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Isometric timelike helicoidal and rotational surfaces in 4–dimensional Minkowski space

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Abstract. In this paper, we examine classical Bour's theorem for timelike helicoidal surfaces in 4-dimensional Minkowski space. Then, we characterize a pair of isometric helicoidal and rotational surface which have same Gauss map. Also, we provide the parametrizations of such isometric surfaces. Finally, we introduce some examples and plot the corresponding graphs by using Wolfram Mathematica 10.4.

1. Introduction

One of the most important knowledge in the surface theory is that the right helicoid and catenoid is only minimal ruled surface and minimal rotational surface, respectively. Also, it is known that they have same Gauss map [16]. In this context, Bour's theorem is a quite popular theorem given as follows.

Bour's theorem.[4] A generalized helicoid is isometric to a rotational surface so that helices on the helicoid correspond to parallel circles on the rotational surface.

By using Bour's theorem, do Carmo and Dajczer [7] investigated helicoidal surfaces with constant mean curvature in $\mathbb{E}^3$. Also, Sasahara [22] studied spacelike helicoidal surfaces with constant mean curvature in 3-dimensional Minkowski space $\mathbb{E}^3_1$. In 2000, Ikawa [16] gave the parametrizations of the pairs of surface of Bour's theorem which have same Gauss map in $\mathbb{E}^3$. Ikawa [17] studied on Bour's theorem for spacelike and timelike generalized helicoid with non–null and null axis in $\mathbb{E}^3_1$. Also, Güler and Vanlı [12] introduced Bour's theorem for generalized helicoid with null axis in $\mathbb{E}^3_1$ and Ji and Kim [18] proved that it holds for cubic screw motion in $\mathbb{E}^3_1$. Güler et al. [13] investigated Bour's theorem for the Gauss map of generalized helicoid in $\mathbb{E}^3$. As a generalization, Güler and Yaylı [14] studied Bour's theorem for helicoidal surfaces in $\mathbb{E}^3$.

In 2017, Hieu and Thang [15] studied Bour's theorem for helicoidal surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ and they proved that if the Gauss maps of isometric surfaces are same, then they are hyperplanar

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and minimal. Also, they gave the parametrizations of such minimal surfaces. Nowadays, Babaarslan et al. [2] studied on Bour's theorem for these three kinds of spacelike helicoidal surfaces in $\mathbb{E}^4_1$ defined in [1].

Apart from Euclidean and pseudo-Euclidean spaces, Bour's theorem has been extended to the different ambient space such as the product spaces $\mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ and Heisenberg group by Earp and Toubiana [10, 11], the spaceforms by Ordónẽs [21], Bianchi-Cartan-Vranceanu (BCV) space by Caddeo, Onnis and Piu [5]. Nowadays, Domingos, Onnis, Piu [8] investigated Bour's theorem for surfaces that are invariant under the action of a one-parameter group of isometries of a Riemannian 3-manifold.

In this paper, we investigate Bour's theorem on four kinds of timelike helicoidal surfaces in 4-dimensional Minkowski space. On the other hand, it is known that a timelike helicoidal surface in $\mathbb{E}^4_1$ could have spacelike or timelike meridian curve which make the classification richer than a spacelike helicoidal surface in $\mathbb{E}^4_1$. In this context, there is also a timelike helicoidal surface obtained by hyperbolic rotation (called as helicoidal surface of type IIb) different from the surfaces in [2]. We get the characterizations of isometric helicoidal and rotational surfaces whose Gauss maps are identical. Also, we present the parametrizations of isometric pair of surfaces having same Gauss map. Finally, we give some examples by using Wolfram Mathematica 10.4.

2. Preliminaries

In this section, we recall some basic definitions and formulas in 4-dimensional Minkowski space $\mathbb{E}^4_1$. For more information, we refer to [20].

A metric tensor g is symmetric, bilinear, non-degenarate and (0,2) tensor field in $\mathbb{E}^4_1$ defined by

$$g(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 \tag{1}$$

for the vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$.

The causal character of a vector $\mathbf{x} \in \mathbb{E}^4_1$ is spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = 0$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike (null) if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq 0$.

A curve in $\mathbb{E}^4_1$ is a smooth mapping $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^4_1$, where I is an open interval. Then, α is a regular curve if $\alpha'(t) \neq 0$ for all $t \in I$. Also, α is spacelike (timelike, lightlike) if all of its tangent vectors $\alpha'(t)$ spacelike (timelike, lightlike). For later use, we mention the definition of a circle in $\mathbb{E}^3_1$ as follows.

Definition 2.1. *[19] We suppose that the plane P involving the circle is the plane of equation, $x_3 = 0$, $x_1 = 0$ or $x_2 - x_3 = 0$, if P is spacelike, timelike or lightlike, respectively. Thus, a circle $C \in \mathbb{E}^3_1$ can be defined as follows:*

- *If $P \equiv \{x_3 = 0\}$, then C is an Euclidean circle $\alpha(s) = p + r(\cos s, \sin s, 0)$ with center $p \in P$ and radius $r > 0$.*
- *If $P \equiv \{x_1 = 0\}$, then C is a spacelike hyperbola $\alpha(s) = p + r(0, \sinh s, \cosh s)$ or C is a timelike hyperbola $\alpha(s) = p + r(0, \cosh s, \sinh s)$, where $p \in P$ and $r > 0$ is the radius.*
- *If $P \equiv \{x_2 - x_3 = 0\}$, then C is spacelike parabola $\alpha(s) = p + (s, rs^2, rs^2)$, where $p \in P$ and $r > 0$.*

Assume that $X : D \subset \mathbb{R}^2 \to \mathbb{E}^4_1$ is a smooth parametric surface in $\mathbb{E}^4_1$ with a coordinate system $\{u, v\}$, where D is an open subset of $\mathbb{R}^2$. The tangent plane of X at p is given by $T_pX = \text{span}\{X_u, X_v\}$. The first fundamental form (or line element) of X is given by

$$g = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{2}$$

where $g_{11} = \langle X_u, X_u \rangle, g_{12} = g_{21} = \langle X_u, X_v \rangle$ and $g_{22} = \langle X_v, X_v \rangle$. When $W = \det(g) = g_{11}g_{22} - g_{12}^2 \neq 0$, the surface X is non-degenerate, namely, when $W > 0$, X is a spacelike surface and when $W < 0$, X is a timelike surface.

Let $\{e_1, e_2, N_1, N_2\}$ be a local orthonormal frame on the surface X in $\mathbb{E}^4_1$ such that e_1, e_2 are tangent to X and N_1, N_2 are normal to X. The coefficients of the second fundamental form tensor according to N_i, $(i = 1, 2)$ are given by

$$b^i_{11} = \langle X_{uu}, N_i \rangle, \ \ b^i_{12} = b^i_{21} = \langle X_{uv}, N_i \rangle, \ \ b^i_{22} = \langle X_{vv}, N_i \rangle. \tag{3}$$

The mean curvature vector H of X in $\mathbb{E}^4_1$ is given by

$$H = \epsilon_1 H_1 N_1 + \epsilon_2 H_2 N_2, \tag{4}$$

where the components H_i of H is $H_i = \dfrac{b^i_{11} g_{22} - 2b^i_{12} g_{12} + b^i_{22} g_{11}}{2W}$ for $i = 1, 2$, $\epsilon_1 = \langle N_1, N_1 \rangle$ and $\epsilon_2 = \langle N_2, N_2 \rangle$. When the mean curvature vector H of X is zero, X is called as a minimal (maximal) surface in $\mathbb{E}^4_1$.

In [6], the definition of the Gauss map was given as follows. Grasmanian manifold $G(2, 4)$ is a space formed by all oriented 2-dimensional planes passing through the origin in $\mathbb{E}^4_1$. Oriented 2-dimensional planes passing through the origin in $\mathbb{E}^4_1$ can be defined by the unit 2-vectors. 2-vectors are elements of space $\bigwedge^2 \mathbb{E}^4_1$, that is, they are obtained with the help of wedge product ($\bigwedge$) of vectors. The Gauss map corresponds to the oriented tangent space of surface X in $\mathbb{E}^4_1$ to every point of X. Thus, it is defined as

$$\nu : X \to G(2, 4) \subset \mathbb{E}^6_t;\ \nu(p) = (e_1 \wedge e_2)(p). \tag{5}$$

Now, we suppose that X is a timelike surface in $\mathbb{E}^4_1$, that is, $W < 0$. Thus, we can choose an orthonormal tangent frame field e_1, e_2 on X as below:

$$e_1 = \frac{1}{\sqrt{\epsilon g_{11}}} X_u, \quad e_2 = \frac{1}{\sqrt{-\epsilon W g_{11}}} (g_{11} X_v - g_{12} X_u), \tag{6}$$

where $\epsilon = \langle e_1, e_1 \rangle = -\langle e_2, e_2 \rangle$. Thus, the Gauss map ν of X at a point p can be given by

$$\nu = \frac{\epsilon}{\sqrt{-W}} X_u \wedge X_v. \tag{7}$$

In [1], the definition of a rotational surface and a helicoidal surface in $\mathbb{E}^4_1$ was given as follows. We suppose that $\beta : I \to \Pi$ is a regular curve in a hyperplane $\Pi \subset \mathbb{E}^4_1$ and P is a 2-plane in Π. When β is rotated about P, then the resulting surface is a rotational surface in $\mathbb{E}^4_1$. As a generalization, we suppose that when β rotates about P, it simultaneously translates along a line l which is parallel to P. Also, the speed of such translation is proportional to the speed of this rotation. Then, the resulting surface is a helicoidal surface in $\mathbb{E}^4_1$.

3. Helicoidal Surface of Type I

In this section, we study on Bour's theorem for timelike helicoidal surface of type I in $\mathbb{E}^4_1$ and we analyse the Gauss maps of isometric pair of surfaces.

Let $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be a standard orthonormal basis of $\mathbb{E}^4_1$, where $\eta_1 = (1, 0, 0, 0)$, $\eta_2 = (0, 1, 0, 0)$, $\eta_3 = (0, 0, 1, 0)$ and $\eta_4 = (0, 0, 0, 1)$. We choose as a timelike 2-plane $P_1 = \text{span}\{\eta_3, \eta_4\}$, a hyperplane $\Pi_1 = \text{span}\{\eta_1, \eta_3, \eta_4\}$ and a line $l_1 = \text{span}\{\eta_4\}$. Also, we suppose that $\beta_1 : I \longrightarrow \Pi_1 \subset \mathbb{E}^4_1$; $\beta_1(u) = (x(u), 0, z(u), w(u))$ is a regular curve, where $x(u) \neq 0$. By using the definition of helicoidal surface, the parametrization of X_1 (called as the helicoidal surface of type I) is given by

$$X_1(u, v) = (x(u) \cos v, x(u) \sin v, z(u), w(u) + \lambda v), \tag{8}$$

where $0 \leq v < 2\pi$ and $\lambda \in \mathbb{R}^+$. When w is a constant function, X_1 is called as right helicoidal surface of type I. Also, when z is a constant function, X_1 is just a helicoidal surface in $\mathbb{E}^3_1$ (see [2]). For $\lambda = 0$, the helicoidal surface given by (8) reduces to the rotational surface of elliptic type in $\mathbb{E}^4_1$ (see [9] and [3]).

By a direct calculation, we get the induced metric of X_1 given as follows.

$$ds^2_{X_1} = (x'^2(u) + z'^2(u) - w'^2(u))du^2 - 2\lambda w'(u) du dv + (x^2(u) - \lambda^2) dv^2 \tag{9}$$

with $W = (x^2(u) - \lambda^2)(x'^2(u) + z'^2(u)) - x^2(u)w'^2(u) < 0$ for all $u \in I \subset \mathbb{R}$. Then, we choose an orthonormal frame field $\{e_1, e_2, N_1, N_2\}$ on X_1 in $\mathbb{E}^4_1$ such that e_1, e_2 are tangent to X_1 and N_1, N_2 are normal to X_1 as follows.

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{\epsilon g_{11}}} X_{1u}, \quad e_2 = \frac{1}{\sqrt{-\epsilon W g_{11}}}(g_{11}X_{1v} - g_{12}X_{1u}), \\ N_1 &= \frac{1}{\sqrt{x'^2 + z'^2}}(z'\cos v, z'\sin v, -x', 0), \\ N_2 &= \frac{1}{\sqrt{-W(x'^2+z'^2)}}(xx'w'\cos v - \lambda(x'^2+z'^2)\sin v, xx'w'\sin v + \lambda(x'^2+z'^2)\cos v, xz'w', x(x'^2+z'^2)), \end{aligned} \tag{10}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \epsilon = \pm 1$ and $\langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 1$. For $\epsilon = 1$, the surface X_1 has a spacelike meridian curve. Otherwise, it has a timelike meridian curve. By direct computations, we get the coefficients of the second fundamental form given as follows.

$$\begin{aligned} b^1_{11} &= \frac{x''z' - x'z''}{\sqrt{x'^2+z'^2}}, \quad b^1_{12} = b^1_{21} = 0, \quad b^1_{22} = -\frac{xz'}{\sqrt{x'^2+z'^2}}, \\ b^2_{11} &= \frac{x(w'(x'x''+z'z'') - w''(x'^2+z'^2))}{\sqrt{-W(x'^2+z'^2)}}, \quad b^2_{12} = b^2_{21} = \frac{\lambda x'\sqrt{x'^2+z'^2}}{\sqrt{-W}}, \\ b^2_{22} &= -\frac{x^2x'w'}{\sqrt{-W(x'^2+z'^2)}}. \end{aligned} \tag{11}$$

Thus, the mean curvature vector H^{X_1} of X_1 in $\mathbb{E}^4_1$ is $H^{X_1} = H^{X_1}_1 N_1 + H^{X_1}_2 N_2$, where N_1, N_2 are normal vector fields in (10), $H^{X_1}_1$ and $H^{X_1}_2$ are given by

$$\begin{aligned} H^{X_1}_1 &= \frac{(x^2-\lambda^2)(x''z' - x'z'') - xz'(x'^2+z'^2-w'^2)}{2W\sqrt{x'^2+z'^2}}, \\ H^{X_1}_2 &= \frac{x'w'(2\lambda^2 - x^2)(x'^2+z'^2) + x^2x'w'^3 - x(x^2-\lambda^2)(x'(x'w''-x''w') + z'(z'w''-w'z''))}{2\sqrt{-W^3(x'^2+z'^2)}}. \end{aligned} \tag{12}$$

3.1. Bour's Theorem and the Gauss map of helicoidal surface of type I

Let define the two subsets I_1 and I_2 of I as $I_1 = \{u \in I \subset \mathbb{R} \mid x^2(u) - \lambda^2 > 0\}$ and $I_2 = \{u \in I \subset \mathbb{R} \mid x^2(u) - \lambda^2 < 0\}$.

