^n - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i.n}| \right) \overline{z}^n$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} \sum_{i=1}^{\infty} t_i \left(|a_{i.n}| + |b_{i.n}|\right) \\
&= \sum_{i=1}^{\infty} t_i \left[\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} \left(|a_{i.n}| + |b_{i.n}|\right) \right] \\
&\leq \sum_{i=1}^{\infty} t_i (1-\beta) = 1 - \beta
\end{aligned}$$

which again by Theorem 2.2 proves that $f = \sum_{i=1}^{\infty} t_i f_i \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$. This proves Theorem 2.5. □

The generalised Bernadi-Libra-Livingston integral operator $\mathcal{I}_c$ for $f = h + \overline{g}$ is defined by

$$\mathcal{I}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) \mathrm{d}t + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t) \mathrm{d}t}, \quad c > -1, z \in \mathbb{U}. \tag{15}$$

Theorem 2.6. *Let $f \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$. $\mathcal{I}_c(f) \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$.*

Proof. Let $f = h + \overline{g} \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$, where h and g are of the form (9). Then, we have

$$\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_n| + |b_n|) \leq 1 - \beta$$

and from (15),

$$\mathcal{I}_c(f) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} |a_n| z^n - \sum_{n=2}^{\infty} \frac{c+1}{c+n} |b_n| \overline{z}^n.$$

Since,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} \frac{c+1}{c+n} (|a_n| + |b_n|) \\
&\leq \sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_n| + |b_n|) \leq 1 - \beta.
\end{aligned}$$

It proves by Theorem 2.2, that $\mathcal{I}_c(f) \in \mathcal{NP}^{\mu}_{H}(\alpha,\beta)$. □

Theorem 2.7. *Let $f \in \mathcal{NP}_H^{\mu}(\alpha,\beta)$ and $F \in \mathcal{NP}_H^{\mu}(\alpha,\beta)$. Then $f * F \in \mathcal{P}_H^{\mu}(\alpha,\beta)$.*

Proof. Let $f = h + \overline{g}$ and $F = H + \overline{G}$ be the harmonic shears of analytic functions $h + g$ and $H + G$, respectively, where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ g(z) = -\sum_{n=2}^{\infty} |b_n| z^n$$

and

$$H(z) = z - \sum_{n=2}^{\infty} |A_n| z^n,\ G(z) = -\sum_{n=2}^{\infty} |B_n| z^n.$$

Then

$$(f * F)(z) = z + \sum_{n=2}^{\infty} |a_n A_n| z^n + \sum_{n=2}^{\infty} |b_n B_n| \overline{z}^n$$

and by Theorem 2.2, we have

$$|A_n| \leq \frac{1-\beta}{n(1-\alpha) + n^{\mu}\alpha} \leq 1,\ n \geq 2$$

and

$$|B_n| \leq \frac{1-\beta}{(1-\alpha)n + \alpha n^{\mu}} \leq 1,\ n \geq 2.$$

Thus, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_n A_n| + |b_n B_n|) \\
&\leq \sum_{n=2}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} (|a_n| + |b_n|) \\
&\leq 1 - \beta,
\end{aligned}$$

by Theorem 2.2, since $f \in \mathcal{NP}_H^{\mu}(\alpha,\beta)$. This proves by Theorem 2.1, that $f * F \in \mathcal{P}_H^{\mu}(\alpha,\beta)$. □

Remark 2.8. *Taking $\mu = 3$ in Theorems 2.1-2.6, we get similar results obtained by Sokół et al. in [19] for functions $f \in S_{\mathcal{H}}^0$. The results in [19] includes the results of Yalcin et al. [27], Ahuja and Jhangiri [1] when $\alpha = 0$ (or $\mu = 1$).*

3. Hypergeometric harmonic functions

For any $\mu \in \mathbb{N}$ and (in case $\mu > 1$) for some positive integers $\lambda_1, ..., \lambda_{\mu-2}$ depend only on μ, the result (6) may also be given by

$$\begin{aligned}
n^{\mu-1} &= 1 + \lambda_1(n-1) + ... + \lambda_{\mu-2}(n-1)...(n-\mu+2) \\
&\quad + (n-1)...(n-\mu+1) \\
&= \sum_{r=0}^{\mu-1} \lambda_r \frac{(1)_{n-1}}{(1)_{n-r-1}} \quad \left(\lambda_0 = 1 = \lambda_{\mu-1}\right).
\end{aligned} \tag{16}$$

