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On gradient Riemann solitons invariant by rotation

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Abstract. In this paper, we consider Riemann solitons that are conformal to a Euclidean space. Assuming that the solutions of the presented system of partial differential equations are invariant under the action of the orthogonal group, we provide all solutions for the gradient Riemann solitons. We show that a gradient Riemann soliton (M, g) is both geodesically complete and rotationally symmetric if and only if g is the canonical product metric on $\mathbb{R} \times \mathbb{S}^{n-1}$. Furthermore, this soliton is shrinking.

1. Introduction and Main Results

Let (M, g) be a smooth manifold of dimension $n \geq 3$. We denote by $G = \frac{1}{2} g \owedge g$ and *Riem* the Riemann curvature tensor associated to the metric g and $\owedge$ is Kulkarni-Nomizu product. We say that (M, g) is a Riemann soliton, or supports a Riemann soliton structure, if there exists a smooth vector field V on M and a smooth function $\lambda : M \longrightarrow \mathbb{R}$ such that

$$2Riem + \lambda(g \owedge g) + (g \owedge \pounds_V g) = 0. \tag{1}$$

Here, $\pounds_V$ denotes the Lie-derivative operator in the direction of the vector field V. If the vector field V is the gradient of a potential function f, then we get the notion of gradient Riemann soliton. In this case, equation (1) becomes

$$Riem + \frac{1}{2}\lambda(g \owedge g) + (g \owedge Hess\ f) = 0, \tag{2}$$

where $Hess\ f$ denotes the Hessian of the smooth function f.

A Riemann soliton is called trivial when the potential vector field V vanishes identically; in this case, the manifold has constant sectional curvature. As usual, the Riemann soliton is called steady for $\lambda = 0$, shrinking for $\lambda > 0$ and expanding for $\lambda < 0$.

Riemann solitons arise as self-similar solutions of the Riemann flow introduced by Udriste [12, 13]. The Riemann flow is an evolution equation for metrics on a Riemannian manifold defined as follows

$$\frac{\partial}{\partial t} G(t) = -2Riem_{g(t)},$$

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where $Riem_{g(t)}$ is the Riemann tensor of the metric $g(t)$.

In [12], the author defines and explores the Riemann flow and establishes a connection to the Ricci flow, showing that if M has dimension $n = 2$, then G is linked to $det(g_{ij})$, while in dimension $n \geq 3$, G determines the metric g. The Riemann flow induces a Ricci flow on the manifold M, but only in some particular cases it is determined by the Ricci flow. Existence and uniqueness theorems, as well as the significance of Riemann flows on constant curvature manifolds, are among the findings. In the compact case, a good result proposed by Hiriča, Udrişte and Petersen, proved that on a compact Riemannian manifold, Ricci and Riemann solitons are gradient solitons (see, [6, 10]). We note that for the short existence of time, the initial value problem associated with the Riemann flow has a unique solution (see, e.g., [5], and references therein).

In a recent work proposed by Tokura et al. [3], the authors provided a complete classification of gradient Riemann soliton whenever $n \geq 4$. In this case, they proved that every gradient Riemann solitons is locally conformally flat. This result states that any gradient Riemann soliton has a null Weyl tensor $W_{ijkl} = 0$.

Example 1.1. *Let (M^n_c, g) be a Riemannian manifold of constant sectional curvature $c \in \{-1, 0, 1\}$. Then (M^n_c, g) is a trivial Riemann soliton for any constant function f on $\mathbb{M}^n_c$ since $Riem = \frac{c}{2} g \owedge g$.*

Example 1.2. *Let $((0,\infty) \times \mathbb{S}^n, g)$ be the Riemannian product manifold of $(0,\infty)$ and the round sphere $\mathbb{S}^n$. By a straight calculation, we deduce that $((0,\infty) \times \mathbb{S}^n, g)$ is a gradient Riemann soliton with potential function*

$$f : (0,\infty) \times \mathbb{S}^n \longrightarrow \mathbb{R}, \quad (r,p) \longmapsto \frac{\lambda}{4r},$$

where $\lambda \in \mathbb{R}$.