Theorem 3.1. *A timelike helicoidal surface of type I in $\mathbb{E}^4_1$ given by (8) is isometric to one of the following timelike rotational surfaces in $\mathbb{E}^4_1$:*

(i)

$$R^1_1(u,v) = \begin{pmatrix} \sqrt{x^2(u)-\lambda^2}\cos\left(v - \int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \\ \sqrt{x^2(u)-\lambda^2}\sin\left(v - \int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \\ \int \frac{a(u)x(u)x'(u)}{\sqrt{x^2(u)-\lambda^2}}du \\ \int \frac{b(u)x(u)x'(u)}{\sqrt{x^2(u)-\lambda^2}}du \end{pmatrix} \tag{13}$$

so that spacelike helices on the timelike helicoidal surface of type I correspond to parallel spacelike circles on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u) - b^2(u) = \frac{x^2(u)(z'^2(u) - w'^2(u)) - \lambda^2(x'^2(u)+z'^2(u))}{x^2(u)x'^2(u)} \tag{14}$$

with $x'(u) \neq 0$ for all $u \in I_1 \subset I$.

(ii)

$$R_1^2(u,v)=\begin{pmatrix} \int \frac{a(u)x(u)x'(u)}{\sqrt{x^2(u)-\lambda^2}}du \\ \int \frac{b(u)x(u)x'(u)}{\sqrt{x^2(u)-\lambda^2}}du \\ \sqrt{x^2(u)-\lambda^2}\sinh\left(v-\int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \\ \sqrt{x^2(u)-\lambda^2}\cosh\left(v-\int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \end{pmatrix} \tag{15}$$

so that spacelike helices on the timelike helicoidal surface of type I correspond to parallel spacelike hyperbolas on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u)+b^2(u)=\frac{(x^2(u)-\lambda^2)(x'^2(u)+z'^2(u))+x^2(u)(x'^2(u)-w'^2(u))}{x^2(u)x'^2(u)} \tag{16}$$

with $x'(u)\neq 0$ for all $u\in I_1\subset I$.

(iii)

$$R_1^3(u,v)=\begin{pmatrix} -\int \frac{a(u)x(u)x'(u)}{\sqrt{\lambda^2-x^2(u)}}du \\ -\int \frac{b(u)x(u)x'(u)}{\sqrt{\lambda^2-x^2(u)}}du \\ \sqrt{\lambda^2-x^2(u)}\cosh\left(v+\int \frac{\lambda w'(u)}{\lambda^2-x^2(u)}du\right) \\ \sqrt{\lambda^2-x^2(u)}\sinh\left(v+\int \frac{\lambda w'(u)}{\lambda^2-x^2(u)}du\right) \end{pmatrix} \tag{17}$$

so that timelike helices on the timelike helicoidal surface of type I correspond to parallel timelike hyperbolas on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u)+b^2(u)=\frac{(\lambda^2-x^2(u))(x'^2(u)+z'^2(u))-x^2(u)(x'^2(u)-w'^2(u))}{x^2(u)x'^2(u)} \tag{18}$$

with $x'(u)\neq 0$ for all $u\in I_2\subset I$.

Proof. Assume that X_1 is a timelike helicoidal surface of type I in $\mathbb{E}_1^4$ defined by (8). Then, we have the induced metric of X_1 given by (9). Now, we will find new coordinates $\bar{u},\bar{v}$ such that the metric becomes $ds^2_{X_1}=F(\bar{u})d\bar{u}^2+G(\bar{u})d\bar{v}^2$ where $F(\bar{u})$ and $G(\bar{u})$ are smooth functions. Set $\bar{u}=u$ and $\bar{v}=v-\displaystyle\int\frac{\lambda w'(u)}{x^2(u)-\lambda^2}du$. Since Jacobian $\dfrac{\partial(\bar{u},\bar{v})}{\partial(u,v)}$ is nonzero, it follows that $\{\bar{u},\bar{v}\}$ are new parameters of X_1. According to the new parameters, the equation (9) becomes

$$ds^2_{X_1}=\left(x'^2(u)+z'^2(u)-w'^2(u)-\frac{\lambda^2 w'^2(u)}{x^2(u)-\lambda^2}\right)du^2+(x^2(u)-\lambda^2)d\bar{v}^2. \tag{19}$$

Then, we consider the following cases.

Case(i.) Assume that I_1 is dense in the interval I. First, we consider a timelike rotational surface R_1 in $\mathbb{E}_1^4$ given by

$$R_1(k,t)=(n(k)\cos t, n(k)\sin t, s(k), r(k)) \tag{20}$$

whose the induced metric is $ds^2_{R_1}=(\dot{n}^2(k)+\dot{s}^2(k)-\dot{r}^2(k))dk^2+n^2(k)dt^2$ with $n(k)>0$. Here, "$\cdot$" denotes the derivative with respect to k. Comparing the first fundamental forms, we get an isometry by taking $\bar{v}=t$, $n(k)=\sqrt{x^2(u)-\lambda^2}$ and

$$\left(x'^2(u)+z'^2(u)-w'^2(u)-\frac{\lambda^2 w'^2(u)}{x^2(u)-\lambda^2}\right)du^2=(\dot{n}^2(k)+\dot{s}^2(k)-\dot{r}^2(k))dk^2. \tag{21}$$

Set $a(u) = \frac{\dot{s}(k)}{\dot{n}(k)}$ and $b(u) = \frac{\dot{r}(k)}{\dot{n}(k)}$. Then, we obtain

$$s = \int \frac{a(u)x(u)x'(u)}{\sqrt{x^2(u) - \lambda^2}}du, \quad r = \int \frac{b(u)x(u)x'(u)}{\sqrt{x^2(u) - \lambda^2}}du. \tag{22}$$

Thus, we get an isometric timelike rotational surface R_1^1 given by (13) satisfying (14). It can be easily seen that a spacelike helix on X_1 which is defined by $u = u_0$ for a constant u_0 corresponds to the parallel spacelike circle on R_1^1 lying on the plane $\{x_3 = c_3, x_4 = c_4\}$ with the radius $\sqrt{x_0^2 - \lambda^2}$ for constants c_3 and c_4, i.e., $R_1^1(u_0, v) = (\sqrt{x_0^2 - \lambda^2}\cos v, \sqrt{x_0^2 - \lambda^2}\sin v, c_3, c_4)$.

Secondly, we consider a timelike rotational surface R_{2a} in $\mathbb{E}_1^4$ given by

$$R_{2a}(k, t) = (n(k), p(k), r(k)\sinh t, r(k)\cosh t) \tag{23}$$

whose the induced metric is

$$ds^2_{R_{2a}} = (\dot{n}^2(k) + \dot{p}^2(k) - \dot{r}^2(k))dk^2 + r^2(k)dt^2 \tag{24}$$

with $r(k) > 0$. Similarly, from the equations (19) and (24), we get an isometry by taking $\bar{v} = t$, $r(k) = \sqrt{x^2(u) - \lambda^2}$ and we have

$$\left(x'^2(u) + z'^2(u) - w'^2(u) - \frac{\lambda^2 w'^2(u)}{x^2(u) - \lambda^2}\right)du^2 = (\dot{n}^2(k) + \dot{p}^2(k) - \dot{r}^2(k))dk^2. \tag{25}$$

Set $a(u) = \frac{\dot{n}(k)}{\dot{r}(k)}$ and $b(u) = \frac{\dot{p}(k)}{\dot{r}(k)}$. Then, we obtain

$$n = \int \frac{a(u)x(u)x'(u)}{\sqrt{x^2(u) - \lambda^2}}du, \quad p = \int \frac{b(u)x(u)x'(u)}{\sqrt{x^2(u) - \lambda^2}}du. \tag{26}$$

Thus, we get an isometric timelike rotational surface R_1^2 given by (15) satisfying (16). It can be easily seen that a spacelike helix on X_1 corresponds to the parallel spacelike hyperbola lying on the plane $\{x_1 = c_1, x_2 = c_2\}$ for constants c_1 and c_2, i.e., $R_1^2(u_0, v) = (c_1, c_2, \sqrt{x_0^2 - \lambda^2}\sinh v, \sqrt{x_0^2 - \lambda^2}\cosh v)$.

Case(ii.) Assume that I_2 is dense in the interval I. We consider a timelike rotational surface R_{2b} in $\mathbb{E}_1^4$ given by

$$R_{2b}(k, t) = (n(k), p(k), s(k)\cosh t, s(k)\sinh t) \tag{27}$$

whose the induced metric is

$$ds^2_{R_{2b}} = (\dot{n}^2(k) + \dot{p}^2(k) + \dot{s}^2(k))dk^2 - s^2(k)dt^2 \tag{28}$$

with $s(k) > 0$. Considering the equations (19) and (28), we get an isometry by taking $\bar{v} = t$, $s(k) = \sqrt{\lambda^2 - x^2(u)}$ and

$$\left(x'^2(u) + z'^2(u) - w'^2(u) - \frac{\lambda^2 w'^2(u)}{x^2(u) - \lambda^2}\right)du^2 = (\dot{n}^2(k) + \dot{p}^2(k) + \dot{s}^2(k))dk^2. \tag{29}$$

Set $a(u) = \frac{\dot{n}(k)}{\dot{s}(k)}$ and $b(u) = \frac{\dot{p}(k)}{\dot{s}(k)}$. Then, we find

$$n = -\int \frac{a(u)x(u)x'(u)}{\sqrt{\lambda^2 - x^2(u)}}du, \quad p = -\int \frac{b(u)x(u)x'(u)}{\sqrt{\lambda^2 - x^2(u)}}du. \tag{30}$$

Thus, we get an isometric timelike rotational surface R_1^3 given by (17) satisfying (18). It can be easily seen that a timelike helix on X_1 corresponds to the parallel timelike hyperbola lying on the plane $\{x_1 = c_1, x_2 = c_2\}$ for constants c_1 and c_2, i.e, $R_1^3(u_0, v) = (c_1, c_2, \sqrt{\lambda^2 - x_0^2}\cosh v, \sqrt{\lambda^2 - x_0^2}\sinh v)$. □

Now, we find the Gauss maps of the surfaces given in Theorem 3.1.

Lemma 3.2. *Let X_1, R_1^1, R_1^2 and R_1^3 be timelike surfaces in $\mathbb{E}_1^4$ given by* (8), (13), (15) *and* (17)*, respectively. Then, the Gauss maps of them are given by the followings*

$$\begin{aligned}\nu_{X_1} = & \frac{\epsilon}{\sqrt{-W}}\Big(xx'\eta_{12} + xz'\sin v\eta_{13} + (\lambda x'\cos v + xw'\sin v)\,\eta_{14} - xz'\cos v\eta_{23} \\ & + (\lambda x'\sin v - xw'\cos v)\,\eta_{24} + \lambda z'\eta_{34}\Big),\end{aligned} \tag{31}$$

$$\begin{aligned}\nu_{R_1^1} = & \frac{\epsilon xx'}{\sqrt{-W}}\Big(\eta_{12} + a\sin\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{13} + b\sin\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{14} \\ & - a\cos\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{23} - b\cos\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{24}\Big),\end{aligned} \tag{32}$$

$$\begin{aligned}\nu_{R_1^2} = & \frac{\epsilon xx'}{\sqrt{-W}}\Big(a\cosh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{13} + a\sinh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{14} \\ & + b\cosh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{23} + b\sinh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{24} - \eta_{34}\Big),\end{aligned} \tag{33}$$

$$\begin{aligned}\nu_{R_1^3} = & -\frac{\epsilon xx'}{\sqrt{-W}}\Big(a\sinh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{13} + a\cosh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{14} \\ & + b\sinh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{23} + b\cosh\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big)\eta_{24} + \eta_{34}\Big),\end{aligned} \tag{34}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ is the standard orthonormal bases of $\mathbb{E}_1^4$ and $\eta_{ij} = \eta_i \wedge \eta_j$ for $i, j = 1, 2, 3, 4$.

Proof. Using the equation (7), the Gauss maps of the surfaces can be calculated directly. □

For later use, we give the following lemma related to the components of the mean curvature vector of the timelike rotational surface R_1^1 in $\mathbb{E}_1^4$ given by (13).

Lemma 3.3. *Let R_1^1 be a timelike rotational surface in $\mathbb{E}_1^4$ defined by* (13)*. Then, the mean curvature vector $H^{R_1^1}$ of R_1^1 in $\mathbb{E}_1^4$ is $H^{R_1^1} = H_1^{R_1^1}N_1 + H_2^{R_1^1}N_2$ with respect to*

$$\begin{aligned}N_1 = & \frac{1}{\sqrt{1+a^2}}\Big(a\cos\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big), a\sin\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big), -1, 0\Big), \\ N_2 = & \frac{1}{\sqrt{(1+a^2)(b^2-a^2-1)}}\Big(b\cos\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big), b\sin\Big(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\Big), ab, 1+a^2\Big),\end{aligned} \tag{35}$$

where $H_1^{R_1^1}$ and $H_2^{R_1^1}$ are given by

$$H_1^{R_1^1} = \frac{(\lambda^2 - x^2)a' + axx'(b^2 - a^2 - 1)}{2xx'(1 + a^2 - b^2)\sqrt{(1+a^2)(x^2-\lambda^2)}}, \tag{36}$$

$$H_2^{R_1^1} = \frac{(x^2-\lambda^2)(a^2b' - aa'b + b') + bxx'(1 + a^2 - b^2)}{2xx'\sqrt{(1+a^2)(x^2-\lambda^2)(b^2-a^2-1)^3}}. \tag{37}$$

Proof. It follows from a direct computation. □

Then, we consider isometric surfaces according to Bour's theorem whose Gauss maps are same.

