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Nonlinear ξ-bi-skew Lie derivations on prime $*$-algebras

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Abstract. Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection and ξ be a nonzero scalar. We prove that a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $\varphi([A,B]^{\xi}_{\diamond}) = [\varphi(A),B]^{\xi}_{\diamond} + [A,\varphi(B)]^{\xi}_{\diamond}$ for all $A,B \in \mathcal{A}$ if and only if φ is an additive $*$-derivation and $\varphi(\xi A) = \xi\varphi(A)$ for all $A \in \mathcal{A}$, where $[A,B]^{\xi}_{\diamond} = AB^* - \xi BA^*$.

1. Introduction

Let $\mathcal{A}$ be a $*$-algebra over the complex field $\mathbb{C}$ and ξ be a nonzero scalar. For $A,B \in \mathcal{A}$, denote by $[A,B]^{\xi}_{*} = AB - \xi BA^*$ the ξ-skew Lie product of A and B. The 1-skew Lie product naturally arose in representing quadratic functionals with sesquilinear functionals and characterizing ideals. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity assumption). φ is called a nonlinear ξ-skew Lie derivation if $\varphi([A,B]^{\xi}_{*}) = [\varphi(A),B]^{\xi}_{*} + [A,\varphi(B)]^{\xi}_{*}$ for all $A,B \in \mathcal{A}$. Also, φ is called an additive derivation if it is additive and $\varphi(AB) = \varphi(A)B + A\varphi(B)$ for all $A,B \in \mathcal{A}$. Moreover, φ is called an additive $*$-derivation if it is an additive derivation and satisfies $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{A}$. Yu and Zhang [17] proved that every nonlinear 1-skew Lie derivation on factor von Neumann algebras is an additive $*$-derivation. Jing [5] considered nonlinear 1-skew Lie derivations on standard operator algebras and obtained the same result. Taghavi et al. [14] proved that every nonlinear (-1)-skew Lie derivation on factor von Neumann algebras is an additive $*$-derivation. Li et al. [9] studied nonlinear $(-\xi)$-skew Lie derivation on von Neumann algebras. In the last decade, there are many results related with ξ-skew Lie product, see for example [10, 12, 15, 16, 19] and their references. For $A,B \in \mathcal{A}$, denote by $[A,B]^{\xi}_{\diamond} = AB^* - \xi BA^*$ the ξ-bi-skew Lie product of A and B. φ is called a nonlinear ξ-bi-skew Lie derivation if $\varphi([A,B]^{\xi}_{\diamond}) = [\varphi(A),B]^{\xi}_{\diamond} + [A,\varphi(B)]^{\xi}_{\diamond}$ for all $A,B \in \mathcal{A}$. Kong and Zhang [8] studied nonlinear 1-bi-skew Lie derivations on factor von Neumann algebras. Ashraf [2] proved that every nonlinear (-1)-bi-skew Lie derivation on factor von Neumann algebras is an additive $*$-derivation. Khan and Alhazmi [6] proved that every nonlinear (-1)-bi-skew Lie triple derivation on prime $*$-algebras is an additive $*$-derivation. Recently, several authors pay more attention to the maps related with ξ-bi-skew Lie product, see for example [1, 3, 4, 7, 11, 13, 18].

An algebra $\mathcal{A}$ is prime if for any $A,B \in \mathcal{A}$, $A\mathcal{A}B = \{0\}$ implies that either $A = 0$ or $B = 0$. Motivated by the above-mentioned works, in this paper, we will completely describe nonlinear ξ-bi-skew Lie derivation on prime $*$-algebras.

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2. Main Result

Theorem 2.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra containing a nontrivial projection and ξ be a nonzero scalar. A map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\varphi([A, B]^{\xi}_{\diamond}) = [\varphi(A), B]^{\xi}_{\diamond} + [A, \varphi(B)]^{\xi}_{\diamond}$$

for all $A, B \in \mathcal{A}$ if and only if φ is an additive $*$-derivation and $\varphi(\xi A) = \xi\varphi(A)$ for all $A \in \mathcal{A}$.

