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On vector variational-like inequalities and vector optimization problems using convexificators

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Abstract. In this paper, we formulate the relationship between the quasi efficient solution of nonsmooth vector optimization problems involving the locally Lipschitz function and the convexificator-based solutions of Stampacchia type vector variational-like inequality problems by using an approximate invex function. We also identify under approximate pseudoinvexity assumptions the weakly quasi efficient points, the vector critical points, and the solutions of the nonsmooth weak vector variational-like inequality problems are equivalent. Our newly proved results generalize some well-known results in the literature.

1. Introduction

Nonsmooth phenomena happens often in optimization theory, leading to the advancement of numerous concepts of subdifferentials and generalized directional derivatives. The concept of convexificator is a generalization of various well-known subdifferentials, such as Mordukhovich, Michel-Penot, and Clarke subdifferentials. In 1994, Demyanov [5] introduced and studied the concept of convexificators as a generalization of the concept of upper convex and lower concave approximations. Convexificators for positively homogenous and locally Lipschitz functions were studied by Demyanov and Jeyakumar [7]. Recently, Golestani and Nobakhtian [9], Li and Zhang [20], Long and Huang [21], and Luu [22] used convexificators to get the best possible conditions for nonsmooth optimization problems. For more information on convexificators, we refer to [6, 19, 23, 27, 32] and the references therein.

Approximation methods are very crucial in optimization theory because finding an exact solution is sometimes unattainable or computationally very expensive. As a result, approximate efficient solutions help in overcoming the difficulties posed by computational imperfections and modeling restrictions. Mishra and Laha [24] gave the concept of approximate efficient solutions for a vector optimization problem by using locally Lipschitz approximately convex functions and describe these approximate efficient solutions by using approximate vector variational inequalities of Minty and Stampacchia form in terms of the Clarke subdifferentials. Later, Wang [28] demonstrated that the solutions of generalized vector variational-like inequalities in terms of the generalized Jacobian are the quasi efficient solutions of nonsmooth multiobjective

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programming problems under higher-order generalized invexity assumptions. We refer to [2, 10, 11, 16, 29] and the references therein for additional literature on approximation and its applications.

In the last three decades, several definitions extending the concept of convex function have been purposed by different researchers. The important generalization of the convex function is an invex function that preserves numerous properties of the convex function. The class of invex functions has recently received a lot of interest since it allows us to relax the smoothness and convex function assumptions for practical applications. In 2013, Bhatia et al. [3] introduced four novel classes of approximate convex functions and established sufficient optimality conditions for quasi efficient solutions to a vector optimization problem involving these functions. Laha et al. [18] formulate vector variational inequalities of Stampacchia and Minty type in terms of convexificators and use these vector variational inequalities as a tool to find out necessary and sufficient conditions for a point to be a vector minimal point of the vector optimization problem. Using the notion of quasi efficiency and strong efficiency, respectively, Mishra and Upadhyay [25] and Upadhyay et al. [30] have established the connections concerning nonsmooth vector optimization problems and vector variational inequalities. Gupta and Mishra [12] gave the idea of generalized approximate convex functions and established some relationship between vector variational inequalities and vector optimization problems in terms of Clarke's subdifferentials. Motivated and inspired by ongoing research work, we introduce a class of approximate invex functions in terms of convexificators and establish some relationship between nonsmooth vector optimization problems and nonsmooth vector variational-like inequality problems.

2. Preliminaries

In this section, we recall some notions of nonsmooth analysis. For more details, see [4]. Suppose $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}^n_+$ be its nonnegative orthant and $int\mathbb{R}^n_+$ be the positive orthant of $\mathbb{R}^n$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ denotes the extended real line and $\langle ., . \rangle$ denote the Euclidean inner product. In the sequel, let X be a nonempty subset of $\mathbb{R}^n$ equipped with the Euclidean norm $\|.\|$.