Theorem 3.4. *Let X_1, R_1^1, R_1^2, R_1^3 be a timelike helicoidal surface of type I and timelike rotational surfaces in $\mathbb{E}_1^4$ given by* (8), (13), (15) *and* (17), *respectively. Then, we have the following statements.*

(i.) If the Gauss maps of X_1 and R_1^1 are same, then they are hyperplanar and minimal. Then, the parametrizations of X_1 and R_1^1 are given by

$$X_1(u,v) = (x(u)\cos v, x(u)\sin v, c_1, w(u) + \lambda v) \tag{38}$$

and

$$R_1^1(u,v) = \begin{pmatrix} \sqrt{x^2(u)-\lambda^2}\cos\left(v - \int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \\ \sqrt{x^2(u)-\lambda^2}\sin\left(v - \int \frac{\lambda w'(u)}{x^2(u)-\lambda^2}du\right) \\ c_2 \\ \pm\frac{1}{\sqrt{-c_3}}\arcsin\sqrt{c_3(\lambda^2 - x^2(u))} + c_4 \end{pmatrix}, \tag{39}$$

where c_1, c_2, c_3, c_4 are arbitrary constants with $c_3 < 0$ and

$$w(u) = \pm\left(\sqrt{\frac{c_3\lambda^2 - 1}{c_3}}\arcsin\left(\sqrt{c_3(\lambda^2 - x^2(u))}\right) - \lambda\arctan\left(\sqrt{\frac{(1-c_3\lambda^2)(x^2(u)-\lambda^2)}{\lambda^2(1+c_3(x^2(u)-\lambda^2))}}\right)\right). \tag{40}$$

(ii.) The Gauss maps of X_1 and R_1^2 or R_1^3 are definitely different.

Proof. Assume that X_1 is a timelike helicoidal surface of type I in $\mathbb{E}_1^4$ defined by (8) and R_1^1, R_1^2, R_1^3 are timelike rotational surfaces in $\mathbb{E}_1^4$ defined by (13), (15) and (17), respectively. From Lemma 3.2, we know the Gauss maps of X_1, R_1^1, R_1^2 and R_1^3 given by (31), (32), (33) and (34), respectively. Then, we consider the Gauss maps of each surfaces.
(i) Suppose that X_1 and R_1^1 have the same Gauss maps. From (31) and (32), we get the following system of equations:

$$xz'\sin v = axx'\sin\left(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\right), \tag{41}$$

$$xz'\cos v = axx'\cos\left(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\right), \tag{42}$$

$$\lambda x'\cos v + xw'\sin v = bxx'\sin\left(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\right), \tag{43}$$

$$\lambda x'\sin v - xw'\cos v = -bxx'\cos\left(v - \int\frac{\lambda w'}{x^2-\lambda^2}du\right), \tag{44}$$

$$\lambda z' = 0. \tag{45}$$

Due to $\lambda \neq 0$, the equation (45) gives $z'(u) = 0$. Then, from the equations (41) and (42) we get $a(u) = 0$. Therefore, it can be easily seen that the timelike surfaces X_1 and R_1^1 are hyperplanar, that is, they are lying in $\mathbb{E}_1^3$. Moreover, the equations (12) and (36) imply that $H_1^{X_1} = H_1^{R_1^1} = 0$. Also, from the equations (12) and (37), we have

$$\begin{aligned} H_2^{X_1} &= \frac{x'^2w'(2\lambda^2 - x^2) + x^2w'^3 + x(x^2-\lambda^2)(x''w' - x'w'')}{2(x^2w'^2 - x'^2(x^2-\lambda^2))^{3/2}}, \\ H_2^{R_1^1} &= \frac{bxx'(1-b^2) + b'(x^2-\lambda^2)}{2xx'\sqrt{(x^2-\lambda^2)(b^2-1)^3}}. \end{aligned} \tag{46}$$

Using $z'(u) = a(u) = 0$, from the equation (14) we have

$$b^2 = \frac{x^2 w'^2 + \lambda^2 x'^2}{x^2 x'^2}. \tag{47}$$

Also, by using the equation (47) in (46), we get

$$H_2^{R_1^1} = -\frac{x^2 w'(x'^2 w'(2\lambda^2 - x^2) + x^2 w'^3 + x(x^2 - \lambda^2)(x'' w' - x' w''))}{2(x^2 w'^2 - x'^2(x^2 - \lambda^2))^{3/2} \sqrt{(x^2 w'^2 + \lambda^2 x'^2)(x^2 - \lambda^2)}}. \tag{48}$$

Thus, we get $H_2^{R_1^1} = -\frac{x^2 w'}{\sqrt{(x^2 w'^2 + \lambda^2 x'^2)(x^2 - \lambda^2)}} H_2^{X_1}$. Moreover, using equations (43) and (44), we obtain the following equations

$$xw' = bxx' \cos\left(\int \frac{\lambda w'}{x^2 - \lambda^2} du\right), \tag{49}$$

$$\lambda x' = -bxx' \sin\left(\int \frac{\lambda w'}{x^2 - \lambda^2} du\right). \tag{50}$$

Considering the equations (49) and (50) together, we have

$$\frac{xw'}{\lambda x'} = -\cot\left(\int \frac{\lambda w'}{x^2 - \lambda^2} du\right). \tag{51}$$

Taking the derivative of (51) with respect to u, we find

$$\lambda^2 (xx' w'' + w'(2x'^2 - xx'')) + x^2 (w'(w'^2 - x'^2) + x(x'' w' - x' w'')) = 0 \tag{52}$$

which implies $H_2^{X_1} = H_2^{R_1^1} = 0$. Thus, we get the desired results. Since R_1^1 is minimal, from the equation (36) we have the following differential equation $(x^2 - \lambda^2)b' + xx'b = xx'b^3$, which is a Bernoulli equation. Then, the general solution of this equation is found as

$$b^2 = \frac{1}{1 + c_3(x^2 - \lambda^2)} \tag{53}$$

for an arbitrary negative constant c_3. Comparing the equations (47) and (53), we get

$$w(u) = \pm \sqrt{1 - c_3 \lambda^2} \int \frac{x'(u)}{x(u)} \sqrt{\frac{x^2(u) - \lambda^2}{1 + c_3(x^2(u) - \lambda^2)}} du \tag{54}$$

whose solution is given by (40) for $c_3 < 0$. Moreover, using the last component of R_1^1 in (13) we have

$$\int \frac{x(u)x'(u)}{\sqrt{(x^2(u) - \lambda^2)(1 + c_3(x^2(u) - \lambda^2))}} du = \pm \frac{1}{\sqrt{-c_3}} \arcsin \sqrt{-c_3(x^2(u) - \lambda^2)} + c_4 \tag{55}$$

for any arbitrary constant c_4.

(ii.) Suppose that X_1 and R_1^2 have the same Gauss maps. Comparing the equations (31) and (33), we get $x(u) = 0$ or $x'(u) = 0$ which give $\nu_{R_1^2} = 0$. That is a contradiction. Thus, their Gauss maps are definitely different. Similarly, we show that the Gauss maps of the surfaces X_1 and R_1^3 are definitely different. □

Remark 3.5. *Taking $x(u) = u$ in Theorem 3.4, we get the cases obtained in [17] and the rotational surface given by* (39) *also has the same form of surface in Proposition 3.4, [17].*

Assume that X_1 is a timelike right helicoidal surface of type I in $\mathbb{E}^4_1$, that is, $w'(u) = 0$ for $u \in I \subset \mathbb{R}$. On the other hand, we know that $W = (x^2(u) - \lambda^2)(x'^2(u) + z'^2(u)) < 0$ when I_2 is dense in I. Thus, from the surface R^3_1 in Theorem 3.1, we get the parametrization of isometric timelike rotational surface. From Theorem 3.4, it can be easily seen that the Gauss maps of these isometric surfaces are definitely different.

Now, we give an example by using Theorem 3.4.

Example 3.6. *If we choose* $x(u) = u$, $\lambda = 1$, $c_3 = -1/2$ *and* $c_4 = 0$, *then isometric surfaces in* (38) *and* (39) *are given as follows*

$$X_1(u,v) = \left(u\cos v, u\sin v, \sqrt{3}\arcsin\sqrt{\frac{u^2-1}{2}} - \arctan\sqrt{\frac{3(u^2-1)}{3-u^2}} + v\right)$$

and

$$R^1_1(u,v) = \begin{pmatrix} \sqrt{u^2-1}\cos\left(v - \frac{1}{2}\arctan\left(\frac{2u^2-3}{\sqrt{-3u^4+12u^2-9}}\right)\right) \\ \sqrt{u^2-1}\sin\left(v - \frac{1}{2}\arctan\left(\frac{2u^2-3}{\sqrt{-3u^4+12u^2-9}}\right)\right) \\ \sqrt{2}\arcsin\sqrt{\frac{u^2-1}{2}} \end{pmatrix}.$$

For $1.32 \leq u \leq 1.72$ *and* $0 \leq v < 2\pi$, *the graphs of timelike helicoidal surface* X_1 *and timelike rotational surface* R^1_1 *in* $\mathbb{E}^3_1$ *can be plotted by using Mathematica 10.4 given in Figure 1 and Figure 2, respectively.*

Figure 1: Timelike helicoidal surface of type I; spacelike helix.

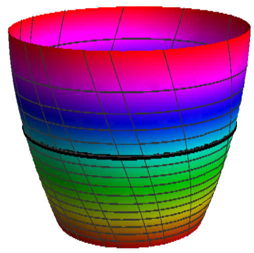

Figure 2: Timelike rotational surface; spacelike circle.

4. Helicoidal Surface of Type IIa

In this section, we study on Bour's theorem for timelike helicoidal surface of type IIa in $\mathbb{E}^4_1$ and we analyse the Gauss maps of isometric pair of surfaces.

Let us choose a spacelike 2-plane $P_2 = span\{\eta_1, \eta_2\}$, a hyperplane $\Pi_{2a} = span\{\eta_1, \eta_2, \eta_4\}$ and a line $l_2 = span\{\eta_1\}$. Also, we suppose that $\beta_{2a} : I \longrightarrow \Pi_{2a} \subset \mathbb{E}^4_1$; $\beta_{2a}(u) = (x(u), y(u), 0, w(u))$ is a regular curve with $w(u) \neq 0$ for all $u \in I$. By using the definition of helicoidal surface, the parametrization of X_{2a} (called as the helicoidal surface of type IIa) is

$$X_{2a}(u, v) = (x(u) + \lambda v, y(u), w(u)\sinh v, w(u)\cosh v), \tag{56}$$

where, $v \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. When x is a constant function, X_{2a} is called as right helicoidal surface of type IIa. Also, when y is a constant function, X_{2a} is just a helicoidal surface in $\mathbb{E}^3_1$ (see [2]). For $\lambda = 0$, the helicoidal surface which is given by (56) reduces to the rotational surface of hyperbolic type in $\mathbb{E}^4_1$ (see [9] and [3]).

By a direct calculation, we get the induced metric of X_{2a} given as follows

$$ds^2_{X_{2a}} = (x'^2(u) + y'^2(u) - w'^2(u))du^2 + 2\lambda x'(u)dudv + (\lambda^2 + w^2(u))dv^2 \tag{57}$$

with $W = (\lambda^2 + w^2(u))(y'^2(u) - w'^2(u)) + x'^2(u)w^2(u) < 0$ for all $u \in I$. Then, we choose an orthonormal frame field $\{e_1, e_2, N_1, N_2\}$ on X_{2a} in $\mathbb{E}^4_1$ such that e_1, e_2 are tangent to X_{2a} and N_1, N_2 are normal to X_{2a} as follows

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{\epsilon g_{11}}} X_{2a_u}, \quad e_2 = \frac{1}{\sqrt{-\epsilon W g_{11}}}(g_{11}X_{2a_v} - g_{12}X_{2a_u}),\\
N_1 &= \frac{1}{\sqrt{w'^2 - y'^2}}(0, w', y'\sinh v, y'\cosh v),\\
N_2 &= -\frac{1}{\sqrt{-W(w'^2 - y'^2)}}(w(w'^2 - y'^2), x'y'w, x'ww'\sinh v - \lambda(w'^2 - y'^2)\cosh v,\\
&\quad x'ww'\cosh v - \lambda(w'^2 - y'^2)\sinh v),
\end{aligned} \tag{58}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \epsilon$ and $\langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 1$. For $\epsilon = 1$, the surface X_{2a} has a spacelike meridian curve. Otherwise, it has a timelike meridian curve. By direct computations, we get the coefficients of the second fundamental form given as follows.

$$
\begin{aligned}
b_{11}^1 &= \frac{y''w' - y'w''}{\sqrt{w'^2 - y'^2}}, \quad b_{12}^1 = b_{21}^1 = 0, \quad b_{22}^1 = -\frac{y'w}{\sqrt{w'^2 - y'^2}},\\
b_{11}^2 &= \frac{w(x'(w'w'' - y'y'') + x''(y'^2 - w'^2))}{\sqrt{W(y'^2 - w'^2)}}, \quad b_{12}^2 = b_{21}^2 = \frac{\lambda w' \sqrt{w'^2 - y'^2}}{\sqrt{-W}},\\
b_{22}^2 &= \frac{x'w^2w'}{\sqrt{W(y'^2 - w'^2)}}.
\end{aligned} \tag{59}
$$