Thus on replacing $\mu-1$ by $\nu\in\mathbb{N}_0=:\mathbb{N}\cup\{0\}$, we get the expansion:

$$\begin{aligned} n^{\nu} &= 1+M_1(n-1)+M_2(n-1)(n-2)+... \\ &\quad +M_{\nu-1}(n-1)...(n-\nu+1)+(n-1)...(n-\nu) \\ &= \sum_{r=0}^{\nu} M_r\frac{(1)_{n-1}}{(1)_{n-r-1}} \quad (M_0=1=M_\nu), \end{aligned} \tag{17}$$

where M_r are positive integers depend on ν.

Remark 3.1. *When we give a fixed value to $\mu\in\mathbb{N}$ (or to $\nu\in\mathbb{N}_0$) in the expansion (16) (or (17)), we may find certain specific values of λ_r (or M_r). In particular we have*

(i) $n^2=1+3(n-1)+(n-1)(n-2)$.

(ii) $n^3=1+7(n-1)+6(n-1)(n-2)+(n-1)(n-2)(n-3)$.

(iii) $n^4=1+15(n-1)+25(n-1)(n-2)+10(n-1)(n-2)(n-3)$
$+(n-1)(n-2)(n-3)(n-4)$.

The Gauss hypergeometric function [10] $F(a,b;c;z)$ is defined by

$$F(a,b;c;z)\ \equiv\ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,\quad z\in\mathbb{U},$$

where a,b,c are complex numbers with $c\neq 0,-.1,-2,...$ and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & \text{if } n=0\\ \lambda(\lambda+1)\dots(\lambda+n-1), & \text{if } n\geq 1.\end{cases}$$

For $k=0,1,2,...$, we have the formula:

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{(a+k)_{n-k}(b+k)_{n-k}}{(c+k)_{n-k}(1)_{n-k}} &= F(a+k,b+k;c+k;1) \\ &= \frac{(c)_k}{(c-a-b-k)_k}F(a,b;c;1),\end{aligned} \tag{18}$$

where $\Re e\,(c-a-b)>k$.

Lemma 3.2. *Let $a,b,c>0$. If $c-a-b>\nu$ for some $\nu\in\mathbb{N}_0$, then*

$$\sum_{n=1}^{\infty}n^{\nu}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}=\sum_{r=0}^{\nu}M_r\frac{(a)_r(b)_r}{(c-a-b-r)_r}F(a,b;c;1) \tag{19}$$

where M_r are some positive integers with $M_0=1=M_\nu$.

Proof. In view of (17), for any $\nu\in\mathbb{N}_0$, we get

$$\begin{aligned}\sum_{n=1}^{\infty}n^{\nu}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} &= \sum_{r=0}^{\nu}M_r\sum_{n=r+1}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-r-1}} \\ &= \sum_{r=0}^{\nu}M_r\frac{(a)_r(b)_r}{(c)_r}\sum_{n=r+1}^{\infty}\frac{(a+r)_{n-r-1}(b+r)_{n-r-1}}{(c+r)_{n-r-1}(1)_{n-r-1}} \\ &= \sum_{r=0}^{\nu}M_r\frac{(a)_r(b)_r}{(c-a-b-r)_r}F(a,b;c;1)\end{aligned}$$

by formula (18), where M_r are positive integers with $M_0=1=M_\nu$.This proves the result (19). □

In particular, from (19), we get following result:

Corollary 3.3. *Let $a, b, c > 0$. If $c - a - b > 1$, then*

$$\sum_{n=1}^{\infty} n \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left(1 + \frac{ab}{c-a-b-1}\right) F(a, b; c; 1). \tag{20}$$

Let $F_1(z) = H_1(z) + \overline{G_1(z)}$, where

$$H_1(z) = z\, F(\alpha_1, \beta_1, \gamma_1; z), \quad G_1(z) = z\,(F(\alpha_2, \beta_2, \gamma_2; z) - 1) \tag{21}$$

$(\alpha_i, \beta_i, \gamma_i \in \mathbb{C}, \Re(\gamma_i) > 0, i = 1, 2)$.

