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Existence of solutions for nonlinear degenerate elliptic equations with L^m-data and Neumann boundary condition

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Abstract. In this paper, we consider the following homogenous Neumann elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u,\nabla u)) + \sum_{i=1}^{N} d(|u|)|D^i u|^{p_i} + |u|^{s_0-2}u = f & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in R^N $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, we prove the existence of solutions in the sense of distributions for our elliptic problem in the anisotropic Sobolev spaces.

1. Introduction

Let Ω be a bounded open subset in R^N $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$. In [11], Boccardo et *al.* have studied the degenerate elliptic equation

$$\begin{cases} -\text{div}\,(A(x,u)\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where the data f is assumed to be in $L^m(\Omega)$ for $m \geq 1$. They have proved the existence of solutions and some regularity results, for more details we refer the reader to [3, 12, 19]. Alvino et *al.* have studied in [2] the following degenerated elliptic equation

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

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they have demonstrated the existence of solutions and some regularity results in the case $f \in L^m(\Omega)$ with $m \geq 1$. In [20], Li has considered the following quasilinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u,D^i u)) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where $f \in L^m(\Omega)$ for $1 < m < \overline{m} = \dfrac{N\overline{p}}{N\overline{p} - N + \overline{p}}$. He has proved the existence of solutions, also some regularity results were concluded. Gao et *al.* have studied in [17] the following degenerate anisotropic elliptic equation :

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u,D^i u(x))) = f(x) & \text{in} \quad \Omega, \\ u(x) = 0 & \text{on} \quad \Omega, \end{cases} \tag{4}$$

They have proved the regularity of global solutions with $f \in L^m(\Omega)$.
In this paper, inspired by [8, 13], we study the existence of solutions in the sense of distributions for the following strongly nonlinear and non-coercive anisotropic Neumann problem with degenerate coercivity

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u,\nabla u)) + H(x,u,\nabla u) + |u|^{s_0-2}u = f & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where $\vec{n} = (n_1, \ldots, n_N)$ is the outward unit normal vector on the boundary $\partial\Omega$, and $H(x,s,\xi)$ verifies only some growth condition, with the data f is assumed to be in $L^\infty(\Omega)$ then in $L^m(\Omega)$.
This paper is structured as follows: In the Section 2, we present some definitions and properties concerning the anisotropic Sobolev spaces. The Section 3 is divided into two essential parts: In the first part, we present the assumptions and we show that our problem has at least one solution in the sense of distributions when $f \in L^\infty(\Omega)$. In the second part, we prove the existence of solutions in the sense of distributions for our nonlinear and non-coercive problem in the case where $f \in L^m(\Omega)$, and we conclude some regularity results.

2. Preliminaries

Let Ω be an open bounded domain in R^N $(N \geq 2)$, with smooth boundary $\partial\Omega$. We set $p_1, \ldots, p_N$ be N real constants numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$ and we denote

$$\vec{p} = (1, p_1, \ldots, p_N) \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \Big\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \Big\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{6}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf [22]).

Proposition 2.1. *(cf. [20, 23]) Let $u \in W^{1,\vec{p}}(\Omega)$, we have*
(i) *Poincaré Wirtinger inequality: there exists a constant $C_p > 0$, such that*

$$\|u - med(u)\|_{L^{p_i}(\Omega)} \leq C_p \, \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \dots, N,$$

with

$$med(u) = \frac{1}{|\Omega|}\int_{\Omega} |u| \, dx.$$

(ii) *Sobolev inequality : there exists an other constant $C_s > 0$, such that*

$$\|u - med(u)\|_q \leq \frac{C_s}{N}\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N-\overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty) & \text{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 2.2. *[18] Let Ω be a bounded open set in R^N $(N \geq 2)$, we set*

$$s = \max(q, \, \underline{p}^+),$$

then, we have the following embeddings :

- *if $\underline{p} < N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s)$,*
- *if $\underline{p} = N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty)$,*
- *if $\underline{p} > N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of the Lemma 2.2 follows from the Proposition 2.1 and the fact that the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and from the compact embedding theorem in classical Sobolev spaces.

Definition 2.3. *Let $k > 0$, we consider the truncation function $T_k(\cdot) : R \longmapsto R$, given by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.4. *Let $u \in \mathcal{T}^{1,\vec{p}}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto R$ such that*

$$\forall k > 0, \qquad D^i T_k(u) = v_i \chi_{\{|u|<k\}} \quad \text{a.e.} \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 2.4 follows the usual techniques developed in [10] for the case of classical Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [5, 9, 14, 15].

We introduce the set $\mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$ as a subset of $\mathcal{T}^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace maybe defined (see also [4] for the case of constant exponent). For more properties about the set $\mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$, we refer the reader to [6].

Lemma 2.5. *(see [1]) Let $g \in L^p(\Omega)$ and let $(g_n)_n$be a sequence uniformly bounded in $L^p(\Omega)$. If $g_n \to g$ almost everywhere in Ω, then $g_n \rightharpoonup g$ weakly in $L^p(\Omega)$.*

3. Main result

We consider the following strongly nonlinear and non-coercive anisotropic elliptic problem :

$$\begin{cases} Au + H(x,u,\nabla u) + |u|^{s_0-2}u = f & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{7}$$

where $A = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u)$ is a Leray-Lions operator acting from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, where $a_i : \Omega \times R \times R^N \longmapsto R$ are Carathéodory functions, for $i = 1,\dots,N$, (measurable with respect to x in Ω for every (s,ξ) in $R \times R^N$, and continuous with respect to (s,ξ) in $R \times R^N$ for almost every x in Ω) which satisfy the following conditions.

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \quad \text{for} \quad \xi_i \neq \xi_i', \tag{8}$$

$$|a_i(x,s,\xi)| \le \beta(K_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{9}$$

with the nonnegative function $K_i(\cdot)$ assumed to be in $L^{p_i'}(\Omega)$ for $i = 1,\dots,N$, where $\beta > 0$.

$$a_i(x,s,\xi)\xi_i \ \ge \ b(|s|)|\xi_i|^{p_i} \qquad \text{with} \qquad \frac{b_0}{(1+|s|)^{\lambda}} \le b(|s|) \qquad \text{for any} \quad s \in R, \tag{10}$$

with $b_0 > 0$ and $0 \le \lambda < min(1, p_i - 1, \frac{1}{p_i - 1})$.

As a consequence of (10) and the continuity of the function $a_i(x,s,\cdot)$ with respect to ξ, one has

$$a_i(x,s,0) = 0.$$

The lower order term $H(x,s,\xi)$ is a Carathéodory function which verifies the following growth condition:

$$|H(x,s,\xi)| \le \sum_{i=1}^{N} d(|s|)|\xi_i|^{p_i}, \tag{11}$$

where $d(|\cdot|) : R^+ \mapsto R^+$ is a decreasing function that satisfying $\frac{d(|\cdot|)}{b(|\cdot|)} \in L^1(R) \cap L^\infty(R)$.

We are going now to recall the following technical lemma, useful to prove our main results.

Lemma 3.1. *(see [10]) Let $h > 0$, assume that* (8) − (10) *hold true, and let $(u_n)_{n\in N}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$ and*

$$\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\,dx + \sum_{i=1}^{N}\int_\Omega (a_i(x, T_h(u_n), \nabla T_h(u_n)) - a_i(x, T_h(u_n), \nabla u))(D^i u_n - D^i u)\,dx \to 0 \quad as \quad n \to \infty, \tag{12}$$

then $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$ for a subsequence.

3.1. Existence of solutions in the sense of distributions for L^∞-data.

We consider the strongly nonlinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x, u, \nabla u)) + H(x, u, \nabla u) + |u|^{s_0-2}u = F & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x, u, \nabla u).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{13}$$

with $F \in L^\infty(\Omega)$.