Thus, the mean curvature vector $H^{X_{2a}}$ of X_{2a} in $\mathbb{E}_1^4$ is $H^{X_{2a}} = H_1^{X_{2a}} N_1 + H_2^{X_{2a}} N_2$ where N_1, N_2 are normal vector fields in (58), $H_1^{X_{2a}}$ and $H_2^{X_{2a}}$ are given by

$$
\begin{aligned}
H_1^{X_{2a}} &= \frac{(w^2 + \lambda^2)(y''w' - y'w'') - y'w(x'^2 + y'^2 - w'^2)}{2W\sqrt{w'^2 - y'^2}},\\
H_2^{X_{2a}} &= \frac{x'w'(w^2 + 2\lambda^2)(y'^2 - w'^2) + x'^3w^2w' + w(\lambda^2 + w^2)(x''(y'^2 - w'^2) + x'(w'w'' - y'y''))}{2\sqrt{W^3(y'^2 - w'^2)}}.
\end{aligned} \tag{60}
$$

4.1. Bour's Theorem and the Gauss map for helicoidal surfaces IIa

Theorem 4.1. *A timelike helicoidal surface of type IIa in $\mathbb{E}_1^4$ given by* (56) *is isometric to one of the following timelike rotational surfaces in $\mathbb{E}_1^4$:*

(i)

$$
R_{2a}^1(u, v) = \begin{pmatrix} \sqrt{\lambda^2 + w^2(u)} \cos\left(v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)} du\right) \\ \sqrt{\lambda^2 + w^2(u)} \sin\left(v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)} du\right) \\ \int \frac{a(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}} du \\ \int \frac{b(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}} du \end{pmatrix} \tag{61}
$$

so that spacelike helices on the timelike helicoidal surface of type IIa correspond to parallel spacelike circles on the timelike rotational surface, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$
a^2(u) - b^2(u) = \frac{\lambda^2(y'^2(u) - w'^2(u)) + w^2(u)(x'^2(u) + y'^2(u) - 2w'^2(u))}{w^2(u)w'^2(u)} \tag{62}
$$

(ii)

$$
R_{2a}^2(u, v) = \begin{pmatrix} \int \frac{a(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}} du \\ \int \frac{b(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}} du \\ \sqrt{\lambda^2 + w^2(u)} \sinh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)} du\right) \\ \sqrt{\lambda^2 + w^2(u)} \cosh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)} du\right) \end{pmatrix} \tag{63}
$$

so that spacelike helices on the timelike helicoidal surface of type IIa correspond to parallel spacelike hyperbolas on the timelike rotational surface, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$
a^2(u) + b^2(u) = \frac{w^2(u)(x'^2(u) + y'^2(u)) + \lambda^2(y'^2(u) - w'^2(u))}{w^2(u)w'^2(u)}. \tag{64}
$$

Proof. Assume that X_{2a} is a timelike helicoidal surface of type IIa in $\mathbb{E}^4_1$ defined by (56). Then, we have the induced metric of X_{2a} given by (57). Now, we will find new coordinates $\bar{u}, \bar{v}$ such that the metric becomes

$$ds^2_{X_{2a}} = F(\bar{u})d\bar{u}^2 + G(\bar{u})d\bar{v}^2, \tag{65}$$

where $F(\bar{u})$ and $G(\bar{u})$ are smooth functions. Set $\bar{u} = u$ and $\bar{v} = v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)}du$. Since Jacobian $\frac{\partial(\bar{u},\bar{v})}{\partial(u,v)}$ is nonzero, it follows that $\{\bar{u}, \bar{v}\}$ are new parameters of X_{2a}. According to the new parameters, the equation (57) becomes

$$ds^2_{X_{2a}} = \left(x'^2(u) + y'^2(u) - w'^2(u) - \frac{\lambda^2 x'^2(u)}{\lambda^2 + w^2(u)}\right)du^2 + (\lambda^2 + w^2(u))d\bar{v}^2. \tag{66}$$

First, we consider a timelike rotational surface R_1 in $\mathbb{E}^4_1$ given by (20). Then, we have the induced metric of R_1. Comparing the metric of R_1 and (66), we take $\bar{v} = t$ and $n(k) = \sqrt{\lambda^2 + w^2(u)}$ and we also have

$$\left(x'^2(u) + y'^2(u) - w'^2(u) - \frac{\lambda^2 x'^2(u)}{\lambda^2 + w^2(u)}\right)du^2 = (\dot{n}^2(k) + \dot{s}^2(k) - \dot{r}^2(k))dk^2. \tag{67}$$

Set $a(u) = \frac{\dot{s}(k)}{\dot{n}(k)}$ and $b(u) = \frac{\dot{r}(k)}{\dot{n}(k)}$. Then, we obtain

$$s = \int \frac{a(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}}du, \quad r = \int \frac{b(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}}du. \tag{68}$$

Thus, we get an isometric timelike rotational surface R^1_{2a} given by (61) satisfying (62). It can be easily seen that a spacelike helix on X_{2a} corresponds to parallel spacelike circle lying on the plane $\{x_3 = c_3, x_4 = c_4\}$ with the radius $\sqrt{\lambda^2 + w_0^2}$ for constants c_3 and c_4, i.e., $R^1_{2a}(u_0, v) = (\sqrt{\lambda^2 + w_0^2}\cos v, \sqrt{\lambda^2 + w_0^2}\sin v, c_3, c_4)$.

Secondly, we consider a timelike rotational surface R_{2a} in $\mathbb{E}^4_1$ given by (23). Then, we know the induced metric given by (24). Comparing the equations (24) and (66), we take $\bar{v} = t$ and $r(k) = \sqrt{\lambda^2 + w^2(u)}$ and we also have

$$\left(x'^2(u) + y'^2(u) - w'^2(u) - \frac{\lambda^2 x'^2(u)}{\lambda^2 + w^2(u)}\right)du^2 = (\dot{n}^2(k) + \dot{p}^2(k) - \dot{r}^2(k))dk^2. \tag{69}$$

Set $a(u) = \frac{\dot{n}(k)}{\dot{r}(k)}$ and $b(u) = \frac{\dot{p}(k)}{\dot{r}(k)}$. Then, we obtain

$$n = \int \frac{a(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}}du, \quad p = \int \frac{b(u)w(u)w'(u)}{\sqrt{\lambda^2 + w^2(u)}}du. \tag{70}$$

Thus, we get an isometric timelike rotational surface R^2_{2a} given by (63) satisfying (64). It can be easily seen that a spacelike helix on X_{2a} which is defined by $u = u_0$ for a constant u_0 corresponds to the parallel spacelike hyperbola lying on the plane $\{x_1 = c_1, x_2 = c_2\}$ for constants c_1 and c_2, i.e., $R^2_{2a}(u_0, v) = (c_1, c_2, \sqrt{\lambda^2 + w_0^2}\sinh v, \sqrt{\lambda^2 + w_0^2}\cosh v)$. □

Lemma 4.2. *Let X_{2a}, R^1_{2a} and R^2_{2a} be timelike surfaces in $\mathbb{E}^4_1$ given by* (56), (61) *and* (63)*, respectively. Then, the Gauss maps of them are given by the followings*

$$\nu_{X_{2a}} = \frac{\epsilon}{\sqrt{-W}}\Big(- \lambda y' \eta_{12} + (x' w \cosh v - \lambda w' \sinh v)\eta_{13} + (x' w \sinh v - \lambda w' \cosh v)\eta_{14} + y' w \cosh v \eta_{23} + y' w \sinh v \eta_{24} - w w' \eta_{34}\Big), \tag{71}$$

$$\nu_{R^1_{2a}} = \frac{\epsilon w w'}{\sqrt{-W}}\left(\eta_{12} + a \sin\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{13} + b \sin\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{14} - a \cos\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{23} - b \cos\left(v + \int \frac{\lambda x'(u)}{\lambda^2 + w^2(u)} du\right)\eta_{24}\right), \tag{72}$$

$$\nu_{R^2_{2a}} = \frac{\epsilon w w'}{\sqrt{-W}}\left(a \cosh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{13} + a \sinh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{14} + b \cosh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{23} + b \sinh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\eta_{24} - \eta_{34}\right), \tag{73}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ is the standard orthonormal bases of $\mathbb{E}^4_1$ and $\eta_{ij} = \eta_i \wedge \eta_j$ for $i, j = 1, 2, 3, 4$.

Proof. Using the equation (7), the Gauss maps of the surfaces can be calculated directly. □

For later use, we give the following lemma related to the components of the mean curvature vector of the timelike rotational surface R^2_{2a} given by (63).

Lemma 4.3. *Let R^2_{2a} be a timelike rotational surface in $\mathbb{E}^4_1$ given by* (63)*. Then, the mean curvature vector $H^{R^2_{2a}}$ of R^2_{2a} in $\mathbb{E}^4_1$ is $H^{R^2_{2a}} = H^{R^2_{2a}}_1 N_1 + H^{R^2_{2a}}_2 N_2$ with respect to*

$$\begin{aligned} N_1 &= \frac{1}{\sqrt{1-b^2}}\left(0, 1, b \sinh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right), b \cosh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\right), \\ N_2 &= \frac{1}{\sqrt{(b^2-1)(a^2+b^2-1)}}\left(1-b^2, ab, a \sinh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right), a \cosh\left(v + \int \frac{\lambda x'}{\lambda^2 + w^2} du\right)\right), \end{aligned} \tag{74}$$

where $H^{R^2_{2a}}_1$ and $H^{R^2_{2a}}_2$ are given by

$$\begin{aligned} H^{R^2_{2a}}_1 &= \frac{(w^2+\lambda^2)b' - (a^2+b^2-1) b w w'}{2 w w' (a^2+b^2-1)\sqrt{(1-b^2)(w^2+\lambda^2)}}, \\ H^{R^2_{2a}}_2 &= \frac{(w^2+\lambda^2)(a'(1-b^2) + a b b') - a w w'(a^2+b^2-1)}{2 w w' \sqrt{(b^2-1)(w^2+\lambda^2)(a^2+b^2-1)^3}}. \end{aligned} \tag{75}$$

Proof. It follows from a direct computation. □

Then, we consider isometric surfaces according to Bour's theorem whose Gauss maps are same.

Theorem 4.4. *Let X_{2a}, R^1_{2a} and R^2_{2a} be a timelike helicoidal surface of type IIa and timelike rotational surfaces in $\mathbb{E}^4_1$ given by* (56), (61) *and* (63)*, respectively. Then, we have the following statements.*

(i.) The Gauss maps of X_{2a} and R^1_{2a} are definitely different.

(ii.) If the surfaces X_{2a} and R^2_{2a} have the same Gauss maps, then they are hyperplanar and minimal. Then, the parametrizations of X_{2a} and R^2_{2a} can be explicitly determined by

$$X_{2a}(u,v) = (x(u) + \lambda v, c_1, w(u)\sinh v, w(u)\cosh v) \tag{76}$$

and

$$R^2_{2a}(u,v) = \begin{pmatrix} \pm\frac{1}{\sqrt{c_3}}\operatorname{arcsinh}\sqrt{c_3(\lambda^2+w^2(u))} + c_4 \\ c_2 \\ \sqrt{\lambda^2+w^2(u)}\sinh\left(v + \int \frac{\lambda x'(u)}{\lambda^2+w^2(u)}du\right) \\ \sqrt{\lambda^2+w^2(u)}\cosh\left(v + \int \frac{\lambda x'(u)}{\lambda^2+w^2(u)}du\right) \end{pmatrix}, \tag{77}$$

where c_1, c_2, c_3, c_4 are arbitrary constants with $c_3 > 0$ and

$$x(u) = \pm\left(\sqrt{1+c_3\lambda^2}\operatorname{arcsinh}\sqrt{c_3(\lambda^2+w^2(u))} - \lambda\sqrt{c_3}\operatorname{arctanh}\left(\frac{\lambda\sqrt{1+c_3(\lambda^2+w^2(u))}}{\sqrt{(1+c_3\lambda^2)(\lambda^2+w^2(u))}}\right)\right). \tag{78}$$

Proof. Assume that X_{2a} is a timelike helicoidal surface of type I in $\mathbb{E}^4_1$ given by (56) and R^1_{2a}, R^2_{2a} are timelike rotational surfaces in $\mathbb{E}^4_1$ given by (61) and (63), respectively. From Lemma 4.2, we have the Gauss maps of X_{2a}, R^1_{2a} and R^2_{2a} given by (71), (72) and (73), respectively.
(i.) Suppose that the Gauss maps of X_{2a} and R^1_{2a} are same. Then, from the equations (71) and (72), we get $w(u) = 0$ or $w'(u) = 0$ for $u \in I$ which implies $\nu_{R^1_{2a}} = 0$. That is a contradiction. Thus, their Gauss maps are definitely different.
(ii) Suppose that the surfaces X_{2a} and R^2_{2a} have the same Gauss maps. From (71) and (73), we get the following system of equations:

$$\lambda y' = 0, \tag{79}$$

$$x'w\cosh v - \lambda w'\sinh v = aww'\cosh\left(v + \int\frac{\lambda x'}{\lambda^2+w^2}du\right), \tag{80}$$

$$x'w\sinh v - \lambda w'\cosh v = aww'\sinh\left(v + \int\frac{\lambda x'}{\lambda^2+w^2}du\right), \tag{81}$$

$$y'w\cosh v = bww'\cosh\left(v + \int\frac{\lambda x'}{\lambda^2+w^2}du\right), \tag{82}$$

$$y'w\sinh v = bww'\sinh\left(v + \int\frac{\lambda x'}{\lambda^2+w^2}du\right). \tag{83}$$

Due to $\lambda \neq 0$, the equation (79) gives $y'(u) = 0$. Then, from the equations (82) and (83) imply $b(u) = 0$. Therefore, it can be easily seen that the surfaces X_{2a} and R^2_{2a} are hyperplanar, that is, they are lying in $\mathbb{E}^3_1$. Moreover, the equations (60) and (75) imply that $H_1^{X_{2a}} = H_1^{R^2_{2a}} = 0$ and

$$\begin{aligned} H_2^{X_{2a}} &= -\frac{x'w'^2(2\lambda^2+w^2) - w^2x'^3 + w(\lambda^2+w^2)(x''w' - x'w'')}{2(w'^2(\lambda^2+w^2) - w^2x'^2)^{3/2}}, \\ H_2^{R^2_{2a}} &= \frac{a'(w^2+\lambda^2) + aww'(1-a^2)}{2ww'\sqrt{(w^2+\lambda^2)(1-a^2)^3}}. \end{aligned} \tag{84}$$