Theorem 3.4. *Let $\mu \in \mathbb{N}, 0 \leq \alpha \leq 1, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, with $\Re(\gamma_i) - |\alpha_i| - |\beta_i| > \mu, \; i = 1, 2$. Then a sufficient condition for the harmonic function $F_1 = H_1 + \overline{G_1}$, where H_1 and G_1 are given by (21), to be in the class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ is that*

$$\begin{aligned}
&\left\{1 + \frac{(1-\alpha)|\alpha_1||\beta_1|}{\Re(\gamma_1) - |\alpha_1| - |\beta_1| - 1} + \alpha \sum_{r=1}^{\mu} M_r \frac{(|\alpha_1|)_r(|\beta_1|)_r}{(\Re(\gamma_1) - |\alpha_1| - |\beta_1| - r)_r}\right\} \\
&\quad F\left(|\alpha_1|, |\beta_1|; \Re e(\gamma_1); 1\right) \\
&+ \left\{1 + \frac{(1-\alpha)|\alpha_2||\beta_2|}{\Re(\gamma_2) - |\alpha_2| - |\beta_2| - 1} + \alpha \sum_{r=1}^{\mu} M_r \frac{(|\alpha_2|)_r(|\beta_2|)_r}{\left(\Re(\gamma_2) - |\alpha_2| - |\beta_2| - r\right)_r}\right\} \\
&\quad F\left(|\alpha_2|, |\beta_2|; \Re e(\gamma_2); 1\right) \\
&\leq 3 - \beta,
\end{aligned} \tag{22}$$

where M_r are some positive integers with $M_\mu = 1$.

Proof. We have

$$F_1(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}} z^n + \overline{\sum_{n=2}^{\infty} \frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}} z^n} \quad (z \in \mathbb{U}).$$

To show $F_1 \in \mathcal{P}_H^{\mu}(\alpha, \beta)$, in view of Theorem 2.1, we need to show that

$$\begin{aligned}
P_1 \quad : \quad &= \sum_{n=1}^{\infty} \{(1-\alpha)n + \alpha n^{\mu}\} \left(\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\right| + \left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\right|\right) \\
&\leq \quad 3 - \beta.
\end{aligned} \tag{23}$$

But we have

$$P_1 \leq \left\{(1-\alpha)\sum_{n=1}^{\infty} n + \alpha \sum_{n=1}^{\infty} n^{\mu}\right\} \left(\frac{(|\alpha_1|)_{n-1}(|\beta_1|)_{n-1}}{(\Re(\gamma_1))_{n-1}(1)_{n-1}} + \frac{(|\alpha_2|)_{n-1}(|\beta_2|)_{n-1}}{(\Re(\gamma_2))_{n-1}(1)_{n-1}}\right)$$

or,

$$\begin{aligned}
P_1 \leq\; &(1-\alpha)\sum_{n=1}^{\infty} n \frac{(|\alpha_1|)_{n-1}(|\beta_1|)_{n-1}}{(\Re(\gamma_1))_{n-1}(1)_{n-1}} + \alpha \sum_{n=1}^{\infty} n^{\mu} \frac{(|\alpha_1|)_{n-1}(|\beta_1|)_{n-1}}{(\Re(\gamma_1))_{n-1}(1)_{n-1}} \\
&+ (1-\alpha)\sum_{n=1}^{\infty} n \frac{(|\alpha_2|)_{n-1}(|\beta_2|)_{n-1}}{(\Re(\gamma_2))_{n-1}(1)_{n-1}} + \alpha \sum_{n=1}^{\infty} n^{\mu} \frac{(|\alpha_2|)_{n-1}(|\beta_2|)_{n-1}}{(\Re(\gamma_2))_{n-1}(1)_{n-1}}
\end{aligned}$$

