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The pseudo-differential operator on local Hardy Morrey space

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Abstract. The main purpose of this paper is to prove the boundedness of pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ from one local Hardy Morrey space to another one by using the atomic decomposition. To overcome the essential difficulty caused by the absence of absolute continuity of quasi-norm in Morrey spaces, the most novelty of the paper exists in that Morrey space is embedded into the weighted Lebesgue space with certain special weight to control the convergence of the sum in the decomposition.

1. Introduction and statement of main results

The real-variable theory of classical Hardy spaces $H^p(\mathbb{R}^n)$ was originally initiated by Stein and Weiss [27] and systematically developed by Fefferman and Stein [11]. The Hardy space $H^p(\mathbb{R}^n)$ is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ when $0 < p \leq 1$. However, the principle of $H^p(\mathbb{R}^n)$ breaks down at some key points, for example, $H^p(\mathbb{R}^n)$ does not contain the Schwartz class of rapidly decreasing test functions and is not well defined on manifolds. Hence, Goldberg [13] introduced the class of local Hardy spaces $h^p(\mathbb{R}^n)$ and established the maximal function characterization of them. The theory of local Hardy spaces $h^p(\mathbb{R}^n)$ plays an important role in harmonic analysis and partial differential equations.

On the other hand, due to the applications in elliptic partial differential equations, Morrey space $M^p_r(\mathbb{R}^n)$ with $0 < r \leq p < \infty$ was introduced by Morrey [24] in 1938. Morrey spaces describe local regularity more precisely than $L^p(\mathbb{R}^n)$ spaces. Moreover, Morrey spaces provide subtle improvements in regularity in elliptic boundary value problems and non-linear evolution equations, for example the Navier–Stokes equations. The applications of Morrey spaces are generalized in various areas of analysis such as partial differential equations, potential theory, and harmonic analysis. For instance, we refer to [1, 12, 23, 25, 37].

Moreover, Jia and Wang [22] introduced Hardy Morrey space $HM^p_r(\mathbb{R}^n)$, which generalize the classical Morrey spaces $M^p_r(\mathbb{R}^n)(r > 1)$ and Hardy spaces $H^p(\mathbb{R}^n)(p \leq 1)$. Then, Sawano [30] investigated Hardy Morrey space $HM^p_r(\mathbb{R}^n)$ and the local version $hM^p_r(\mathbb{R}^n)$ from the viewpoint of Littlewood–Paley characterization. Wang and Jia [35] proved the boundedness of the singular integral operator and the Riesz potential on $HM^p_r(\mathbb{R}^n)$. In particular, Wang et al. [36] proved the

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boundedness of the pseudo-differential operators with symbols in $S^0_{1,0}$ on $hM^p_r(\mathbb{R}^n)$. Hoepfner [16] proved the boundedness of the pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ on local Hardy space $h^p(\mathbb{R}^n)$. Then, Tan et al. [33] obtained the continuity of the pseudo-differential operators on local variable Hardy space $h^{p(\cdot)}(\mathbb{R}^n)$.

In this paper, we obtain the boundedness of pseudo-differential operators with symbols in $S^{-\alpha}_{1,\delta}$ on local Hardy Morrey space $hM^p_r(\mathbb{R}^n)$. The novelty of this paper is as follow: Since the property of absolutely continuous quasi-norm fails in $M^p_r(\mathbb{R}^n)$, the density argument can not be applied. Hence, we apply the fact that $M^p_r(\mathbb{R}^n)$ embeds continuously into $L^{r_0}_w(\mathbb{R}^n)$ to overcome this issue. By product, we obtain the continuity of T_σ on Morrey spaces.

First we recall the definition of Morrey space $M^p_r(\mathbb{R}^n)$ with $0 < r \le p < \infty$. Here and hereafter, for any $x \in \mathbb{R}^n$ and $l \in (0,\infty)$, let $B(x,l) := \{y \in \mathbb{R}^n : |x-y| < l\}$ and $\mathbb{B}(\mathbb{R}^n) := \{B(x,l) : x \in \mathbb{R}^n \text{ and } l \in (0,\infty)\}$.

Definition 1.1. *Let $0 < r \le p < \infty$. The Morrey space $M^p_r(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that*

$$\|f\|_{M^p_r(\mathbb{R}^n)} := \sup_{B \in \mathbb{B}(\mathbb{R}^n)} |B|^{\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(B)} < \infty.$$

Sawano et al. in [28] introduced the ball-quasi Banach function spaces $X(\mathbb{R}^n)$ and their related Hardy spaces $H_X(\mathbb{R}^n)$, which generalized the theory of Hardy spaces built on general function spaces. From the definition of ball-quasi Banach function space [28, Definition 2.2], we easily know that Morrey space $M^p_r(\mathbb{R}^n)$ is a ball-quasi Banach function space. Hence, according to [28, Definition 5.2], we obtain the definition of local Hardy Morrey spaces. In what follows, denote by $\mathcal{S}(\mathbb{R}^n)$ the class of Schwartz functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the class of tempered functions.

Definition 1.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi_t(x) = t^{-n}\phi(t^{-1}x)$, $x \in \mathbb{R}^n$. The local grand maximal operator $\mathcal{M}_{loc} f(x) := \sup\{|\phi_t * f(x)| : t \in (0,1), \phi \in \mathcal{F}_N(\mathbb{R}^n)\}$ for any fixed large integer N, where $\mathcal{F}_N(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \int \phi(x)dx = 1, \sum_{|\alpha|\leqslant N} \sup(1+|x|)^N \left|\partial^\alpha \phi(x)\right| \leqslant 1\}$. The local Hardy Morrey space is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying the quantity*

$$\|f\|_{hM^p_r(\mathbb{R}^n)} := \|\mathcal{M}_{loc} f\|_{M^p_r(\mathbb{R}^n)} < \infty.$$

We recall the Hörmander class of pseudo-differential operators [20].

Definition 1.3. *Suppose that $m \in \mathbb{R}$ and $\rho, \delta \in [0,1]$. Let $f \in \mathcal{S}$, then a classical pseudo-differential operator T_σ is defined by setting, for any $x \in \mathbb{R}^n$,*

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \sigma(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi,$$

where $\sigma \in S^m_{\rho,\sigma}$, that is, $\sigma(x,\xi)$ is a smooth function for $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and

$$\left|\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)\right| \leqslant C(1+|\xi|)^{m-\rho|\beta|+\sigma|\alpha|}. \tag{1}$$

Álvarez et al. [2] obtained the boundedness of pseudo-differential operator with $\sigma \in \mathcal{S}^{-\alpha}_{1,\delta}$ on Lebesgue spaces.

