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Offset linear canonical wavelet transforms

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Abstract. This study extends the one-dimensional offset linear canonical transform (OLCT) to n-dimensional OLCT and establishes a significant connection between the traditional Fourier transform and the OLCT. We present the Parseval identity for the OLCT in n-dimensions. Furthermore, the offset linear canonical wavelet (OLCW) with special rotation and the offset linear canonical wavelet transform (OLCWT) with special rotation are introduced. The OLCT of the OLCW is computed and, using it, the inner-product relation of the OLCWT and the corresponding reconstruction formula are obtained. A relationship between the OLCT of a function and OLCT of its derivative is derived, which is then applied to differential equations and an inequality of the uncertainty principle type.

1. Introduction

Wavelet transforms have emerged as powerful tools in both theoretical and applied analysis, with diverse applications in signal processing, differential equations, and mathematical physics. Among the various wavelet frameworks, the linear canonical wavelet transform (LCWT) has received significant attention. For instance, Wei *et al.* [34], Gupta *et al.* [13], and Srivastava *et al.* [29] have made substantial contributions to its theoretical development. Studies by Guo *et al.* [11], Prasad *et al.* [23], Rejini *et al.* [24] and Guo *et al.* [10] have explored the applications of LCWT in function spaces, watermarking, and matrix decomposition, while MRA structures have been examined in [12]. The quaternionic extension of LCWT was considered by Shah *et al.* [25].

Despite this progress, the offset linear canonical wavelet transform (OLCWT), an extension of the LCWT, has received comparatively little attention. Kaur *et al.* [16] initiated its study, but the OLCWT with special rotation, remained largely unexplored. The OLCT, also known as the Special Affine Fourier Transform was originally introduced by Abe and Sheridan [1, 2], the OLCT generalizes many classical non-windowed integral transforms. Pei and Dang studied its eigenfunctions [22], while Stern [32] and Xiang *et al.* [36, 37] developed sampling theorems. Wei *et al.* [33, 35] proposed new product and sampling formulas, and the uncertainty principle has been widely studied by Zhu *et al.* [39], Huo *et al.* [15], El Haoui *et al.* [7], and

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others. In addition, extensions of Pitt's inequality [3] and quaternionic frameworks [5, 6] demonstrate its theoretical richness. Various uncertainty principles for windowed OLCT are documented in [9, 14], while the extrapolation for signals associated with OLCT is studied by X. Shuiqing*et al.* [38].

Parallel to the OLCT and LCWT developments, there has been a growing interest in fractional and generalized wavelet transforms. Srivastava *et al.* [26] laid foundational work on fractional wavelet transforms, including Parseval's identity, inversion, and multiresolution analysis. Subsequent contributions of H. M. Srivastava with his collaborators include the Maxican hat wavelet transform [30], the fractional Bessel wavelet transform with weighted Sobolev estimates [27], and kernel-based canonical wavelet transforms inspired by quantum mechanics [31]. Analytical insights such as Abelian theorems [28] and practical techniques for signal denoising using diffusive wavelets [20] illustrate the growing relevance of wavelet theory in functional spaces and applications. Together, these developments motivate the current investigation into the OLCWT with special rotation and its mathematical properties on function spaces. Our goal is to bridge existing gaps by enriching the wavelet framework through the lens of generalized transforms like the OLCT and its wavelet analogues.

The aim of the paper is to address this research gap by defining OLCT in n-dimensions and introducing the Offset Linear Canonical Wavelet Transform (OLCWT) with a special class of rotation. The uncertainty principle for OLCT and OLCWT is derived, and their application in differential equations is discussed.

This paper is summarized in four sections. Section 1 lays the foundation for understanding OLCT and discusses the existing literature. Section 2 introduces a novel offset linear canonical wavelet with a special rotation based on Shinya Moritoh's wavelet and also the corresponding offset linear canonical wavelet transform with special rotation is introduced. The section concludes with the derivation of the inner-product relation and the reconstruction formula for this OLCWT, along with an exploration of the relationship between the OLCT of a function and its derivative. Section 3 presents the applications of OLCWT. A novel differential operator has been introduced and explored. Furthermore, its application in differential equations has been illustrated with an example. The uncertainty principles for OLCT and OLCWT have been obtained. Section 4 provides the conclusion of the paper.

The OLCT is characterized by six parameters: a, b, c, d, $\boldsymbol{v}$, and $\boldsymbol{\omega_0}$. In these six parameters $a, b, c, d \in \mathbb{R}$, while $\boldsymbol{v}, \boldsymbol{\omega_0} \in \mathbb{R}^n$. It serves as a generalization of various familiar non-windowed Fourier transform-type integral transforms. Before we explore the intricate connection between the OLCT and the classical Fourier transform, it's essential to introduce the fundamental definition of the OLCT in n-dimensions and gain insight into its signal processing capabilities.