which on applying results (20) and (19) and then on simplifying, yields that

$$
\begin{aligned}
P_1 \leq & \left\{1+\frac{(1-\alpha)\left|\alpha_1\right|\left|\beta_1\right|}{\Re\left(\gamma_1\right)-\left|\alpha_1\right|-\left|\beta_1\right|-1}+\alpha \sum_{r=1}^{\mu} M_r \frac{\left(\left|\alpha_1\right|\right)_r\left(\left|\beta_1\right|\right)_r}{\left(\Re\left(\gamma_1\right)-\left|\alpha_1\right|-\left|\beta_1\right|-r\right)_r}\right\} F\left(\left|\alpha_1\right|,\left|\beta_1\right| ; \Re e\left(\gamma_1\right) ; 1\right) \\
& \left\{1+\frac{(1-\alpha)\left|\alpha_2\right|\left|\beta_2\right|}{\Re\left(\gamma_2\right)-\left|\alpha_2\right|-\left|\beta_2\right|-1}+\alpha \sum_{r=1}^{\mu} M_r \frac{\left(\left|\alpha_2\right|\right)_r\left(\left|\beta_2\right|\right)_r}{\left(\Re\left(\gamma_2\right)-\left|\alpha_2\right|-\left|\beta_2\right|-r\right)_r}\right\} F\left(\left|\alpha_2\right|,\left|\beta_2\right| ; \Re e\left(\gamma_2\right) ; 1\right) \\
& \leq 3-\beta
\end{aligned}
$$

if the inequality (22) holds. Thus the inequality (22) implies (23). Now to show that $F_1 \in S_{\mathcal{H}}$, we show first that $|\omega_{F_1}| < 1$ in $\mathbb{U}$ or $\left|H_1'(z)\right| > \left|G_1'(z)\right|$, $z \in \mathbb{U}$. We have

$$
\begin{aligned}
\left|H_1'(z)\right| & = \left|1+\sum_{n=2}^{\infty} n \frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}} z^{n-1}\right| \\
& > 1-\sum_{n=2}^{\infty} n\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\right| \\
& \geq 1-\beta \\
& -\sum_{n=2}^{\infty}\left\{(1-\alpha) n+\alpha n^{\mu}\right\}\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\right| \\
& \geq \sum_{n=2}^{\infty}\left\{(1-\alpha) n+\alpha n^{\mu}\right\}\left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\right| \quad \text{(by (23))} \\
& \geq \sum_{n=2}^{\infty} n\left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\right| \\
& > \left|\sum_{n=2}^{\infty} n \frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}} z^{n-1}\right| = \left|G_1'(z)\right| .
\end{aligned}
$$

Since, $n(1-\alpha)+n^{\mu}\alpha \geq n$, $n \geq 2$ and $0 \leq \beta < 1$. Now we show that F_1 is univalent in $\mathbb{U}$. For this we consider any two points $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$. Then we obtain

$$
\begin{aligned}
& \left|F_1\left(z_1\right)-F_1\left(z_2\right)\right| \\
& \geq \left|H_1\left(z_1\right)-H_1\left(z_2\right)\right|-\left|G_1\left(z_1\right)-G_1\left(z_2\right)\right| \\
& = \left|z_1-z_2+\sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\left(z_1^n-z_2^n\right)\right| \\
& -\left|\sum_{n=2}^{\infty} \frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\left(z_1^n-z_2^n\right)\right| \\
& > \left|z_1-z_2\right|\left[1-\sum_{n=2}^{\infty} n\left(\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\right|+\left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\right|\right)\right] \\
& \geq \left|z_1-z_2\right|\left[1-\beta\right. \\
& \left.-\sum_{n=2}^{\infty}\left\{(1-\alpha) n+\alpha n^{\mu}\right\}\left(\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_{n-1}}\right|+\left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_{n-1}}\right|\right)\right] \\
& \geq 0,
\end{aligned}
$$

if (23) holds, which shows that F_1 is univalent in $\mathbb{U}$. This proves Theorem 3.4. □

Similar to the proof of Theorem 3.4, as an application of Theorem 2.2, we get following result for the function F_2 defined by

$$F_2(z) = z\left(2 - \frac{H_1(z)}{z}\right) - \overline{G_1(z)}, \tag{24}$$

where H_1 and G_1 are given by (21).