Using $y'(u) = b(u) = 0$, from the equation (64) we have

$$a^2(u) = \frac{x'^2(u)w^2(u) - \lambda^2 w'^2(u)}{w^2(u)w'^2(u)}. \tag{85}$$

By using the equation (85) in (84), we get

$$H_2^{R_{2a}^2} = \frac{x'w^2(x'w'^2(2\lambda^2 + w^2) - x'^3w^2 + w(\lambda^2 + w^2)(x''w' - x'w''))}{2(w'^2(\lambda^2 + w^2) - w^2x'^2)^{3/2}\sqrt{(\lambda^2 + w^2)(w^2x'^2 - \lambda^2 w'^2)}} \tag{86}$$

which implies $H_2^{R_{2a}^2} = -\frac{x'w^2}{\sqrt{(w^2x'^2 - \lambda^2w'^2)(\lambda^2 + w^2)}} H_2^{X_{2a}}$. Moreover, using equations (80) and (81), we obtain the following equations

$$x'w = aww' \cosh\left(\int \frac{\lambda x'}{\lambda^2 + w^2} du\right), \tag{87}$$

$$\lambda w' = -aww' \sinh\left(\int \frac{\lambda x'}{\lambda^2 + w^2} du\right). \tag{88}$$

Considering the equations (87) and (88) together, we have

$$-\frac{x'w}{\lambda w'} = \coth\left(\int \frac{\lambda x'}{\lambda^2 + w^2} du\right). \tag{89}$$

If we take the derivative of the equation (89) with respect to u, the equation (89) becomes

$$\lambda(x'w'^2(2\lambda^2 + w^2) - w^2x'^3 + w(\lambda^2 + w^2)(x''w' - x'w'')) = 0 \tag{90}$$

which implies $H_2^{X_{2a}} = H_2^{R_{2a}^2} = 0$. Now, we determine the parametrizations of the isometric surfaces X_{2a} and R_{2a}^2. Since R_{2a}^2 is minimal, from the equation (84) we have the following Bernoulli differential equation

$$(\lambda^2 + w^2)a' + ww'a = ww'a^3 \tag{91}$$

whose solution is given by

$$a^2 = \frac{1}{1 + c_3(\lambda^2 + w^2)} \tag{92}$$

for an arbitrary positive constant c_3. Comparing the equations (85) and (92), we get

$$x(u) = \pm\sqrt{1 + c_3\lambda^2} \int \frac{w'(u)}{w(u)} \sqrt{\frac{w^2(u) + \lambda^2}{1 + c_3(w^2(u) + \lambda^2)}} du \tag{93}$$

whose solution is given by (78) for $c_3 > 0$. Moreover, using the first component of $R_{2a}^2(u, v)$ in (63), we have

$$\int \frac{w(u)w'(u)}{\sqrt{(\lambda^2 + w^2(u))(1 + c_3(\lambda^2 + w^2(u))}} du = \pm\frac{1}{\sqrt{c_3}} \operatorname{arcsinh} \sqrt{c_3(\lambda^2 + w^2(u))} + c_4 \tag{94}$$

for any arbitrary constant c_4. □

Remark 4.5. *Taking $w(u) = u$ in Theorem 4.4, we get isometric surfaces obtained in [17] and the rotational surface given by* (77) *also has the same form of minimal rotational surface in Proposition 3.2, [17].*

Assume that X_{2a} is a timelike right helicoidal surface of type IIa in $\mathbb{E}_1^4$, that is, $x'(u) = 0$ for all $u \in I$. Then, from Theorem 4.1, we get the parametrizations of isometric timelike rotational surfaces in $\mathbb{E}_1^4$. On the other hand, Theorem 4.4 implies that their Gauss maps can not be same.

Now, we give an example by using Theorem 4.4.

Example 4.6. *If we choose* $w(u) = u$, $\lambda = c_3 = 1$ *and* $c_4 = 0$, *then isometric surfaces in* (76) *and* (77) *are given as follows*

$$X_{2a}(u,v) = \left(\sqrt{2}\operatorname{arcsinh}\sqrt{1+u^2} - \operatorname{arctanh}\sqrt{\frac{2+u^2}{2+2u^2}} + v, u\sinh v, u\cosh v \right)$$

and

$$R^2_{2a}(u,v) = \begin{pmatrix} \operatorname{arcsinh}\sqrt{1+u^2} \\ \sqrt{1+u^2}\sinh\left(v + \ln\frac{u}{\sqrt{3u^2+4+\sqrt{8u^4+24u^2+16}}}\right) \\ \sqrt{1+u^2}\cosh\left(v + \ln\frac{u}{\sqrt{3u^2+4+\sqrt{8u^4+24u^2+16}}}\right) \end{pmatrix}.$$

For $1.19 \leq u \leq 10$ *and* $-1.5 \leq v < 1.5$, *the graphs of timelike helicoidal surface* X_{2a} *and timelike rotational surface* R^2_{2a} *in* $\mathbb{E}^3_1$ *can be plotted by using Mathematica 10.4 given in Figure 3 and Figure 4, respectively.*

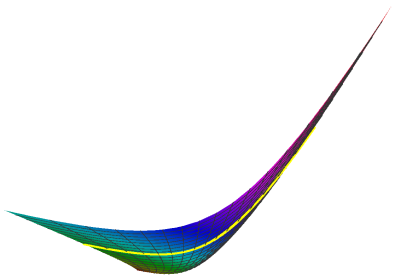

Figure 3: Timelike helicoidal surface of type IIa; spacelike helix.

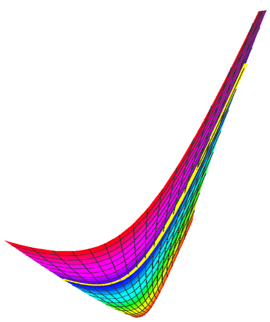

Figure 4: Timelike rotational surface; spacelike hyperbola.

5. Helicoidal Surface of Type IIb

Let us choose a spacelike 2-plane $P_2 = span\{\eta_1, \eta_2\}$, a hyperplane $\Pi_{2b} = \{\eta_1, \eta_2, \eta_3\}$ and a line $l_2 = span\{\eta_1\}$. Also, we suppose that $\beta_{2b} : I \longrightarrow \Pi_{2b} \subset \mathbb{E}^4_1$; $\beta_{2b}(u) = (x(u), y(u), z(u), 0)$ is a regular spacelike curve with $z(u) \neq 0$ for all $u \in I$. By using the definition of helicoidal surface, the parametrization of X_{2b} (called as the timelike helicoidal surface of type IIb) is

$$X_{2b}(u, v) = (x(u) + \lambda v, y(u), z(u)\cosh v, z(u)\sinh v), \tag{95}$$

where, $v \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. When x is a constant function, X_{2b} is called as timelike right helicoidal surface of type IIb. Also, when y is a constant function, X_{2b} is just a timelike helicoidal surface in $\mathbb{E}^3_1$. For $\lambda = 0$, the helicoidal surface which is given by (95) reduces to the rotational surface of hyperbolic type in $\mathbb{E}^4_1$.

By a direct calculation, we get the induced metric of X_{2b} given as follows.

$$ds^2_{X_{2b}} = (x'^2(u) + y'^2(u) + z'^2(u))du^2 + 2\lambda x'(u)dudv + (\lambda^2 - z^2(u))dv^2 \tag{96}$$

with $W = (\lambda^2 - z^2(u))(y'^2(u) + z'^2(u)) - x'^2(u)z^2(u) < 0$. Then, we choose an orthonormal frame field $\{e_1, e_2, N_1, N_2\}$ on X_{2b} in $\mathbb{E}^4_1$ such that e_1, e_2 are tangent to X_{2b} and N_1, N_2 are normal to X_{2b} as follows

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{g_{11}}} X_{2b_u}, \quad e_2 = \frac{1}{\sqrt{-Wg_{11}}}(g_{11}X_{2b_v} - g_{12}X_{2b_u}),\\
N_1 &= \frac{1}{\sqrt{y'^2 + z'^2}}(0, -z', y'\cosh v, y'\sinh v),\\
N_2 &= \frac{1}{\sqrt{-W(y'^2 + z'^2)}}(-z(y'^2 + z'^2), zx'y', x'zz'\cosh v - \lambda(y'^2 + z'^2)\sinh v, x'zz'\sinh v - \lambda(y'^2 + z'^2)\cosh v),
\end{aligned} \tag{97}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = 1$ and $\langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 1$. By direct computations, we get the coefficients of the second fundamental form given as follows.

$$\begin{aligned}
b^1_{11} &= \frac{y'z'' - y''z'}{\sqrt{y'^2 + z'^2}}, \quad b^1_{12} = b^1_{21} = 0, \quad b^1_{22} = \frac{y'z}{\sqrt{y'^2 + z'^2}},\\
b^2_{11} &= \frac{z(x'(y'y'' + z'z'') - x''(y'^2 + z'^2))}{\sqrt{-W(y'^2 + z'^2)}}, \quad b^2_{12} = b^2_{21} = \frac{\lambda z'\sqrt{y'^2 + z'^2}}{\sqrt{-W}},\\
b^2_{22} &= \frac{x'z^2z'}{\sqrt{-W(y'^2 + z'^2)}}.
\end{aligned} \tag{98}$$

Thus, the mean curvature vector $H^{X_{2b}}$ of X_{2b} in $\mathbb{E}^4_1$ is $H^{X_{2b}} = H^{X_{2b}}_1 N_1 + H^{X_{2b}}_2 N_2$, where N_1, N_2 are normal vector fields in (97), $H^{X_{2b}}_1$ and $H^{X_{2b}}_2$ are given by

$$\begin{aligned}
H^{X_{2b}}_1 &= \frac{(\lambda^2 - z^2)(y'z'' - z'y'') + zy'(x'^2 + y'^2 + z'^2)}{2W\sqrt{y'^2 + z'^2}},\\
H^{X_{2b}}_2 &= \frac{1}{2\sqrt{-W^3(y'^2 + z'^2)}}\Big(x'z'((z^2 - 2\lambda^2)(y'^2 + z'^2) + z^2(x'^2 - zz''))\\
&\quad + \lambda^2 z(x'(z'z'' + y'y'') - x''(y'^2 + z'^2)) + z^3(x''(y'^2 + z'^2) - x'y'y'')\Big).
\end{aligned} \tag{99}$$

5.1. Bour's Theorem and the Gauss map for helicoidal surfaces of type IIb

In this section, we study on Bour's theorem for timelike helicoidal surface of type IIb in $\mathbb{E}^4_1$ and we analyse the Gauss maps of isometric pair of surfaces.
Let define the two subsets I_1 and I_2 of I as $I_1 = \{u \in I \mid z^2(u) - \lambda^2 < 0\}$ and $I_2 = \{u \in I \mid z^2(u) - \lambda^2 > 0\}$.

Theorem 5.1. *A timelike helicoidal surface of type IIb in $\mathbb{E}^4_1$ given by* (95) *is isometric to one of the following timelike rotational surfaces in $\mathbb{E}^4_1$:*

(i)

$$R^1_{2b}(u,v) = \begin{pmatrix} \sqrt{\lambda^2 - z^2(u)}\cos\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \\ \sqrt{\lambda^2 - z^2(u)}\sin\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \\ -\int \frac{a(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du \\ -\int \frac{b(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du \end{pmatrix} \tag{100}$$

so that spacelike helices on the timelike helicoidal surface of type IIb correspond to parallel spacelike circles on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u) - b^2(u) = \frac{\lambda^2(y'^2(u) + z'^2(u)) - z^2(u)(x'^2(u) + y'^2(u) + 2z'^2(u))}{z^2(u)z'^2(u)} \tag{101}$$

for all $u \in I_1 \subset \mathbb{R}$,

(ii)

$$R^2_{2b}(u,v) = \begin{pmatrix} -\int \frac{a(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du \\ -\int \frac{b(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du \\ \sqrt{\lambda^2 - z^2(u)}\sinh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \\ \sqrt{\lambda^2 - z^2(u)}\cosh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \end{pmatrix} \tag{102}$$

so that spacelike helices on the timelike helicoidal surface of type IIb correspond to parallel spacelike hyperbolas on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u) + b^2(u) = \frac{\lambda^2(y'^2(u) + z'^2(u)) - z^2(u)(x'^2(u) + y'^2(u))}{z^2(u)z'^2(u)} \tag{103}$$

for all $u \in I_1 \subset \mathbb{R}$,

(iii)

$$R^3_{2b}(u,v) = \begin{pmatrix} \int \frac{a(u)z(u)z'(u)}{\sqrt{z^2(u) - \lambda^2}}du \\ \int \frac{b(u)z(u)z'(u)}{\sqrt{z^2(u) - \lambda^2}}du \\ \sqrt{z^2(u) - \lambda^2}\cosh\left(v - \int \frac{\lambda x'(u)}{z^2(u) - \lambda^2}du\right) \\ \sqrt{z^2(u) - \lambda^2}\sinh\left(v - \int \frac{\lambda x'(u)}{z^2(u) - \lambda^2}du\right) \end{pmatrix} \tag{104}$$

so that timelike helices on the timelike helicoidal surface of type IIb correspond to parallel timelike hyperbolas on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u) + b^2(u) = \frac{z^2(u)(x'^2(u) + y'^2(u)) - \lambda^2(y'^2(u) + z'^2(u))}{z^2(u)z'^2(u)} \tag{105}$$

with $z'(u) \neq 0$ for all $u \in I_2 \subset \mathbb{R}$.