Definition 1.1 (OLCT). *The offset linear canonical transform of a function f with $A \equiv (a, b, c, d, \boldsymbol{v}, \boldsymbol{\omega_0})$ is denoted by $\tilde{\mathscr{F}}^A[f](\boldsymbol{u})$ and defined by*

$$\tilde{\mathscr{F}}^A[f](\boldsymbol{u}) = \begin{cases} \int_{\mathbb{R}^n} f(\boldsymbol{t}) h_A(\boldsymbol{t}, \boldsymbol{u}) d^n\boldsymbol{t}, & b \neq 0 \\ \sqrt{d} e^{j(\frac{cd}{2})\|\boldsymbol{u}-\boldsymbol{v}\|^2 + j\langle \boldsymbol{\omega_0}, \boldsymbol{u}\rangle} f(d(\boldsymbol{u}-\boldsymbol{v})), & b = 0, \end{cases}$$

where

$$h_A(\boldsymbol{t}, \boldsymbol{u}) = \left(\frac{1}{j2\pi|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}[a\|\boldsymbol{t}\|^2 + 2\langle \boldsymbol{t}, \boldsymbol{v}-\boldsymbol{u}\rangle - 2\langle \boldsymbol{u}, d\boldsymbol{v} - b\boldsymbol{\omega_0}\rangle + d(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2)]} \tag{1}$$

and $ad - cb = 1$, provided the integral exists. Here, $\|\boldsymbol{t}\|^2 = t_1^2 + t_2^2 + \ldots + t_n^2$. To define the inverse OLCT of function f whose OLCT is well defined, the kernel is taken as $h_{A^{-1}}$ where $A^{-1} = (d, -b, -c, a, b\boldsymbol{\omega_0} - d\boldsymbol{v}, c\boldsymbol{v} - a\boldsymbol{\omega_0})$. Here, $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{\omega_0} \in \mathbb{R}^n$ while $a, b, c, d \in \mathbb{R}$ and $b \neq 0$.

Let us establish a fundamental relationship between the OLCT and the classical Fourier transform.

Theorem 1.2 (Relation Between Classical Fourier Transform and OLCT). *For $b \neq 0$, the OLCT of function f and the classical Fourier transform has the following relation:*

$$\tilde{\mathscr{F}}^A[f](\boldsymbol{u}) = \left(\frac{1}{j\,|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}(d(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2) - 2\langle \boldsymbol{u}, d\boldsymbol{v} - b\boldsymbol{\omega_0}\rangle)} \mathscr{F}[f_{a,b}]\left(\frac{\boldsymbol{u}}{b}\right),$$

where $f_{a,b}(\boldsymbol{t}) = e^{\frac{j}{2b}[a\|\boldsymbol{t}\|^2+2\langle \boldsymbol{t},\boldsymbol{v}\rangle]}f(\boldsymbol{t})$ *and* $\mathscr{F}[f]$ *denotes the classical Fourier transform of* f*.*

We present the Parseval identity for the OLCT in order to have a deeper understanding of its characteristics in sense of the energy preservation and inner product properties.

Theorem 1.3 (The Parseval's Identity for OLCT). *Let* f *and* g *be two functions such that their OLCTs, i.e.,* $\tilde{\mathscr{F}}^A[f]$ *and* $\tilde{\mathscr{F}}^A[g]$ *exist. Then*

$$\langle f, g\rangle = \left(\frac{|b|}{b}\right)^n \langle \tilde{\mathscr{F}}^A[f], \tilde{\mathscr{F}}^A[g]\rangle, \text{ for } b \neq 0. \tag{2}$$

Proof. By the definition of inner-product and $f_{a,b}(\boldsymbol{t}) = e^{\frac{j}{2b}[a\|\boldsymbol{t}\|^2+2\langle \boldsymbol{t},\boldsymbol{v}\rangle]}f(\boldsymbol{t})$, we have

$$\langle f, g\rangle = \langle f_{a,b}, g_{a,b}\rangle = \langle \mathscr{F}[f_{a,b}], \mathscr{F}[g_{a,b}]\rangle = \int_{\mathbb{R}^n} \mathscr{F}[f_{a,b}](\boldsymbol{\omega})\overline{\mathscr{F}[g_{a,b}]}(\boldsymbol{\omega})d^n\boldsymbol{\omega}.$$

Putting $\boldsymbol{\omega} = \frac{\boldsymbol{u}}{b}$, $b \neq 0$, we get

$$\langle f, g\rangle = \frac{1}{b^n}\int_{\mathbb{R}^n} \mathscr{F}[f_{a,b}]\left(\frac{\boldsymbol{u}}{b}\right)\overline{\mathscr{F}[g_{a,b}]}\left(\frac{\boldsymbol{u}}{b}\right)d^n\boldsymbol{u}.$$

Now, using Theorem 1.2, we get

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{b^n}\int_{\mathbb{R}^n} \frac{\tilde{\mathscr{F}}^A[f](\boldsymbol{u})}{(\frac{1}{j|b|})^{\frac{n}{2}}\, e^{j(d\|\boldsymbol{v}\|^2)/2b}e^{j(d|\boldsymbol{u}^2|-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle)/(2b)}}\left(\frac{\overline{\tilde{\mathscr{F}}^A[g](\boldsymbol{u})}}{(\frac{1}{j|b|})^{\frac{n}{2}}\, e^{-j(d\|\boldsymbol{v}\|^2)/2b}\, e^{-j(d\|\boldsymbol{u}\|^2-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle)/(2b)}}\right)d^n\boldsymbol{u}\\ &= \left(\frac{|b|}{b}\right)^n \langle \tilde{\mathscr{F}}^A[f](\boldsymbol{u}), \tilde{\mathscr{F}}^A[g](\boldsymbol{u})\rangle.\end{aligned}$$

□

This theorem states that for $b > 0$, the L^2-norm of a function is preserved under the OLCT, i.e., the L^2-norm of the function equals the L^2-norm of its OLCT. The Parseval identity for the OLCT offers crucial insights into energy preservation and orthogonality in the transformed domain. Now, we wish to establish the convolution theorem-type result for OLCT.