Let $P \in \mathcal{A}$ be a nontrivial projection. Write $P_1 = P, P_2 = I - P_1, \mathcal{A}_{ij} = P_i\mathcal{A}P_j$ $(i, j = 1, 2)$. Then $\mathcal{A}$=$\mathcal{A}_{11}$+$\mathcal{A}_{12}$+$\mathcal{A}_{21}$+$\mathcal{A}_{22}$, and for $A \in \mathcal{A}$, $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{A}_{ij}$ $(i, j = 1, 2)$. Let $\mathcal{M} = \{A \in \mathcal{A} : A^* = A\}$ and $\mathcal{N} = \{A \in \mathcal{A} : A^* = -A\}$.

Proof. Clearly, we only need to prove the necessity. If $\xi = \pm 1$, then Theorem 2.1 holds by the results of [18] and [4]. In what follows, assume that $\xi \in \mathbb{C}\backslash\{0, \pm 1\}$. We will complete the proof by a series of claims.

Claim 1 *For every $A_{12} \in \mathcal{A}_{12}, B_{21} \in \mathcal{A}_{21}$, we have*

$$\varphi(A_{12} + B_{21}) = \varphi(A_{12}) + \varphi(B_{21}).$$

It is clear that $\varphi(0) = 0$. Let

$$T = \varphi(A_{12} + B_{21}) - \varphi(A_{12}) - \varphi(B_{21}).$$

We next prove that $T = 0$. Since $[P_1, A_{12}]^{\xi}_{\diamond} = 0$ and $\varphi(0) = 0$, we have

$$\begin{aligned}
[\varphi(P_1), A_{12} + B_{21}]^{\xi}_{\diamond} + [P_1, \varphi(A_{12} + B_{21})]^{\xi}_{\diamond} &= \varphi([P_1, A_{12} + B_{21}]^{\xi}_{\diamond}) \\
&= \varphi([P_1, A_{12}]^{\xi}_{\diamond}) + \varphi([P_1, B_{21}]^{\xi}_{\diamond}) \\
&= [\varphi(P_1), A_{12} + B_{21}]^{\xi}_{\diamond} + [P_1, \varphi(A_{12}) + \varphi(B_{21})]^{\xi}_{\diamond},
\end{aligned}$$

which implies that $[P_1, T]^{\xi}_{\diamond} = 0$, that is

$$P_1T^* - \xi TP_1 = 0. \tag{2.1}$$

It follows from $[\mathrm{i}P_1, A_{12}]^{\xi}_{\diamond} = 0$ that

$$\begin{aligned}
[\varphi(\mathrm{i}P_1), A_{12} + B_{21}]^{\xi}_{\diamond} + [\mathrm{i}P_1, \varphi(A_{12} + B_{21})]^{\xi}_{\diamond} &= \varphi([\mathrm{i}P_1, A_{12} + B_{21}]^{\xi}_{\diamond}) \\
&= \varphi([\mathrm{i}P_1, A_{12}]^{\xi}_{\diamond}) + \varphi([\mathrm{i}P_1, B_{21}]^{\xi}_{\diamond}) \\
&= [\varphi(\mathrm{i}P_1), A_{12} + B_{21}]^{\xi}_{\diamond} + [\mathrm{i}P_1, \varphi(A_{12}) + \varphi(B_{21})]^{\xi}_{\diamond}.
\end{aligned}$$

This implies that $[\mathrm{i}P_1, T]^{\xi}_{\diamond} = 0$, and so

$$P_1T^* + \xi TP_1 = 0. \tag{2.2}$$

From Eq. (2.1) and Eq. (2.2), we obtain that $T_{11} = T_{21} = 0$. Similarly, we can show that $T_{12} = T_{22} = 0$. Hence $T = 0$.