Proof. Assume that X_{2b} is a timelike helicoidal surface of type IIb in $\mathbb{E}^4_1$ defined by (95). Then, we have the induced metric of X_{2b} given by (96). Now, we will find new coordinates $\bar{u}, \bar{v}$ such that the metric becomes

$$ds^2_{X_{2b}} = F(\bar{u})d\bar{u}^2 + G(\bar{u})d\bar{v}^2, \tag{106}$$

where $F(\bar{u})$ and $G(\bar{u})$ are smooth functions. Set $\bar{u} = u$ and $\bar{v} = v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du$. Since Jacobian $\dfrac{\partial(\bar{u}, \bar{v})}{\partial(u, v)}$ is nonzero, it follows that $\{\bar{u}, \bar{v}\}$ are new parameters of X_{2b}. According to the new parameters, the equation (96) becomes

$$ds^2_{X_{2b}} = \left(x'^2(u) + y'^2(u) + z'^2(u) + \frac{\lambda^2 x'^2(u)}{z^2(u) - \lambda^2}\right)du^2 + (\lambda^2 - z^2(u))d\bar{v}^2. \tag{107}$$

Then, we consider the following cases.

Case(i) Assume that I_1 is dense in I. First, we consider a timelike rotational surface R_1 in $\mathbb{E}^4_1$ given by (20). Comparing the induced metric of R_1 and (107), we take $\bar{v} = t$ and $n(k) = \sqrt{\lambda^2 - z^2(u)}$ and we also have

$$\left(x'^2(u) + y'^2(u) + z'^2(u) + \frac{\lambda^2 x'^2(u)}{z^2(u) - \lambda^2}\right)du^2 = (\dot{n}^2(k) + \dot{s}^2(k) - \dot{r}^2(k))dk^2. \tag{108}$$

Set $a(u) = \frac{\dot{s}(k)}{\dot{n}(k)}$ and $b(u) = \frac{\dot{r}(k)}{\dot{n}(k)}$. Then, we obtain

$$s = -\int \frac{a(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du, \quad r = -\int \frac{b(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du. \tag{109}$$

Thus, we get an isometric timelike rotational surface R^1_{2b} given by (100) satisfying (101). It can be easily seen that a spacelike helix on X_{2b} which is defined by $u = u_0$ for a constant u_0 corresponds to the parallel spacelike circle lying on the plane $\{x_3 = c_3, x_4 = c_4\}$ with the radius $\sqrt{\lambda^2 - z_0^2}$ for constants c_3 and c_4, i.e., $R^1_{2b}(u_0, v) = (\sqrt{\lambda^2 - z_0^2}\cos v, \sqrt{\lambda^2 - z_0^2}\sin v, c_3, c_4)$.

Secondly, we consider a timelike rotational surface R_{2a} in $\mathbb{E}^4_1$ given by (23). Then, we have the equation (24). Comparing the equations (24) and (107), we take $\bar{v} = t$ and $r(k) = \sqrt{\lambda^2 - z^2(u)}$ and we also have

$$\left(x'^2(u) + y'^2(u) + z'^2(u) + \frac{\lambda^2 x'^2(u)}{z^2(u) - \lambda^2}\right)du^2 = (\dot{n}^2(k) + \dot{p}^2(k) - \dot{r}^2(k))dk^2. \tag{110}$$

Set $a(u) = \frac{\dot{n}(k)}{\dot{r}(k)}$ and $b(u) = \frac{\dot{p}(k)}{\dot{r}(k)}$. We find

$$n = -\int \frac{a(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du, \quad p = -\int \frac{b(u)z(u)z'(u)}{\sqrt{\lambda^2 - z^2(u)}}du. \tag{111}$$

Thus, we get an isometric timelike rotational surface R^2_{2b} given by (102) satisfying (103). It can be easily seen that a spacelike helix on X_{2b} which is defined by $u = u_0$ for a constant u_0 corresponds to parallel spacelike hyperbola lying on the plane $\{x_1 = c_1, x_2 = c_2\}$ for constants c_1 and c_2 i.e., $R^2_{2b}(u_0, v) = (c_1, c_2, \sqrt{\lambda^2 - z_0^2}\sinh v, \sqrt{\lambda^2 - z_0^2}\cosh v)$.

Case (ii) Assume that I_2 is dense in I. Then, we consider a timelike rotational surface R_{2b} in $\mathbb{E}^4_1$ given by (27). Comparing the equations (28) and (107), we take $\bar{v} = t$ and $s(k) = \sqrt{z^2(u) - \lambda^2}$ and we also have

$$\left(x'^2(u) + y'^2(u) + z'^2(u) + \frac{\lambda^2 x'^2(u)}{z^2(u) - \lambda^2}\right)du^2 = (\dot{n}^2(k) + \dot{p}^2(k) + \dot{s}^2(k))dk^2. \tag{112}$$

Set $a(u) = \frac{\dot{n}(k)}{\dot{s}(k)}$ and $b(u) = \frac{\dot{p}(k)}{\dot{s}(k)}$. Then, we obtain

$$n = \int \frac{a(u)z(u)z'(u)}{\sqrt{z^2(u) - \lambda^2}}du, \quad p = \int \frac{b(u)z(u)z'(u)}{\sqrt{z^2(u) - \lambda^2}}du. \tag{113}$$

Thus, we get the timelike isometric rotational surface R^3_{2b} given by (104) satisfying (105). It can be easily seen that a timelike helix on X_{2b} corresponds to the parallel timelike hyperbola lying on the plane $\{x_1 = c_1, x_2 = c_2\}$ for constants c_1 and c_2, i.e., $R^3_{2b}(u_0, v) = (c_1, c_2, \sqrt{z_0^2 - \lambda^2}\cosh v, \sqrt{z_0^2 - \lambda^2}\sinh v)$. □

Lemma 5.2. *Let $X_{2b}, R^1_{2b}, R^2_{2b}$ and R^3_{2b} be timelike surfaces in $\mathbb{E}^4_1$ given by* (95), (100), (102) *and* (104), *respectively. The Gauss maps of them are given by*

$$\begin{aligned}\nu_{X_{2b}} = \frac{\epsilon}{\sqrt{-W}}\Big(&- \lambda y'\eta_{12} + (x'z\sinh v - \lambda z'\cosh v)\,\eta_{13} + (x'z\cosh v - \lambda z'\sinh v)\,\eta_{14} \\ &+ y'z\sinh v\eta_{23} + y'z\cosh v\eta_{24} + zz'\eta_{34}\Big),\end{aligned} \tag{114}$$

$$\begin{aligned}\nu_{R^1_{2b}} = -\frac{\epsilon zz'}{\sqrt{-W}}\Bigg(&\eta_{12} + a\sin\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{13} + b\sin\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{14} \\ &- a\cos\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{23} - b\cos\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{24}\Bigg),\end{aligned} \tag{115}$$

$$\begin{aligned}\nu_{R^2_{2b}} = -\frac{\epsilon zz'}{\sqrt{-W}}\Bigg(&a\cosh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{13} + a\sinh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{14} \\ &+ b\cosh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{23} + b\sinh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\eta_{24} - \eta_{34}\Bigg),\end{aligned} \tag{116}$$

$$\begin{aligned}\nu_{R^3_{2b}} = \frac{\epsilon zz'}{\sqrt{-W}}\Bigg(&a\sinh\left(v - \int \frac{\lambda x'}{z^2 - \lambda^2}du\right)\eta_{13} + a\cosh\left(v - \int \frac{\lambda x'}{z^2 - \lambda^2}du\right)\eta_{14} \\ &+ b\sinh\left(v - \int \frac{\lambda x'}{z^2 - \lambda^2}du\right)\eta_{23} + b\cosh\left(v - \int \frac{\lambda x'}{z^2 - \lambda^2}du\right)\eta_{24} + \eta_{34}\Bigg),\end{aligned} \tag{117}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ is the standard orthonormal bases of $\mathbb{E}^4_1$ and $\eta_{ij} = \eta_i \wedge \eta_j$ for $i, j = 1, 2, 3, 4$.

Proof. Using the equation (7), the Gauss maps of the surfaces can be calculated directly. □

For later use, we find the mean curvature vectors of the timelike rotational surfaces R^2_{2b} and R^3_{2b} as follows.

Lemma 5.3. *Let R^2_{2b} and R^3_{2b} be timelike rotational surfaces in $\mathbb{E}^4_1$ given by* (102) *and* (104).

(i.) The mean curvature vector $H^{R^2_{2b}}$ of R^2_{2b} in $\mathbb{E}^4_1$ is $H^{R^2_{2b}} = H_1^{R^2_{2b}}N_1 + H_2^{R^2_{2b}}N_2$ with respect to

$$\begin{aligned}N_1 &= \frac{1}{\sqrt{1 - b^2}}\left(0, 1, b\sinh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right), b\cosh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\right), \\ N_2 &= \frac{1}{\sqrt{(b^2 - 1)(a^2 + b^2 - 1)}}\left(1 - b^2, ab, a\sinh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right), a\cosh\left(v + \int \frac{\lambda x'}{\lambda^2 - z^2}du\right)\right),\end{aligned} \tag{118}$$

where $H_1^{R^2_{2b}}$ *and* $H_2^{R^2_{2b}}$ *are given by*

$$\begin{aligned} H_1^{R^2_{2b}} &= \frac{b'(z^2-\lambda^2) - bzz'(a^2+b^2-1)}{2zz'(a^2+b^2-1)\sqrt{(1-b^2)(\lambda^2-z^2)}}, \\ H_2^{R^2_{2b}} &= \frac{(z^2-\lambda^2)(a'(b^2-1)-abb') + azz'(a^2+b^2-1)}{2zz'\sqrt{(b^2-1)(\lambda^2-z^2)(a^2+b^2-1)^3}}. \end{aligned} \tag{119}$$

(ii.) The mean curvature vector $H^{R^3_{2b}}$ *of* R^3_{2b} *in* $\mathbb{E}^4_1$ *is* $H^{R^3_{2b}} = H_1^{R^3_{2b}} N_1 + H_2^{R^3_{2b}} N_2$ *with respect to*

$$\begin{aligned} N_1 &= -\frac{1}{\sqrt{1+b^2}}\left(0, -1, b\cosh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right), b\sinh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right)\right), \\ N_2 &= -\frac{1}{\sqrt{(1+b^2)(1+a^2+b^2)}}\left(-1-b^2, ab, a\cosh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right), a\sinh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right)\right), \end{aligned} \tag{120}$$

where $H_1^{R^3_{2b}}$ *and* $H_2^{R^3_{2b}}$ *are given by*

$$\begin{aligned} H_1^{R^3_{2b}} &= \frac{b'(z^2-\lambda^2) + bzz'(a^2+b^2+1)}{2zz'(a^2+b^2+1)\sqrt{(1+b^2)(z^2-\lambda^2)}}, \\ H_2^{R^3_{2b}} &= \frac{(z^2-\lambda^2)(a'(1+b^2)-abb') + azz'(a^2+b^2+1)}{2zz'\sqrt{(1+b^2)(z^2-\lambda^2)(a^2+b^2+1)^3}}. \end{aligned} \tag{121}$$

Proof. It follows from a direct calculation. □

Then, we consider isometric surfaces according to Bour's theorem whose Gauss maps are same.

Theorem 5.4. *Let* $X_{2b}, R^1_{2b}, R^2_{2b}$ *and* R^3_{2b} *be a timelike helicoidal surface of type IIb and timelike rotational surfaces in* $\mathbb{E}^4_1$ *given by* (95), (100), (102) *and* (104), *respectively. Then, we have the following statements.*

(i.) The Gauss maps of X_{2b} *and* R^1_{2b} *are definitely different.*

(ii.) If the Gauss maps of the surfaces X_{2b} *and* R^2_{2b} *are same, then they are hyperplanar and minimal. Then, the parametrizations of the surfaces* X_{2b} *and* R^2_{2b} *can be explicitly determined by*

$$X_{2b}(u,v) = (x(u) + \lambda v, c_1, z(u)\cosh v, z(u)\sinh v) \tag{122}$$

and

$$R^2_{2b}(u,v) = \begin{pmatrix} \pm\frac{1}{\sqrt{c_3}}\,\mathrm{arcsinh}\,\sqrt{c_3(\lambda^2 - z^2(u))} + c_4 \\ c_2 \\ \sqrt{\lambda^2 - z^2(u)}\sinh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \\ \sqrt{\lambda^2 - z^2(u)}\cosh\left(v + \int \frac{\lambda x'(u)}{\lambda^2 - z^2(u)}du\right) \end{pmatrix}, \tag{123}$$

where c_1, c_2, c_3 *and* c_4 *are arbitrary constants with* $c_3 > 0$ *and*

$$x(u) = \pm\left(\sqrt{\frac{1+c_3\lambda^2}{c_3}}\,\mathrm{arcsinh}\,\sqrt{c_3(\lambda^2 - z^2(u))} - \lambda\,\mathrm{arctanh}\left(\frac{\sqrt{(1+c_3\lambda^2)(\lambda^2 - z^2(u))}}{\lambda\sqrt{1 + c_3(\lambda^2 - z^2(u))}}\right)\right). \tag{124}$$

(iii) If the Gauss maps of the surfaces X_{2b} *and* R^3_{2b} *are same, then they are hyperplanar and minimal. Then, the parametrizations of the surfaces* X_{2b} *and* R^3_{2b} *can be explicitly determined by*

$$X_{2b}(u,v) = (x(u) + \lambda v, c_1, z(u)\cosh v, z(u)\sinh v) \tag{125}$$

and

$$R^3_{2b}(u,v) = \begin{pmatrix} \pm\frac{1}{\sqrt{c_3}} \operatorname{arccosh} \sqrt{c_3\,(z^2(u)-\lambda^2)} + c_4 \\ c_2 \\ \sqrt{z^2(u)-\lambda^2}\cosh\left(v - \int \frac{\lambda x'(u)}{z^2(u)-\lambda^2}du\right) \\ \sqrt{z^2(u)-\lambda^2}\sinh\left(v - \int \frac{\lambda x'(u)}{z^2(u)-\lambda^2}du\right) \end{pmatrix}, \tag{126}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants with $c_3 > 0$ and

$$x(u) = \pm\frac{1}{\sqrt{2}}\Big(\sqrt{1+c_3\lambda^2}\operatorname{arcsinh}\sqrt{c_3(z^2(u)-\lambda^2)-1} - \lambda\sqrt{c_3}\operatorname{arctanh}\left(\frac{\lambda\sqrt{c_3(z^2(u)-\lambda^2)-1}}{\sqrt{(1+c_3\lambda^2)(z^2(u)-\lambda^2)}}\right)\Big). \tag{127}$$