Using the relation between OLCT and Fourier transform, we have

$$\tilde{\mathscr{F}}^A[f * g](\boldsymbol{u}) = \left(\frac{1}{j|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}(d(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle)}\mathscr{F}[(f * g)(\boldsymbol{t})e^{\frac{j}{2b}[a\|\boldsymbol{t}\|^2+2\langle \boldsymbol{t},\boldsymbol{v}\rangle]}]\left(\frac{\boldsymbol{u}}{b}\right).$$

This motivated Wei *et al.* [35] to define a new convolution operator and its generalization in n dimensions is presented below.

Definition 1.4 (The Generalized Convolution). *Let* f *and* g *be functions defined on* $\mathbb{R}^n$*. Let* $f * g$ *denote the usual convolution of functions* f *and* g*. Then the generalized convolution of functions* f *and* g *is denoted by* $f *_A g$ *and defined as:*

$$(f *_A g)(\boldsymbol{t}) := e^{-\frac{j}{2b}[a\|\boldsymbol{t}\|^2+2\langle \boldsymbol{t},\boldsymbol{v}\rangle]}(f_{a,b} * g)(\boldsymbol{t}),$$

where $f_{a,b}(\boldsymbol{t}) = f(\boldsymbol{t})e^{\frac{j}{2b}[a\|\boldsymbol{t}\|^2+2\langle \boldsymbol{t},\boldsymbol{v}\rangle]}$.

Theorem 1.5 (OLCT of Generalized Convolution). *Let* f *and* g *be the functions in* $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*. Then, the OLCT of* $f *_A g$ *is given by*

$$\tilde{\mathscr{F}}^A[f *_A g](\boldsymbol{u}) = \tilde{\mathscr{F}}^A[f](\boldsymbol{u})\mathscr{F}[g]\left(\frac{\boldsymbol{u}}{b}\right).$$

Proof. Using the relation between OLCT and the Fourier transform with the definition of new convolution, we have

$$\tilde{\mathscr{F}}^A[f *_A g](\boldsymbol{u}) = \left(\frac{1}{j\,|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}(d(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle)} \mathscr{F}[(f_{a,b} * g)]\left(\frac{\boldsymbol{u}}{b}\right).$$

Using the convolution theorem for classical Fourier transform in the above equation, we get

$$\tilde{\mathscr{F}}^A[f *_A g](\boldsymbol{u}) = \left(\frac{1}{j\,|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}(d(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle)} \mathscr{F}[f_{a,b}]\left(\frac{\boldsymbol{u}}{b}\right)\mathscr{F}[g]\left(\frac{\boldsymbol{u}}{b}\right).$$

Hence, the proof. □

This theorem is a kind of convolution theorem for the generalized convolution. Furthermore, we wish to extend the concept of OLCT to wavelets.

2. Offset Linear Canonical Wavelets

The choice of chirp modulation in defining linear canonical wavelets (LCWs) is not arbitrary but rather motivated by a combination of mathematical considerations, such as maintaining basic properties, and applications, such as effective signal representation and time-frequency analysis. S. Moritoh [19] has defined a new wavelet transform with special rotation. The quaternionic Moritoh transform is explored by Kumar *et al.* [17] and the octonionic Moritoh transform is studied by Kumar *et al.* [18]. It's offset version is introduced here.

Definition 2.1 (Offset Linear Canonical Wavelet Family). *Let ψ be the wavelet in $L^2(\mathbb{R}^n)$, then the offset linear canonical wavelet family is denoted by $\Psi_{x,\xi,A}$ and defined by*

$$\Psi_{x,\xi,A} := \left\{\psi_{x,\xi,A}(\boldsymbol{t}) = |\boldsymbol{\xi}|^{\frac{n}{2}}\psi(|\boldsymbol{\xi}|r_{\xi}(\boldsymbol{t}-\boldsymbol{x}))e^{-\frac{ja}{2b}[\|\boldsymbol{t}\|^2-\|\boldsymbol{x}\|^2+2\langle \boldsymbol{v},\boldsymbol{t}-\boldsymbol{x}\rangle]} \mid \boldsymbol{t},\boldsymbol{x},\boldsymbol{\xi}\in\mathbb{R}^n \text{ and } A\equiv(a,b,c,d,\boldsymbol{v},\boldsymbol{\omega_0})\right\}. \quad (3)$$

Proposition 2.2. *The offset linear canonical transform of the wavelet $\psi_{x,\xi,A}(\boldsymbol{t})$ is given by*

$$\tilde{\mathscr{F}}^A[\psi_{x,\xi,A}](\boldsymbol{u}) = e^{\frac{j}{2b}[a\|\boldsymbol{x}\|^2+2\langle \boldsymbol{v}-\boldsymbol{u},\boldsymbol{x}\rangle+2\langle|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u}-\boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle+d\|\boldsymbol{u}\|^2(1-|\boldsymbol{\xi}|^{-2})]}|\boldsymbol{\xi}|^{-\frac{n}{2}}\tilde{\mathscr{F}}^A[\psi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u}),$$

where $b \neq 0$.