Claim 2 *For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have*

$$\varphi(A_{11} + B_{12} + C_{21}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21})$$

and

$$\varphi(B_{12} + C_{21} + D_{22}) = \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22}).$$

Let

$$T = \varphi(A_{11} + B_{12} + C_{21}) - \varphi(A_{11}) - \varphi(B_{12}) - \varphi(C_{21}).$$

Since $[A_{11}, P_2]^{\xi}_{\diamond} = [C_{21}, P_2]^{\xi}_{\diamond} = 0$, we have

$$\begin{aligned}
&[\varphi(A_{11} + B_{12} + C_{21}), P_2]^{\xi}_{\diamond} + [A_{11} + B_{12} + C_{21}, \varphi(P_2)]^{\xi}_{\diamond} \\
&= \varphi([A_{11} + B_{12} + C_{21}, P_2]^{\xi}_{\diamond}) \\
&= \varphi([A_{11}, P_2]^{\xi}_{\diamond}) + \varphi([B_{12}, P_2]^{\xi}_{\diamond}) + \varphi([C_{21}, P_2]^{\xi}_{\diamond}) \\
&= [\varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}), P_2]^{\xi}_{\diamond} + [A_{11} + B_{12} + C_{21}, \varphi(P_2)]^{\xi}_{\diamond},
\end{aligned}$$

which implies that $[T,P_2]_\diamond^\xi=0$. It follows that

$$TP_2-\xi P_2T^*=0. \tag{2.3}$$

From $[A_{11},\mathrm{i}P_2]_\diamond^\xi=[C_{21},\mathrm{i}P_2]_\diamond^\xi=0$, we have

$$\begin{aligned}&[\varphi(A_{11}+B_{12}+C_{21}),\mathrm{i}P_2]_\diamond^\xi+[A_{11}+B_{12}+C_{21},\varphi(\mathrm{i}P_2)]_\diamond^\xi\\ =&\varphi([A_{11}+B_{12}+C_{21},\mathrm{i}P_2]_\diamond^\xi)\\ =&\varphi([A_{11},\mathrm{i}P_2]_\diamond^\xi)+\varphi([B_{12},\mathrm{i}P_2]_\diamond^\xi)+\varphi([C_{21},\mathrm{i}P_2]_\diamond^\xi)\\ =&[\varphi(A_{11})+\varphi(B_{12})+\varphi(C_{21}),\mathrm{i}P_2]_\diamond^\xi+[A_{11}+B_{12}+C_{21},\varphi(\mathrm{i}P_2)]_\diamond^\xi.\end{aligned}$$

This gives that $[T,\mathrm{i}P_2]_\diamond^\xi=0$, and so

$$TP_2+\xi P_2T^*=0. \tag{2.4}$$

By Eq. (2.3) and Eq. (2.4), we obtain that $T_{12}=T_{22}=0$.

Let

$$Q_{A_{11},B_{12},C_{21}}=T_{21}, R_{A_{11},B_{12},C_{21}}=T_{11}.$$

Then $Q_{A_{11},B_{12},C_{21}}\in\mathcal{A}_{21}, R_{A_{11},B_{12},C_{21}}\in\mathcal{A}_{11}$, and so

$$\varphi(A_{11}+B_{12}+C_{21})=\varphi(A_{11})+\varphi(B_{12})+\varphi(C_{21})+Q_{A_{11},B_{12},C_{21}}+R_{A_{11},B_{12},C_{21}}. \tag{2.5}$$