Definition 3.2. *A measurable function u is a solution in sense of distribution for the problem* (13) *if $u \in W^{1,\vec{p}}(\Omega)$, with $|u|^{s_0-1} \in L^1(\Omega)$ and $H(x, u, \nabla u) \in L^1(\Omega)$, such that u verifies the following equality*

$$\sum_{i=1}^{N}\int_\Omega a_i(x, u, \nabla u)D^i v\,dx + \int_\Omega H(x, u, \nabla u)v\,dx + \int_\Omega |u|^{s_0-2}uv\,dx = \int_\Omega Fv\,dx, \tag{14}$$

for any $v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.

Theorem 3.3. *Assume that* (8) − (11) *hold true. Then, there exists at least one solution u in the sense of distributions for the strongly nonlinear elliptic problem* (13)*. Moreover, we have $u \in L^\infty(\Omega) \cap W^{1,\vec{p}}(\Omega)$.*

Proof of Theorem 3.3

The proof of the Theorem 3.3 will be divided into several steps.

Step 1: Approximate problem

For any $m \in N^*$, we consider the following approximate problem.

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x, T_m(u_m), \nabla u_m)) + H_m(x, u_m, \nabla u_m) + \frac{1}{m}|u_m|^{\underline{p}-2}u_m + |T_m(u_m)|^{s_0-2}T_m(u_m) = F & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x, T_m(u_m), \nabla u_m).n_i = 0, & \text{in} \quad \partial\Omega, \end{cases} \tag{15}$$

with $H_m(x, s, \xi) = T_m(H(x, s, \xi))$.
We define the operator B_m acting from $W^{1,\vec{p}}(\Omega)$ into $(W^{1,\vec{p}}(\Omega))'$ by

$$\begin{aligned} \langle B_m u, v\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x, T_m(u), \nabla u)D^i v\,dx + \int_\Omega H_m(x, u, \nabla u)v\,dx \\ &\quad + \int_\Omega |T_m(u)|^{s_0-2}T_m(u)v\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,dx. \end{aligned} \tag{16}$$

Lemma 3.4. *The operator B_m, acting from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, is bounded and pseudo-monotone. Moreover, the operator B_m is coercive in the following sense: there exists $v \in W^{1,\vec{p}}(\Omega)$ such that*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{W^{1,\vec{p}}(\Omega)}} \to \infty \quad as \quad \|v\|_{W^{1,\vec{p}}(\Omega)} \to \infty \quad for \quad v \in W^{1,\vec{p}}(\Omega). \tag{17}$$

For the proof of Lemma 3.4, see Appendix.
In view of Lemma 3.4 (see [21], Theorem 8.2), there exists at least one weak solution $u_m \in W^{1,\vec{p}}(\Omega)$ for the approximate problem (15) as follows

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_m),\nabla u_m)D^i v\,dx \;+\; \int_\Omega H_m(x,u_m,\nabla u_m)v\;dx \;+\; \int_\Omega |T_m(u_m)|^{s_0-2}T_m(u_m)v\,dx\\ +\frac{1}{m}\int_\Omega |u_m|_-^{p-2}u_m v\,dx \;=\; \int_\Omega F(x)v\,dx\\ for \quad any \quad v\in W^{1,\vec{p}}(\Omega).\end{aligned} \tag{18}$$

Step 2 : Some regularity results

We set $B(|s|) = \displaystyle\int_0^s \frac{d(|\tau|)}{b(|\tau|)}d\tau$ and since $\dfrac{d(|\cdot|)}{b(|\cdot|)} \in L^\infty(\Omega)\cap L^1(\Omega)$ then $1 \le e^{B(|s|)} \le e^{B(\infty)} < \infty$.
Let $\delta \ge 1$, by taking $v_m = (T_{k+\delta}(u_m) - T_k(u_m))e^{B(|u_m|)} \in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ as a test function for the approximate problem (15), it follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{k<|u_m|\le k+\delta\}} a_i(x,T_m(u_m),\nabla u_m)D^i u_m\,dx\\ &+2\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_m),\nabla u_m)D^i u_m\frac{d(|u_m|)}{b(|u_m|)}|T_{k+\delta}(u_m)-T_k(u_m)|\,dx\\ &+\int_\Omega H_m(x,u_m,\nabla u_m)(T_{k+\delta}(u_m)-T_k(u_m))e^{B(|u_m|)}\,dx + \int_\Omega |T_m(u_m)|^{s_0-1}|T_{k+\delta}(u_m)-T_k(u_m)|e^{B(|u_m|)}\,dx\\ &+\frac{1}{m}\int_\Omega |u_m|_-^{p-1}|T_{k+\delta}(u_m)-T_k(u_m)|e^{B(|u_m|)}\,dx = \int_\Omega F(x)(T_{k+\delta}(u_m)-T_k(u_m)e^{B(|u_m|)}\,dx.\end{aligned} \tag{19}$$

In view of (10) and (11), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{k<|u_m|\le k+\delta\}} b(|u_m|)|D^i u_m|^{p_i}\,dx + \sum_{i=1}^{N}\int_\Omega |D^i u_m|^{p_i} d(|u_m|)|T_{k+\delta}(u_m)-T_k(u_m)|\,dx\\ &+\int_\Omega |T_m(u_m)|^{s_0-1}|T_{k+\delta}(u_m)-T_k(u_m)|\,dx + \frac{1}{m}\int_\Omega |u_m|_-^{p-1}|T_{k+\delta}(u_m)-T_k(u_m)|\,dx\\ &\le \int_\Omega |F(x)||T_{k+\delta}(u_m)-T_k(u_m)|e^{B(|u_m|)}\,dx.\end{aligned} \tag{20}$$

Let m be large enough, by choosing $k \le m$ we have

$$\begin{aligned}\delta k^{s_0-1}\text{meas}(\{|u_m| > k+\delta\}) &\le \delta\int_{\{|u_m|>k+\delta\}} |T_m(u_m)|^{s_0-1}\,dx\\ &\le \delta e^{B(\infty)}\int_{\{|u_m|>k\}} |F(x)|\,dx\\ &\le \delta e^{B(\infty)}\|F\|_{L^\infty(\Omega)}\text{meas}(\{|u_m| > k\}).\end{aligned} \tag{21}$$

By letting δ tends to 0, we conclude that

$$k^{s_0-1}\text{meas}(\{|u_m| > k\}) \le e^{B(\infty)}\|F\|_{L^\infty(\Omega)}\text{meas}(\{|u_m| > k\}). \tag{22}$$

By choosing k large enough, for example $e^{B(\infty)}\|F\|_{L^\infty(\Omega)} < k^{s_0-1}$, we deduce necessarily that meas$(\{|u_m| > k\}) = 0$. Consequently

$$\|u_m\|_{L^\infty(\Omega)} \leq k \leq (e^{B(\infty)}\|F\|_{L^\infty(\Omega)})^{\frac{1}{s_0-1}}. \tag{23}$$

Step 3: Weak convergence of $(u_m)_m$

We denote by $C_1, C_2, C_3, \ldots$ some positive constants of real number that don't depend on m and n.
By taking $v_m = u_m e^{B(|u_m|)} \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in (15) with $B(|s|) = 2\int_0^s \frac{d(|\tau|)}{b(|\tau|)}d\tau$, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m) D^i u_m e^{B(|u_m|)}\, dx + 2\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m) D^i u_m |u_m| \frac{d(|u_m|)}{b(|u_m|)} e^{B(|u_m|)}\, dx \\
&+ \int_\Omega H_m(x, u_m, \nabla u_m) u_m e^{B(|u_m|)}\, dx + \int_\Omega |T_m(u_m)|^{s_0-1}|u_m| e^{B(|u_m|)}\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}} e^{B(|u_m|)}\, dx \\
&\leq \int_\Omega |F(x)||u_m| e^{B(|u_m|)}\, dx.
\end{aligned} \tag{24}$$