Example 1.3. *Consider the product manifold $(0,\infty) \times \mathbb{S}^n$ equipped with the metric tensor given by*

$$g_{(r,p)}(t_1 \oplus v_1, t_2 \oplus v_2) = r\,(t_1 t_2 + v_1 v_2)\,,$$

for every $(r,p) \in (0,\infty) \times \mathbb{S}^n$ and $t_1 \oplus v_1, t_2 \oplus v_2 \in \mathbb{R} \oplus T_p\mathbb{S}^n \cong T_{(r,p)}\left((0,\infty) \times \mathbb{S}^n\right)$. Then, by a straight calculation, we have $((0,\infty) \times \mathbb{S}^n, g)$ as a gradient Riemann soliton for

$$f : (0,\infty) \times \mathbb{S}^n \longrightarrow \mathbb{R}, \quad (r,p) \longmapsto \ln\left(r^2\right) + \frac{r^2}{4}.$$

Examples (1.2) and (1.3) are both locally conformally flat gradient Riemann solitons invariant by the left action of

$$\mathrm{O}(n,\mathbb{R}) = \left\{A \in \mathrm{GL}(n,\mathbb{R}) : AA^t = Id_n\right\},$$

on $(0,\infty) \times \mathbb{S}^n$, given by

$$\mathrm{O}(n,\mathbb{R}) \times ((0,\infty) \times \mathbb{S}^n) \longrightarrow (0,\infty) \times \mathbb{S}^n, \quad (A,(r,p)) \longmapsto (r, A \cdot p).$$

The previous rotationally symmetric examples are particular instances of a characterization result established by the present authors in [3].

Inspired by this examples, an interesting question arises: what is the classification of rotationally symmetric gradient Riemann solitons? In this paper, we consider spaces that are conformal to the Euclidean space and, under certain assumptions, give a complete description of rotational gradient Riemann solitons.

In the next result, we show that $\left(\mathbb{R}^n \setminus \{0\}, g = \delta/\varphi^2\right)$, $n \geq 4$, where $\varphi(r) = \sqrt{r}$, is isometric to the standard product $\mathbb{R} \times \mathbb{S}^{n-1}$.

Theorem 1.4. *Let g be a geodesically complete, rotationally invariant metric on $M = \mathbb{R}^n \setminus \{0\}$, $n \geq 4$, where $g_{ij} = \delta_{ij}/\varphi^2$. Then (M, g) supports a gradient Riemann soliton if and only if g is (the pullback via Ψ of) the natural product metric on $\mathbb{R} \times \mathbb{S}^{n-1}$, under the identificaction $\Psi : M = \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$ given by*

$$\Psi(x) = (\log\|x\|, x/\|x\|)\,. \tag{3}$$

Furthermore, this soliton is shirinking.

Let $(\mathbb{R}^n, g_0)$, where $n \geq 3$, be the standard Euclidean space with metric g_0 and coordinates $(x_1, ..., x_n)$, where $(g_0)_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$, and δ_{ij} is the Kronecker delta. Let $r = \sum_{i=1}^{n} x_i^2$ be a basic invariant for an $(n-1)$−dimensional orthogonal group. Initially, we find a system of differential equations to be satisfied by the functions f and φ, such that the metric $g = g_0/\varphi^2$ satisfies the soliton equation (2) (see Theorem 1.5). Note that if the solutions are invariant under the action of the orthogonal group, the system of partial differential equations given in Theorem 1.5 can be reduced to a system of ordinary differential equations (see Corollary 1.6). In Lemma 2.2, we provide necessary conditions for the functions to be solutions of the system of ODE in Corollary 1.6 . Finally, we obtain the necessary and sufficient conditions on $f(r)$ and $\varphi(r)$ for the existence of gradient Riemann solitons. In this case, all solutions are given explicitly.