Since $[P_1,A_{11}+B_{12}+C_{21}]_\diamond^\xi=A_{11}^*-\xi A_{11}+C_{21}^*-\xi C_{21}$, we have from Eq. (2.5) and Claim 1 that

$$\begin{aligned}&[\varphi(P_1),A_{11}+B_{12}+C_{21}]_\diamond^\xi+[P_1,\varphi(A_{11}+B_{12}+C_{21})]_\diamond^\xi\\ =&\varphi([P_1,A_{11}+B_{12}+C_{21}]_\diamond^\xi)\\ =&\varphi(A_{11}^*-\xi A_{11})+\varphi(C_{21}^*)+\varphi(-\xi C_{21})+Q_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}+R_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}\\ =&\varphi([P_1,A_{11}]_\diamond^\xi)+\varphi([P_1,B_{12}]_\diamond^\xi)+\varphi([P_1,C_{21}]_\diamond^\xi)+Q_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}+R_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}\\ =&[\varphi(P_1),A_{11}+B_{12}+C_{21}]_\diamond^\xi+[P_1,\varphi(A_{11})+\varphi(B_{12})+\varphi(C_{21})]_\diamond^\xi+Q_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}+R_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}.\end{aligned}$$

It follows that

$$[P_1,T]_\diamond^\xi=Q_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}+R_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}. \tag{2.6}$$

Multiplying Eq. (2.6) by P_1 from the left and by P_2 from the right, then by $Q_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}\in\mathcal{A}_{21}$, $R_{A_{11}^*-\xi A_{11},C_{21}^*,-\xi C_{21}}\in\mathcal{A}_{11}$, we obtain $P_1T^*P_2=0$, and so $Q_{A_{11},B_{12},C_{21}}=T_{21}=0$. Hence we have from Eq. (2.5) that

$$\varphi(A_{11}+B_{12}+C_{21})=\varphi(A_{11})+\varphi(B_{12})+\varphi(C_{21})+R_{A_{11},B_{12},C_{21}},$$

where $R_{A_{11},B_{12},C_{21}}\in\mathcal{A}_{11}$.

Similarly, there exists $U_{B_{12},C_{21},D_{22}}\in\mathcal{A}_{22}$ such that

$$\varphi(B_{12}+C_{21}+D_{22})=\varphi(B_{12})+\varphi(C_{21})+\varphi(D_{22})+U_{B_{12},C_{21},D_{22}}. \tag{2.7}$$

For any $X_{21}\in\mathcal{A}_{21}$, since

$$[A_{11}+B_{12}+C_{21},X_{21}]_\diamond^\xi=A_{11}X_{21}^*-\xi X_{21}A_{11}^*+C_{21}X_{21}^*-\xi X_{21}C_{21}^*,$$

we have from Eq. (2.7) and Claim 1 that

$$\begin{aligned}&[\varphi(A_{11}+B_{12}+C_{21}),X_{21}]_\diamond^\xi+[A_{11}+B_{12}+C_{21},\varphi(X_{21})]_\diamond^\xi\\ =&\varphi([A_{11}+B_{12}+C_{21},X_{21}]_\diamond^\xi)\\ =&\varphi(A_{11}X_{21}^*)+\varphi(-\xi X_{21}A_{11}^*)+\varphi(C_{21}X_{21}^*-\xi X_{21}C_{21}^*)+U_{A_{11}X_{21}^*,-\xi X_{21}A_{11}^*,C_{21}X_{21}^*-\xi X_{21}C_{21}^*}\\ =&\varphi([A_{11},X_{21}]_\diamond^\xi)+\varphi([B_{12},X_{21}]_\diamond^\xi)+\varphi([C_{21},X_{21}]_\diamond^\xi)+U_{A_{11}X_{21}^*,-\xi X_{21}A_{11}^*,C_{21}X_{21}^*-\xi X_{21}C_{21}^*}\\ =&[\varphi(A_{11})+\varphi(B_{12})+\varphi(C_{21}),X_{21}]_\diamond^\xi+[A_{11}+B_{12}+C_{21},\varphi(X_{21})]_\diamond^\xi+U_{A_{11}X_{21}^*,-\xi X_{21}A_{11}^*,C_{21}X_{21}^*-\xi X_{21}C_{21}^*}.\end{aligned}$$