Since $T_m(u_m) = u_m$ for $m \geq \|u_m\|_{L^\infty(\Omega)}$, and in view of (10) and (11) we conclude that

$$\begin{aligned}
&b_0 \sum_{i=1}^N \int_\Omega \frac{|D^i u_m|^{p_i}}{(1+|u_m|)^\lambda}\, dx \; + \; \sum_{i=1}^N \int_\Omega d(|u_m|)|D^i u_m|^{p_i}|u_m|\, dx \\
&+ \int_\Omega |u_m|^{s_0}\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\, dx \; \leq \; e^{B(\infty)} \int_\Omega |F(x)||u_m|\, dx.
\end{aligned} \tag{25}$$

Thanks to (25) and (23), we get

$$\begin{aligned}
b_0 \sum_{i=1}^N \int_\Omega \frac{|D^i u_m|^{p_i}}{(1+|u_m|)^\lambda}\, dx + \int_\Omega |u_m|^{s_0}\, dx \quad &\leq e^{B(\infty)} \int_\Omega |F(x)||u_m|\, dx \\
&\leq \; e^{B(\infty)}\|u_m\|_{L^\infty(\Omega)}\|F\|_{L^\infty(\Omega)}\text{meas}(\Omega) \\
&\leq \; (e^{B(\infty)}\|F\|_{L^\infty(\Omega)})^{\frac{s_0}{s_0-1}}\text{meas}(\Omega).
\end{aligned} \tag{26}$$

Therefore, it follows that

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\, dx \quad &\leq C_1 \|F\|_{L^\infty(\Omega)}^{\frac{s_0}{s_0-1}} (1 + \|u_m\|_{L^\infty(\Omega)})^\lambda \\
&\leq C_2 \|F\|_{L^\infty(\Omega)}^{\frac{s_0+\lambda}{s_0-1}} \\
&\leq C_3.
\end{aligned} \tag{27}$$

Thus, we obtain

$$\begin{aligned}
\|u_m\|_{1,\vec{p}} \quad &= \|u_m\|_{1,1} + \sum_{i=1}^N \|D^i u_m\|_{L^{p_i}(\Omega)} \\
&\leq \|u_m\|_{L^1(\Omega)} + 2\sum_{i=1}^N \int_\Omega |D^i u_m|\, dx + \sum_{i=1}^N \Big(\int_\Omega |D^i u_m|^{p_i}\, dx\Big)^{\frac{1}{p_i}} \\
&\leq |\Omega|\|u_m\|_{L^\infty(\Omega)} + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\, dx + N(1 + |\Omega|) \\
&\leq C_4,
\end{aligned} \tag{28}$$

with C_4 is a positive constant that doesn't depend on m. Thus, the sequence $(u_m)_{m\in N}$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and we deduce that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ u_m \longrightarrow u & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega, \\ u_m \rightharpoonup u & \text{weakly in } L^1(\partial\Omega) \text{ and a.e. on } \partial\Omega, \\ u_m \rightharpoonup u & \text{weak} - * \ \ in \ \ L^\infty(\Omega). \end{cases} \tag{29}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^1(\Omega). \tag{30}$$

Furthermore, in view of (23), we have $|T_m(u_m)| \le (e^{B(\infty)}\|F\|_{L^\infty(\Omega)})^{\frac{1}{s_0-1}}$, and since $T_m(u_m) \longrightarrow u$ almost everywhere in Ω, then in view of Lebesgue dominated convergence theorem, we obtain

$$|T_m(u_m)|^{s_0-1} \longrightarrow |u|^{s_0-1} \qquad \text{strongly in} \quad L^1(\Omega). \tag{31}$$

Step 4: The convergence almost everywhere of the gradient

In this step, we will denote by $\varepsilon_i(m)$ for $i = 0, 1, \ldots$ some various functions of real numbers which converges to 0 as m tends to infinity.
We set $\varphi(s) = s.exp(\frac{\delta^2 s^2}{2})$ with $\delta = 3\left\|\frac{d(|.|)}{b(|.|)}\right\|_{L^\infty(\Omega)}$ then we have $\varphi'(s) - \delta\varphi(s) \ge \frac{1}{2}$ for any $s \in R$.
By taking $v_m = \varphi(u_m - u)e^{B(|u_m|)} \in W^{1,\vec{p}} \cap L^\infty(\Omega)$ as a test function for the approximate problem (15), we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m)(D^i u_m - D^i u)\varphi'(u_m - u)e^{B(|u_m|)}\, dx \\ &+2\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m)D^i u_m \frac{d(|u_m|)}{b(|u_m|)} sign(u_m)\varphi(u_m - u)e^{B(|u_m|)}\, dx \\ &+\int_\Omega H_m(x, u_m, \nabla u_m)\varphi(u_m - u)e^{B(|u_m|)}\, dx + \int_\Omega |T_m(u_m)|^{s_0-2}T_m(u_m)\varphi(u_m - u)e^{B(|u_m|)}\, dx \\ &+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m\varphi(u_m - u)e^{B(|u_m|)}\, dx \le \int_\Omega |F(x)||\varphi(u_m - u)|e^{B(|u_m|)}\, dx. \end{aligned} \tag{32}$$

In view of (10) and (11), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m)\varphi'(u_m - u)D^i(u_m - u)e^{B(|u_m|)}\, dx \\ &-3\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m)D^i u_m \frac{d(|u_m|)}{b(|u_m|)}|\varphi(u_m - u)|e^{B(|u_m|)}\, dx \\ &\le e^{B(\infty)}\int_\Omega |F(x)||\varphi(u_m - u)|\, dx + e^{B(\infty)}\int_\Omega |T_m(u_m)|^{s_0-1}|\varphi(u_m - u)|\, dx \\ &+\frac{1}{m}e^{B(\infty)}\int_\Omega |u_m|^{\underline{p}-1}|\varphi(u_m - u)|\, dx. \end{aligned} \tag{33}$$

For the first term of the right-hand side of (33), we have Ω is an open bounded subset, then $F(x) \in L^\infty(\Omega) \subset L^1(\Omega)$ and since $\varphi(u_m - u) \rightharpoonup 0$ weak-$*$ in $L^\infty(\Omega)$, it follows that

$$\varepsilon_1(m) = e^{B(\infty)}\int_\Omega |F(x)|\, |\varphi(u_m - u)|\, dx \longrightarrow 0 \quad as \quad m \to \infty. \tag{34}$$

Similarly, in view of (30) and (31) we get

$$\varepsilon_2(m) = e^{B(\infty)} \int_\Omega |T_m(u_m)|^{s_0-1}|\varphi(u_m-u)|\,dx \longrightarrow 0 \quad as \quad m\to\infty \tag{35}$$

and

$$\varepsilon_3(m) = \frac{e^{B(\infty)}}{m} \int_\Omega |u_m|_-^{p-1}|\varphi(u_m-u)|\,dx \longrightarrow 0 \quad as \quad m\to\infty. \tag{36}$$