Theorem 3.5. *Let $\mu \in \mathbb{N}, \alpha_i, \beta_i, \gamma_i > 0$, with $\gamma_i - \alpha_i - \beta_i > \mu$ for $i = 1, 2$. Then the function F_2 defined by (24) belongs to the class $N\mathcal{P}_H^{\mu}(\alpha, \beta)$ if and only if*

$$\begin{aligned}&\left\{1 + \frac{(1-\alpha)\alpha_1\beta_1}{\gamma_1 - \alpha_1 - \beta_1 - 1} + \alpha \sum_{r=1}^{\mu} M_r \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1 - \alpha_1 - \beta_1 - r)_r}\right\} \\ &\quad F(\alpha_1, \beta_1; \gamma_1; 1) \\ &+ \left\{1 + \frac{(1-\alpha)\alpha_2\beta_2}{\gamma_2 - \alpha_2 - \beta_2 - 1} + \alpha \sum_{r=1}^{\mu} M_r \frac{(\alpha_2)_r(\beta_2)_r}{(\gamma_2 - \alpha_2 - \beta_2 - r)_r}\right\} \\ &\quad F(\alpha_2, \beta_2; \gamma_2; 1) \\ &\le 3 - \beta\end{aligned}$$

holds, where M_r are some positive integers with $M_\mu = 1$.

Theorem 3.6. *Let $\mu \in \mathbb{N}$. If $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, with $\Re(\gamma_i) - |\alpha_i| - |\beta_i| > \mu - 1, \;\; i = 1, 2$, then a sufficient condition for the function F_3 defined by*

$$F_3(z) = \int_0^z F(\alpha_1, \beta_1; \gamma_1; t)\,\mathrm{d}t + \overline{\int_0^z [F(\alpha_2, \beta_2; \gamma_2; t) - 1]\,\mathrm{d}t} \tag{25}$$

to be in the class $\mathcal{P}_H^{\mu}(\alpha, \beta)$ is that

$$\begin{aligned}&\left\{1 + \alpha \sum_{r=1}^{\mu-1} \lambda_r \frac{(|\alpha_1|)_r(|\beta_1|)_r}{\left(\Re(\gamma_1) - |\alpha_1| - |\beta_1| - r\right)_r}\right\} F\left(|\alpha_1|, |\beta_1|; \Re e(\gamma_1); 1\right) \\ &+ \left\{1 + \alpha \sum_{r=1}^{\mu-1} \lambda_r \frac{(|\alpha_2|)_r(|\beta_2|)_r}{\left(\Re(\gamma_2) - |\alpha_2| - |\beta_2| - r\right)_r}\right\} F\left(|\alpha_2|, |\beta_2|; \Re e(\gamma_2); 1\right) \\ &\le 3 - \beta,\end{aligned} \tag{26}$$

where λ_r are some positive integers with $\lambda_{\mu-1} = 1$.

Proof. We have

$$F_3(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_n} z^n + \overline{\sum_{n=2}^{\infty} \frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_n} z^n}.$$

By Theorem 2.1, $F_3 \in \mathcal{P}_H^{\mu}(\alpha, \beta)$ if

$$\begin{aligned}P_2 := &\sum_{n=1}^{\infty} \{n(1-\alpha) + n^{\mu}\alpha\} \left(\left|\frac{(\alpha_1)_{n-1}(\beta_1)_{n-1}}{(\gamma_1)_{n-1}(1)_n}\right| + \left|\frac{(\alpha_2)_{n-1}(\beta_2)_{n-1}}{(\gamma_2)_{n-1}(1)_n}\right|\right) \\ &\le 3 - \beta,\end{aligned}$$