In what follows, we will use the following convention for the derivatives of a function $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, in a coordinate system $(x_1, ..., x_n)$

$$u_{,x_i} = \frac{\partial u}{\partial x_i} \quad \text{and} \quad u_{,x_i x_j} = \frac{\partial u}{\partial x_i \partial x_j}.$$

Theorem 1.5. *Let $(\mathbb{R}^n, g_0)$, $n \geq 4$, be a Euclidean space with coordinates $x = (x_1, ..., x_n)$ and metric components $(g_0)_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$. Let $\Omega \subseteq \mathbb{R}^n$ be an open subset and consider a smooth function $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. Then, there exists a metric $g = \frac{1}{\varphi^2} g_0$ for a smooth now here vanishing function $\varphi : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ such that (Ω, g) is a gradient Riemann soliton with f as a potential function if and only if the functions φ and f satisfy*

$$\varphi_{,x_i x_j} + \varphi f_{,x_i x_j} + \varphi_{,x_i} f_{,x_j} + \varphi_{,x_j} f_{,x_i} = 0, \quad i \neq j, \tag{4}$$

and for each i

$$\varphi \left[\varphi_{,x_i x_i} + \varphi f_{,x_i x_i} + 2\varphi_{,x_i} f_{,x_i} \right] - \sum_{k=1}^{n} \left[\frac{1}{2} \left(\varphi_{,x_k} \right)^2 + \varphi \varphi_{,x_k} f_{,x_k} \right] = \frac{\overline{\lambda}}{n-2}, \tag{5}$$

where $\overline{\lambda} = \frac{(n-2)}{2} \lambda$.

Assuming that the Riemann soliton allows rotation invariant solutions, the following result reduces the system of partial differential equations (4) and (5) to a system of ordinary differential equations. Given that, occasionally, the system solutions are undefined only for $r = 0$, we focus on the case of positive r. Thus, we seek for smooth functions $h, \phi : (0, +\infty) \longrightarrow \mathbb{R}$, such that $f = h \circ r$, $\varphi = \phi \circ r : \Omega = r^{-1}(0, +\infty) \subseteq \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, where $r : \Omega \subseteq \mathbb{R}^n \setminus \{0\} \longrightarrow (0, +\infty)$ is given by $r(x_1, ..., x_n) = \sum_{i=1}^{n} x_i^2$.

Corollary 1.6. *Let $(\mathbb{R}^n, g_0)$, $n \geq 4$, be a Euclidean space with coordinates $x = (x_1, ..., x_n)$ and metric components $(g_0)_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$. Consider smooth functions $\varphi(r)$ and $f(r)$ with $r = \sum_{k=1}^{n} x_k^2$. Then there exists a metric $g = \frac{1}{\varphi^2} g_0$ for a smooth now here vanishing function φ, with $\varphi, f : \Omega \subseteq \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, such that (Ω, g) is a gradient Riemann soliton with f as a potential function if and only if the functions φ and f satisfy*

$$\varphi'' + \varphi f'' + 2\varphi' f' = 0, \tag{6}$$

and

$$\varphi \left(\varphi' + \varphi f' \right) - r \left((\varphi')^2 + 2\varphi \varphi' f' \right) = \frac{\overline{\lambda}}{2(n-2)}. \tag{7}$$

Next, we found all conformal metrics, invariant under the action of the orthogonal group, that are gradient Riemann solitons.