Theorem 1.4. *Let T_σ is a pseudo-differential operator with $\sigma \in \mathcal{S}^{-\alpha}_{1,\delta}$ and $0 \le \delta < 1$. Then T_σ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ when $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $1 < p \le q < \infty$.*

The main goal of this paper is to prove the following result:

Theorem 1.5. *Let $\alpha \in [0,n)$, $0 \le \delta < 1$, $0 < r_1 \le p < \frac{n}{\alpha}$, $0 < r_2 \le q < \infty$ satisfying $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\alpha}{n}$. Then the T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ maps continuously $hM^p_{r_1}(\mathbb{R}^n)$ to $hM^q_{r_2}(\mathbb{R}^n)$.*

Remark 1.6. *From the definition of Morrey spaces, we find that, if $p = r_1$ and $q = r_2$, then $hM^p_{r_1}(\mathbb{R}^n) = h^p(\mathbb{R}^n)$, $hM^q_{r_2}(\mathbb{R}^n) = h^q(\mathbb{R}^n)$. In this case, Theorem 1.5 coincides with [16, Theorem 1.1]. Moreover, if $\alpha = \delta = 0$, then Theorem 1.5 is the consequence of [36, Corollary 4.14(e)].*

Chiarenza and Frasca [5] obtained the boundedness of Hardy–Littilewood maximal operator on Morrey spaces, we know that $hM^p_r(\mathbb{R}^n) = M^p_r(\mathbb{R}^n)$ when $1 < r \le p < \infty$. Also, Iida et al. [21] provide the relation between $HM^p_r(\mathbb{R}^n)$ and $M^p_r(\mathbb{R}^n)$ when $1 < r \le p < \infty$. Then we can obtain the following corallary.

Corollary 1.7. *Let $\alpha \in [0, n)$, $0 \le \delta < 1$, $1 < r_1 \le p < \frac{n}{\alpha}$, $1 < r_2 \le q < \infty$ satisfying $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\alpha}{n}$. Then the T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ maps continuously $M^p_{r_1}(\mathbb{R}^n)$ to $M^q_{r_2}(\mathbb{R}^n)$.*

Throughout this paper, C will denote a positive constant that may vary at each occurrence but is independent to the essential variables, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the essential variables such that $C_1 B \leqslant A \leqslant C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function. For $\tau > 0$, τQ denote the cube with the same center such that $l(\tau Q) = \tau l(Q)$. We denote by $\lfloor r \rfloor$(resp.,$\lceil r \rceil$) the *maximal*(resp., *minimal*) *integer not greater*(resp., *less*) *than* r, $\mathbb{N} := \{0, 1, 2, \cdots\}$. The operator $\mathcal{M}$ always denotes the Hardy–Littlewood maximal operator, which is defined by setting, for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{r \in (0,\infty)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $B(x, r)$ denotes the ball with the center x and the radius r.

2. Preliminaries

In this section, we present some known results that will be used in the next section. First we give the atomic decomposition of $hM^p_r(\mathbb{R}^n)$.

To establish the atom decomposition of $hM^p_r(\mathbb{R}^n)$, we need lemmas about the boundedness of the Hardy–Littilewood maximal operator $\mathcal{M}$ on Morrey space and its predual. From [31, Lemma 2.5], we can easily obtain the following lemma.

Lemma 2.1. *Let $0 < r \le p < \infty$. For some $\theta, u \in (0, 1]$ and $\theta \in (0, \min\{u, r\})$, there exists a positive constant C such that, for any $\{f_j\}_{j=1}^{\infty} \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$\left\| \left\{ \sum_{j=1}^{\infty} \left[\mathcal{M}^{(\theta)}(f_j) \right]^u \right\}^{\frac{1}{u}} \right\|_{M^p_r(\mathbb{R}^n)} \leqslant C \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^u \right\}^{\frac{1}{u}} \right\|_{M^p_r(\mathbb{R}^n)},$$

where the powered Hardy–Littilewood maximal operator $\mathcal{M}^{(\theta)}$ is defined by setting, for any $\theta \in (0, \infty)$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^{(\theta)}(f)(x) := \left\{ \mathcal{M}\left(|f|^\theta\right)(x) \right\}^{\frac{1}{\theta}}.$$

Now we give the boundedness of the Hardy–Littilewood maximal operator on block space, which is the predual of Morrey space. Let $1 < r \le p < \infty$ and p', r' are conjugate numbers of p, r. A function b on $\mathbb{R}^n$ is called a (p', r')-block if supp(b) $\subset$ Q with $Q \in \mathcal{Q}$, and

$$\left(\int_Q |b(x)|^{r'} dx \right)^{\frac{1}{r'}} \le |Q|^{\frac{1}{p} - \frac{1}{r}}, \tag{2}$$

where $\mathcal{Q}$ denotes the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. We write $b \in \mathfrak{b}_{r'}^{p'}$, if $b(x)$ is a (p', r')-block with supp(b) $\subset$ Q. The space $\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)$ is defined by the set of all functions f locally in $L^{r'}(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)} := \inf\left\{\|\{\lambda_k\}\|_{l^1} : f = \sum_k \lambda_k b_k\right\} < \infty,$$

where $\left\|\{\lambda_k\}_{k=1}^{\infty}\right\|_{l^1} = \sum_k |\lambda_k| < \infty$ and b_k is a (p', r')-block, and the infimum is taken over all possible decompositions of f (see,for instance [29, p.666]). From [29, Theorem 4.1], we can get that

$$[(M_{r/r_0}^{p/r_0})']^{1/(p_0/r_0)'}(\mathbb{R}^n) = \mathcal{B}_{(r/r_0)'/(p_0/r_0)'}^{(p/r_0)'/(p_0/r_0)'}(\mathbb{R}^n).$$

Hence, from this and the fact that $\mathcal{M}$ is bounded on $\mathcal{B}_r^p(\mathbb{R}^n)$ for any $1 < p \le r < \infty$(see , for instance [6, Theorem 3.1]), we easily get the following lemma, which is verified in [36, Remark 2.7(e)].

Lemma 2.2. *Let $0 < r \le p < \infty$. There exists an $r_0 \in (0, \min\{1, r\})$ and a $p_0 \in (\max\{1, p\}, \infty)$ such that $M_{r/r_0}^{p/r_0}$ is a ball Banach function space and there exists a positive constant C such that, for any $f \in \mathcal{B}_{(r/r_0)'/(p_0/r_0)'}^{(p/r_0)'/(p_0/r_0)'}(\mathbb{R}^n)$,*

$$\left\|\mathcal{M}(f)\right\|_{\mathcal{B}_{(r/r_0)'/(p_0/r_0)'}^{(p/r_0)'/(p_0/r_0)'}(\mathbb{R}^n)} \leqslant C\|f\|_{\mathcal{B}_{(r/r_0)'/(p_0/r_0)'}^{(p/r_0)'/(p_0/r_0)'}(\mathbb{R}^n)}.$$

Via borrowing some ideas from [17], we obtain the following three lemmas. First we present the Hölder inequality for $M_r^p(\mathbb{R}^n)$ and $\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)$.