Throughout this paper, for inequalities we use the following conventions:
If $u, v \in \mathbb{R}^n$, then
$u \geqq v \;\Leftrightarrow\; u_i \geq v_i,\ i = 1, 2, 3, ..., n \;\Leftrightarrow\; u - v \in \mathbb{R}^n_+;$
$u > v \;\Leftrightarrow\; u_i > v_i,\ i = 1, 2, 3, ..., n \;\Leftrightarrow\; u - v \in int\mathbb{R}^n_+;$
$u \geq v \;\Leftrightarrow\; u_i \geq v_i,\ i = 1, 2, 3, ..., n$, but $u \neq v \;\Leftrightarrow\; u - v \in int\mathbb{R}^n_+ \setminus \{0\}$.

First of all, we recall some definitions.

Definition 2.1. *Suppose $g : X \to \overline{\mathbb{R}}$ be an extended real valued function, $u \in X$ and $g(u)$ be finite. Then, the lower and upper Dini derivatives of g at $u \in X$ in the direction $r \in \mathbb{R}^n$, are denoted by $g^-(u, r)$ and $g^+(u, r)$, respectively and are defined as follows:*

$$g^-(u, r) = \liminf_{\lambda \to 0} \frac{g(u + \lambda r) - g(u)}{\lambda}$$

and

$$g^+(u, r) = \limsup_{\lambda \to 0} \frac{g(u + \lambda r) - g(u)}{\lambda}.$$

Definition 2.2. *[15] Suppose $g : X \to \overline{\mathbb{R}}$ be an extended real valued function, $u \in X$ and $g(u)$ be finite. Then g is said to be:*

(i) an upper convexificator $\partial^* g(u) \subseteq \mathbb{R}^n$ at $u \in X$, if and only if $\partial^* g(u)$ is closed and for every $r \in \mathbb{R}^n$, we have

$$g^-(u, r) \leq \sup_{\zeta \in \partial^* g(u)} \langle \zeta, r \rangle,$$

(ii) a lower convexificator $\partial_* g(u) \subseteq \mathbb{R}^n$ at $u \in X$, if and only if $\partial_* g(u)$ is closed and for every $r \in \mathbb{R}^n$, we have

$$g^+(u,r) \geq \inf_{\zeta \in \partial_* g(u)} \langle \zeta, r \rangle ,$$

(iii) a convexificator $\partial^*_* g(u) \subseteq \mathbb{R}^n$ at $u \in X$, if and only if $\partial^*_* g(u)$ is both upper and lower convexificator of g at u. That is, for every $r \in \mathbb{R}^n$, we have

$$g^-(u,r) \leq \sup_{\zeta \in \partial^*_* g(u)} \langle \zeta, r \rangle , \quad g^+(u,r) \geq \inf_{\zeta \in \partial^*_* g(u)} \langle \zeta, r \rangle .$$

The definitions and properties presented above can be extended to a locally Lipschitz vector-valued function $g : X \to \mathbb{R}^p$. Denote by g_i, $i \in K = \{1,2,3,...,p\}$, the components of g. The convexificator of g at $u \in X$ is the set

$$\partial^*_* g(u) = \partial^*_* g_1(u) \times \partial^*_* g_2(u) \times \partial^*_* g_3(u) \times ... \times \partial^*_* g_p(u).$$

From now onwards, we suppose $\eta : X \times X \to \mathbb{R}^n$ be a continuous map and X be a nonempty, closed and invex set unless, otherwise specified.

In terms of convexificators, we define the ∂^*_*-invex function, ∂^*_*-approximate invex function and ∂^*_*-approximate pseudoinvex function of type I as follows:

Definition 2.3. *Suppose $g : X \to \mathbb{R}$ be a locally Lipschitz function at $v \in X$ and admits a bounded convexificator $\partial^*_* g(v)$ at v. Then g is said to be ∂^*_*-invex function at $v \in X$, if*

$$g(u) - g(v) \geq \langle \zeta, \eta(u,v) \rangle , \ \forall \ \zeta \in \partial^*_* g(v), \ \forall \ u \in X.$$

The function g is said to be $\partial^*_* g(u)$-invex on X, if the above condition is satisfied for all $v \in X$.