Proof. By definition of OLCT, we get

$$\tilde{\mathscr{F}}^A[\psi_{x,\xi,A}(\boldsymbol{t})](\boldsymbol{u}) = \left(\frac{|\boldsymbol{\xi}|}{j2\pi|b|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}[a\|\boldsymbol{x}\|^2-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle+2\langle \boldsymbol{v},\boldsymbol{x}\rangle+d(\|\boldsymbol{v}\|^2+\|\boldsymbol{u}\|^2)]}\int_{\mathbb{R}^n}\psi(|\boldsymbol{\xi}|r_{\xi}(\boldsymbol{t}-\boldsymbol{x}))\,e^{-j\langle \boldsymbol{t},\frac{\boldsymbol{u}}{b}\rangle}d^n\boldsymbol{t}.$$

Now, changing the variable $|\boldsymbol{\xi}|r_{\xi}(\boldsymbol{t}-\boldsymbol{x})$ by the variable $\boldsymbol{y}$, we get

$$\tilde{\mathscr{F}}^A[\psi_{x,\xi,A}(\boldsymbol{t})](\boldsymbol{u}) = \left(\frac{1}{j\,|b|\,|\boldsymbol{\xi}|}\right)^{\frac{n}{2}} e^{\frac{j}{2b}[a\|\boldsymbol{x}\|^2-2\langle \boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle+2\langle \boldsymbol{v}-\boldsymbol{u},\boldsymbol{x}\rangle+d(\|\boldsymbol{v}\|^2+\|\boldsymbol{u}\|^2)]}\mathscr{F}[\psi]\left(|\boldsymbol{\xi}|^{-1}r_{\xi}\left(\frac{\boldsymbol{u}}{b}\right)\right).$$

Now, adding and subtracting the necessary terms to make the OLCT kernel and then simplifying, we get

$$\tilde{\mathscr{F}}^A[\psi_{x,\xi,A}](\boldsymbol{u}) = e^{\frac{j}{2b}[a\|\boldsymbol{x}\|^2+2\langle \boldsymbol{v}-\boldsymbol{u},\boldsymbol{x}\rangle+2\langle(|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})-\boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle+d\|\boldsymbol{u}\|^2(1-|\boldsymbol{\xi}|^{-2})]}|\boldsymbol{\xi}|^{-\frac{n}{2}}\tilde{\mathscr{F}}^A[\psi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u}).$$

□

Definition 2.3 (Offset Linear Canonical Wavelet Transform). *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\psi_{x,\xi,A}$ be the offset linear canonical wavelet (OLCW) as defined in the equation* (3). *Then, the OLCWT is denoted by $\mathscr{W}_\psi^A f(x,\xi)$ and defined by*

$$\mathscr{W}_\psi^A f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{\psi_{x,\xi,A}(t)}d^n t, \tag{4}$$

provided the integral exists.

Proposition 2.4 (Representation of OLCWT in terms of OLCT). *For $b > 0$, the OLCWT can be written as :*

$$\mathscr{W}_\psi^A f(x,\xi) = \int_{\mathbb{R}^n} \tilde{\mathscr{F}}^A[f](u)|\xi|^{\frac{n}{2}}\, \overline{\chi_A(x,u,\xi)}\, \overline{\tilde{\mathscr{F}}^A[\psi](|\xi|^{-1}r_\xi u)}d^n u,$$

where $\chi_A(x,u,\xi) = e^{\frac{j}{2b}[a\|x\|^2+2\langle v-u,x\rangle+2\langle|\xi|^{-1}r_\xi u-u,dv-b\omega_0\rangle+d\|u\|^2(1-|\xi|^{-2})]}$.

Proof. The proof is an easy application of Parseval's identity for OLCT and Proposition 2.2. □

Theorem 2.5 (Inner Product Relation for OLCWT). *Let $f,g \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $\psi_{x,\xi,A}, \phi_{x,\xi,A}$ be offset linear canonical wavelets (OLCWs) as defined in equation* (3). *Let $\mathscr{W}_\psi^A f(x,\xi)$ and $\mathscr{W}_\psi^A g(x,\xi)$ be OLCWTs of the functions f and g, respectively. Then the inner product relation for the OLCWTs can be stated as*

$$\langle \mathscr{W}_\psi^A f, \mathscr{W}_\psi^A g\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n;\, d^n x\, d^n \xi)} = C_{\phi,\psi}^A \langle f,g\rangle_{L^2(\mathbb{R}^n; d^n x)},$$

where $C_{\phi,\psi}^A = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \overline{\tilde{\mathscr{F}}^A[\psi]}(|\xi|^{-1}r_\xi u)\tilde{\mathscr{F}}^A[\phi](|\xi|^{-1}r_\xi u)|\xi|^n d^n\xi$. *Furthermore, the signal $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ whose OLCWT exists can be reconstructed by the formula:*

$$f(t) = \frac{1}{C_\psi^A}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathscr{W}_\psi^A f(x,\xi)\psi_{x,\xi,A}(t)d^n x d^n\xi.$$