Lemma 2.3. *Let $1 < r \leqslant p < \infty$, $f \in M_r^p(\mathbb{R}^n)$ and $g \in \mathcal{B}_{r'}^{p'}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant C\|f\|_{M_r^p(\mathbb{R}^n)}\|g\|_{\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)}$$

for some $C > 0$ independent of f and g.

Proof. From the definition of Morrey space, we have

$$\begin{aligned}\|f\chi_Q\|_{L^r(\mathbb{R}^n)} &= |Q|^{\frac{1}{r}-\frac{1}{p}}|Q|^{\frac{1}{p}-\frac{1}{r}}\|f\|_{L^r(Q)} \\ &\leqslant |Q|^{\frac{1}{r}-\frac{1}{p}}\|f\|_{M_r^p(\mathbb{R}^n)}.\end{aligned} \tag{3}$$

For any (p', r')-block $b(x)$ with supp(b) $\subset$ Q, by using Hölder inequality, (2) and (3), we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)b(x)|dx &\leqslant \|f\chi_Q\|_{L^r(\mathbb{R}^n)}\|b\|_{L^{r'}(\mathbb{R}^n)} \\ &\leqslant \|f\|_{M_r^p(\mathbb{R}^n)}.\end{aligned} \tag{4}$$

For any $g \in \mathcal{B}_{r'}^{p'}(\mathbb{R}^n)$, we have a family of (p', r')-block $\{b_k\}_{k=1}^{\infty}$ and sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that $g = \sum_{k=1}^{\infty} \lambda_k b_k$ and

$$\sum_k |\lambda_k| \leqslant C\|g\|_{\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)}. \tag{5}$$

Therefor, (4) and (5) give

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx = \int_{\mathbb{R}^n} |f(x)\sum_k \lambda_k b_k(x)|dx$$

$$\leqslant C \sum_k |\lambda_k| \int_{\mathbb{R}^n} |f(x)b_k(x)|dx$$
$$\leqslant C\|f\|_{M_r^p(\mathbb{R}^n)}\|g\|_{\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)}.$$

Hence, we have completed the proof of this lemma. □

The following lemma is the norm conjugate formula for $M_r^p(\mathbb{R}^n)$ and $\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)$.

Lemma 2.4. *Let $1 < r \leqslant p < \infty$. Then for any $f \in M_r^p(\mathbb{R}^n)$, we have constants $C_0, C_1 > 0$ such that*

$$C_0\|f\|_{M_r^p(\mathbb{R}^n)} \leqslant \sup_{b \in \mathfrak{b}_{r'}^{p'}} \int_{\mathbb{R}^n} |f(x)b(x)|dx \leqslant C_1\|f\|_{M_r^p(\mathbb{R}^n)}. \tag{6}$$

Proof. The inequality on the right hand side of (6) follows from (4). Next, we show the inequality on the left hand side of (6). According to the definition of Morrey space $M_r^p(\mathbb{R}^n)$, there exists a $B \in \mathbb{B}(\mathbb{R}^n)$ such that

$$\frac{1}{2}\|f\|_{M_r^p(\mathbb{R}^n)} < |B|^{\frac{1}{p}-\frac{1}{r}}\|f\chi_B\|_{L^r(\mathbb{R}^n)}.$$

From the norm conjugate formula for $L^r(\mathbb{R}^n)$, we have

$$\|f\chi_B\|_{L^r(\mathbb{R}^n)} = \sup_{\|g\|_{L^{r'}(\mathbb{R}^n)}\leqslant 1} \int_{\mathbb{R}^n} |f(x)\chi_B(x)g(x)|dx,$$

hence,

$$\frac{1}{2}\|f\|_{M_r^p(\mathbb{R}^n)} < |B|^{\frac{1}{p}-\frac{1}{r}} \int_{\mathbb{R}^n} |f(x)\chi_B(x)g(x)|dx = \int_{\mathbb{R}^n} |f(x)G(x)|dx,$$

where $G(x) = |B|^{\frac{1}{p}-\frac{1}{r}}\chi_B(x)g(x)$. Obviously, $G(x)$ is a (p', r')-block. Therefore, the inequality on the left hand side of (6) follows. □

The subsequent lemma gives an estimate of the action of the Hardy–Littlewood operator on blocks.

Lemma 2.5. *Let $1 < r \leqslant p < \infty$. For any $b \in \mathfrak{b}_{r'}^{p'}$, if $q > r$, we have*

$$\|(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}}\|_{\mathcal{B}_{r'}^{p'}(\mathbb{R}^n)} \leqslant C \tag{7}$$

for some $C > 0$ independent of b.

Proof. Let $b \in \mathfrak{b}_{r'}^{p'}$ with support $Q(x_0, l)$, $x_0 \in \mathbb{R}^n$, $l > 0$. For any $k \in \mathbb{N}$, let $Q_k = Q(x_0, 2^k l)$. Define $m_k = \chi_{Q_{k+1}\backslash Q_k}(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}}$, where $k \in \mathbb{N}\backslash\{0\}$ and $m_0 = \chi_{Q(x_0,l)}(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}}$. We have $\text{supp}(m_k) \subset Q_{k_1}\backslash Q_k$ and

$$(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}} = \sum_{k\in\mathbb{N}} m_k.$$

By the boundedness of Hardy–Littilewood maximal operator $\mathcal{M}$ on $L^{r'/q'}(\mathbb{R}^n)$ and (2), we have

$$\|m_0\|_{L^{r'}(\mathbb{R}^n)} = \|\chi_{Q(x_0,l)}(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}}\|_{L^{r'}(\mathbb{R}^n)}$$
$$\leqslant C\|\mathcal{M}(|b|^{q'})\|_{L^{r'/q'}(\mathbb{R}^n)}^{\frac{1}{q'}}$$
$$\leqslant C\||b|^{q'}\|_{L^{r'/q'}(\mathbb{R}^n)}^{\frac{1}{q'}}$$

$$= C\|b\|_{L^{r'}(\mathbb{R}^n)}$$
$$\leqslant C|Q(x_0,l)|^{\frac{1}{p}-\frac{1}{r}}.$$