Definition 2.4. *Suppose $g : X \to \mathbb{R}$ be a locally Lipschitz function at $v \in X$ and admits a bounded convexificator $\partial^*_* g(v)$ at v. Then g is said to be ∂^*_*-approximate invex (strictly $\partial^*_* g(u)$-approximate invex) at $v \in X$, if for every $e > 0$, there exists $\delta > 0$ such that*

$$g(u) - g(v) \geq (>) \langle \zeta, \eta(u,v) \rangle - e\|\eta(u,v)\|, \ \forall \ \zeta \in \partial^*_* g(v), \ \forall \ u \in B(v,\delta) \cap X, (u \neq v).$$

Definition 2.5. *Suppose $g : X \to \mathbb{R}$ be a locally Lipschitz function at $v \in X$ and admits a bounded convexificator $\partial^*_* g(v)$ at v. Then g is said to be ∂^*_*-approximate pseudoinvex function of type I at $v \in X$, if for every $e > 0$, there exists $\delta > 0$ such that $u \in B(v,\delta) \cap X$, we have*

$$\exists \ \zeta \in \partial^*_* g(v) \text{ such that } \ \langle \zeta, \eta(u,v) \rangle \geq 0 \ \Rightarrow \ g(u) - g(v) \geq -e\|\eta(u,v)\|.$$

or equivalently,

$$g(u) - g(v) < -e\|\eta(u,v)\| \ \Rightarrow \ \langle \zeta, \eta(u,v) \rangle < 0, \ \forall \ \zeta \in \partial^*_* g(v).$$

Remark 2.6. *If $\eta(u,v) = u - v$, then we see that ∂^*_*-convex function, ∂^*_*-approximate convex function and ∂^*_*-approximate pseudoconvex function of type I [31] is a special case of ∂^*_*-invex function, ∂^*_*-approximate invex function and ∂^*_*-approximate pseudoinvex function of type I, respectively.*

We consider the following nonsmooth vector optimization problem (for short, $NVOP$)

$$\text{Min } \left\{ g(u) = (g_1(u), g_2(u), ..., g_p(u)) \right\} \text{ such that } u \in X,$$

where $g_i : X \to \mathbb{R}$, $i = 1,2,3,...,p$ are nondifferentiable, bounded below and locally Lipschitz functions on X.

Now, we define local quasi efficient and local weak quasi efficient solutions to the ($NVOP$) as follows:

Definition 2.7. *[14] A point $v \in X$ is said to be a local quasi efficient solution to the $(NVOP)$, if there exists $e \in int\mathbb{R}^p_+$ and a neighborhood U of v such that*

$$g(u) - g(v) + e\|\eta(u,v)\| \notin -\mathbb{R}^p_+ \setminus \{0\}, \ \forall\ u \in X \cap U.$$

Definition 2.8. *A point $v \in X$ is said to be a local weak quasi efficient solution to the $(NVOP)$, if there exists $e \in int\mathbb{R}^p_+$ and a neighborhood U of v such that*

$$g(u) - g(v) + e\|\eta(u,v)\| \notin -int\mathbb{R}^p_+, \ \forall\ u \in X \cap U.$$

We consider the following Stampacchia type vector variational-like inequality problems in terms of convexificators as follows:

$(\partial^*_* - SVVLIP)$: Find $v \in X$ such that for any $\zeta_i \in \partial^*_* g_i(v),\ i \in\ K$, there does not exist $u \in X$ such that

$$\left(\langle \zeta_1, \eta(u,v)\rangle, \langle \zeta_2, \eta(u,v)\rangle, ..., \left\langle \zeta_p, \eta(u,v)\right\rangle\right) \in -\mathbb{R}^p_+ \setminus \{0\}.$$

$(\partial^*_* - WSVVLIP)$: Find $v \in X$ such that for any $\zeta_i \in \partial^*_* g_i(v),\ i \in\ K$, there does not exist $u \in X$ such that

$$\left(\langle \zeta_1, \eta(u,v)\rangle, \langle \zeta_2, \eta(u,v)\rangle, ..., \left\langle \zeta_p, \eta(u,v)\right\rangle\right) \in -int\mathbb{R}^p_+.$$

3. Relationships between vector variational-like inequality problems and nonsmooth vector optimization problems

In this section, using local weak quasi efficiency and local quasi efficiency, we establish relationships between $(NVOP)$, $(\partial^*_* - WSVVLIP)$ and $(\partial^*_* - SVVLIP)$.