where

$$P_2 \le \sum_{n=1}^{\infty} \left\{1-\alpha+n^{\mu-1}\alpha\right\} \left(\frac{(|\alpha_1|)_{n-1}(|\beta_1|)_{n-1}}{(\Re(\gamma_1))_{n-1}(1)_{n-1}} + \frac{(|\alpha_2|)_{n-1}(|\beta_2|)_{n-1}}{(\Re(\gamma_2))_{n-1}(1)_{n-1}} \right) \tag{27}$$

which on applying result (19) for $\nu = 0$ and for $\nu = \mu - 1$, yields

$$\begin{aligned} P_2 \le & \left\{1-\alpha+\alpha\sum_{r=0}^{\mu-1}\lambda_r \frac{(|\alpha_1|)_r(|\beta_1|)_r}{\left(\Re(\gamma_1)-|\alpha_1|-|\beta_1|-r\right)_r}\right\} F\left(|\alpha_1|,|\beta_1|;\Re e(\gamma_1);1\right) \\ & + \left\{1-\alpha+\alpha\sum_{r=0}^{\mu-1}\lambda_r \frac{(|\alpha_2|)_r(|\beta_2|)_r}{\left(\Re(\gamma_2)-|\alpha_2|-|\beta_2|-r\right)_r}\right\} F\left(|\alpha_2|,|\beta_2|;\Re e(\gamma_2);1\right) \\ & \le 3-\beta \end{aligned}$$

if (26) holds. This proves Theorem 3.6. □

Similar to the proof of Theorem 3.6, as an application of Theorem 2.2, we get following result for the function F_4 defined by

$$F_4(z) = z\left(2-\frac{1}{z}\int_0^z F(\alpha_1,\beta_1;\gamma_1;t)\,dt\right) - \overline{\int_0^z \left[F(\alpha_2,\beta_2;\gamma_2;t)-1\right]dt}. \tag{28}$$

Theorem 3.7. *Let $\mu \in \mathbb{N}, \alpha_i, \beta_i, \gamma_i > 0$, with $\gamma_i - \alpha_i - \beta_i > \mu - 1$ for $i = 1, 2$. Then the function F_4 defined by (28) belongs to the class $N\mathcal{P}_H^{\mu}(\alpha,\beta)$ if and only if*

$$\begin{aligned} & \left\{1+\alpha\sum_{r=1}^{\mu-1}\lambda_r \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1-\alpha_1-\beta_1-r)_r}\right\} F(\alpha_1,\beta_1;\gamma_1;1) \\ & + \left\{1+\alpha\sum_{r=1}^{\mu-1}\lambda_r \frac{(\alpha_2)_r(\beta_2)_r}{(\gamma_2-\alpha_2-\beta_2-r)_r}\right\} F(\alpha_2,\beta_2;\gamma_2;1) \\ & \le 3-\beta \end{aligned}$$

where λ_r are some positive integers with $\lambda_{\mu-1} = 1$.

Remark 3.8. *Certain hypergeometric harmonic functions $f = h + \overline{g} \in S_{\mathcal{H}}$, where $F = h + g$ satisfy condition (8) when $\mu = 1, 2$ and 3 were studied in [2], [5] and [19], respectively.*

Here we give our results proved in Section 3 for $\mu = 4$ which in view of the Remark 3.1 are as follows:

Corollary 3.9. *Let $0 \le \alpha \le 1, \alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, with $\Re(\gamma_i) - |\alpha_i| - |\beta_i| > 4,\ \ i = 1, 2$. Then a sufficient condition for the harmonic function $F_1 = H_1 + \overline{G_1}$, where H_1 and G_1 are given by (21), to be in the class $\mathcal{P}_H^4(\alpha,\beta)$ is that*