Proof. Assume that X_{2b} is a timelike helicoidal surface of type I in $\mathbb{E}^4_1$ given by (95) and $R^1_{2b}, R^2_{2b}, R^3_{2b}$ are timelike rotational surfaces $\mathbb{E}^4_1$ given by (100), (102) and (104), respectively. From Lemma 5.2, we have the Gauss maps of $X_{2b}, R^1_{2b}, R^2_{2b}$ and R^3_{2b} given by (114), (115), (116) and (117), respectively.
(i.) Suppose that the Gauss maps of X_{2b} and R^1_{2b} are same. From the equations (114) and (115), we get $z(u) = 0$ or $z'(u) = 0$ which implies $\nu_{R^1_{2b}} = 0$. That is a contradiction. Hence, their Gauss maps are definitely different.
(ii.) Suppose that the surfaces X_{2b} and R^2_{2b} have the same Gauss maps. Comparing (114) and (116), we have the following system of equations:

$$\lambda y' = 0, \tag{128}$$

$$x'z\sinh v - \lambda z'\cosh v = -azz'\cosh\left(v + \int\frac{\lambda x'}{\lambda^2 - z^2}du\right), \tag{129}$$

$$x'z\cosh v - \lambda z'\sinh v = -azz'\sinh\left(v + \int\frac{\lambda x'}{\lambda^2 - z^2}du\right), \tag{130}$$

$$y'z\sinh v = -bzz'\cosh\left(v + \int\frac{\lambda x'}{\lambda^2 - z^2}du\right), \tag{131}$$

$$y'z\cosh v = -bzz'\sinh\left(v + \int\frac{\lambda x'}{\lambda^2 - z^2}du\right). \tag{132}$$

Due to $\lambda \neq 0$, the equation (128) gives $y'(u) = 0$ for all $u \in I_1$. Then, from the equations (131) and (132) imply $b(u) = 0$. Therefore, it can be easily seen that the surfaces X_{2b} and R^2_{2b} are hyperplanar, that is, they are lying in $\mathbb{E}^3_1$. Moreover, the equations (99) and (119) imply that $H_1^{X_{2b}} = H_1^{R^2_{2b}} = 0$. Also, from the equations (99) and (119), we have

$$\begin{aligned} H_2^{X_{2b}} &= \frac{x'z'^2(z^2-2\lambda^2) + z^2x'^3 + z(\lambda^2 - z^2)(x'z'' - x''z')}{2(z^2(x'^2+z'^2) - \lambda^2 z'^2)^{3/2}}, \\ H_2^{R^2_{2b}} &= \frac{azz'(a^2-1) - a'(z^2-\lambda^2)}{2zz'\sqrt{(\lambda^2-z^2)(1-a^2)^3}}. \end{aligned} \tag{133}$$

Using $y'(u) = b(u) = 0$, from the equation (103) we have

$$a^2(u) = \frac{\lambda^2 z'^2(u) - z^2(u)x'^2(u)}{z^2(u)z'^2(u)}. \tag{134}$$

Also, by using the equation (134) in (133), we get

$$H_2^{R^2_{2b}} = -\frac{x'z^2(x'z'^2(2\lambda^2 - z^2) - z^2x'^3 + z(\lambda^2 - z^2)(z'x'' - x'z''))}{2(z^2(x'^2+z'^2) - \lambda^2 z'^2)^{3/2}\sqrt{(\lambda^2 z'^2 - z^2x'^2)(\lambda^2 - z^2)}} \tag{135}$$

which implies $H_2^{R^2_{2b}} = \frac{x'z^2}{\sqrt{(\lambda^2 z'^2 - z^2 x'^2)(\lambda^2 - z^2)}} H_2^{X_{2b}}$. Moreover, using equations (129) and (130), we obtain the following equations

$$-x'z = azz' \sinh\left(\int \frac{\lambda x'}{\lambda^2 - z^2} du\right), \tag{136}$$

$$\lambda z' = azz' \cosh\left(\int \frac{\lambda x'}{\lambda^2 - z^2} du\right). \tag{137}$$

Considering the equations (136) and (137) together, we have

$$-\frac{x'z}{\lambda z'} = \tanh\left(\int \frac{\lambda x'}{\lambda^2 - z^2} du\right). \tag{138}$$

If we take the derivative of the equation (137) with respect to u, the equation (137) becomes

$$x'z'^2(2\lambda^2 - z^2) - z^2 x'^3 + z(\lambda^2 - z^2)(z'x'' - x'z'') = 0 \tag{139}$$

which implies $H_2^{X_{2b}} = H_2^{R^2_{2b}} = 0$. Thus, we get the desired results. Now, we determine the parametrizations of the isometric surfaces X_{2b} and R^2_{2b}. Since the surface R^2_{2b} is minimal, from the equation (139), we have the following differential equation

$$(z^2 - \lambda^2)a' + zz'a = zz'a^3 \tag{140}$$

which is a Bernoulli equation. Then, the general solution of this equation is found as

$$a^2 = \frac{1}{1 + c_3(\lambda^2 - z^2)} \tag{141}$$

for an arbitrary positive constant c_3. Comparing the equations (134) and (141), we get

$$x(u) = \pm\sqrt{1 + c_3\lambda^2} \int \frac{z'(u)}{z(u)} \sqrt{\frac{\lambda^2 - z^2(u)}{1 + c_3(\lambda^2 - z^2)}} du. \tag{142}$$

whose solution is given by (127) for $c_3 > 0$. Moreover, using the last component of $R^2_{2b}(u, v)$ in (126), we have

$$\int \frac{z(u)z'(u)}{\sqrt{(\lambda^2 - z^2(u))(1 + c_3(\lambda^2 - z^2(u)))}} du = \pm\frac{1}{\sqrt{c_3}} \operatorname{arcsinh}\left(\sqrt{c_3(\lambda^2 - z^2(u))}\right) + c_4 \tag{143}$$

for any arbitrary constant c_4.

(iii.) Suppose that the surfaces X_{2b} and R^3_{2b} have the same Gauss maps. From (114) and (117), we get the following system of equations:

$$\lambda y' = 0, \tag{144}$$

$$x'z\sinh v - \lambda z'\cosh v = azz'\sinh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right), \tag{145}$$

$$x'z\cosh v - \lambda z'\sinh v = azz'\cosh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right), \tag{146}$$

$$y'z\sinh v = bzz'\sinh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right), \tag{147}$$

$$y'z\cosh v = bzz'\cosh\left(v - \int \frac{\lambda x'}{z^2-\lambda^2}du\right). \tag{148}$$

Due to $\lambda \neq 0$, the equation (144) gives $y'(u) = 0$ for $u \in I_2$. Then, from the equations (147) and (148) imply $b(u) = 0$. Therefore, it can be easily seen that the surfaces X_{2b} and R^3_{2b} are hyperplanar, that is, they are lying in $\mathbb{E}^3_1$. Moreover, the equations (99) and (121) imply that $H_1^{X_{2b}} = H_1^{R^3_{2b}} = 0$. Also, from the equations (99) and (121), we have

$$\begin{aligned} H_2^{X_{2b}} &= \frac{x'z'^2(z^2-2\lambda^2) + z^2x'^3 + z(\lambda^2 - z^2)(x'z'' - x''z')}{2(z^2(x'^2+z'^2) - \lambda^2 z'^2)^{3/2}}, \\ H_2^{R^3_{2b}} &= \frac{a'(z^2-\lambda^2) + azz'(1+a^2)}{2zz'\sqrt{(z^2-\lambda^2)(1+a^2)^3}}. \end{aligned} \tag{149}$$

Using $y'(u) = b(u) = 0$ for all $u \in I_2$, from the equation (105) we have

$$a^2(u) = \frac{z^2(u)x'^2(u) - \lambda^2 z'^2(u)}{z^2(u)z'^2(u)}. \tag{150}$$

Using the equation (150) in (149), we get

$$H_2^{R^3_{2b}} = \frac{z^2x'(x'z'^2(z^2-2\lambda^2) + z^2x'^3 + z(\lambda^2-z^2)(x'z''-x''z'))}{2(z^2(x'^2+z'^2)-\lambda^2z'^2)^{3/2}\sqrt{(z^2-\lambda^2)(z^2x'^2-\lambda^2z'^2)}} \tag{151}$$

which implies $H_2^{R^3_{2b}} = \frac{z^2x'}{\sqrt{(z^2x'^2-\lambda^2z'^2)(z^2-\lambda^2)}} H_2^{X_{2b}}$. Moreover, using equations (145) and (146), we obtain the following equations

$$x'z = azz'\cosh\left(\int \frac{\lambda x'}{z^2-\lambda^2}du\right), \tag{152}$$

$$\lambda z' = azz'\sinh\left(\int \frac{\lambda x'}{z^2-\lambda^2}du\right). \tag{153}$$

Considering the equations (152) and (153) together, we have

$$\frac{x'z}{\lambda z'} = \coth\left(\int \frac{\lambda x'}{z^2-\lambda^2}du\right). \tag{154}$$

If we take the derivative of the equation (154) with respect to u, (154) becomes

$$x'z'^2(z^2-2\lambda^2) + z^2x'^3 + z(\lambda^2-z^2)(x'z''-x''z') = 0 \tag{155}$$

which implies $H_2^{X_{2b}} = H_2^{R^3_{2b}} = 0$ in the equation (149). Thus, we get the desired results. Since R^3_{2b} is minimal, from the equation (149) we have the following differential equation

$$(z^2 - \lambda^2)a' + zz'a = -zz'a^3 \tag{156}$$

which is a Bernoulli equation. Then, the general solution of this equation is found as

$$a^2 = \frac{1}{c_3(z^2 - \lambda^2) - 1} \tag{157}$$

for an arbitrary positive constant c_3. Comparing the equations (150) and (157), we get

$$x(u) = \pm\sqrt{1 + c_3\lambda^2} \int \frac{z'(u)}{z(u)} \sqrt{\frac{z^2(u) - \lambda^2}{c_3(z^2 - \lambda^2) - 1}} du \tag{158}$$

whose solution is given by (127) for $c_3 > 0$. Moreover, using the last component of $R^3_{2b}(u, v)$ in (126), we have

$$\int \frac{z(u)z'(u)}{\sqrt{(z^2(u) - \lambda^2)(c_3(z^2(u) - \lambda^2) - 1)}} du = \pm\frac{1}{\sqrt{c_3}} \operatorname{arccosh}\left(\sqrt{c_3(z^2(u) - \lambda^2)}\right) + c_4 \tag{159}$$

for any arbitrary constant c_4. □

Remark 5.5. *Taking $z(u) = u$ in Theorem 5.4, we get isometric surfaces obtained in [17] and the rotational surface given by* (126) *also has the same form of minimal surface in Proposition 3.1, [17].*

Assume that X_{2b} is a timelike right helicoidal surface of type IIb in $\mathbb{E}^4_1$, that is, $x'(u) = 0$ for all $u \in I$. On the other hand, we know that $W = (\lambda^2 - z^2(u))(y'^2(u) + z'^2(u)) < 0$ when I_2 is dense in I. Then, from Theorem 5.1, we get the parametrizations of isometric timelike rotational surfaces in $\mathbb{E}^4_1$. Then, Theorem 4.4 implies that if their Gauss maps are same, $y'(u) = b(u) = 0$ for all $u \in I_2$. Hence, we get $a^2(u) = -\frac{\lambda^2}{z^2(u)}$ which gives a contradiction. Thus, they have the different Gauss maps.

Now, we give an example by using Theorem 5.4.

Example 5.6. *If we choose $z(u) = u$, $c_3 = \frac{1}{2}$, $\lambda = 1$ and $c_4 = 0$, then isometric surfaces in* (125) *and* (126) *are given as follows*

$$X_{2b}(u, v) = \left(\frac{\sqrt{3}}{2} \operatorname{arcsinh} \sqrt{\frac{u^2 - 3}{2}} - \frac{1}{2} \operatorname{arctanh} \sqrt{\frac{u^2 - 3}{3u^2 - 3}} + v, u\cosh v, u\sinh v\right)$$

and

$$R^3_{2b}(u, v) = \begin{pmatrix} \sqrt{2}\operatorname{arccosh}\sqrt{\frac{u^2-1}{2}} \\ \sqrt{u^2 - 1}\cosh\left(v - \operatorname{arctanh}\sqrt{\frac{u^2-3}{3u^2-3}}\right) \\ \sqrt{u^2 - 1}\sinh\left(v - \operatorname{arctanh}\sqrt{\frac{u^2-3}{3u^2-3}}\right) \end{pmatrix}.$$

For $2 \leq u \leq 8$ and $-1 \leq v \leq 1$, the graphs of timelike helicoidal surface X_{2b} and timelike rotational surface R^3_{2b} in $\mathbb{E}^3_1$ can be plotted by using Wolfram Mathematica 10.4 given in Figure 5 and Figure 6, respectively.

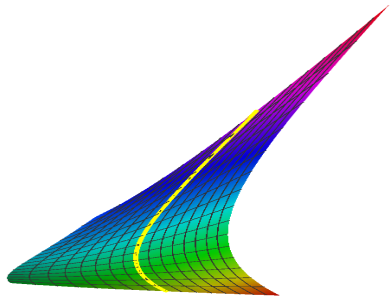

Figure 5: Timelike helicoidal surface of type IIb; timelike helix.

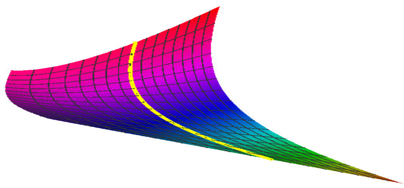

Figure 6: Timelike rotational surface; timelike hyperbola.