Theorem 3.1. *Suppose $X \subseteq \mathbb{R}^n$ be a nonempty set and $g : \mathbb{R}^n \to \mathbb{R}^p$ be a vector valued function such that $g_i : X \to \mathbb{R}$ be a locally Lipschitz function on X and admit bounded convexificators $\partial^*_* g_i(u)$, for all $u \in X$ and $i \in K$. Let $g_i : X \to \mathbb{R},\ i \in K$ be ∂^*_*-approximate invex function at $v \in X$. If $v \in X$ solves the $(\partial^*_* - SVVLIP)$, then v is a local quasi efficient solution to the $(NVOP)$.*

Proof. Suppose that v is not a local quasi efficient solution to the $(NVOP)$. Then for any $e \in int\mathbb{R}^p_+$ and any $\delta > 0$, there exists $u \in B(v,\delta) \cap X$ such that

$$g(u) - g(v) + e\|\eta(u,v)\| \in -\mathbb{R}^p_+ \setminus \{0\}.$$

That is, for every $i \in K$,

$$g_i(u) - g_i(v) \leq -e_i\|\eta(u,v)\|,$$

with the strict inequality hold for some $i \in K$.
Using ∂^*_*-approximate invexity of g_i at v, in particular for $e_i > 0$, $i \in K$, there exist $\delta > 0$ such that

$$\langle \zeta_i, \eta(u,v)\rangle \leq 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v), \text{ and } u \in B(v,\delta) \cap X,$$

with the strict inequality hold for some $i \in K$.
Therefore, there exist $u \in X$ such that

$$\left(\langle \zeta_1, \eta(u,v)\rangle, \langle \zeta_2, \eta(u,v)\rangle, ..., \left\langle \zeta_p, \eta(u,v)\right\rangle\right) \in -\mathbb{R}^p_+ \setminus \{0\}, \text{ for all } \zeta_i \in \partial^*_* g_i(v),\ i \in K.$$

This means that $v \in X$ is not a solution of $(\partial^*_* - SVVLIP)$, which is a contradiction.
Hence v must be a local quasi efficient solution of $(NVOP)$. □

Remark 3.2. *We see that $(\partial^*_* - SVVLIP)$ is not a sufficient condition for a local quasi efficient solution to the $(NVOP)$. The following example can be used to demonstrate this point.*

Example 3.3. *Consider the following nonsmooth vector optimization problem* ($NVOP$) *as follows:*

$$min\ g(u) = (g_1(u), g_2(u)),\ such\ that\ u \in X = [-1, 1],$$

where $g_1, g_2 : [-1, 1] \to \mathbb{R}$ *are defined by*

$$g_1(u) = \begin{cases} u^2 + e^u, & if\ u \geq 0, \\ u^2 + e^{-u}, & if\ u < 0, \end{cases}$$

$$g_2(u) = \begin{cases} u^2 - u, & if\ u \geq 0, \\ u^2 - 3u, & if\ u < 0. \end{cases}$$

Suppose $\eta : X \times X \to \mathbb{R}$ *be defined by*

$$\eta(u, v) = \begin{cases} 1 - v, & if\ u \geq 0\ \ v < 0, \\ u - v, & elsewhere. \end{cases}$$

For every $e = (e_1, e_2) \in int\mathbb{R}^2_+$ *and* $\delta > 0$, *we can show that* $g = (g_1, g_2)$ *is* ∂^*_**-approximate invex at* $v = 0$. *Moreover, we can easily check that* $v = 0$ *is local quasi efficient solution of the* ($NVOP$), *as for every* $e = (e_1, e_2) \in int\mathbb{R}^2_+$, *there exist* $\delta > 0$ *such that*

$$(g_1(u) - g_1(v) + e_1\|\eta(u, v)\|, g_2(u) - g_2(v) + e_2\|\eta(u, v)\|) \notin -\mathbb{R}^2_+ \setminus \{0\}.$$

Suppose $v = 0$, *then for* $u > 0$, *we have*

$$(\langle\zeta_1, \eta(u, v)\rangle, \langle\zeta_2, \eta(u, v)\rangle) \leq (0, 0),\ for\ all\ \zeta_1 \in \partial^*_* g_1(v) = [-1, 1]\ and\ \zeta_2 \in \partial^*_* g_2(v) = [-3, -1].$$

Hence $v = 0$ *is not a solution of* ($\partial^*_* - SVVLIP$).