$$\begin{aligned} & \left\{1+\frac{(1-\alpha)|\alpha_1||\beta_1|}{\Re(\gamma_1)-|\alpha_1|-|\beta_1|-1}+\alpha\sum_{r=1}^{4}M_r\,\frac{(|\alpha_1|)_r(|\beta_1|)_r}{(\Re(\gamma_1)-|\alpha_1|-|\beta_1|-r)_r}\right\} \\ & \quad F\left(|\alpha_1|,|\beta_1|;\Re e(\gamma_1);1\right) \\ & + \left\{1+\frac{(1-\alpha)|\alpha_2||\beta_2|}{\Re(\gamma_2)-|\alpha_2|-|\beta_2|-1}+\alpha\sum_{r=1}^{4}M_r\,\frac{(|\alpha_2|)_r(|\beta_2|)_r}{\left(\Re(\gamma_2)-|\alpha_2|-|\beta_2|-r\right)_r}\right\} \\ & \quad F\left(|\alpha_2|,|\beta_2|;\Re e(\gamma_2);1\right) \\ & \le 3-\beta, \end{aligned}$$

where $M_1 = 15, M_2 = 25, M_3 = 10, M_4 = 1$.

Corollary 3.10. *Let $\alpha_i, \beta_i, \gamma_i > 0$, with $\gamma_i - \alpha_i - \beta_i > 4$ for $i = 1, 2$. Then the function F_2 defined by (24) belongs to the class $N\mathcal{P}_H^4(\alpha, \beta)$ if and only if*

$$\left\{1 + \frac{(1-\alpha)\alpha_1\beta_1}{\gamma_1 - \alpha_1 - \beta_1 - 1} + \alpha \sum_{r=1}^{4} M_r \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1 - \alpha_1 - \beta_1 - r)_r}\right\} F(\alpha_1, \beta_1; \gamma_1; 1)$$
$$+ \left\{1 + \frac{(1-\alpha)\alpha_2\beta_2}{\gamma_2 - \alpha_2 - \beta_2 - 1} + \alpha \sum_{r=1}^{4} M_r \frac{(\alpha_2)_r(\beta_2)_r}{(\gamma_2 - \alpha_2 - \beta_2 - r)_r}\right\} F(\alpha_2, \beta_2; \gamma_2; 1)$$
$$\leq 3 - \beta,$$

where $M_1 = 15, M_2 = 25, M_3 = 10, M_4 = 1$.

Corollary 3.11. *If $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$, with $\Re(\gamma_i) - |\alpha_i| - |\beta_i| > 3,\ \ i = 1, 2$, then a sufficient condition for the function F_3 defined by (25) to be in the class $\mathcal{P}_H^4(\alpha, \beta)$ is that*

$$\left\{1 + \alpha \sum_{r=1}^{3} \lambda_r \frac{(|\alpha_1|)_r(|\beta_1|)_r}{\left(\Re(\gamma_1) - |\alpha_1| - |\beta_1| - r\right)_r}\right\} F\left(|\alpha_1|, |\beta_1|; \Re e(\gamma_1); 1\right)$$
$$+ \left\{1 + \alpha \sum_{r=1}^{3} \lambda_r \frac{(|\alpha_2|)_r(|\beta_2|)_r}{\left(\Re(\gamma_2) - |\alpha_2| - |\beta_2| - r\right)_r}\right\} F\left(|\alpha_2|, |\beta_2|; \Re e(\gamma_2); 1\right)$$
$$\leq 3 - \beta,$$

where $\lambda_1 = 7, \lambda_2 = 6, \lambda_3 = 1$.

Corollary 3.12. *Let $\alpha_i, \beta_i, \gamma_i > 0$, with $\gamma_i - \alpha_i - \beta_i > 3$ for $i = 1, 2$. Then the function F_4 defined by (28) belongs to the class $N\mathcal{P}_H^4(\alpha, \beta)$ if and only if*

$$\left\{1 + \alpha \sum_{r=1}^{3} \lambda_r \frac{(\alpha_1)_r(\beta_1)_r}{(\gamma_1 - \alpha_1 - \beta_1 - r)_r}\right\} F(\alpha_1, \beta_1; \gamma_1; 1)$$
$$+ \left\{1 + \alpha \sum_{r=1}^{3} \lambda_r \frac{(\alpha_2)_r(\beta_2)_r}{(\gamma_2 - \alpha_2 - \beta_2 - r)_r}\right\} F(\alpha_2, \beta_2; \gamma_2; 1)$$
$$\leq 3 - \beta$$

where $\lambda_1 = 7, \lambda_2 = 6, \lambda_3 = 1$.

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