By combining (34) − (36), we conclude that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,T_m(u_m),\nabla u_m)\varphi'(u_m-u)D^i(u_m-u)e^{B(|u_m|)}\,dx\\ &-3\sum_{i=1}^N \int_\Omega a_i(x,T_m(u_m),\nabla u_m)\frac{d(|u_m|)}{b(|u_m|)}D^iu_m|\varphi(u_m-u)|e^{B(|u_m|)}\,dx\\ &\le \varepsilon_4(m).\end{aligned} \tag{37}$$

such that $\varepsilon_4(m) = \varepsilon_1(m)+\varepsilon_2(m)+\varepsilon_3(m) \longrightarrow 0$ as m goes to ∞. It follows that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega (a_i(x,T_m(u_m),\nabla u_m)-a_i(x,T_m(u_m),\nabla u))(D^iu_m-D^iu)\\ &\qquad\times\Big(\varphi'(u_m-u)-3\frac{d(|u_m|)}{b(|u_m|)}|\varphi(u_m-u)|\Big)e^{B(|u_m|)}\,dx\\ &\le 3\sum_{i=1}^N \int_\Omega a_i(x,T_m(u_m),\nabla u_m)D^iu\frac{d(|u_m|)}{b(|u_m|)}|\varphi(u_m-u)|e^{B(|u_m|)}\,dx\\ &\quad-\sum_{i=1}^N \int_\Omega a_i(x,T_m(u_m),\nabla u)(D^iu_m-D^iu)\varphi'(u_m-u)e^{B(|u_m|)}\,dx\\ &\quad+3\sum_{i=1}^N \int_\Omega a_i(x,T_m(u_m),\nabla u)(D^iu_m-D^iu)\frac{d(|u_m|)}{b(|u_m|)}|\varphi(u_m-u)|e^{B(|u_m|)}\,dx+\varepsilon_4(m)\\ &\le 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(R)}e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x,T_m(u_m),\nabla u_m)|\,|D^iu|\,|\varphi(u_m-u)|\,dx\\ &\quad+\Big(\varphi'(2k_0)+\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(R)}|\varphi(2k_0)|\Big)e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x,T_m(u_m),\nabla u)|\,|D^iu_m-D^iu|\,dx+\varepsilon_4(m),\end{aligned} \tag{38}$$

with $k_0 = (e^{B(\infty)}\|F\|_{L^\infty(\Omega)})^{\frac{1}{s_0-1}}$.

For the first term on the right-hand side of (38), thanks to (9) we have $(a_i(x,T_m(u_m),\nabla u_m))_{m\in N}$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that $a_i(x,T_m(u_m),\nabla u_m) \rightharpoonup \varphi_i$ weakly in $L^{p_i'}(\Omega)$, and since $|D^iu|\,|\varphi(u_m-u)| \to 0$ strongly in $L^{p_i}(\Omega)$, it follows that

$$\varepsilon_5(m) = \sum_{i=1}^N \int_\Omega |a_i(x,T_m(u_m),\nabla u_m)|\,|D^iu|\,|\varphi(u_m-u)|\,dx \longrightarrow 0 \quad \text{as} \quad m\to\infty. \tag{39}$$

Concerning the second term on the right-hand side of (38), using (29) we have $T_m(u_m) \to u$ strongly in $L^{p_i}(\Omega)$ then $a_i(x,T_m(u_m),\nabla u) \longrightarrow a_i(x,u,\nabla u)$ strongly in $L^{p_i'}(\Omega)$, and since $D^iu_m \rightharpoonup D^iu$ weakly in $L^{p_i}(\Omega)$, we conclude that

$$\varepsilon_6(m) = \sum_{i=1}^N \int_\Omega |a_i(x,T_m(u_m),\nabla u)|\,|D^iu_m-D^iu|\,dx \longrightarrow 0 \quad \text{as} \quad m\to\infty. \tag{40}$$

Having in mind that $\varphi'(s) - \delta\varphi(s) \geq \frac{1}{2}$ for any $s \in R$. Thus, by combining (38) and (39) − (40) we conclude that

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_m(u_m),\nabla u_m) - a_i(x,T_m(u_m),\nabla u))(D^iu_m - D^iu)\,dx \\ &\leq \sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_m(u_m),\nabla u_m) - a_i(x,T_m(u_m),\nabla u))(D^iu_m - D^iu) \\ &\qquad\qquad \times\Big(\varphi'(u_m-u) - 3\frac{d(|u_m|)}{b(|u_m|)}|\varphi(u_m-u)|\Big)e^{B(|u_m|)}\,dx \\ &\leq \varepsilon_7(m) \end{aligned} \tag{41}$$

where $\varepsilon_7(m) = \varepsilon_4(m) + \varepsilon_5(m) + \varepsilon_6(m) \longrightarrow 0$ as $m \longrightarrow \infty$..
By letting m goes to infinity in the previous inequality, and since $u_m \longrightarrow u$ strongly in $L^{\underline{p}}(\Omega)$, we conclude that

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_m(u_m),\nabla u_m) - a_i(x,T_m(u_m),\nabla u))(D^iu_m - D^iu)\,dx \\ &\quad + \int_{\Omega}(|u_m|^{\underline{p}-2}u_m - |u|^{\underline{p}-2}u)(u_m-u)\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \end{aligned} \tag{42}$$

In view of Lemma 3.1, we deduce that

$$\begin{cases} u_n \longrightarrow u & \text{strongly in} \quad W^{1,\vec{p}}(\Omega) \\ D^iu_n \longrightarrow D^iu & \text{a.e. in} \quad \Omega. \end{cases} \tag{43}$$

Step 5: The equi-integrability of the sequence $(H_m(x,u_m,\nabla u_m))_{m\in N}$

In view of (43) we have

$$a_i(x,T_m(u_m),\nabla u_m) \longrightarrow a_i(x,u,\nabla u) \qquad \text{a.e.} \quad \Omega, \tag{44}$$

and

$$H_m(x,u_m,\nabla u_m) \longrightarrow H(x,u,\nabla u) \qquad \text{a.e.} \quad \Omega. \tag{45}$$

On the one hand, thanks to (9), we have $(a_i(x,T_m(u_m),\nabla u_m))_{m\in N}$ is bounded in $L^{p_i'}(\Omega)$ and in view of (44) and Lemma 2.5, we conclude that

$$a_i(x,T_m(u_m),\nabla u_m) \rightharpoonup a_i(x,u,\nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega). \tag{46}$$

Now, we show that

$$H_m(x,u_m,\nabla u_m) \longrightarrow H(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega), \tag{47}$$

using Vitali's theorem, it's sufficient to prove that the sequence $(H_m(x,u_m,\nabla u_m))_m$ is uniformly equi-integrable.
We have $\|u_m\|_{L^\infty(\Omega)} \leq k_0 = (e^{B(\infty)}\|F\|_{L^\infty(\Omega)})^{\frac{1}{s_0-1}}$. Thus, for any measurable subset $E \subset \Omega$, it follows that

$$\begin{aligned} \int_E |H_m(x,u_m,\nabla u_m)|\,dx &= \int_E |H_m(x,T_{k_0}(u_m),\nabla T_{k_0}(u_m))|\,dx \\ &\leq \sum_{i=1}^{N}\int_E d(T_{k_0}(u_m))|D^iT_{k_0}(u_m)|^{p_i}\,dx \\ &\leq \|d(T_{k_0}(s))\|_{L^\infty(R)}\sum_{i=1}^{N}\int_E |D^iu_m|^{p_i}\,dx. \end{aligned} \tag{48}$$

In view of (43), we conclude that : for any $\eta > 0$, there exists $\beta(\eta) > 0$ such that

$$\int_E H_m(x, u_m, \nabla u_m)\, dx \le \eta \qquad \text{for any} \quad E \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \beta(\eta). \tag{49}$$

Thus, the sequence $(H_m(x, u_m, \nabla u_m))_{m\in N}$ is uniformly equi-integrable. In view of (47) and Vitali's theorem we conclude that $H_m(x, u_m, \nabla u_m)$ tends strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$.