Proof. By Proposition 2.4, we have

$$\begin{aligned}
&\mathscr{W}_\psi^A f(x,\xi)\overline{\mathscr{W}_\psi^A g(x,\xi)} \\
&= \left(\int_{\mathbb{R}^n} \tilde{\mathscr{F}}^A[f](u)|\xi|^{\frac{n}{2}}\, \overline{\chi_A(x,u,\xi)}\, \overline{\tilde{\mathscr{F}}^A[\psi]}(|\xi|^{-1}r_\xi u)d^n u\right)\overline{\left(\int_{\mathbb{R}^n} \tilde{\mathscr{F}}^A[g](u')|\xi|^{\frac{n}{2}}\, \overline{\chi_A(x,u',\xi)}\, \overline{\tilde{\mathscr{F}}^A[\phi]}(|\xi|^{-1}r_\xi u')d^n u'\right)} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{\mathscr{F}}^A[f](u)\overline{\tilde{\mathscr{F}}^A[g](u')}e^{-\frac{j}{2b}[2\langle u'-u,x\rangle+2\langle(|\xi|^{-1}r_\xi(u'-u))+u'-u,dv-b\omega_0\rangle+d(\|u\|^2-\|u'\|^2)(1-|\xi|^{-2})]} \\
&\quad\times \overline{\tilde{\mathscr{F}}^A[\psi]}(|\xi|^{-1}r_\xi u)\tilde{\mathscr{F}}^A[\phi](|\xi|^{-1}r_\xi u')|\xi|^n d^n u\, d^n u'.
\end{aligned}$$

Integrating both sides with respect to $d^n x d^n \xi$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathscr{W}_\psi^A f(x,\xi)\overline{\mathscr{W}_\psi^A g(x,\xi)}d^n x d^n\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{j}{2b}[2\langle u'-u,x\rangle+2\langle(|\xi|^{-1}r_\xi(u'-u))+u'-u,dv-b\omega_0\rangle+d(\|u\|^2-\|u'\|^2)(1-|\xi|^{-2})]} \\
&\quad\times \tilde{\mathscr{F}}^A[f](u)\overline{\tilde{\mathscr{F}}^A[g](u')}\; \overline{\tilde{\mathscr{F}}^A[\psi]}(|\xi|^{-1}r_\xi u)\tilde{\mathscr{F}}^A[\phi](|\xi|^{-1}r_\xi u')|\xi|^n d^n u\, d^n u' d^n x d^n\xi.
\end{aligned}$$

Using Dirac delta function for variable $\boldsymbol{x}$, we get

$$
\begin{aligned}
&\langle \mathscr{W}_{\psi}^{A} f(\boldsymbol{x},\boldsymbol{\xi}), \mathscr{W}_{\psi}^{A} g(\boldsymbol{x},\boldsymbol{\xi})\rangle \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi|^n \tilde{\mathscr{F}}^A[f](\boldsymbol{u})\overline{\tilde{\mathscr{F}}^A[g](\boldsymbol{u}')}(2\pi)^{\frac{n}{2}}\delta\left(\frac{-\boldsymbol{u}+\boldsymbol{u}'}{b}\right)\overline{\tilde{\mathscr{F}}^A[\psi]}(|\xi|^{-1}r_{\xi}\boldsymbol{u}) \\
&\quad\times e^{-\frac{j}{2b}[2\langle(|\xi|^{-1}r_{\xi}(\boldsymbol{u}'-\boldsymbol{u}))+\boldsymbol{u}'-\boldsymbol{u}, d\boldsymbol{v}-b\boldsymbol{\omega_0}\rangle + d(\|\boldsymbol{u}\|^2-\|\boldsymbol{u}'\|^2)(1-|\xi|^{-2})]}\tilde{\mathscr{F}}^A[\phi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u}')d^n\boldsymbol{u}d^n\boldsymbol{u}'d^n\boldsymbol{\xi} \\
&= (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\tilde{\mathscr{F}}^A[f](\boldsymbol{u})\overline{\tilde{\mathscr{F}}^A[g]}(\boldsymbol{u})\,\overline{\tilde{\mathscr{F}}^A[\psi]}(|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})\tilde{\mathscr{F}}^A[\phi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})d^n\boldsymbol{u}|\boldsymbol{\xi}|^n d^n\boldsymbol{\xi} \\
&= C_{\phi,\psi}^A\int_{\mathbb{R}^n}\tilde{\mathscr{F}}^A[f](\boldsymbol{u})\overline{\tilde{\mathscr{F}}^A[g](\boldsymbol{u})}d^n\boldsymbol{u} = C_{\phi,\psi}^A\langle f,g\rangle_{L^2(\mathbb{R}^n)},
\end{aligned}
$$

where $C_{\phi,\psi}^A = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\overline{\tilde{\mathscr{F}}^A[\psi]}(|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})\tilde{\mathscr{F}}^A[\phi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})|\boldsymbol{\xi}|^n d^n\boldsymbol{\xi}$.
In the above theorem taking $\phi = \psi$, we have