That is, m_0 is a constant-multiple of an (p', r')-block. From the definition of Hardy–Littilewood maximal operator $\mathcal{M}$ and the Hölder inequality, we have

$$\begin{aligned}|m_k|^{q'} &= \chi_{Q_{k+1}\setminus Q_k}|\mathcal{M}(|b|^{q'})| \\ &\leqslant \frac{\chi_{Q_{k+1}\setminus Q_k}}{2^{kn}l^n}\int_{Q(x_0,l)}|b(x)|^{q'}dx \\ &\leqslant C\frac{\chi_{Q_{k+1}\setminus Q_k}}{2^{kn}l^n}\||b|^{q'}\|_{L^{r'/q'}(\mathbb{R}^n)}\|\chi_{Q(x_0,l)}\|_{L^{(r'/q')'}(\mathbb{R}^n)}\end{aligned} \tag{8}$$

for some $C > 0$ independent o f k. The fact that $\|\chi_Q\|_{L^r(\mathbb{R}^n)}\|\chi_Q\|_{L^{r'}(\mathbb{R}^n)} = |Q|$ and (8) assert that

$$\begin{aligned}\|m_k\|_{L^{r'}(\mathbb{R}^n)} &= \||m_k|^{q'}\|_{L^{r'/q'}(\mathbb{R}^n)}^{\frac{1}{q'}} \\ &\leqslant C\left(\frac{\|\chi_{Q_{k+1}\setminus Q_k}\|_{L^{r'/q'}(\mathbb{R}^n)}}{2^{kn}l^n}\frac{l^n}{\|\chi_{Q(x_0,l)}\|_{L^{r'/q'}(\mathbb{R}^n)}}\right)^{\frac{1}{q'}}\|b\|_{L^{r'}(\mathbb{R}^n)} \\ &\leqslant C\frac{\|\chi_{Q_{k+1}}\|_{L^{r'}(\mathbb{R}^n)}}{2^{\frac{kn}{q'}}\|\chi_{Q(x_0,l)}\|_{L^{r'}(\mathbb{R}^n)}}|Q(x_0,l)|^{\frac{1}{p}-\frac{1}{r}}.\end{aligned}$$

Denote $m_k = \delta_k b_k$, where

$$\delta_k = \frac{\|\chi_{Q_{k+1}}\|_{L^{r'}(\mathbb{R}^n)}}{2^{\frac{kn}{q'}}\|\chi_{Q(x_0,l)}\|_{L^{r'}(\mathbb{R}^n)}}.$$

Hence, b_k is a constant-multiple of an (p', r')-block and this constant is independent of k. From the definition of (p', r')-block and $\mathcal{B}^{p'}_{r'}(\mathbb{R}^n)$, a simple consequence that for any $b \in \mathfrak{b}^{p'}_{r'}$, $\|b\|_{\mathcal{B}^{p'}_{r'}(\mathbb{R}^n)} \leqslant 1$ holds true, which yields that

$$\|b_k\|_{\mathcal{B}^{p'}_{r'}(\mathbb{R}^n)} \leqslant C. \tag{9}$$

When $q > r$, we have

$$\sum_{k\in\mathbb{N}}\delta_k = \sum_{k\in\mathbb{N}}\frac{\|\chi_{Q_{k+1}}\|_{L^{r'}(\mathbb{R}^n)}}{2^{\frac{kn}{q'}}\|\chi_{Q(x_0,l)}\|_{L^{r'}(\mathbb{R}^n)}} \sim \sum_{k\in\mathbb{N}}2^{kn(\frac{1}{r'}-\frac{1}{q'})}$$

is finite. Combine this and (9), we obtain

$$\|(\mathcal{M}(|b|^{q'}))^{\frac{1}{q'}}\|_{\mathcal{B}^{p'}_{r'}(\mathbb{R}^n)} \leqslant \sum_{k\in\mathbb{N}}\delta_k\|b_k\|_{\mathcal{B}^{p'}_{r'}(\mathbb{R}^n)} \leqslant C.$$

Therefore, we have completed the proof of Lemma 2.5. □

According to [36, Definition 4.6], we give the definition of local-(M^p_r, q, d)-atom.

Definition 2.6. *Let $q \in [1,\infty]$. Assume that $d \in \mathbb{N}$ satisfies $d \geq d_X$, where $d_X := \lceil n(1/\theta - 1)\rceil$ with the same $0 < \theta < u \leq 1$ in Proposition 2.1. Then a measurable function a is called a local-(M^p_r, q, d)-atom if*

(i) *there exists a cube $Q \subset \mathbb{R}^n$ such that* $\mathrm{supp}(a) := \{x \in \mathbb{R}^n : a(x) \neq 0\} \subset Q$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq \frac{|Q|^{1/q}}{\|\chi_Q\|_{M^p_r(\mathbb{R}^n)}}$;

(iii) *if* $|Q| < 1$, *then* $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ *for any multi-index* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \leq d$.

Iida et al. proved the atomic decomposition of $M_r^p(\mathbb{R}^n)$ and $HM_r^p(\mathbb{R}^n)$ in [21]. Now we give the atomic decomposition of $hM_r^p(\mathbb{R}^n)$ due to [36, Theorem 4.8].

Lemma 2.7. *Let* $s \in (0,1]$, *and the same* d *in Definition 2.6. Then* $f \in hM_r^p(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'$ *and there exists a sequence* $\{a_j\}_{j=1}^\infty$ *of local-*(M_r^p, ∞, d)*-atoms supported, respectively, in cubes* $\{Q_j\}_{j=1}^\infty$ *and a sequence* $\{\lambda_j\}_{j=1}^\infty$ *of non-negative numbers such that*

$$f = \sum_{j=1}^\infty \lambda_j a_j \quad in \quad \mathcal{S}'(\mathbb{R}^n) \tag{10}$$

and

$$\left\| \left[\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{M_r^p(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right]^{1/s} \right\|_{M_r^p(\mathbb{R}^n)} < \infty.$$

Moreover,

$$\|f\|_{hM_r^p(\mathbb{R}^n)} \sim \inf \left\{ \left\| \left[\sum_{j=1}^\infty \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{M_r^p(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right]^{1/s} \right\|_{M_r^p(\mathbb{R}^n)} \right\},$$

where the infimum is taken over all decompositions of f *as in* (10) *and the positive equivalence constant are independent of* f *but may depend on* s.

The proof of Lemma 2.7 is just to replace the $X(\mathbb{R}^n)$ in [36, Theorem 4.8] with $M_r^p(\mathbb{R}^n)$. The following lemma is the vector-valued inequality of $\mathcal{M}_\alpha$ on Morrey space, which is a special case of [34, Theorem 2.6]. When $\alpha = 0$, the vector-valued inequality of Hardy–Littlewood maximal function can be found in [17].