Step 6: Passage to the limit

By taking $v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in (15), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m) D^i v\, dx + \int_\Omega H_m(x, u_m, \nabla u_m) v\, dx + \int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m) v\, dx \\ &+ \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m v\, dx \; = \int_\Omega F(x) v\, dx. \end{aligned} \tag{50}$$

For the first term on the left hand side of (50), in view of (46) and since $v \in W^{1,\vec{p}}(\Omega)$ then

$$\sum_{i=1}^N \int_\Omega a_i(x, T_m(u_m), \nabla u_m) D^i v\, dx \longrightarrow \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u) D^i v\, dx \qquad \text{as} \quad m \to \infty. \tag{51}$$

Moreover, thanks to (46) and since $v \in L^\infty(\Omega)$, then

$$\int_\Omega H_m(x, u_m, \nabla u_m) v\, dx \longrightarrow \int_\Omega H(x, u, \nabla u) v\, dx \qquad \text{as} \quad m \to \infty. \tag{52}$$

Similarly, using (30) and (31), we get

$$\int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m) v\, dx \longrightarrow \int_\Omega |u|^{p_0-2} u v\, dx \qquad \text{as} \quad m \to \infty, \tag{53}$$

and

$$\frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m v\, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty. \tag{54}$$

In view of (50) and (51) – (54) we conclude that

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u) D^i v\, dx + \int_\Omega H(x, u, \nabla u) v\, dx + \int_\Omega |u|^{p_0-2} u v\, dx \; = \int_\Omega F v\, dx, \\ &\text{for any} \quad v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega), \end{aligned} \tag{55}$$

which conclude the proof of Theorem 3.3.

3.2. Existence of solutions in the sense of distributions for L^m-data

In this part of the paper, we consider the strongly nonlinear and non-coercive Neumann problem:

$$\begin{cases} Au + H(x, u, \nabla u) + |u|^{s_0-2} u = f(x) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x, u, \nabla u).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{56}$$

with $f \in L^m(\Omega)$ with $m \ge \dfrac{s_0 + \lambda}{s_0 - 1}$.

Definition 3.5. *A measurable function u is a solution in the sense of distribution for the problem* (56), *if* $u \in W^{1,\vec{p}}(\Omega)$ *and* $|u|^{s_0-2}u \in L^1(\Omega)$ *with* $H(x,u,\nabla u) \in L^1(\Omega)$, *such that u verifies the following equality*

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)D^i v\,dx + \int_{\Omega}|u|^{s_0-2}uv\,dx + \int_{\Omega}H(x,u,\nabla u)v\,dx = \int_{\Omega} fv\,dx, \quad \text{for any} \quad v \in W^{1,\vec{p}}(\Omega)\cap L^{\infty}(\Omega). \tag{57}$$

Theorem 3.6. *Let* $f \in L^m(\Omega)$ *with* $m \geq \dfrac{s_0+\lambda}{s_0-1}$. *Under the assumptions* (8) − (11), *there exists at least one solution in the sense of distributions for the problem* (56).

Proof of Theorem 3.6

Step 1: Approximate problem

For any $n \in N^*$, we consider the following approximate problem

$$\begin{cases} \displaystyle\sum_{i=1}^{N} D^i a_i(x,u_n,\nabla u_n) + H(x,u_n,\nabla u_n) + |u_n|^{s_0-2}u_n = f_n(x) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u_n,\nabla u_n).n_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{58}$$

with $f_n = T_n(f) \in L^m(\Omega)\cap L^{\infty}(\Omega)$.
According to the Theorem 3.3, there exists at least one solution in the sense of distributions for the problem (58), i. e.

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)D^i v\,dx + \int_{\Omega}H_n(x,u_n,\nabla u_n)v\,dx + \int_{\Omega}|u_n|^{s_0-2}u_n v\,dx = \int_{\Omega} f_n(x)v\,dx, \quad \text{for any} \quad v \in W^{1,\vec{p}}(\Omega)\cap L^{\infty}(\Omega). \tag{59}$$

Step 2: Weak convergence of truncations

Let $k > 0$, by taking $v = T_k(u_n)(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)} \in W^{1,\vec{p}}(\Omega)\cap L^{\infty}(\Omega)$ as a test function in the approximate problem (58) with $B(|s|) = 2\displaystyle\int_0^s \frac{d(|\tau|)}{b(|\tau|)}d\tau$, we obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)D^iT_k(u_n)(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)}\,dx \\ &+\lambda\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)D^iT_k(u_n)|T_k(u_n)|(1+|T_k(u_n)|)^{\lambda-1}e^{B(|u_n|)}\,dx \\ &+2\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)D^iu_n|T_k(u_n)|\frac{d(|u_n|)}{b(|u_n|)}(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)}\,dx \\ &+\int_{\Omega}|u_n|^{s_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)}\,dx + \int_{\Omega}H_n(x,u_n,\nabla u_n)T_k(u_n)(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)}\,dx \\ &\leq \int_{\Omega}|f_n||T_k(u_n)|(1+|T_k(u_n)|)^{\lambda}e^{B(|u_n|)}\,dx. \end{aligned} \tag{60}$$

In view of (10) and (11), we conclude that

$$\begin{aligned} &b_0\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iT_k(u_n)|^{p_i}\,dx+b_0\lambda\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\frac{|D^iT_k(u_n)|^{p_i}}{(1+|u_n|)}|T_k(u_n)|e^{B(|u_n|)}\,dx\\ &+2\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx+\int_\Omega |u_n|^{s_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\ &\le e^{B(\infty)}\int_\Omega |f(x)||T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,dx+\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx. \end{aligned} \tag{61}$$

It follows that,

$$\begin{aligned} &b_0\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iT_k(u_n)|^{p_i}\,dx+b_0\lambda\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\frac{|D^iT_k(u_n)|^{p_i}}{(1+|u_n|)}|T_k(u_n)|\,dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda\,dx+\int_\Omega |T_k(u_n)|^{s_0+\lambda}\,dx\\ &\le C_0\int_\Omega |f(x)|\,|T_k(u_n)|\,dx+C_0\int_\Omega |f(x)|\,|T_k(u_n)|^{1+\lambda}\,dx\\ &\le C_1\int_\Omega |f(x)|^{\frac{s_0+\lambda}{s_0+\lambda-1}}\,dx+\int_\Omega |f(x)|^{\frac{s_0+\lambda}{s_0-1}}\,dx+\frac{1}{2}\int_\Omega |T_k(u_n)|^{s_0+\lambda}\,dx. \end{aligned} \tag{62}$$

Having in mind that $f\in L^m(\Omega)$ for $m\ge\dfrac{s_0+\lambda}{s_0-1}$, we get

$$b_0\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D^iT_k(u_n)|^{p_i}\,dx+k(1+k)^\lambda\sum_{i=1}^{N}\int_{\{|u_n|>k\}}d(|u_n|)|D^iu_n|^{p_i}\,dx+\frac{1}{2}\int_\Omega |T_k(u_n)|^{s_0+\lambda}\,dx\le C_2, \tag{63}$$

where C_2 is a positive constant that doesn't depend on k and n. Thus, by letting k tends to infinity, we deduce that

$$b_0\sum_{i=1}^{N}\int_\Omega |D^iu_n|^{p_i}\,dx+\frac{1}{2}\int_\Omega |u_n|^{s_0+\lambda}\,dx\le C_2. \tag{64}$$

Thus, in view of Young's inequality, we obtain

$$\begin{aligned} \|u_n\|_{W^{1,\vec{p}}(\Omega)} &= \|u_n\|_{W^{1,1}(\Omega)}+\sum_{i=1}^{N}\|D^iu_n\|_{L^{p_i}(\Omega)},\\ &= \int_\Omega |u_n|\,dx+\sum_{i=1}^{N}\int_\Omega |D^iu_n|\,dx+\sum_{i=1}^{N}\Big(\int_\Omega |D^iu_n|^{p_i}\,dx\Big)^{\frac{1}{p_i}}\\ &= \int_\Omega |u_n|^{s_0+\lambda}\,dx+2\sum_{i=1}^{N}\int_\Omega |D^iu_n|^{p_i}\,dx+(N+1)(1+\text{meas}(\Omega))\\ &\le C_3, \end{aligned} \tag{65}$$

with C_3 is a constant that doesn't depend on n and k. we conclude that the sequence $(u_n)_n$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and we get

$$\begin{cases} u_n\rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega),\\ u_n\longrightarrow u & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega,\\ u_n\rightharpoonup u & \text{weakly in } L^1(\partial\Omega) \text{ and a.e. in } \partial\Omega. \end{cases} \tag{66}$$