This implies that

$$[T, X_{21}]_\diamond^\xi = U_{A_{11}X_{21}^*, -\xi X_{21}A_{11}^*, C_{21}X_{21}^* - \xi X_{21}C_{21}^*}. \tag{2.8}$$

Multiplying Eq. (2.8) by P_1 from the right, then by $U_{A_{11}X_{21}^*, -\xi X_{21}A_{11}^*, C_{21}X_{21}^* - \xi X_{21}C_{21}^*} \in \mathcal{A}_{22}$, we have $X_{21}T^*P_1 = 0$, and so $T_{11} = 0$ by the primeness of $\mathcal{A}$. Therefore $T = 0$.

Similarly, we can show that $\varphi(B_{12} + C_{21} + D_{22}) = \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22})$.

Claim 3 *For every $A_{11} \in \mathcal{A}_{11}, B_{12} \in \mathcal{A}_{12}, C_{21} \in \mathcal{A}_{21}, D_{22} \in \mathcal{A}_{22}$, we have*

$$\varphi(A_{11} + B_{12} + C_{21} + D_{22}) = \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22}).$$

Let

$$T = \varphi(A_{11} + B_{12} + C_{21} + D_{22}) - \varphi(A_{11}) - \varphi(B_{12}) - \varphi(C_{21}) - \varphi(D_{22}).$$

From $[P_1, A_{11}]_\diamond^\xi = A_{11}^* - \xi A_{11}, [P_1, B_{12}]_\diamond^\xi = 0, [P_1, C_{21}]_\diamond^\xi = C_{21}^* - \xi C_{21}, [P_1, D_{22}]_\diamond^\xi = 0$ and Claim 2, we have

$$\begin{aligned}
&[\varphi(P_1), A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi + [P_1, \varphi(A_{11} + B_{12} + C_{21} + D_{22})]_\diamond^\xi \\
&=\varphi([P_1, A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi) \\
&=\varphi(A_{11}^* - \xi A_{11} + C_{21}^* - \xi C_{21}) \\
&=\varphi(A_{11}^* - \xi A_{11}) + \varphi(C_{21}^* - \xi C_{21}) \\
&=\varphi([P_1, A_{11}]_\diamond^\xi) + \varphi([P_1, B_{12}]_\diamond^\xi) + \varphi([P_1, C_{21}]_\diamond^\xi) + \varphi([P_1, D_{22}]_\diamond^\xi) \\
&=[\varphi(P_1), A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi + [P_1, \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22})]_\diamond^\xi.
\end{aligned}$$

This implies that

$$[P_1, T]_\diamond^\xi = P_1T^* - \xi TP_1 = 0. \tag{2.9}$$

From

$$[\mathrm{i}P_1, A_{11}]_\diamond^\xi = \mathrm{i}A_{11}^* + \mathrm{i}\xi A_{11}, [\mathrm{i}P_1, B_{12}]_\diamond^\xi = 0, [\mathrm{i}P_1, C_{21}]_\diamond^\xi = \mathrm{i}C_{21}^* + \mathrm{i}\xi C_{21}, [\mathrm{i}P_1, D_{22}]_\diamond^\xi = 0$$

and Claim 2, we have

$$\begin{aligned}
&[\varphi(\mathrm{i}P_1), A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi + [\mathrm{i}P_1, \varphi(A_{11} + B_{12} + C_{21} + D_{22})]_\diamond^\xi \\
&=\varphi([\mathrm{i}P_1, A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi) \\
&=\varphi(\mathrm{i}A_{11}^* + \mathrm{i}\xi A_{11} + \mathrm{i}C_{21}^* + \mathrm{i}\xi C_{21}) \\
&=\varphi(\mathrm{i}A_{11}^* + \mathrm{i}\xi A_{11}) + \varphi(\mathrm{i}C_{21}^* + \mathrm{i}\xi C_{21}) \\
&=\varphi([\mathrm{i}P_1, A_{11}]_\diamond^\xi) + \varphi([\mathrm{i}P_1, B_{12}]_\diamond^\xi) + \varphi([\mathrm{i}P_1, C_{21}]_\diamond^\xi) + \varphi([\mathrm{i}P_1, D_{22}]_\diamond^\xi) \\
&=[\varphi(\mathrm{i}P_1), A_{11} + B_{12} + C_{21} + D_{22}]_\diamond^\xi + [\mathrm{i}P_1, \varphi(A_{11}) + \varphi(B_{12}) + \varphi(C_{21}) + \varphi(D_{22})]_\diamond^\xi.
\end{aligned}$$