Theorem 1.7. *Let $(\mathbb{R}^n, g_0)$, $n \geq 4$, be a Euclidean space with coordinates $x = (x_1, ..., x_n)$ and metric components $(g_0)_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$. Consider smooth functions $\varphi(r)$ and $f(r)$ such that $r = \sum_{k=1}^{n} x_k^2$. Then there exists a metric $g = \frac{1}{\varphi^2} g_0$ for a smooth now here vanishing function φ, with $\varphi, f : \Omega \subseteq \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, such that (Ω, g) is a gradient Riemannian soliton with f as a potential function if and only if the functions φ and f are given by*

$$\begin{cases} \varphi(r) = kr^s, \\ f(r) = -\frac{(1-s)s}{(1-2s)}\ln r + \frac{\overline{\lambda} r^{1-2s}}{2(n-2)k^2(1-2s)^2} + k_1, \quad for \quad s \neq \frac{1}{2}, \\ f(r) = k\ln r + \frac{r^2}{4} + k_1, \quad for \quad s = \frac{1}{2}, \end{cases}$$

where $s, k \in \mathbb{R}_+^*$ *and* $k_1 \in \mathbb{R}$.

In the case $n = 3$, the equation (2) describes a Ricci almost solitons structure. In this case, not all Riemann solitons are locally conformally flat. Explicit examples of pseudo-Riemannian steady gradient Ricci solitons that are not locally conformally flat were constructed by Souza and Pina in [8]. Fernández-López and García-Río [4], using the local decomposition of a Ricci soliton metric into a warped product metric, proved that a locally conformally flat gradient Ricci soliton is rotationally symmetric. Therefore, when the three–dimensional Riemann soliton manifold is locally conformally flat, all of the previous results follow.

2. Proof of the main results

Let us begin this section with an observation that will be fundamental in the proof of the main result (Theorem 1.4). From point 2) of the Remark below and Theorem 1.7, we can see the metric (unique).

Remark 2.1. *Here we will provide the formula for the scalar curvature for a conformal metric. Consider the metric* $g_{ij} = \frac{(g_0)_{ij}}{\varphi^2}$ *in* $\Omega \subseteq \mathbb{R}^n$ *an open subset, where* φ *is a smooth now here vanishing function. We can write* $g^{ij} = \varphi^2 (g_0)_{ij}$ *for the inverse metric* g_{ij}*. In these conditions, we have:*

$$R_g = (n-1)\left(2\varphi\sum_{i=1}^{n}\varphi_{,x_ix_i} - n\sum_{i=1}^{n}(\varphi_{,x_i})^2\right). \tag{8}$$

Let $r = \sum_{i=1}^{n} x_i^2$*, consider* $\varphi(r)$ *a function of* r*. Since*

$$\varphi_{,x_i} = 2x_i\varphi', \qquad \varphi_{,x_ix_i} = 4x_i^2\varphi'' + 2\varphi', \qquad \varphi_{,x_ix_j} = 4x_ix_j\varphi''.$$

Substituting these expressions into (8)*, we have:*

$$R_g = -4(n-1)(n-2)k^2 s(s-1)r^{2s-1}.$$

Corresponding to special values of the parameter s*, we have the following classification:*

1) *If* $s = 1$*, then the gradient Riemann soliton* $(\Omega,\ g,\ \varphi)$ *is flat.*
2) *If* $s = \frac{1}{2}$*, then the gradient Riemann soliton* $(\Omega,\ g,\ \varphi)$ *is complete (see [11], for instance).*

Proof. [Proof of Theorem 1.5] Let us remember the Ricci and scalar curvature for a conformal metric of the form $g = \frac{g_0}{\varphi^2}$ (cf. [1], [7] or [9]):

$$Ric_g = \frac{1}{\varphi^2}\{(n-2)\varphi Hess_{g_0}(\varphi) + [\varphi\Delta_{g_0}\varphi - (n-1)|\nabla_{g_0}\varphi|^2]g_0\}, \tag{9}$$

and

$$R_g = (n-1)\left(2\varphi\Delta_{g_0}\varphi - n|\nabla_{g_0}\varphi|^2\right). \tag{10}$$

Taking the trace in (2), we obtain

$$Ric_g + (n-2)Hess_g f = \left[\frac{(n-2)}{2}\lambda + \frac{\bar{R}}{2(n-1)}\right]g. \tag{11}$$