6. Helicoidal Surface of Type III

Let $\{\eta_1, \eta_2, \xi_3, \xi_4\}$ be the pseudo-orthonormal basis of $\mathbb{E}^4_1$ such that $\xi_3 = \frac{1}{\sqrt{2}}(\eta_4 - \eta_3)$ and $\xi_4 = \frac{1}{\sqrt{2}}(\eta_3 + \eta_4)$. We choose as a lightlike 2-plane $P_3 = span\{\eta_1, \xi_3\}$, a hyperplane $\Pi_3 = span\{\eta_1, \xi_3, \xi_4\}$ and a line $l_3 = span\{\xi_3\}$. Then, the orthogonal transformation T_3 of $\mathbb{E}^4_1$ which leaves the lightlike plane P_3 invariant is given by $T_3(\eta_1) = \eta_1$, $T_3(\eta_2) = \eta_2 + \sqrt{2}v\xi_3$, $T_3(\xi_3) = \xi_3$ and $T_3(\xi_4) = \sqrt{2}v\eta_2 + v^2\xi_3 + \xi_4$. We suppose that $\beta_3(u) = x(u)\eta_1 + z(u)\xi_3 + w(u)\xi_4$ is a regular curve, where $w(u) \neq 0$. By using the definition of helicoidal surface, the parametrization of X_3 (called as the helicoidal surface of type III) is given by

$$X_3(u, v) = x(u)\eta_1 + \sqrt{2}vw(u)\eta_2 + (z(u) + v^2w(u) + \lambda v)\xi_3 + w(u)\xi_4, \tag{160}$$

where, $v \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. When w is a constant function, X_3 is called as right helicoidal surface of type III (see [2]). For $\lambda = 0$, the helicoidal surface which is given by (160) reduces to the rotational surface of parabolic type in $\mathbb{E}^4_1$ (see [9] and [3]).

By a direct calculation, we get the induced metric of X_3 given as follows.

$$ds^2_{X_3} = (x'^2(u) - 2w'(u)z'(u))du^2 - 2\lambda w'(u)dudv + 2w^2(u)dv^2. \tag{161}$$

Due to the fact that X_3 is a timelike helicoidal surface in $\mathbb{E}^4_1$, we have $W = 2w^2(u)(x'^2(u) - 2w'(u)z'(u)) - \lambda^2 w'^2(u) < 0$ for all $u \in I \subset \mathbb{R}$. Then, we choose an orthonormal frame field $\{e_1, e_2, N_1, N_2\}$ on X_3 in $\mathbb{E}^4_1$ such that e_1, e_2 are tangent to X_3 and N_1, N_2 are normal to X_3 as follows.

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{\epsilon g_{11}}} X_{3u}, \quad e_2 = \frac{1}{\sqrt{-\epsilon W g_{11}}}(g_{11}X_{3v} - g_{12}X_{3u}),\\ N_1 &= \eta_1 + \frac{x'}{w'}\xi_3,\\ N_2 &= \frac{1}{w'\sqrt{-W}}\Big(\sqrt{2}x'ww'\eta_1 + (\lambda w'^2 + 2vww'^2)\eta_2 + \sqrt{2}(\lambda v w'^2 + v^2 ww'^2 + wx'^2 - ww'z')\xi_3 + \sqrt{2}ww'^2\xi_4\Big), \end{aligned} \tag{162}$$

where $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = \epsilon$ and $\langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 1$. It can be easily seen that X_3 has a spacelike meridian curve for $\epsilon = 1$. Otherwise, it has a timelike meridian curve. By direct computations, we get the coefficients of the second fundamental form given as follows.

$$\begin{aligned} b^1_{11} &= \frac{x''w' - x'w''}{w'}, \quad b^1_{12} = b^1_{21} = b^1_{22} = 0,\\ b^2_{11} &= \frac{\sqrt{2}w(x'x''w' - x'^2w'' + w'(z'w'' - w'z''))}{w'\sqrt{-W}}, \quad b^2_{12} = b^2_{21} = \frac{\sqrt{2}\lambda w'^2}{\sqrt{-W}}, \quad b^2_{22} = -\frac{2\sqrt{2}w^2w'}{\sqrt{-W}}. \end{aligned} \tag{163}$$

Thus, the mean curvature vector H^{X_3} of X_3 in $\mathbb{E}^4_1$ is

$$H^{X_3} = H^{X_3}_1 N_1 + H^{X_3}_2 N_2, \tag{164}$$

where N_1, N_2 are normal vector fields in (162), $H^{X_3}_1$ and $H^{X_3}_2$ are given by

$$\begin{aligned} H^{X_3}_1 &= \frac{w^2(x''w' - x'w'')}{w'W},\\ H^{X_3}_2 &= \frac{-\sqrt{2}(\lambda^2 w'^4 + 2w^2w'^3z' - w^3x'^2w'' + w^3w'(z'w'' + x'x'') - w^2w'^2(x'^2 + wz''))}{w'(-W)^{3/2}}. \end{aligned} \tag{165}$$

Note that $w'(u) \neq 0$ for all $u \in I \subset \mathbb{R}$ because X_3 is a timelike surface in $\mathbb{E}^4_1$.

6.1. Bour's Theorem and the Gauss map for helicoidal surface of type III

In this section, we study on Bour's theorem for timelike helicoidal surface of type III in $\mathbb{E}^4_1$ and we analyse the Gauss maps of isometric pair of surfaces.

Theorem 6.1. *A timelike helicoidal surface of type III in $\mathbb{E}^4_1$ given by* (160) *is isometric to a timelike rotational surface in $\mathbb{E}^4_1$:*

$$R_3(u,v) = \int a(u)w'(u)du\,\eta_1 + \sqrt{2}w(u)\left(v + \frac{\lambda}{2w(u)}\right)\eta_2 + \left(\int b(u)w'(u)du + w(u)\left(v + \frac{\lambda}{2w(u)}\right)^2\right)\xi_3 + w(u)\xi_4 \tag{166}$$

so that spacelike helices on the timelike helicoidal surface of type III correspond to parallel spacelike parabolas on the timelike rotational surfaces, where $a(u)$ and $b(u)$ are differentiable functions satisfying the following equation:

$$a^2(u) - 2b(u) = \frac{x'^2(u) - 2w'(u)z'(u)}{w'^2(u)} - \frac{\lambda^2}{2w^2(u)}. \tag{167}$$

Proof. Assume that X_3 is a timelike helicoidal surface of type III in $\mathbb{E}^4_1$ defined by (160). Then, we have the induced metric of X_3 given by (161). Now, we will find new coordinates $\bar{u}, \bar{v}$ such that the metric becomes $ds^2_{X_3} = F(\bar{u})d\bar{u}^2 + G(\bar{u})d\bar{v}^2$, where $F(\bar{u})$ and $G(\bar{u})$ are some smooth functions. Set $\bar{u} = u$ and $\bar{v} = v + \frac{\lambda}{2w(u)}$. Since Jacobian $\dfrac{\partial(\bar{u},\bar{v})}{\partial(u,v)}$ is nonzero, it follows that $\{\bar{u},\bar{v}\}$ are new parameters of X_3. According to the new parameters, the equation (161) becomes

$$ds^2_{X_3} = \left(x'^2(u) - 2w'(u)z'(u) - \frac{\lambda^2 w'^2(u)}{2w^2(u)}\right)du^2 + 2w^2(u)d\bar{v}^2. \tag{168}$$

On the other hand, the timelike rotational surface R_3 in $\mathbb{E}^4_1$ related to X_3 is given by

$$R_3(k,t) = n(k)\eta_1 + \sqrt{2}tr(k)\eta_2 + (s(k) + t^2 r(k))\xi_3 + r(k)\xi_4. \tag{169}$$

We know that the induced metric of R_3 is given by

$$ds^2_{R_3} = (\dot{n}^2(k) - 2\dot{r}(k)\dot{s}(k))dk^2 + 2r^2(k)dt^2 \tag{170}$$

with $r(k) > 0$. From the equations (168) and (170), we get an isometry between X_3 and R_3 by taking $\bar{v} = t$, $r(k) = w(u)$ and

$$\left(x'^2(u) - 2w'(u)z'(u) - \frac{\lambda^2 w'^2(u)}{2w^2(u)}\right)du^2 = (\dot{n}^2(k) - 2\dot{r}(k)\dot{s}(k))dk^2. \tag{171}$$

Let define $a(u) = \frac{\dot{n}(k)}{\dot{r}(k)}$ and $b(u) = \frac{\dot{s}(k)}{\dot{r}(k)}$. Using these in the equation (171), we obtain the equation (167). Then, we have

$$n = \int a(u)w'(u)du \quad \text{and} \quad s = \int b(u)w'(u)du. \tag{172}$$

Thus, we get an isometric timelike rotational surface R_3 given by (166). Moreover, if we choose a spacelike helix $X_3(u_0, v)$ on X_3 for an arbitrary constant u_0, then it corresponds to $R_3(u_0, v) = c_1\eta_1 + \sqrt{2}w_0\left(v + \frac{\lambda}{2w_0}\right)\eta_2 + \left(c_2 + w_0\left(v + \frac{\lambda}{2w_0}\right)^2\right)\xi_3 + w_0\xi_4$, where c_1 and c_2 arbitrary constant. If we take $t = v + \frac{\lambda}{2w_0}$, then it can be rewritten $\alpha(t) = \sqrt{2}w_0\left(0, t, -\frac{t^2}{2}, \frac{t^2}{2}\right) + \frac{1}{\sqrt{2}}\left(\sqrt{2}c_1, 0, w_0 - c_2, w_0 + c_2\right)$. From Definition 2.1, it can be seen that $\alpha(t)$ is a spacelike parabola. □

Lemma 6.2. *Let X_3 and R_3 be timelike surfaces in $\mathbb{E}^4_1$ given by* (160) *and* (166)*, respectively. Then, the Gauss maps of them*

$$\begin{aligned}\nu_{X_3} =& \frac{\epsilon}{\sqrt{-W}}\Big(\sqrt{2}x'w\eta_1\wedge\eta_2 + x'(\lambda + 2vw)\eta_1\wedge\xi_3 + \sqrt{2}(v^2ww' - z'w + \lambda vw')\eta_2\wedge\xi_3 \\ &- \sqrt{2}ww'\eta_2\wedge\xi_4 - w'(\lambda + 2vw)\xi_3\wedge\xi_4\Big),\end{aligned} \tag{173}$$

$$\begin{aligned}\nu_{R_3} =& \frac{\epsilon ww'}{\sqrt{-W}}\left(\sqrt{2}a\eta_1\wedge\eta_2 + 2a\left(v + \frac{\lambda}{2w}\right)\eta_1\wedge\xi_3 + \sqrt{2}\left(\left(v + \frac{\lambda}{2w}\right)^2 - b\right)\eta_2\wedge\xi_3\right. \\ &\left.- \sqrt{2}\eta_2\wedge\xi_4 - 2\left(v + \frac{\lambda}{2w}\right)\xi_3\wedge\xi_4\right).\end{aligned} \tag{174}$$

Proof. It follows from a direct calculation. □

Theorem 6.3. *A timelike helicoidal surface of type III and a timelike rotational surface in* $\mathbb{E}^4_1$ *given by* (160) *and* (166)*, respectively have the same Gauss map.*

Proof. Assume that the surfaces X_3 and R_3 have the same Gauss map. Comparing (173) and (174), we get the following system of equations

$$x' = aw', \tag{175}$$

$$x'(\lambda + 2vw) = 2aww'\left(v + \frac{\lambda}{2w}\right), \tag{176}$$

$$wz' = bww' - \frac{\lambda^2 w'}{4w}. \tag{177}$$

From the equations (175) and (177), we find $a(u)$ and $b(u)$. Using these in (167), we can see that they have the same Gauss map. □

We note that T. Ikawa [17] studied Bour's theorem for helicoidal surfaces in $\mathbb{E}^3_1$ with lightlike axis and he showed that they have the same Gauss map.

Now, we give an example by using Theorem 6.1.

Example 6.4. *If we choose* $x(u) = a(u) = 0$*,* $w(u) = z(u) = u$ *and* $\lambda = 5$*, then isometric surfaces in* (160) *and* (166) *are given as follows*

$$X_3(u, v) = \sqrt{2}uv\eta_2 + \left(u + uv^2 + 5v\right)\xi_3 + u\xi_4$$

and

$$R_3(u, v) = \left(\sqrt{2}uv + \frac{5}{\sqrt{2}}\right)\eta_2 + \left(u - \frac{25}{4u} + u\left(v + \frac{5}{2u}\right)^2\right)\xi_3 + u\xi_4.$$

For $-4 \leq u \leq 4$ *and* $-3 \leq v \leq 3$*, the graphs of timelike helicoidal surface* X_3 *and timelike rotational surface* R_3 *in* $\mathbb{E}^3_1$ *can be plotted by using Mathematica 10.4 given in Figure 7 and Figure 8, respectively.*

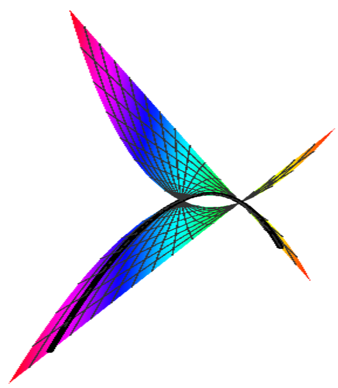

Figure 7: Timelike helicoidal surface of type III; spacelike helix.

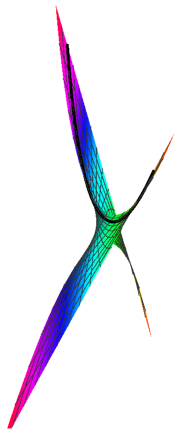

Figure 8: Timelike rotational surface; spacelike parabola.

7. Conclusion

In this paper, we study on Bour's theorem for four kinds of timelike helicoidal surfaces in 4-dimensional Minkowski space. Then, we find the Gauss maps of these isometric pair of surfaces. We get the characterizations of isometric helicoidal and rotational surfaces whose Gauss maps are identical. Also, we determine the parametrizations of such isometric pair of surfaces. Thus, these results are a kind of generalization of right helicoidal and catenoid in $\mathbb{E}^3$. Finally, we give some examples by using Wolfram Mathematica 10.4.

In the future, we will try to determine the helicoidal and rotational surfaces which are isometric according to Bour's theorem whose the mean curvature vectors or their lengths are zero and the Gaussian curvatures are zero, respectively.

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