It follows that $[\mathrm{i}P_1, T]_\diamond^\xi = \mathrm{i}P_1T^* + \mathrm{i}\xi TP_1 = 0$, and so

$$P_1T^* + \xi TP_1 = 0. \tag{2.10}$$

From Eq. (2.9) and Eq. (2.10), we obtain that $T_{11} = T_{21} = 0$. Similarly, we can show that $T_{12} = T_{22} = 0$. Consequently, $T = 0$.

Claim 4 *For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$, we have*

$$\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij}).$$

Let $T = \varphi(A_{ij} + B_{ij}) - \varphi(A_{ij}) - \varphi(B_{ij})$. It follows from $[P_i, A_{ij}]_\diamond^\xi = 0$ that

$$\begin{aligned}
[\varphi(P_i), A_{ij} + B_{ij}]_\diamond^\xi + [P_i, \varphi(A_{ij} + B_{ij})]_\diamond^\xi &=\varphi([P_i, A_{ij} + B_{ij}]_\diamond^\xi) \\
&=\varphi([P_i, A_{ij}]_\diamond^\xi) + \varphi([P_i, B_{ij}]_\diamond^\xi) \\
&=[\varphi(P_i), A_{ij} + B_{ij}]_\diamond^\xi + [P_i, \varphi(A_{ij}) + \varphi(B_{ij})]_\diamond^\xi,
\end{aligned}$$

which implies that

$$[P_i, T]_\diamond^\xi = P_i T^* - \xi T P_i = 0. \tag{2.11}$$

From $[\mathrm{i}P_i, A_{ij}]_\diamond^\xi = 0$, we have

$$\begin{aligned}[\varphi(\mathrm{i}P_i), A_{ij} + B_{ij}]_\diamond^\xi + [\mathrm{i}P_i, \varphi(A_{ij} + B_{ij})]_\diamond^\xi =&\varphi([\mathrm{i}P_i, A_{ij} + B_{ij}]_\diamond^\xi)\\ =&\varphi([\mathrm{i}P_i, A_{ij}]_\diamond^\xi) + \varphi([\mathrm{i}P_i, B_{ij}]_\diamond^\xi)\\ =&[\varphi(\mathrm{i}P_i), A_{ij} + B_{ij}]_\diamond^\xi + [\mathrm{i}P_i, \varphi(A_{ij}) + \varphi(B_{ij})]_\diamond^\xi.\end{aligned}$$

It follows that $[\mathrm{i}P_i, T]_\diamond^\xi = 0$. Then

$$P_i T^* + \xi T P_i = 0. \tag{2.12}$$

From Eq. (2.11) and Eq. (2.12), we obtain that $T_{ii} = T_{ji} = 0$. Let

$$H_{A_{ij},B_{ij}} = T_{ij}, F_{A_{ij},B_{ij}} = T_{jj}.$$

Then $H_{A_{ij},B_{ij}} \in \mathcal{A}_{ij}, F_{A_{ij},B_{ij}} \in \mathcal{A}_{jj}$, and so

$$\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij}) + H_{A_{ij},B_{ij}} + F_{A_{ij},B_{ij}}. \tag{2.13}$$