Writing $\overline{\lambda} = \frac{(n-2)}{2}\lambda$ and substituting (9) and (10) into (11), we have

$$\begin{aligned} &\frac{1}{\varphi^2}\{(n-2)\varphi Hess_{g_0}(\varphi)_{ij} + [\varphi\Delta_{g_0}\varphi - (n-1)|\nabla_{g_0}\varphi|^2](g_0)_{ij}\} + (n-2)Hess_g(f)_{ij} \\ &= \left[\frac{(n-1)}{2(n-1)}(2\varphi\Delta_{g_0}\varphi - n|\nabla_{g_0}\varphi|^2) + \overline{\lambda}\right]\frac{1}{\varphi^2}(g_0)_{ij}. \end{aligned} \tag{12}$$

Now, recall that:

$$Hess_g(f)_{ij} = f_{,x_ix_j} - \sum_{k=1}^{n}\bar{\Gamma}_{ij}^k f_{,x_k},$$

where $\bar{\Gamma}_{ij}^k$ are the Christoffel symbols of the metric g. For a distinct i, j, k, we have

$$\bar{\Gamma}_{ij}^k = 0, \qquad \bar{\Gamma}_{ij}^i = -\frac{\varphi_{,x_j}}{\varphi}, \qquad \bar{\Gamma}_{ii}^k = \frac{\varphi_{,x_k}}{\varphi}, \qquad \bar{\Gamma}_{ii}^i = -\frac{\varphi_{,x_i}}{\varphi},$$

therefore,

$$Hess_g(f)_{ij} = f_{,x_ix_j} + \frac{\varphi_{,x_j}f_{,x_i}}{\varphi} + \frac{\varphi_{,x_i}f_{,x_j}}{\varphi}, \quad i \neq j. \tag{13}$$

Similarly, by considering $i = j$, we have

$$Hess_g(f)_{ii} = f_{,x_ix_i} + \frac{2\varphi_{,x_i}f_{,x_i}}{\varphi} - \sum_{k=1}^{n}\frac{\varphi_{,x_k}f_{,x_k}}{\varphi}. \tag{14}$$

However, we note that

$$|\nabla_{g_0}\varphi|^2 = \sum_{k=1}^{n}\left(\frac{\partial\varphi}{\partial x_k}\right)^2, \qquad \Delta_{g_0}\varphi = \sum_{k=1}^{n}\varphi_{,x_kx_k}, \qquad Hess_{g_0}(\varphi)_{ij} = \varphi_{,x_ix_j}. \tag{15}$$

If $i \neq j$ in (12), we obtain

$$\frac{Hess_{g_0}(\varphi)_{ij}}{\varphi} + Hess_g(f)_{ij} = 0. \tag{16}$$

Substituting the expressions found in (13), and (15) into (16), we obtain

$$\varphi_{,x_ix_j} + \varphi f_{,x_ix_j} + \varphi_{,x_i}f_{,x_j} + \varphi_{,x_j}f_{,x_i} = 0, \quad i \neq j.$$

Similarly, if $i = j$ in (12), we have

$$\begin{aligned} &(n-2)\varphi Hess_{g_0}(\varphi)_{ii} + \varphi\Delta_{g_0}\varphi - (n-1)|\nabla_{g_0}\varphi|^2 + (n-2)\varphi^2 Hess_g(f)_{ii} \\ &= \Delta_{g_0}\varphi - \frac{n}{2}|\nabla_{g_0}\varphi|^2 + \overline{\lambda}. \end{aligned} \tag{17}$$

Inserting the expressions found in (14), and (15) into (17), we obtain

$$\varphi\left(\varphi_{,x_ix_i} + \varphi f_{,x_ix_i} + 2\varphi_{,x_i}f_{,x_i}\right) - \sum_{k=1}^{n}\left(\frac{1}{2}\left(\varphi_{,x_k}\right)^2 + \varphi\varphi_{,x_k}f_{,x_k}\right) = \frac{\overline{\lambda}}{(n-2)}.$$