$$
\langle \mathscr{W}_{\psi}^{A} f, \mathscr{W}_{\psi}^{A} g\rangle_{L^2(\mathbb{R}^n\times\mathbb{R}^n;\, d^n\boldsymbol{x}\, d^n\boldsymbol{\xi})} = C_{\psi}^A\langle f,g\rangle_{L^2(\mathbb{R}^n;\, d^n\boldsymbol{x})}, \tag{5}
$$

where $C_{\psi}^A = (2\pi)^{\frac{n}{2}}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}^A[\psi](|\boldsymbol{\xi}|^{-1}r_{\xi}\boldsymbol{u})\right|^2|\boldsymbol{\xi}|^n d^n\boldsymbol{\xi}$. Therefore, we get

$$
f(\boldsymbol{t}) = \frac{1}{C_{\psi}^A}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathscr{W}_{\psi}^{A} f(\boldsymbol{x},\boldsymbol{\xi})\psi_{\boldsymbol{x},\boldsymbol{\xi},A}(\boldsymbol{t})d^n\boldsymbol{x}d^n\boldsymbol{\xi}.
$$

□

Lemma 2.6. *If $f \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, then the relation between the OLCT of derivative of function f and the OLCT of the function f is given by*

$$
\tilde{\mathscr{F}}^A[f'(\boldsymbol{t})](\boldsymbol{u}) + \frac{j}{b}\tilde{\mathscr{F}}^A\left[a\boldsymbol{t}f(\boldsymbol{t})\right](\boldsymbol{u}) + \frac{j}{b}(\boldsymbol{v}-\boldsymbol{u})\tilde{\mathscr{F}}^A[f(\boldsymbol{t})](\boldsymbol{u}) = 0.
$$

Proof. The derivative of the kernel $h_A(\boldsymbol{t},\boldsymbol{u})$ defined in (1) is given by

$$
\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}h_A(\boldsymbol{t},\boldsymbol{u}) = h_A(\boldsymbol{t},\boldsymbol{u})\frac{j}{b}\left(a\boldsymbol{t}+\boldsymbol{v}-\boldsymbol{u}\right).
$$

By the chain rule of differentiation, we have

$$
\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}\left(f(\boldsymbol{t})h_A(\boldsymbol{t},\boldsymbol{u})\right) = f'(\boldsymbol{t})h_A(\boldsymbol{t},\boldsymbol{u}) + f(\boldsymbol{t})\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}}h_A(\boldsymbol{t},\boldsymbol{u}).
$$

Using this relation, integration by parts and finiteness of the integral of f in the definition of OLCT, we have the desired result. □

3. Applications of OLCWT

In this section, we introduce a novel offset differential operator, and using it, we develop a method for the solution of differential equations using OLCT and OLCWT. Furthermore, using Lemma 2.6 the uncertainty principle for the OLCT is established which paved the path for the uncertainty principle for the OLCWT with special rotation.

3.1. OLCWT-Based Solution of Differential Equation

Proposition 3.1. *Let $\mathcal{D}_{\boldsymbol{t}}$ denote the differential operator: $\mathcal{D}_{\boldsymbol{t}} = \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} - j\frac{a\boldsymbol{t}}{b}$ and $h_A(\boldsymbol{t},\boldsymbol{u})$ denote the kernel of the offset linear canonical transform. Then for $\boldsymbol{t},\boldsymbol{u} \in \mathbb{R}^n$, we have*

(i) $\mathcal{D}^m_{\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) = \left(\frac{j(\boldsymbol{v}-\boldsymbol{u})}{b}\right)^m h_A(\boldsymbol{t},\boldsymbol{u})$.

(ii) $\int \mathcal{D}^m_{\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} = (-1)^m \int h_A(\boldsymbol{t},\boldsymbol{u})(\mathcal{D}^*_{\boldsymbol{t}})^m f(\boldsymbol{t}) d^n\boldsymbol{t}$*, for all* $m \in \mathbb{N}$ *and* f *in* $\mathscr{S}(\mathbb{R}^n)$*, the Schwartz space [21], here* $\mathcal{D}^*_{\boldsymbol{t}} = \left[\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} + \frac{ja\boldsymbol{t}}{b}\right]$.

Proof. (i) Taking the derivative of the kernel of the offset linear canonical transform, we get

$$\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) = h_A(\boldsymbol{t},\boldsymbol{u}) \frac{j}{2b}[2at_1 + 2(v_1 - u_1), \ldots, 2at_n + 2(v_n - u_n)].$$

Equivalently,

$$\left[\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} - j\frac{a\boldsymbol{t}}{b}\right] h_A(\boldsymbol{t},\boldsymbol{u}) = \mathcal{D}_t h_A(\boldsymbol{t},\boldsymbol{u}) = \frac{j(\boldsymbol{v}-\boldsymbol{u})}{b} h_A(\boldsymbol{t},\boldsymbol{u}).$$

Similarly, we can easily find that $\mathcal{D}^m_{\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) = \left(\frac{j(\boldsymbol{v}-\boldsymbol{u})}{b}\right)^m h_A(\boldsymbol{t},\boldsymbol{u})$.