Lemma 2.8. *Given* $0 < \alpha < n$, $1 < u < \infty$, $1 < r_1 \leqslant p < \frac{n}{\alpha}$, $1 < r_2 \leqslant q < \infty$ *satisfying* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ *and* $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\alpha}{n}$. *Then we have*

$$\left\| \left(\sum_j \left(\mathcal{M}_\alpha f_j \right)^u \right)^{\frac{1}{u}} \right\|_{M_{r_2}^q(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_j |f_j|^u \right)^{\frac{1}{u}} \right\|_{M_{r_1}^p(\mathbb{R}^n)},$$

where

$$\mathcal{M}_\alpha f(x) := \sup_{B \ni x} |B|^{\frac{\alpha}{n}-1} \int_B |f(y)| dy.$$

Moreover, when $1 < u < \infty$ *and* $1 < r \leqslant p < \infty$, *then we have*

$$\left\| \left(\sum_j \left(\mathcal{M} f_j \right)^u \right)^{\frac{1}{u}} \right\|_{M_r^p(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_j |f_j|^u \right)^{\frac{1}{u}} \right\|_{M_r^p(\mathbb{R}^n)}.$$

By using Lemma 2.8, we can get the following lemma.

Lemma 2.9. *Given some collection of cubes* $\{Q_j\}_{j=1}^\infty$, *let* $0 < r \leqslant p < \infty$ *and* $\tau > 1$, *there exists* $u > 1$ *and a constant* C *such that*

$$\left\| \sum_j \chi_{\tau Q_j} \right\|_{M_r^p(\mathbb{R}^n)} \leqslant C \left\| \sum_j \chi_{Q_j} \right\|_{M_r^p(\mathbb{R}^n)}.$$

To prove our result, we also need the concept of weights. For more information, see [4, 10, 14].

Definition 2.10. *Let $p \in [1, \infty)$ and w be a non-negative locally integrable function on $\mathbb{R}^n$. Then w is called an $A_p(\mathbb{R}^n)$ weight, denoted by $w \in A_p(\mathbb{R}^n)$, if, when $p \in (1, \infty)$,*

$$[w]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \left[\int_Q w(x)dx \right] \left\{ \frac{1}{|Q|} \int_B [w(x)]^{-\frac{1}{p-1}} dx \right\}^{p-1} < \infty,$$

and

$$[w]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \left[\int_Q w(x)dx \right] \left\{ \operatorname*{ess\,sup}_{x \in Q} [w(x)]^{-1} \right\} < \infty,$$

where the suprema are taken over all cubes $Q \in \mathcal{Q}(\mathbb{R}^n)$. Moreover, the class $A_\infty(\mathbb{R}^n)$ is defined by setting

$$A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1,\infty)} A_p(\mathbb{R}^n).$$

Now we recall the definition of weighted Lebesgue spaces and local weighted Hardy spaces. For more details, for example we refer to [15, Section 7].

Definition 2.11. *Let $p \in (0, \infty)$ and $w \in A_\infty(\mathbb{R}^n)$. The weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$*

$$\|f\|_{L^p_w(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} |f(x)|^p w(x)dx \right]^{\frac{1}{p}} < \infty.$$

Definition 2.12. *[7] Let $0 < p < \infty, w \in A_\infty(\mathbb{R}^n)$. Then the weighted local Hardy space $h^p_w(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'$ satisfying the quantity*

$$\|f\|_{h^p_w(\mathbb{R}^n)} := \|\mathcal{M}_{loc} f\|_{L^p_w(\mathbb{R}^n)} < \infty,$$

where the local grand maximal operator $\mathcal{M}_{loc}$ is same as in Definition 1.2.

3. Proof of Theorem 1.5

In this section, we will show the boundedness of T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ for $0 \le \delta < 1$ on local Hardy Morrey spaces by applying the atomic decomposition theory. The following lemmas provide the boundedness of pseudo-differential operator with $\sigma \in \mathcal{S}^{-\alpha}_{1,\delta}$ on Lebesgue spaces and local weighted Hardy spaces..

Lemma 3.1. *[8] Let T_σ is a pseudo-differential operator with $\sigma \in \mathcal{S}^{-\alpha}_{1,\delta}$ and $0 \le \delta < 1$. Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If a weight w is such that $w^q \in A_\infty$, then T_σ is bounded from $h^p_{w^p}(\mathbb{R}^n)$ to $h^q_{w^q}(\mathbb{R}^n)$.*

Now we give the definition of the absolutely continuous quasi-norm, which is given in [18, 28].

Definition 3.2. *A ball quasi-Banach function space $X(\mathbb{R}^n)$ is said to have an absolutely continuous quasi-norm if $\|\chi_{E_j}\|_{X(\mathbb{R}^n)} \downarrow 0$ whenever ${E_j}_{j=1}^\infty$ is a sequence of measurable sets that satisfies $E_j \supset E_{j+1}$ for all $j \in \mathbb{N}$ and $\bigcap_{j=1}^\infty E_j = \emptyset$.*

Since Morrey space $M^p_r(\mathbb{R}^n)$ has no absolutely continuous quasi-norm, to prove our main result we need the following lemma, which is proved in [19].

Lemma 3.3. *Let $r_0 \in (0, \infty)$ and a $p_0 \in (r_0, \infty)$ be same as Lemma 2.2. Then for any $\varepsilon \in (1 - \frac{r_0}{p_0}, 1)$, $M^p_r(\mathbb{R}^n)$ embeds continuously into $L^{r_0}_w(\mathbb{R}^n)$ with $w := [\mathcal{M}(\chi_{Q(\mathbf{0},1)})]^\varepsilon$.*

Now, we prove Theorem 1.5.

Proof of Theorem 1.5 For any $f \in hM^p_{r_1}(\mathbb{R}^n)$, using the atomic decomposition of local Hardy Morrey space in Lemma 2.7, we get that there exists a sequence $\{a_j\}_{j=1}^\infty$ of local-$(M^p_{r_1}, \infty, d)$-atoms supported, respectively, in cubes $\{Q_j\}_{j=1}^\infty$ and a sequence $\{\lambda_j\}_{j=1}^\infty$ of non-negative numbers such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n)$$

and

$$\left\| \left[\sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right]^{1/s} \right\|_{M^p_{r_1}(\mathbb{R}^n)} \leqslant C \|f\|_{hM^p_{r_1}(\mathbb{R}^n)}. \tag{11}$$

We can find $r_0 \in (0, \infty)$ and $p_0 \in (r_0, \frac{r_0}{1-r_0})$ which satisfies Lemma 2.2. Then $r_0 > 1 - \frac{r_0}{p_0}$, hence there exists a $\gamma \in (0, 1)$ such that $r_0\gamma \in (1 - \frac{r_0}{p_0}, 1)$. From this and Lemma 3.3, we get that $M^p_{r_1}(\mathbb{R}^n)$ embeds continuously into $L^{r_0}_{w^{r_0}}(\mathbb{R}^n)$ with $w := [\mathcal{M}(\chi_{Q(\mathbf{0},1)})]^\gamma$.