This concludes the proof of Theorem 1.5. □

Proof. [Proof of corollary 1.6] Let $g = \varphi^{-2}g_0$ be a conformal metric of g_0. We are assuming that $\varphi(r)$ and $f(r)$ are functions of r, where $r = \sum_{k=1}^{n} x_k^2$. Hence, we have

$$\varphi_{,x_i} = 2x_i\varphi', \quad \varphi_{,x_ix_i} = 4x_i^2\varphi'' + 2\varphi', \quad \varphi_{,x_ix_j} = 4x_ix_j\varphi'',$$

and

$$f_{,x_i} = 2x_if', \quad f_{,x_ix_i} = 4x_i^2f'' + 2f', \quad f_{,x_ix_j} = 4x_ix_jf''.$$

Substituting these expressions into (4), we obtain

$$4x_ix_j\varphi'' + 4x_ix_j\varphi f'' + (2x_i\varphi').(2x_jf') + (2x_j\varphi').(2x_if') = 0,$$

which is equivalent to

$$4\left(\varphi'' + \varphi f'' + 2\varphi' f'\right)x_ix_j = 0.$$

Since $x_ix_j \neq 0$ in an open subset $\Omega \subseteq \mathbb{R}^n \setminus \{0\}$, whenever $i \neq j$, we have

$$\varphi'' + f''\varphi + 2\varphi' f' = 0.$$

Similarly, considering the equation (5), we obtain

$$4\varphi\left(\varphi'' + \varphi f'' + 2\varphi' f'\right)x_i^2 + 2\varphi\left(\varphi' + \varphi f'\right) + 2n(1 - 2(n-1)\rho)\varphi\varphi'$$
$$-\sum_{k=1}^{n}\left(2x_k^2\left(\varphi'\right)^2 + 4x_k^2\varphi\varphi' f'\right) = \frac{\overline{\lambda}}{n-2}.$$

Note that $\varphi'' + \varphi f'' + 2\varphi' f' = 0$ and $r = \sum_{k=1}^{n} x_k^2$. Therefore, we obtain

$$\varphi\left(\varphi' + \varphi f'\right) - r\left(\left(\varphi'\right)^2 + 2\varphi\varphi' f'\right) = \frac{\overline{\lambda}}{2(n-2)}.$$

This concludes the proof of Corollary 1.6. □

The following lemma is the key to proving of Theorem 1.7.

Lemma 2.2. *Consider smooth functions $\varphi(r)$ and $f(r)$ such that $r = \sum_{i=1}^{n} x_i^2$. If $\varphi(r)$ and $f(r)$ satisfy* (6) *and* (7)*, then $\varphi(r)$ satisfies the following ordinary differential equation:*

$$r\varphi\varphi'' = r\left(\varphi'\right)^2 - \varphi\varphi'.$$

Proof. Differentiating both sides of the equation (7), we obtain

$$\varphi'\left(\varphi' + \varphi f'\right) + \varphi\left(\varphi'' + \varphi' f' + \varphi f''\right) - \left(\left(\varphi'\right)^2 + 2\varphi\varphi' f'\right)$$
$$-r\left(2\varphi'\varphi'' + 2\left(\varphi'\right)^2 f' + 2\varphi\varphi'' f' + 2\varphi\varphi' f''\right) = 0.$$

This is equivalent to

$$\varphi\varphi'' + \varphi^2 f'' - 2r\varphi'\varphi'' - 2r\left(\varphi'\right)^2 f' - 2r\varphi\varphi'' f' - 2r\varphi\varphi' f'' = 0. \tag{18}$$

By the first equation in Corollary 1.6, we obtain

$$\varphi f'' = -\left(\varphi'' + 2\varphi' f'\right). \tag{19}$$

Substituting (19) into (18), we obtain

$$2\left(\varphi\varphi' + r\left(\varphi'\right)^2 + r\varphi\varphi'' - 2r\left(\varphi'\right)^2\right)f' = 0.$$

Since $f' \neq 0$, we have that

$$r\varphi\varphi'' = r(\varphi')^2 - \varphi\varphi'.$$

This concludes the proof of Lemma 2.2.