(ii) Employing integration by parts, we have

$$\int \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} = (-1) \int h_A(\boldsymbol{t},\boldsymbol{u}) \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} f(\boldsymbol{t}) d^n\boldsymbol{t}.$$

Utilizing it, we have

$$\begin{aligned}\int \mathcal{D}_t h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} &= \int \left[\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} - j\frac{a\boldsymbol{t}}{b}\right] h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} \\ &= -\int h_A(\boldsymbol{t},\boldsymbol{u}) \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{t}} f(\boldsymbol{t}) d^n\boldsymbol{t} - \int j\frac{a\boldsymbol{t}}{b} h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} \\ &= (-1) \int h_A(\boldsymbol{t},\boldsymbol{u}) \mathcal{D}^*_t f(\boldsymbol{t}) d^n\boldsymbol{t}.\end{aligned}$$

Now, applying principle of mathematical induction, we conclude:

$$\int \mathcal{D}^m_{\boldsymbol{t}} h_A(\boldsymbol{t},\boldsymbol{u}) f(\boldsymbol{t}) d^n\boldsymbol{t} = (-1)^m \int h_A(\boldsymbol{t},\boldsymbol{u}) (\mathcal{D}^*_t)^m f(\boldsymbol{t}) d^n\boldsymbol{t}.$$

□

This proposition paves the path for the solution of non-homogeneous ordinary differential equations. Let $(a_n(\mathcal{D}^*_{\boldsymbol{t}})^n + a_{n-1}(\mathcal{D}^*_{\boldsymbol{t}})^{n-1} + \ldots + a_1\mathcal{D}^*_{\boldsymbol{t}} + a_0)g(\boldsymbol{t}) = f(\boldsymbol{t})$ be an n^{th} order differential equation with $a'_\lambda s$ as constant for all $\lambda = 1, 2, \ldots, n$. Utilizing the both results of the Proposition 3.1, we have

$$C_{b,v}(\boldsymbol{u})\tilde{\mathscr{F}}^A[g](\boldsymbol{u}) = \tilde{\mathscr{F}}^A[f](\boldsymbol{u}), \tag{6}$$

where, $C_{b,v}(\boldsymbol{u})$ is a polynomial of degree having value:

$$a_n\left(\frac{j(\boldsymbol{u}-\boldsymbol{v})}{b}\right)^n + a_{n-1}\left(\frac{j(\boldsymbol{u}-\boldsymbol{v})}{b}\right)^{n-1} + \ldots + a_1\left(\frac{j(\boldsymbol{u}-\boldsymbol{v})}{b}\right) + a_0.$$

Now, the solution to the differential equation can be expressed as:

$$g(\boldsymbol{t}) = \tilde{\mathscr{F}}^{A^{-1}}\left[\frac{\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})}{C_{b,v}(\boldsymbol{u})}\right](\boldsymbol{t}).$$

To analyse the behaviour of the solution at specific time, we take the inner-product with $\psi_{\boldsymbol{x},\boldsymbol{\xi},A}(\boldsymbol{t})$ in both side and get

$$\mathscr{W}_{\psi}^{A} g(\boldsymbol{x},\boldsymbol{\xi}) = \left\langle \tilde{\mathscr{F}}^{A^{-1}}\left[\frac{\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})}{C_{b,v}(\boldsymbol{u})}\right](\boldsymbol{t}), \psi_{\boldsymbol{x},\boldsymbol{\xi},A}(\boldsymbol{t})\right\rangle.$$

Example 3.2. *Consider the one-dimensional differential equation* $\left(1-\mu(\mathcal{D}_t^*)^4\right)g(t) = f(t)$ *such that* $\tilde{\mathscr{F}}^{A}[f](u) = (b^4-\mu(v-u)^4)\tilde{\mathscr{F}}^{A}[h](u)$, *where* μ *is constant. Then the general solution of the one-dimensional differential equation is given by:*

$$g(t) = b^4\tilde{\mathscr{F}}^{A^{-1}}\left[\frac{\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})}{(b^4-\mu(v-u)^4)}\right](t).$$

Utilizing the specific condition, the solution becomes:

$$\mathscr{W}_{\psi}^{A} g(x,\xi) = (b^4)\mathscr{W}_{\psi}^{A} h(x,\xi).$$

3.2. Uncertainty Principle Type Inequality

Using Lemma 2.6 in the uncertainty principle given in Corollary 2.8 [8], we have the following uncertainty principle for the offset linear canonical transform.