Theorem 3.4. *Suppose* $g_i : X \to \mathbb{R}$ *be a locally Lipschitz function on* X *and admit bounded convexificators* $\partial^*_* g_i(u)$, *for all* $u \in X$ *and* $i \in K$. *Let* $-g_i$ *be strictly* ∂^*_**-approximate invex function at* $v \in X$. *If* $v \in X$ *is a local weak quasi efficient solution to the* ($NVOP$), *then* v *solves the* ($\partial^*_* - SVVLIP$).

Proof. Suppose that v does not solves the ($\partial^*_* - SVVLIP$). Then there exists $u \in X$ such that

$$\Big(\langle\zeta_1, \eta(u, v)\rangle, \langle\zeta_2, \eta(u, v)\rangle, ..., \Big\langle\zeta_p, \eta(u, v)\Big\rangle\Big) \in -\mathbb{R}^p_+ \setminus \{0\},\ \zeta_i \in \partial^*_* g_i(v),\ i \in K.$$

That is, for every $i \in K$,

$$\langle\zeta_i, \eta(u, v)\rangle \leq 0,\ \text{for all}\ \zeta_i \in \partial^*_* g_i(v).$$

Using strictly ∂^*_*-approximate invexity of $-g_i$ at v, in particular for $e_i > 0$, $i \in K$, there exist $\delta > 0$ such that

$$g_i(u) - g_i(v) + e_i\|\eta(u, v)\| < \langle\zeta_i, \eta(u, v)\rangle \leq 0,\ \text{for all}\ \zeta_i \in \partial^*_* g_i(v).$$

Therefore, there exist $u \in X$ such that

$$g_i(u) - g_i(v) < -e_i\|\eta(u, v)\|,\ \text{for all}\ i \in K.$$

which contradicts that $v \in X$ being a local weak quasi efficient solution of ($NVOP$). Hence v must be a solution of ($\partial^*_* - SVVLIP$). □

Theorem 3.5. *Suppose* $g_i : X \to \mathbb{R}$ *be a locally Lipschitz function on* X *and admit bounded convexificators* $\partial^*_* g_i(u)$, *for all* $u \in X$ *and* $i \in K$. *Let* $g_i : X \to \mathbb{R}$, $i \in K$ *be* ∂^*_**-approximate pseudoinvex function of type I at* $v \in X$. *If* $v \in X$ *solves the* ($\partial^*_* - WSVVLIP$), *then* v *is a local weak quasi efficient solution to the* ($NVOP$).

Proof. Suppose that v is not a local weak quasi efficient solution to the $(NVOP)$. Then for any $e \in int\mathbb{R}^p_+$ and any $\delta > 0$, there exists $u \in B(v, \delta) \cap X$ such that

$$g(u) - g(v) + e\|\eta(u, v)\| \in -int\mathbb{R}^p_+.$$

That is, for every $i \in K$,

$$g_i(u) - g_i(v) < -e_i\|\eta(u, v)\|,$$

with the strict inequality hold for some $i \in K$.
Using ∂^*_*-approximate pseudoinvexity of type I of g_i at v, in particular for $e_i > 0$, $i \in K$, there exist $\delta > 0$ such that

$$\langle \zeta_i, \eta(u, v)\rangle < 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v), \text{ and } u \in B(v, \delta) \cap X,$$

with the strict inequality hold for some $i \in K$.
Therefore, there exist $u \in X$ such that

$$\left(\langle \zeta_1, \eta(u, v)\rangle, \langle \zeta_2, \eta(u, v)\rangle, ..., \left\langle \zeta_p, \eta(u, v)\right\rangle\right) \in -int\mathbb{R}^p_+, \text{ for all } \zeta_i \in \partial^*_* g_i(v), \ i \in K.$$

This means that $v \in X$ is not a solution of $(\partial^*_* - WSVVLIP)$, which is a contradiction.
Hence v must be a local weak quasi efficient solution of $(NVOP)$. □