□

This result deals with a necessary condition satisfied by the conformal factor φ in order that (Ω, g, φ), $\Omega \subseteq \mathbb{R}^n$, be a solutions for the system (6) and (7) with $g = g_0/\varphi^2$.

Proof. [Proof of Theorem 1.7] We know by Lemma 2.2, that the conformal factor necessarily satisfies the equation

$$r\varphi\varphi'' = r(\varphi')^2 - \varphi\varphi'.$$

Dividing the above equation by $r\varphi^2$, we can verify that

$$\left(\frac{\varphi'}{\varphi}\right)' = -\frac{1}{r}\frac{\varphi'}{\varphi}.$$

Writing $y = \frac{\varphi'}{\varphi}$, we obtain

$$y = sr^{-1}.$$

Therefore

$$\varphi(r) = kr^s, \tag{20}$$

where s and $k \in \mathbb{R} \setminus \{0\}$.

Since φ and f satisfy equation (7), by substituting (20) into (7), we obtain

$$kr^s ksr^{s-1} + k^2r^{2s}f' - rk^2s^2r^{2s-2} - 2rkr^s ksr^{s-1}f' = \frac{\overline{\lambda}}{2(n-2)},$$

which is equivalent to,

$$k^2\,(1-2s)\,r^{2s}f' + k^2s\,(1-s)\,r^{2s-1} = \frac{\overline{\lambda}}{2(n-2)}. \tag{21}$$

If $s \neq \frac{1}{2}$, then

$$f'(r) = -\frac{(1-s)s}{(1-2s)r} + \frac{\overline{\lambda}}{2(n-2)k^2(1-2s)r^{2s}}.$$

Consequently,

$$f(r) = -\frac{(1-s)s}{(1-2s)}\ln r + \frac{\overline{\lambda}r^{1-2s}}{2(n-2)k^2(1-2s)^2} + k_1. \tag{22}$$

Reciprocally, the function $\varphi(r)$ and $f(r)$ given in (20) and (22) satisfy (6) and (7). We notice that

$$\varphi(r) = kr^s, \quad \varphi'(r) = ksr^{s-1}, \quad \varphi''(r) = ks(s-1)r^{s-2}$$

and

$$f'(r) = -\frac{(1-s)s}{(1-2s)r} + \frac{\overline{\lambda}}{2(n-2)k^2(1-2s)r^{2s}}, \quad f''(r) = \frac{(1-s)s}{(1-2s)r^2} - \frac{s\overline{\lambda}}{(n-2)k^2(1-2s)r^{2s+1}}.$$

In this case

$$\varphi'' + \varphi f'' + 2\varphi' f' = ks(s-1)r^{s-2} + kr^s\left(\frac{(1-s)s}{(1-2s)r^2} - \frac{s\overline{\lambda}}{(n-2)k^2(1-2s)r^{2s+1}}\right) + 2ksr^{s-1}\left(-\frac{(1-s)s}{(1-2s)r} + \frac{\overline{\lambda}}{2(n-2)k^2(1-2s)r^{2s}}\right)$$

equivalently,

$$\varphi'' + \varphi f'' + 2\varphi' f' = ks(s-1)r^{s-2} + \frac{(1-s)sk}{(1-2s)r^{2-s}} - \frac{sk\overline{\lambda}}{(n-2)k^2(1-2s)r^{s+1}} - \frac{2(1-s)s^2k}{(1-2s)r^{2-s}} + \frac{sk\overline{\lambda}}{(n-2)k^2(1-2s)r^{s+1}}.$$

Therefore,

$$\varphi'' + \varphi f'' + 2\varphi' f' = 0$$

By other hand, the equation (7) is trivially satisfied.