From this, we deduce that

$$\begin{aligned}
&\left\| \left\{ \sum_j \left(\lambda_j \frac{\|\chi_{Q_j}\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)}}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \frac{1}{\|\chi_{Q_j}\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)} \\
&= \left\| \left\{ \sum_j \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)} \\
&\leqslant C \left\| \left\{ \sum_j \left(\frac{\lambda_j}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_{M^p_{r_1}(\mathbb{R}^n)} \leqslant C\|f\|_{hM^p_{r_1}(\mathbb{R}^n)}.
\end{aligned}$$

Since for any $j \in \mathbb{N}$, a_j is a local-$(M^p_{r_1}, \infty, d)$-atom, we deduce that for any $j \in \mathbb{N}$, $\frac{\|\chi_{Q_j}\|_{M^p_{r_1}}}{\|\chi_{Q_j}\|_{L^{r_0}_{w^{r_0}}}} a_j$ is a local-$(L^{r_0}_{w^{r_0}}, \infty, d)$-atom. Similar to [36, (3.10)], we obtain that

$$\begin{aligned}
&\sum_j \left(\lambda_j \frac{\|\chi_{Q_j}\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)}}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \right) \left(\frac{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}{\|\chi_{Q_j}\|_{L^{r_0}_{w^{r_0}}(\mathbb{R}^n)}} a_j \right) \\
&= \sum_j \lambda_j a_j = f \quad \text{in} \quad \mathcal{S}' \quad \text{and} \quad h^{r_0}_{w^{r_0}}(\mathbb{R}^n).
\end{aligned}$$

From the definition of w and the range of $r_0\gamma$, we deduce that $w^{\widetilde{r_0}} \in A_\infty$, where $\frac{1}{\widetilde{r_0}} = \frac{1}{r_0} - \frac{\alpha}{n}$. Hence, using this and Lemma 3.1, we conclude that T_σ is bounded from $h^{r_0}_{w^{r_0}}(\mathbb{R}^n)$ to $h^{\widetilde{r_0}}_{w^{\widetilde{r_0}}}(\mathbb{R}^n)$. Thus, we deduce that $T_\sigma(f) = \sum_j \lambda_j T_\sigma(a_j)$ in $h^{\widetilde{r_0}}_{w^{\widetilde{r_0}}}(\mathbb{R}^n)$.

From this, we can obtain that, for $x \in \mathbb{R}^n$,

$$\left| \mathcal{M}_{loc} T_\sigma(f)(x) \right| \leqslant \sum_j |\lambda_j| |\mathcal{M}_{loc} T_\sigma \left(a_j \right)(x)|$$

$$\leqslant \sum_j |\lambda_j||\mathcal{M}_{loc}T_\sigma\left(a_j\right)(x)|\chi_{Q_j^*}(x) + \sum_j |\lambda_j||\mathcal{M}_{loc}T_\sigma\left(a_j\right)(x)|\chi_{Q_j^{*,c}}(x)$$
$$:= J_1 + J_2,$$

where $Q_j^* = 2\sqrt{n}Q_j$.

First we estimate J_1.

For any $s_0 \in (0, \min\{p, r_1, 1\})$, we have

$$\begin{aligned}\|J_1\|_{M^q_{r_2}(\mathbb{R}^n)} &= \left\|\sum_j |\lambda_j||Q_j^*|^{\frac{\alpha}{n}}\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma\left(a_j\right)|\chi_{Q_j^*}\right\|_{M^q_{r_2}(\mathbb{R}^n)}\\ &\leqslant C\left\|\left\{\sum_j\left(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}}\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma\left(a_j\right)|\chi_{Q_j^*}\right)^{s_0}\right\}^{\frac{1}{s_0}}\right\|_{M^q_{r_2}(\mathbb{R}^n)}\\ &= C\left\|\sum_j\left(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}}\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma\left(a_j\right)|\chi_{Q_j^*}\right)^{s_0}\right\|^{\frac{1}{s_0}}_{M^{q/s_0}_{r_2/s_0}(\mathbb{R}^n)}.\end{aligned}$$

We claim that

$$\begin{aligned}&\left\|\sum_j\left(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}}\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma\left(a_j\right)|\chi_{Q_j^*}\right)^{s_0}\right\|_{M^{q/s_0}_{r_2/s_0}(\mathbb{R}^n)}\\ &\leqslant C\left\|\sum_j\left(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}}\frac{1}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\chi_{Q_j^*}\right)^{s_0}\right\|_{M^{q/s_0}_{r_2/s_0}(\mathbb{R}^n)}.\end{aligned} \tag{12}$$

Now we prove our claim. Due to the size condition of a_j, the fact that $\mathcal{M}_{loc}$ is bounded on $L^s(\mathbb{R}^n)$ for all $1 < s \leqslant \infty$ and T_σ is bounded from $L^{p_0}(\mathbb{R}^n)$ to $L^{q_0}(\mathbb{R}^n)$, where $1 < p_0 \leqslant q_0 \leqslant \infty$, $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$, then for $q_0 > \frac{r_2}{s_0}$, $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$, we get that

$$\begin{aligned}\left\|\left(\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma a_j(x)|\right)^{s_0}\right\|_{L^{q_0/s_0}(\mathbb{R}^n)} &= \frac{1}{|Q_j^*|^{\frac{\alpha s_0}{n}}}\left\|\mathcal{M}_{loc}T_\sigma a_j\right\|^{s_0}_{L^{q_0}(\mathbb{R}^n)}\\ &\leqslant C\frac{1}{|Q_j^*|^{\frac{\alpha s_0}{n}}}\left\|T_\sigma a_j\right\|^{s_0}_{L^{q_0}(\mathbb{R}^n)}\\ &\leqslant C\frac{1}{|Q_j^*|^{\frac{\alpha s_0}{n}}}\left\|a_j\right\|^{s_0}_{L^{p_0}(\mathbb{R}^n)}\\ &\leqslant C\frac{1}{|Q_j^*|^{\frac{\alpha s_0}{n}}}\left(\frac{|Q_j|^{\frac{1}{p_0}}}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\right)^{s_0}\\ &\leqslant C\frac{|Q_j^*|^{\frac{s_0}{q_0}}}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}.\end{aligned} \tag{13}$$

We denote that $F_j = \left(\frac{1}{|Q_j^*|^{\frac{\alpha}{n}}}|\mathcal{M}_{loc}T_\sigma a_j(x)|\chi_{Q_j^*}\right)^{s_0}$. Observe that, $\frac{q_0}{s_0} > 1$, hence, for any $b \in \mathfrak{b}^{(q/s_0)'}_{(r_2/s_0)'}$, by the