Finally, by using Lemma 2.5 and the Lebesgue dominated convergence theorem, we conclude that for any $k > 0$

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ T_k(u_n) \longrightarrow T_k(u) & \text{strongly in } L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } L^1(\partial\Omega) \quad \text{and} \quad \text{a.e. in} \quad \partial\Omega. \end{cases} \tag{67}$$

Now, we will show that the sequence $(|u_n|^{s_0-2}u_n)_n$ is uniformly equi-integrable.
Let E be a measurable subset of Ω. By using (64), we have for any $h > 0$

$$\begin{aligned} \int_E |u_n|^{s_0-1}\,dx &\le \int_E |T_h(u_n)|^{s_0-1}\,dx + \int_{E\cap\{|u_n|>h\}} |u_n|^{s_0-1}\,dx, \\ &\le meas(E)h^{s_0-1} + \frac{2C_2}{h^{\lambda+1}}. \end{aligned} \tag{68}$$

Thus, for any $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that

$$\int_E |u_n|^{s_0-1}\,dx \le \varepsilon \qquad \text{for any} \quad E \subset \Omega \quad \text{with} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{69}$$

Then, we deduce that the sequence $(|u_n|^{s_0-2}u_n)_n$ is equi-integrable in $L^1(\Omega)$, and since $u_n \longrightarrow u$ a.e. in Ω, then the Vitali's theorem implies that

$$|u_n|^{s_0-2}u_n \longrightarrow |u|^{s_0-2}u \qquad \text{in} \quad L^1(\Omega). \tag{70}$$

Step 3: Strong convergence of gradients

In this step, one will denote by $\varepsilon_i(n)$, $i = 0, 1, \ldots$ some various functions of real numbers which converges to 0 as n tends to infinity.
Let $k \ge 1$, by taking $\phi_n = \varphi(T_k(u_n) - T_k(u))e^{B(|u_n|)} \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function for the approximate problem (58), we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\varphi'(T_k(u_n) - T_k(u))e^{B(|u_n|)}\,dx \\ &+2\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}sign(u_n)\varphi(T_k(u_n) - T_k(u))e^{B(|u_n|)}\,dx \\ &+\int_\Omega H_n(x,u_n,\nabla u_n)\varphi(T_k(u_n) - T_k(u))e^{B(|u_n|)}\,dx + \int_\Omega |u_n|^{s_0-2}u_n\varphi(T_k(u_n) - T_k(u))e^{B(|u_n|)}\,dx \\ &= \int_\Omega f_n\varphi(T_k(u_n) - T_k(u))e^{B(|u_n|)}\,dx. \end{aligned} \tag{71}$$

It's clear that $\varphi(T_k(u_n)-T_k(u))$ has the same sign as u_n on the set $\{|u_n|\geq k\}$. In view of (10) and (11), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,u_n,\nabla u_n)D^iT_k(u)\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&+2\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,u_n,\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&-2\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} a_i(x,T_k(u_n),\nabla u_n)D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&+\int_{\Omega}|u_n|^{s_0-2}u_n\varphi(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&\leq \int_{\Omega}|f_n|\,|\varphi(T_k(u_n)-T_k(u))e^{B(|u_n|)}|\,dx+\int_{\Omega}|H_n(x,u_n,\nabla u_n)|\,|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&\leq e^{B(|u_n|)}\int_{\Omega}|f_n|\,|\varphi(T_k(u_n)-T_k(u))|\,dx+\sum_{i=1}^{N}\int_{\Omega}d(|u_n|)|D^iu_n|^{p_i}|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&\leq e^{B(|u_n|)}\int_{\Omega}|f|\,|\varphi(T_k(u_n)-T_k(u))|\,dx+\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx.
\end{aligned}
\tag{72}
$$

It follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,u_n,\nabla u_n)D^iT_k(u)\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-3\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}} a_i(x,T_k(u_n),\nabla u_n)D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&\leq e^{B(\infty)}\int_{\Omega}|u_n|^{s_0-1}\,|\varphi(T_k(u_n)-T_k(u))|\,dx+e^{B(|u_n|)}\int_{\Omega}|f|\,|\varphi(T_k(u_n)-T_k(u))|\,dx.
\end{aligned}
\tag{73}
$$

For the second term on the left-hand side of (73), in view of (9) and (64), the sequence $(a_i(x,u_n,\nabla u_n))_n$ is uniformly bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function ξ_i such that $|a_i(x,u_n,\nabla u_n)|\rightharpoonup\xi_i$ weakly in $L^{p_i'}(\Omega)$, it follows that

$$
\begin{aligned}
\varepsilon_1(n) &= \sum_{i=1}^{N}\Big|\int_{\{|u_n|>k\}} a_i(x,u_n,\nabla u_n)D^iT_k(u)\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\Big|\\
&\leq \varphi'(2k)e^{B(\infty)}\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|a_i(x,u_n,\nabla u_n)|\,|D^iT_k(u)|\,dx\\
&\longrightarrow \varphi'(2k)e^{B(\infty)}\sum_{i=1}^{N}\int_{\{|u|>k\}}\xi_i\,|D^iT_k(u)|\,dx=0 \quad\text{as}\quad n\to\infty.
\end{aligned}
\tag{74}
$$

Concerning the first term on the right-hand side of (73), thanks to (70) we have $|u_n|^{s_0-1}\longrightarrow|u|^{s_0-1}$ strongly in $L^1(\Omega)$, and since $\varphi(T_k(u_n)-T_k(u))\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, then

$$
\varepsilon_2(n)=\int_{\Omega}|u_n|^{s_0-1}\,|\varphi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0 \quad\text{as}\quad n\to\infty. \tag{75}
$$

Similarly, we have $f \in L^m(\Omega)$ with $m \geq 1$, then

$$\varepsilon_3(n) = \int_\Omega |f|\, |\varphi(T_k(u_n) - T_k(u))|\, dx \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{76}$$

By combining (73) and (74) – (76), we conclude that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))\varphi'(T_k(u_n) - T_k(u))e^{B(|u_n|)}\, dx \\ &-3\sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))D^i T_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\varphi(T_k(u_n) - T_k(u))|e^{B(|u_n|)}\, dx \\ &\leq \varepsilon_4(n)\end{aligned} \tag{77}$$

where $\varepsilon_4(n) = \varepsilon_1(n) + \varepsilon_2(n) + \varepsilon_3(n) \longrightarrow 0$ when n goes to ∞. Therefore, we deduce that

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u))\varphi'(T_k(u_n) - T_k(u))e^{B(|u_n|)}\, dx \\ &-3\sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u)) \\ &\qquad \times|\varphi(T_k(u_n) - T_k(u))|\frac{d(|u_n|)}{b(|u_n|)}e^{B(|u_n|)}\, dx \\ &\leq \varepsilon_4(n) + 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}\varphi(2k)e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx \\ &+3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u_n))|\, |D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))|\, dx \\ &+\varphi'(2k)e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))||D^i T_k(u_n) - D^i T_k(u)|\, dx.\end{aligned} \tag{78}$$

Thus, we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u)) \\ &\qquad\qquad \times\Big(\varphi'(T_k(u_n) - T_k(u))e^{B(|u_n|)} - 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}|\varphi(T_k(u_n) - T_k(u))|\Big)e^{B(|u_n|)}\, dx \\ &\leq 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u_n))|\, |D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))|\, dx \\ &+\Big(\varphi'(2k) + 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}\varphi(2k)\Big)e^{B(\infty)}\sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx + \varepsilon_4(n).\end{aligned} \tag{79}$$