If $s = \frac{1}{2}$, we have $\overline{\lambda} = \frac{(n-2)k^2}{2}$, and (7) is satisfied for all f. Substituting (20) in (6), we obtain $f(r) = k\ln r + \frac{r^2}{4} + k_1$, where k and $k_1 \in \mathbb{R}$.
This concludes the proof of Theorem 1.7.
□

Proof. [Proof of Theorem 1.4]
We split the proof into three cases $s > \frac{1}{2}$, $s = \frac{1}{2}$ and $s < \frac{1}{2}$.

Case 1: If $s > \frac{1}{2}$, consider a divergent curve $\alpha(t) = (t, 0, ..., 0)$. Note that $\alpha'(t) = (1, 0, ..., 0)$, $\varphi(\alpha(t)) = t^{2s}$ and $g(\alpha'(t), \alpha'(t)) = \frac{1}{t^{4s}}$. Consequently,

$$l(\alpha) = \lim_{t\longrightarrow\infty} \int_2^t |\alpha'(t)|dt = \lim_{t\longrightarrow\infty} \int_2^t \frac{1}{t^{2s}}dt.$$

Therefore,

$$l(\alpha) = \lim_{t\longrightarrow\infty} \left(\frac{t^{1-2s}}{1-2s}\right) + k = k,$$

where k is a constant (finite). Therefore $(\mathbb{R}^n \setminus \{0\}, g)$ is not complete.

Case 2: If $s < \frac{1}{2}$, we obtain

$$l(\alpha) = \lim_{t\longrightarrow 0} \int_t^2 |\alpha'(t)|dt = \lim_{t\longrightarrow 0} \int_2^t \frac{1}{t^{2s}}dt.$$

Therefore,

$$l(\alpha) = \lim_{t\longrightarrow 0} \left(\frac{t^{1-2s}}{1-2s}\right) + k = k,$$

where k is a finite constant. Therefore $(\mathbb{R}^n \setminus \{0\}, g)$ is not complete.

Case 3: If $s = \frac{1}{2}$ the authors in [2] showed that $\left(\mathbb{R}^3 \setminus \{0\}, g = \frac{1}{r} g_0\right)$ is isometric to $\mathbb{R} \times \mathbb{S}^2$. Similarly, the diffeomorphism

$$\Psi : M = \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \times \mathbb{S}^{n-1}, \qquad \Psi(x) = \left(\log \|x\|, x/\|x\|\right),$$

transforms M into to an open subset of $\mathbb{R} \times \mathbb{S}^{n-1}$, which is geodesically complete. Since (M, g) is geodesically complete, M must be connected. Hence $\Psi(M) = I \times S^{n-1}$ for some open interval I. Up to a (rotationally symmetric) reparametrization, we can assume that $\Psi(M) = \mathbb{R} \times S^{n-1}$ and then $M = \mathbb{R}^n \setminus \{0\}$. The (unique) metric can be read off from Theorem 1.7 and remark above point 2). We see that the metric on $\mathbb{R}^n \setminus \{0\}$ is the one with conformal factor $1/r$, up to a constant. In spherical coordinates, this gives

$$g = (d \log r)^2 + g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the round metric on the unit sphere. With reparametrization $t = \log r$, the metric on $\mathbb{R} \times S^{n-1}$ becomes the natural product metric. Furthermore, by expression (7), we obtain

$$\varphi \left(\varphi' + \varphi f'\right) - r\left((\varphi')^2 + 2\varphi\varphi' f'\right) = \frac{\overline{\lambda}}{2(n-2)}.$$

Note that $\varphi(r) = kr^{\frac{1}{2}}$, therefore

$$\overline{\lambda} = \frac{(n-2)\,k^2}{2} > 0.$$

Since $n \geq 4$, the proof is complete.

□

Conflict of interest

The authors declare that there is no conflict of interest.

Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the currentstudy.

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