Hölder inequality, (13) and the definition of Hardy–Littilewood maximal operator, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^n} F_j(x)b(x)dx\right| &\leqslant \|F_j\|_{L^{q_0/s_0}(\mathbb{R}^n)}\|b\chi_{Q_j^*}\|_{L^{(q_0/s_0)'}(\mathbb{R}^n)} \\ &\leqslant \frac{|Q_j^*|^{\frac{s_0}{q_0}}}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}\left(\int_{Q_j^*}|b(x)|^{(q_0/s_0)'}dx\right)^{\frac{1}{(q_0/s_0)'}} \\ &\leqslant \frac{|Q_j^*|}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}\left(\frac{1}{|Q_j^*|}\int_{Q_j^*}|b(x)|^{(q_0/s_0)'}dx\right)^{\frac{1}{(q_0/s_0)'}} \\ &\leqslant C\frac{|Q_j^*|}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}\inf_{x\in Q_j^*}(\mathcal{M}(|b|^{(q_0/s_0)'})(x))^{\frac{1}{(q_0/s_0)'}} \\ &\leqslant C\frac{1}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}\int_{Q_j^*}(\mathcal{M}(|b|^{(q_0/s_0)'})(x))^{\frac{1}{(q_0/s_0)'}}dx\end{aligned}$$

for some $C > 0$.

By the Hölder inequality and Lemma 2.3, the above inequalities yield that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\left(\sum_j(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}})^{s_0}F_j(x)\right)b(x)dx\right| \\ &\leqslant C\sum_j\frac{(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}})^{s_0}}{\|\chi_{Q_j}\|_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}}\int_{Q_j^*}(\mathcal{M}(|b|^{(q_0/s_0)'})(x))^{\frac{1}{(q_0/s_0)'}}dx \\ &\leqslant C\int_{\mathbb{R}^n}\left(\sum_j\frac{(|\lambda_j||Q_j^*|^{\frac{\alpha}{n}})^{s_0}}{\|\chi_{Q_j}\|_{M^p_r(\mathbb{R}^n)}}\chi_{Q_j^*}\right)(\mathcal{M}(|b|^{(q_0/s_0)'})(x))^{\frac{1}{(q_0/s_0)'}}dx \\ &\leqslant \left\|\sum_j\left(\frac{|\lambda_j||Q_j^*|^{\frac{\alpha}{n}}}{\|\chi_{Q_j}\|_{M^p_r(\mathbb{R}^n)}}\right)^{s_0}\chi_{Q_j^*}\right\|_{M^{q/s_0}_{r_2/s_0}(\mathbb{R}^n)}\left\|\mathcal{M}(|b|^{(q_0/s_0)'}))^{\frac{1}{(q_0/s_0)'}}\right\|_{\mathcal{B}^{(q/s_0)'}_{(r_2/s_0)'}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, Lemma 2.4 and 2.5 yield the (12). From the definition of $\mathcal{M}_\alpha$, it is easy to verify that

$$\left|Q_j^*\right|^{\frac{\alpha}{n}}\chi_{Q_j^*} \leq M_{\alpha s_0/2}(\chi_{Q_j^*})^{\frac{2}{s_0}}(x).$$

Combining this, (12), Lemma 2.8 and Lemma 2.9, we get that

$$\begin{aligned}\|J_1\|_{M^q_{r_2}(\mathbb{R}^n)} &\leqslant C\left\|\sum_j\left(\frac{|\lambda_j|M_{\alpha s_0/2}(\chi_{Q_j^*})^{\frac{2}{s_0}}}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\right)^{s_0}\right\|^{\frac{1}{s_0}}_{M^{q/s_0}_{r_2/s_0}(\mathbb{R}^n)} \\ &\leqslant C\left\|\sum_j\left(\frac{|\lambda_j|^{s_0}M_{\alpha s_0/2}(\chi_{Q_j^*})^2}{\|\chi_{Q_j}\|^{s_0}_{M^p_{r_1}(\mathbb{R}^n)}}\right)^{\frac{1}{2}}\right\|^{\frac{2}{s_0}}_{M^{2q/s_0}_{2r_2/s_0}(\mathbb{R}^n)} \\ &\leqslant C\left\|\sum_j\left(\frac{|\lambda_j|^{s_0}\chi_{Q_j^*}}{\|\chi_{Q_j}\|^{s_0}_{M^p_{r_1}(\mathbb{R}^n)}}\right)^{\frac{1}{2}}\right\|^{\frac{2}{s_0}}_{M^{2p/s_0}_{2r_1/s_0}(\mathbb{R}^n)}\end{aligned}$$

$$= C\left\|\sum_j \frac{|\lambda_j|^{s_0}\chi_{Q_j^*}}{\|\chi_{Q_j}\|^{s_0}_{M^p_{r_1}(\mathbb{R}^n)}}\right\|^{\frac{1}{s_0}}_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}$$
$$\leqslant C\left\|\sum_j \frac{|\lambda_j|^{s_0}\chi_{Q_j}}{\|\chi_{Q_j}\|^{s_0}_{M^p_{r_1}(\mathbb{R}^n)}}\right\|^{\frac{1}{s_0}}_{M^{p/s_0}_{r_1/s_0}(\mathbb{R}^n)}$$
$$= C\left\|\left\{\sum_j \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\right)^{s_0}\chi_{Q_j}\right\}^{\frac{1}{s_0}}\right\|_{M^p_{r_1}(\mathbb{R}^n)}.$$

Hence, $\|J_1\|_{M^q_{r_2}(\mathbb{R}^n)} \leqslant C\|f\|_{hM^p_{r_1}(\mathbb{R}^n)}$.

Now we estimate J_2. We denote T^ε_σ the composition operator $a \to \phi_\varepsilon * T_\sigma\left(a_j\right)$ with the kernel K_ε for some $\phi \in \mathcal{S}$, where the K_ε is a kernel associated with ϕ_ε. Indeed, $\phi_\varepsilon * f$ can be written as

$$\phi_\varepsilon * f(x) = \int e^{2\pi i x\cdot\xi}\hat{\phi}(\varepsilon\xi)\hat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n)$$

and regarded as a pseudo-differential operator with the symbol $\hat{\phi}(\varepsilon\xi)$. Moreover, $\xi \to \hat{\phi}(\varepsilon\xi)$ belongs to $S^0_{1,0}(\mathbb{R}^N)$ uniformly in $0 < \varepsilon \leq 1$ and T^ε_σ is obtained by composing on the left the pseudo-differential operator $f \to \phi_\varepsilon * f$ with T_σ. For more details, see [16, Remark 3.1]. From [16, Remark 3.1], we also know that if $M \in \mathbb{N}$ and $M - \alpha + n > 0$, then for the multi-index α, β, the kernel K_ε satisfies

$$\sup_{|\alpha|+|\beta|=M}\left|\partial^\alpha_x\partial^\beta_y K_\varepsilon(x,y)\right| \leqslant C\frac{1}{|x-y|^{M-\alpha+n}}, \quad x \neq y. \tag{14}$$