For the first term on the right-hand side of (79), according to assumption (9) we have $(a_i(x, T_k(u_n), \nabla T_k(u_n)))_{n\in N}$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\vartheta_{k,i}$ in $L^{p_i'}(\Omega)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \vartheta_{k,i}$ weakly in $L^{p_i'}(\Omega)$, and since $|D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))| \to 0$ strongly in $L^{p_i}(\Omega)$, then we get

$$\varepsilon_5(n) = \sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u_n))|\, |D^i T_k(u)|\, |\varphi(T_k(u_n) - T_k(u))|\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{80}$$

For the second term on the right-hand side of (79), we have $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ weakly in $L^{p_i}(\Omega)$, it follows that

$$\varepsilon_6(n) = \sum_{i=1}^N \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\, |D^i T_k(u_n) - D^i T_k(u)|\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{81}$$

By combining (79) and (80) − (81), we conclude that

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\\&\leq\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\\&\qquad\times\Big(\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}-3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\Omega)}|\varphi(T_k(u_n)-T_k(u))|\Big)e^{B(|u_n|)}\,dx\\&\leq\varepsilon_7(n)\end{aligned}\tag{82}$$

where $\varepsilon_7(n)=\varepsilon_4(n)+\varepsilon_5(n)+\varepsilon_6(n)\longrightarrow 0$ as n goes to ∞.
By letting n tends to infinity, we conclude that

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\tag{83}$$

Thanks to (67) we have $T_k(u_n)\to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$, and in view of Lemma 3.1, we conclude that

$$\begin{cases}T_k(u_n)\longrightarrow T_k(u) & \text{in}\quad W^{1,\vec{p}}(\Omega),\\ D^iT_k(u_n)\longrightarrow D^iT_k(u)\quad a.e. & \text{in}\quad\Omega.\end{cases}\tag{84}$$

Having in mind (66) by letting k tends to infinity, we obtain

$$\begin{cases}u_n\longrightarrow u & \text{in}\quad W^{1,\vec{p}}(\Omega)\\ D^iu_n\longrightarrow D^iu\quad a.e. & \text{in}\quad\Omega.\end{cases}\tag{85}$$

Step 4 : Equi-integrability of $(H_n(x,u_n,\nabla u_n))_{n\in N}$

Let $h\geq 1$ and $E\subset\Omega$ be a measurable subset. It's clear that

$$\int_{\Omega}|H_n(x,u_n,\nabla u_n)|\,dx\leq\int_{E\cap\{|u_n|\leq h\}}|H_n(x,T_h(u_n),\nabla T_h(u_n))|\,dx+\int_{\{|u_n|>h\}}|H_n(x,u_n,\nabla u_n)|\,dx.\tag{86}$$

On the one hand, according to (84), we have : For any $\varepsilon>0$, there exists $\beta(\varepsilon,h)>0$ such that

$$\int_{E\cap\{|u_n|\leq h\}}|H_n(x,T_h(u_n),\nabla T_h(u_n))|\,dx\leq\frac{\varepsilon}{2}\qquad\text{for any}\quad E\subset\Omega\quad\text{with}\quad\text{meas}(E)\leq\beta(\varepsilon,h).\tag{87}$$

On the other hand, thanks to (63) we have

$$h(1+h)^{\lambda}\int_{\{|u_n|>h\}}d(|u_n|)|D^iu_n|^{p_i}\,dx\leq C_2.$$

Using (11) we deduce that

$$\begin{aligned}\int_{\{|u_n|>h\}}|H_n(x,u_n,\nabla u_n)|\,dx\quad&\leq\sum_{i=1}^{N}\int_{\{|u_n|>h\}}d(|u_n|)|D^iu_n|^{p_i}\,dx,\\&\leq\frac{C_2}{h(1+h)^{\lambda}}\longrightarrow 0\quad as\quad h\to\infty.\end{aligned}\tag{88}$$

Thus, for all $\varepsilon\geq 0$, there exists $h_0(\varepsilon)\geq 0$ such that

$$\int_{\{|u_n|>h\}}|H_n(x,u_n,\nabla u_n)|\,dx\leq\frac{\varepsilon}{2}\qquad\text{for}\quad h\geq h_0(\varepsilon).\tag{89}$$

By combining (86), (87) and (89) we conclude that : For any $\varepsilon>0$ there exists $\beta(\varepsilon)>0$ such that

$$\int_{E}|H_n(x,u_n,\nabla u_n)|\,dx\leq\varepsilon\qquad\text{for any}\quad E\subseteq\Omega\quad\text{with}\quad meas(E)\leq\beta(\varepsilon).\tag{90}$$

Thus, the sequence $(H_n(x,u_n,\nabla u_n))_{n\in N}$ is uniformly equi-integrable and thanks to (85), we have

$$H_n(x,u_n,\nabla u_n)\longrightarrow H(x,u,\nabla u)\quad\text{a.e. in}\quad\Omega.\tag{91}$$

In view of Vitali's theorem, we deduce that

$$H_n(x,u_n,\nabla u_n)\longrightarrow H(x,u,\nabla u)\qquad\text{strongly in}\quad L^1(\Omega).\tag{92}$$

Step 5: Passage to the limit

Thanks to (85), we have $a_i(x,u_n,\nabla u_n) \longrightarrow a_i(x,u,\nabla u)$ a.e. in Ω and since $(a_i(x,u_n,\nabla u_n))_{n\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$, in view of Lemma 2.5, we obtain

$$a_i(x,u_n,\nabla u_n) \rightharpoonup a_i(x,u,\nabla u) \quad in \quad L^{p_i'}(\Omega). \tag{93}$$

Also, we have $f_n \longrightarrow f$ strongly in $L^m(\Omega)$.
By taking $v \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in (59) and in view of (70), (92) and (93), easily we pass to the limit in the following equality

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)D^i v\,dx + \int_\Omega H_n(x,u_n,\nabla u_n)v\,dx + \int_\Omega |u_n|^{s_0-2}u_n v\,dx = \int_\Omega f_n v\,dx, \tag{94}$$

to obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)D^i v\,dx + \int_\Omega H(x,u,\nabla u)v\,dx + \int_\Omega |u|^{s_0-2}uv\,dx = \int_\Omega fv\,dx, \tag{95}$$

which conclude the proof of the Theorem 3.6.

4. Appendix

Proof of Lemma 3.4.
We denote by C_1,C_2,C_3,...some positive constants which depend on m.
For any u and v in $W^{1,\vec{p}}(\Omega)$, we have

$$\begin{aligned}\langle B_m u, v\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u),\nabla u)D^i v\,dx + \int_\Omega |T_m(u)|^{s_0-2}T_m(u)v\,dx \\ &\quad + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,dx + \int_\Omega H_m(x,u,\nabla u)v\,dx.\end{aligned} \tag{96}$$

In view of Hölder's inequality we get

$$\begin{aligned}|\langle B_m u, v\rangle| &\leq \sum_{i=1}^{N}\int_\Omega |a_i(x,T_m(u),\nabla u)|\,|D^i v|\,dx + \int_\Omega |H_m(x,u,\nabla u)|\,|v|\,dx \\ &\quad + \int_\Omega |T_m(u)|^{s_0-1}\,|v|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-1}|v|\,dx \\ &\leq \sum_{i=1}^{N}\int_\Omega \beta(K_i(x) + |T_m(u)|^{p_i-1} + |D^i u|^{p_i-1})\,|D^i v|\,dx + m\int_\Omega |v|\,dx \\ &\quad + m^{s_0-1}\int_\Omega |v|\,dx + \frac{1}{m}\|u\|^{\underline{p}-1}_{L^{\underline{p}}(\Omega)}\|v\|_{L^{\underline{p}}(\Omega)} \\ &\leq \sum_{i=1}^{N}\beta\Big(\|K_i(x)\|_{L^{p_i'}(\Omega)} + \|T_m(u)\|^{p_i-1}_{L^{p_i}(\Omega)} + \|D^i u\|^{p_i-1}_{L^{p_i}(\Omega)}\Big)\|v\|_{1,\vec{p}} + m\|v\|_{1,\vec{p}} \\ &\quad + m^{s_0-1}\|v\|_{1,\vec{p}} + \frac{C_0}{m}\|u\|^{\underline{p}-1}_{1,\vec{p}}\|v\|_{1,\vec{p}} \\ &\leq \Big(C_1 + \|u\|^{p_0-1}_{1,\vec{p}}\Big)\|v\|_{1,\vec{p}}.\end{aligned} \tag{97}$$