Theorem 3.6. *Suppose $g_i : X \to \mathbb{R}$ be a locally Lipschitz function on X and admit bounded convexificators $\partial^*_* g_i(u)$, for all $u \in X$ and $i \in K$. Let $g_i : X \to \mathbb{R}$, $i \in K$ be ∂^*_*-approximate invex function at $v \in X$. If $v \in X$ is a local weak quasi efficient solution to the $(NVOP)$, then v is a local quasi efficient solution to the $(NVOP)$.*

Proof. Suppose v is a local quasi efficient solution to the $(NVOP)$, but not a local weak quasi efficient solution.
Then for any $e \in int\mathbb{R}^p_+$ and any $\delta > 0$, there exists $u \in B(v, \delta) \cap X$ such that

$$g(u) - g(v) + e\|\eta(u, v)\| \in -\mathbb{R}^p_+ \setminus \{0\}.$$

That is, for every $i \in K$,

$$g_i(u) - g_i(v) + e_i\|\eta(u, v)\| \leq 0,$$

with the strict inequality hold for some $i \in K$.
Using ∂^*_*-approximate invexity of g_i, $i \in K$ at v, in particular for $e_i > 0$, there exist $\delta > 0$ such that

$$0 \geq g_i(u) - g_i(v) + e_i\|\eta(u, v)\| > \langle \zeta_i, \eta(u, v)\rangle,$$

for every $\zeta_i \in \partial^*_* g_i(v)$ and $u \in B(v, \delta) \cap X$, $u \neq v$.
Therefore, there exist $u \in X$ such that

$$\left(\langle \zeta_1, \eta(u, v)\rangle, \langle \zeta_2, \eta(u, v)\rangle, ..., \left\langle \zeta_p, \eta(u, v)\right\rangle\right) \in -int\mathbb{R}^p_+, \text{ for all } \zeta_i \in \partial^*_* g_i(v), \ i \in K.$$

This means that $v \in X$ is not a solution of $(\partial^*_* - WSVVLIP)$. Then by Theorem 3.5, we get a contradiction.
Hence v must be a local weak quasi efficient solution of $(NVOP)$. □

The following definition gives the notion of nonsmooth case of vector critical point in terms of convexificators.

Definition 3.7. *[31] A feasible solution $v \in X$ is said to be a vectorial critical point to the $(NVOP)$, if there exist a vector $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that*

$$\langle \lambda, \zeta\rangle = 0, \textit{ for all } \zeta \in \partial^*_* g(v).$$

Lemma 3.8. *(Gordan's Theorem) [1]*
If A is a $n \times m$ matrix, then one of the following holds:

(i) $Au < 0$, *for some* $u \in \mathbb{R}^m$,

(ii) $\langle A, v\rangle = 0,\ v \geq 0$, *for some nonzero solution* $v \in \mathbb{R}^n$,

but not both.

Theorem 3.9. *Suppose* $X \subseteq \mathbb{R}^n$ *be a nonempty set and* $g : \mathbb{R}^n \to \mathbb{R}^p$ *be a vector valued function such that* $g_i : X \to \mathbb{R}$ *be a locally Lipschitz function on X, and admit bounded convexificators* $\partial^*_* g_i(u)$, *for all* $u \in X$ *and* $i \in K$. *Let* $g_i : X \to \mathbb{R}$ *be* ∂^*_**-approximate pseudoinvex function of type I at* $v \in X$. *If* $v \in X$ *be a vector critical point to the* $(NVOP)$, *then* v *is a local weak quasi efficient solution to the* $(NVOP)$.

Proof. Suppose $v \in X$ be a vector critical point to the $(NVOP)$, then there exists $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$, and a vector $\zeta \in \partial^*_* g(v)$ such that

$$\langle \lambda, \zeta \rangle = 0.$$

Suppose on the contrary that v is not a local weak quasi efficient solution to the $NVOP$. Then for any $e \in int\mathbb{R}^p_+$ and any $\delta > 0$, there exists $u \in B(v, \delta) \cap X$ such that

$$g(u) - g(v) + e\|\eta(u, v)\| \in -int\mathbb{R}^p_+.$$

That is, for every $i \in K$,

$$g_i(u) - g_i(v) < -e_i\|\eta(u, v)\|.$$

Using the assumption of ∂^*_*-approximate pseudoinvex function of type I of g at $v \in X$, therefore, in particular for $e_i > 0$ and $\delta > 0$, there exists $u \in B(v, \delta) \cap X,\ u \neq v$ such that

$$\left\langle \zeta_i, \eta(u, v) \right\rangle < 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v),\ i \in K.$$