Furthermore, for each $L > L_0$, there exists an $L_0 \in \mathbb{N}$ such that

$$\sup_{|x-y|\geqslant 1/2}|x-y|^L\left|\partial^\alpha_x\partial^\beta_y K_\varepsilon(x,y)\right| \leqslant C. \tag{15}$$

We divide J_2 in two cases. When $|Q_j| \leqslant 1$, a_j has zero vanishing moment up to the order d. Let $P_N(x,y)$ be the Taylor polynomial of degree d of the kernel of T^ε_σ centered at z_j, we have

$$\begin{aligned}
T^\varepsilon_\sigma\left(a_j\right)(x) &= \int_{Q_j} K_\varepsilon(x,y)a_j(y)dy \\
&= \int_{Q_j}\left[K_\varepsilon(x,y) - P_N(x,y)\right]a_j(y)dy \\
&= \int_{Q_j}\sum_{|\gamma|=d+1}\left(\partial^\gamma_y K_\varepsilon\right)(x,\xi)\frac{\left(y-z_j\right)^\gamma}{\gamma!}a_j(y)dy
\end{aligned}$$

for some ξ on the line segment joining y to z_j.

Since $x \in (Q_j^*)^c$, we know that $|x-\xi| \geqslant \frac{1}{2}\left|x - z_j\right|$ and $\left|y - z_j\right| \leqslant \ell(Q_j)$. By this, the estimate of K_ε in (14), the size condition of local-$(M^p_{r_1}, \infty, d)$-atom and the Hölder inequality, we get that

$$|T^\varepsilon_\sigma a_j(x)| \leqslant C\int_{Q_j}\sum_{|\gamma|=d+1}|\left(\partial^\gamma_y K_\varepsilon\right)(x,\xi)|\frac{|y-z_j|^\gamma}{\gamma!}|a_j(y)|dy$$

$$
\begin{aligned}
&\leqslant C\int_{Q_j}\frac{\left|y-z_j\right|^{d+1}}{(|x-\xi|)^{n+d+1-\alpha}}|a_j(y)|dy\\
&\leqslant C\frac{|Q_j|^{\frac{d+1}{n}+1}}{|x-z_j|^{n+d+1-\alpha}\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\\
&\leqslant C\frac{|Q_j|^{\frac{d+1}{n}+1}}{(|x-z_j|+l(Q_j))^{n+d+1-\alpha}\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}.
\end{aligned}\tag{16}
$$

When $|Q_j|>1$, for $x\in Q_j^{*,c}$ and $y\in Q_j$, we have $|x-y|\sim|x-z_j|$ and $|x-y|\geqslant 1/2$. By this, the estimate of K_ε in (15), give a sufficiently large $L>L_0$ and $x\in Q_j^{*,c}$, we get that

$$
\begin{aligned}
|T^\varepsilon_\sigma a_j(x)| &= \left|\int_{Q_j}K_\varepsilon(x,y)a_j(y)dy\right|\\
&\leqslant C\int_{Q_j}|K_\varepsilon(x,y)||c_j(y)|dy\\
&\leqslant C\frac{|Q_j|}{|x-z_j|^L\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\\
&\leqslant C\frac{|Q_j|^{1+\frac{d+1}{n}}}{(|x-z_j|+l(Q_j))^L\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}.
\end{aligned}\tag{17}
$$

Choose $L=n+d+1-\alpha$. Then from (16) and (17), we obtain

$$
\|J_2\|_{M^q_{r_2}(\mathbb{R}^n)}\leqslant C\left\|\sum_j|\lambda_j|\frac{|Q_j|^{1+\frac{d+1}{n}}\chi_{Q_j^{*,c}}}{(|x-z_j|+l(Q_j))^{n+d+1-\alpha}\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\right\|_{M^q_{r_2}(\mathbb{R}^n)}.
$$

Moreover,

$$
\frac{|Q_j|^{1+\frac{d+1}{n}}}{(|x-z_j|+l(Q_j))^{n+d+1-\alpha}}\leqslant C(\mathcal{M}_{\alpha/\tau}(\chi_{Q_j}))^\tau,\tag{18}
$$

where $\tau=\frac{n+d+1}{n}$. Note that $\frac{1}{\tau p}-\frac{1}{\tau q}=\frac{\alpha/\tau}{n}$ and $\frac{1}{\tau r_1}-\frac{1}{\tau r_2}=\frac{\alpha/\tau}{n}$. Then we get that

$$
\begin{aligned}
\|J_2\|_{M^q_{r_2}(\mathbb{R}^n)} &\leqslant C\left\|\sum_j\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\left[\mathcal{M}_{\alpha/\tau}(\chi_{Q_j})\right]^\tau\right\|_{M^q_{r_2}(\mathbb{R}^n)}\\
&\leqslant C\left\|\left(\sum_j\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\left[\mathcal{M}_{\alpha/\tau}(\chi_{Q_j})\right]^\tau\right)^{\frac{1}{\tau}}\right\|^\tau_{M^{\tau q}_{\tau r_2}(\mathbb{R}^n)}\\
&\leqslant C\left\|\left(\sum_j\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\chi_{Q_j}\right)^{\frac{1}{\tau}}\right\|^\tau_{M^{\tau p}_{\tau r_1}(\mathbb{R}^n)}\\
&= C\left\|\sum_j\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}}\chi_{Q_j}\right\|_{M^p_{r_1}(\mathbb{R}^n)}
\end{aligned}
$$

$$\leqslant C \left\| \left\{ \sum_j \left(\frac{|\lambda_j|}{\|\chi_{Q_j}\|_{M^p_{r_1}(\mathbb{R}^n)}} \right)^s \chi_{Q_j} \right\}^{\frac{1}{s}} \right\|_{M^p_{r_1}(\mathbb{R}^n)},$$

where the first inequality follows by (18) and the third one follows by Lemma 2.8. Hence, $\|J_2\|_{M^q_{r_2}(\mathbb{R}^n)} \leqslant C\|f\|_{hM^p_{r_1}(\mathbb{R}^n)}$.

This finishes the proof of Theorem 1.5. □

By employing a similar but simpler argument to that used in the proof of Theorem 1.5, we obtain the following corollary.

Corollary 3.4. *Let $\alpha \in [0, n)$, $0 \le \delta < 1$, $0 < r_1 \le p < \frac{n}{\alpha}$, $0 < r_2 \le q < \infty$ satisfying $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{1}{r_1} - \frac{1}{r_2} = \frac{\alpha}{n}$. Then the T_σ with $\sigma \in S^{-\alpha}_{1,\delta}$ maps continuously $hM^p_{r_1}(\mathbb{R}^n)$ to $M^q_{r_2}(\mathbb{R}^n)$.*

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