Theorem 3.3. *For the function* $f \in L^2(\mathbb{R}^n)$, *the uncertainty principle in the offset linear canonical transform domain is given by*

$$\frac{b^2}{16}\|f\|_2^4 \le \left(\int \boldsymbol{t}^2|f(\boldsymbol{t})|^2 d^n\boldsymbol{t}\right)\left(\int \left|\tilde{\mathscr{F}}^{A}\left[a\boldsymbol{t}f(\boldsymbol{t})\right](\boldsymbol{u})\right|^2 d^n\boldsymbol{u}\right) + \left(\int \boldsymbol{t}^2|f(\boldsymbol{t})|^2 d^n\boldsymbol{t}\right)\left(\int |\boldsymbol{v}-\boldsymbol{u}|^2|\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})|^2 d^n\boldsymbol{u}\right).$$

Proof. Using integral by parts and the fact that $f \in L^2(\mathbb{R}^n)$, we have

$$2Re\int_{\mathbb{R}^n} \boldsymbol{x}f(\boldsymbol{x})\overline{f'(\boldsymbol{x})}d^n\boldsymbol{x} = -\int_{\mathbb{R}^n}|f(\boldsymbol{x})|^2 d^n\boldsymbol{x}.$$

Applying Parseval's relation, we have

$$\|f\|_2^4 \le 4\left(\int \boldsymbol{x}^2|f(\boldsymbol{x})|^2 d^n\boldsymbol{x}\right)\left(\int |\tilde{\mathscr{F}}^{A}[f'](\boldsymbol{u})|^2 d^n\boldsymbol{u}\right).$$

Now, employing Lemma 2.6, we obtain

$$|\tilde{\mathscr{F}}^{A}[f'](\boldsymbol{u})|^2 = \left|\frac{1}{b}\right|^2\left|\tilde{\mathscr{F}}^{A}\left[a\boldsymbol{t}f(\boldsymbol{t})\right](\boldsymbol{u}) + (\boldsymbol{v}-\boldsymbol{u})\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})\right|^2.$$

Using inequality $|a+b| \le 4(|a|^2+|b|^2)$, the above equation becomes

$$|\tilde{\mathscr{F}}^{A}[f'](\boldsymbol{u})|^2 \le \left|\frac{2}{b}\right|^2\left[\left|\tilde{\mathscr{F}}^{A}\left[a\boldsymbol{t}f(\boldsymbol{t})\right](\boldsymbol{u})\right|^2 + \left|(\boldsymbol{v}-\boldsymbol{u})\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})\right|^2\right].$$

Therefore, we have

$$\frac{b^2}{16}\|f\|_2^4 \le \left(\int \boldsymbol{t}^2|f(\boldsymbol{t})|^2 d^n\boldsymbol{t}\right)\left(\int \left|\tilde{\mathscr{F}}^{A}\left[a\boldsymbol{t}f(\boldsymbol{t})\right](\boldsymbol{u})\right|^2 d^n\boldsymbol{u}\right) + \left(\int \boldsymbol{t}^2|f(\boldsymbol{t})|^2 d^n\boldsymbol{t}\right)\left(\int |\boldsymbol{v}-\boldsymbol{u}|^2|\tilde{\mathscr{F}}^{A}[f](\boldsymbol{u})|^2 d^n\boldsymbol{u}\right).$$

□

Using the inner product relation for the OLCWT in Theorem 3.3, the uncertainty principle for the offset linear canonical wavelet transform can be established.

Theorem 3.4. *For the function $f \in L^2(\mathbb{R}^n)$ the uncertainty principle for the OLCWT is given by*

$$\left(\iiint \boldsymbol{x}^2 |\mathscr{W}_\psi^A f(\boldsymbol{x},\boldsymbol{\xi})|^2 d^n\boldsymbol{x} d^n\boldsymbol{\xi}\right)\left(\iiint \left|\tilde{\mathscr{F}}^A\left[a\boldsymbol{x}\mathscr{W}_\psi^A f(\boldsymbol{x},\boldsymbol{\xi})\right](\boldsymbol{u})\right|^2 d^n\boldsymbol{u} d^n\boldsymbol{\xi}\right)$$
$$+\left(\iiint \boldsymbol{x}^2 |\mathscr{W}_\psi^A f(\boldsymbol{x},\boldsymbol{\xi})|^2 d^n\boldsymbol{x} d^n\boldsymbol{\xi}\right)\left(C_\psi^A \int |\boldsymbol{v}-\boldsymbol{u}|^2 \left|\tilde{\mathscr{F}}^A[f](\boldsymbol{u})\right|^2 d^n\boldsymbol{u} d^n\boldsymbol{\xi}\right) \geq \frac{b^2 (C_\psi^A)^2}{16}\|f\|^4.$$

4. Conclusion

In this paper, we extend the one-dimensional offset linear canonical transform (OLCT) to its n-dimensional counterpart and establish its relationship with the classical Fourier transform (CFT). Additionally, we demonstrated that the inner product of the OLCT of two functions is proportional to the inner product of the original functions. A novel offset wavelet family based on Moritoh's wavelet was introduced, and its inner product properties, along with a reconstruction formula, were derived. Applications for solving differential equations using the offset linear canonical wavelet transform (OLCWT) with special rotation are investigated, and uncertainty principle-type inequalities are obtained. The offset linear canonical transform (OLCT) is the most generalized non-window integral transform, and by imposing restrictions on its parameters, one can obtain the Fourier transform (FT), fractional FT, linear canonical transform, and other special cases (see [4, Table 1, pp. 137]). Similarly, the results for the fractional wavelet transform with special rotation and the linear canonical wavelet transform with special rotation can be obtained by appropriate restriction on the results presented in this article. These findings contribute to both the theoretical development and practical applications of offset linear canonical transforms and their associated wavelet transforms.

Declarations

The authors declare that they have no competing interests.

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