Therefore, there exist $u \in X$ such that

$$\Big(\left\langle \zeta_1, \eta(u, v) \right\rangle, \left\langle \zeta_2, \eta(u, v) \right\rangle, ..., \Big\langle \zeta_p, \eta(u, v) \Big\rangle \Big) < 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v),\ i \in K.$$

Applying the Gordan's theorem, we deduce that there is no $\lambda \geq 0$ such that

$$\langle \lambda, \zeta \rangle = 0, \text{ for all } \zeta \in \partial^*_* g(v),$$

which is a contradiction to the fact that $v \in X$ is a vector critical point to the $(NVOP)$. □

Theorem 3.10. *Suppose* $g_i : X \to \mathbb{R}$ *be a locally Lipschitz function on X and admit bounded convexificators* $\partial^*_* g_i(u)$, *for all* $u \in X$ *and* $i \in K$. *All the vector critical points are local weak quasi efficient solutions to the NVOP iff* $g_i : X \to \mathbb{R},\ i \in K$ *is* ∂^*_**-approximate pseudoinvex function of type I on X.*

Proof. The sufficient condition can be shown from Lemma 2.1 [26]. Next, we prove the necessary condition, that is, if every vector critical point is a local weak quasi efficient solution to the $(NVOP)$, then $g_i,\ i \in K$ is ∂^*_*-approximate pseudoinvex function of type I on X.
Suppose $v \in X$ be a local weak quasi efficient solution to the $(NVOP)$. Then there exist $e \in int\mathbb{R}^p_+$ and $\delta > 0$ such that

$$g(u) - g(v) + e\|\eta(u, v)\| \in -int\mathbb{R}^p_+.$$

Therefore, for every $i \in K$, the following system

$$g_i(u) - g_i(v) < -e_i\|\eta(u, v)\|$$

has no solution, for $u \in B(v, \delta) \cap X$.
On the other hand, if $v \in X$ is a vector critical point to the $(NVOP)$. Then there exists $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that

$$\langle \lambda, \zeta \rangle = 0, \text{ for all } \zeta \in \partial^*_* g(v).$$

From the Gordan theorem, there is no η such that

$$\langle \lambda, \zeta \rangle = \left(\langle \lambda, \zeta_1 \rangle, \langle \lambda, \zeta_2 \rangle, ..., \left\langle \lambda, \zeta_p \right\rangle \right) < 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v).$$

Since every vector critical point is a local weak quasi efficient solution, so for every $e \in int\ \mathbb{R}^p_+$ and $\delta > 0$, there exists $u \in B(v, \delta) \cap X$ such that

$$g_i(u) - g_i(v) < -e_i \|\eta(u, v)\|, \text{ for all } i \in K.$$

Then there exist η such that

$$\langle \zeta_i, \eta(u, v) \rangle < 0, \text{ for all } \zeta_i \in \partial^*_* g_i(v),\ i \in K.$$

This is precisely the ∂^*_*-approximate pseudoinvex function of type I of $g_i,\ i \in K$. □

The following corollary can link the vector critical points to the solutions of the $(\partial^*_* - WSVVLIP)$ using Theorem 3.5 and Theorem 3.10.

Corollary 3.11. *Suppose $g_i : X \to \mathbb{R}$ be a locally Lipschitz function, ∂^*_*-approximate pseudoinvex function of type I on X and admit bounded convexificators $\partial^*_* g_i(u)$, for all $u \in X$ and $i \in K$. Then the vector critical points, the local weak quasi efficient solutions to the $(NVOP)$ and the solutions of the $(\partial^*_* - WSVVLIP)$ are equivalent.*

4. Conclusion

In this paper, we formulate the relationship between the quasi efficient solution of nonsmooth vector optimization problems involving the locally Lipschitz function and the convexificator-based solutions of Stampacchia type vector variational-like inequality problems by using an approximate invex function. Further, the results of this paper may be extended on Hadamard manifolds by using general locally Lipschitz functions.

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