Thus, the operator B_m is bounded.
For the coercivity, we have for any $u, v \in W^{1,\vec{p}}(\Omega)$,

$$\begin{aligned} \langle B_m v, v\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_m(v),\nabla v).D^i v\,dx + \int_\Omega H_m(x,v,\nabla v)v\,dx \\ &\quad + \int_\Omega |T_m(v)|^{s_0-2}T_m(v)v\,dx + \frac{1}{m}\int_\Omega |v|^{\underline{p}}\,dx \\ &\geq \frac{b_0}{(1+m)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx + \frac{1}{m}\int_\Omega |v|^{\underline{p}}\,dx - m\int_\Omega |v|\,dx \\ &\geq \frac{b_0}{(1+m)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i v|^{p_i}\,dx + \frac{C_2}{m}\|v\|^{\underline{p}}_{L^1(\Omega)} - m\|v\|_{1,\vec{p}} \\ &\geq C_3\|v\|^{\underline{p}}_{1,\vec{p}} - m\|v\|_{1,\vec{p}}\,. \end{aligned} \tag{98}$$

Since $\underline{p}-1>0$, then

$$\frac{|\langle B_m v, v\rangle|}{\|v\|_{1,\vec{p}}} \to \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \to \infty.$$

Therefore, the operator B_m is coercive.
Now, it remains to show that the operator B_m is pseudo-monotone. Let $(u_k)_{k\in N}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in} \ \ W^{1,\vec{p}}(\Omega), \\ B_m u_k \rightharpoonup \chi \quad \text{in} \ \ (W^{1,\vec{p}}(\Omega))', \\ \limsup\limits_{k\to\infty}\langle B_m u_k, u_k\rangle \leq \langle \chi, u\rangle. \end{cases} \tag{99}$$

We will prove that $\chi = B_m u$ and $\langle B_m u_k, u_k\rangle \to \langle \chi, u\rangle$ as $k\to\infty$.
We have $W^{1,\vec{p}}(\Omega) \hookrightarrow L^{\underline{p}}(\Omega)$, then $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$ and *a.e.* in Ω for a subsequence denoted again $(u_k)_k$.
In view of (9), the sequence $(a_i(x,T_m(u_k),\nabla u_k))_{k\in N}$ is uniformly bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x,T_m(u_k),\nabla u_k) \rightharpoonup \varphi_i \quad \textit{weakly} \quad \textit{in} \quad L^{p_i'}(\Omega). \tag{100}$$

On the other hand, we have

$$|u_n|^{\underline{p}-2}u_n \to |u|^{\underline{p}-2}u \quad \textit{strongly} \quad \textit{in} \quad L^{\underline{p}'}(\Omega). \tag{101}$$

Also, in view of Lebesgue's dominated convergence theorem we obtain

$$|T_m(u_k)|^{s_0-2}T_m(u_k) \longrightarrow |T_m(u)|^{s_0-2}T_m(u) \quad \textit{strongly} \quad \textit{in} \quad L^{\underline{p}'}(\Omega). \tag{102}$$

Moreover, $(H_m(x,u_k,\nabla u_k))_k$ is uniformly bounded sequence in $L^{\underline{p}}(\Omega)$, then there exists a measurable function $\psi \in L^{\underline{p}}(\Omega)$ such that

$$H_m(x,u_k,\nabla u_k) \rightharpoonup \psi \quad \textit{weakly} \quad \textit{in} \quad L^{\underline{p}}(\Omega). \tag{103}$$

By combining (100) − (103) we conclude that for any $v \in W^{1,\vec{p}}(\Omega)$,

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to\infty}\langle B_m u_k, v\rangle \\ &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega H_m(x,u_k,\nabla u_k)v\,dx \\ &\quad + \lim_{k\to\infty}\int_\Omega |T_m(u_k)|^{s_0-2}T_m(u_k)v\,dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}-2}u_k v\,dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i D^i v\,dx + \int_\Omega \psi v\,dx + \int_\Omega |T_m(u)|^{s_0-2}T_m(u)v\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,dx. \end{aligned} \tag{104}$$

Using (99) and (104), we deduce that

$$\begin{aligned}\limsup_{k\to\infty}\langle B_m u_k, u_k\rangle &= \limsup_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u_k\,dx + \int_\Omega H_m(x,u_k,\nabla u_k)u_k\,dx \\ &\quad + \int_\Omega |T_m(u_k)|^{s_0-1}|u_k|\,dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}}\,dx\Big) \\ &\leq \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx + \int_\Omega \psi u\,dx + \int_\Omega |T_m(u)|^{s_0-1}|u|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx.\end{aligned} \tag{105}$$

Having in mind (99), (104) and (105), we get

$$\limsup_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u_k\,dx \leq \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx. \tag{106}$$

On the other hand, thanks to (8) we have

$$\sum_{i=1}^{N}\int_\Omega (a_i(x,T_m(u_k),\nabla u_k) - a_i(x,T_m(u_k),\nabla u))(D^i u_k - D^i u)\,dx \geq 0, \tag{107}$$

then,

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u_k\,dx &\geq \sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u)(D^i u_k - D^i u)\,dx \\ &\quad + \sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u\,dx.\end{aligned} \tag{108}$$

Thanks to (100), we get

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u_k\,dx \geq \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u dx. \tag{109}$$

Combining (106) and (109), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k)D^i u_k\,dx = \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u dx. \tag{110}$$

Thus, in view of (110), (103), (102) and (101), we obtain

$$\begin{aligned}\lim_{k\to\infty}\langle B_m u_k, u_k\rangle &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_m(u_k),\nabla u_k).D^i u_k\,dx \\ &\quad + \lim_{k\to\infty}\int_\Omega H_m(x,u_k,\nabla u_k).u_k\,dx + \int_\Omega |T_m(u_k)|^{s_0-1}|u|\,dx + \frac{1}{m}\lim_{k\to\infty}\int_\Omega |u_k|^{\underline{p}}\,dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,dx + \int_\Omega \psi u\,dx + \int_\Omega |T_m(u)|^{s_0-1}|u|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx, \\ &= \langle \chi, u\rangle.\end{aligned} \tag{111}$$

Moreover, thanks to (110) we have

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_m(u_k),\nabla u_k) - a_i(x,T_m(u),\nabla u))(D^i u_k - D^i u)\,dx = 0. \tag{112}$$

Having in mind that $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$. Thus, by applying Lemma 3.1 we deduce that

$$\begin{cases} u_k \longrightarrow u \quad \textit{strongly} \quad \text{in} \quad W^{1,\vec{p}}(\Omega) \\ D^i u_k \longrightarrow D^i u \quad \textit{a.e.} \quad \text{in} \quad \Omega. \end{cases}$$

Then, $a_i(x, T_m(u_k), \nabla u_k) \longrightarrow a_i(x, T_m(u), \nabla u)$ and $H_m(x, u_k, \nabla u_k) \to H_m(x, u, \nabla u)$ almost everywhere in Ω, and thanks to (9) we conclude that

$$a_i(x, T_m(u_k), \nabla u_k) \rightharpoonup a_i(x, T_m(u), \nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega) \tag{113}$$

and

$$H_m(x, u_k, \nabla u_k) \rightharpoonup H_m(x, u, \nabla u) \qquad \text{weakly in} \quad L^{\underline{p}}(\Omega). \tag{114}$$

Then, in view of (101) and (102) we deduce that $B_m u = \chi$. Thus, the proof of Lemma 3.4 is concluded.

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