Similarly, for $A_{ji}, B_{ji} \in \mathcal{A}_{ji}$ with $1 \le i \ne j \le 2$, there exist $K_{A_{ji},B_{ji}} \in \mathcal{A}_{ii}$ and $G_{A_{ji},B_{ji}} \in \mathcal{A}_{ji}$ such that

$$\varphi(A_{ji} + B_{ji}) = \varphi(A_{ji}) + \varphi(B_{ji}) + K_{A_{ji},B_{ji}} + G_{A_{ji},B_{ji}}. \tag{2.14}$$

By Eq. (2.14), there exist $K_{-\xi A_{ij}^*,-\xi B_{ij}^*} \in \mathcal{A}_{ii}$ and $G_{-\xi A_{ij}^*,-\xi B_{ij}^*} \in \mathcal{A}_{ji}$ such that

$$\varphi(-\xi A_{ij}^* - \xi B_{ij}^*) = \varphi(-\xi A_{ij}^*) + \varphi(-\xi B_{ij}^*) + K_{-\xi A_{ij}^*,-\xi B_{ij}^*} + G_{-\xi A_{ij}^*,-\xi B_{ij}^*}. \tag{2.15}$$

Since

$$[P_i + A_{ij}, P_j + B_{ij}^*]_\diamond^\xi = A_{ij} + B_{ij} - \xi A_{ij}^* - \xi B_{ij}^*,$$

we have from Claim 3 and Eq. (2.15) that

$$\begin{aligned}&\varphi(A_{ij} + B_{ij}) + \varphi(-\xi A_{ij}^*) + \varphi(-\xi B_{ij}^*) + K_{-\xi A_{ij}^*,-\xi B_{ij}^*} + G_{-\xi A_{ij}^*,-\xi B_{ij}^*}\\ =&\varphi(A_{ij} + B_{ij}) + \varphi(-\xi A_{ij}^* - \xi B_{ij}^*)\\ =&\varphi([P_i + A_{ij}, P_j + B_{ij}^*]_\diamond^\xi)\\ =&[\varphi(P_i) + \varphi(A_{ij}), P_j + B_{ij}^*]_\diamond^\xi + [P_i + A_{ij}, \varphi(P_j) + \varphi(B_{ij}^*)]_\diamond^\xi\\ =&\varphi([P_i, P_j]_\diamond^\xi) + \varphi([P_i, B_{ij}^*]_\diamond^\xi) + \varphi([A_{ij}, P_j]_\diamond^\xi) + \varphi([A_{ij}, B_{ij}^*]_\diamond^\xi)\\ =&\varphi(B_{ij} - \xi B_{ij}^*) + \varphi(A_{ij} - \xi A_{ij}^*)\\ =&\varphi(A_{ij}) + \varphi(B_{ij}) + \varphi(-\xi A_{ij}^*) + \varphi(-\xi B_{ij}^*),\end{aligned}$$

which implies that

$$\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij}) - K_{-\xi A_{ij}^*,-\xi B_{ij}^*} - G_{-\xi A_{ij}^*,-\xi B_{ij}^*}. \tag{2.16}$$

It follows from Eq. (2.13) and Eq. (2.16) that

$$H_{A_{ij},B_{ij}} + F_{A_{ij},B_{ij}} = -K_{-\xi A_{ij}^*,-\xi B_{ij}^*} - G_{-\xi A_{ij}^*,-\xi B_{ij}^*}.$$

This together with the fact that $H_{A_{ij},B_{ij}} \in \mathcal{A}_{ij}, F_{A_{ij},B_{ij}} \in \mathcal{A}_{jj}, K_{-\xi A_{ij}^*,-\xi B_{ij}^*} \in \mathcal{A}_{ii}$ and $G_{-\xi A_{ij}^*,-\xi B_{ij}^*} \in \mathcal{A}_{ji}$ yields that $H_{A_{ij},B_{ij}} = F_{A_{ij},B_{ij}} = 0$. Hence by Eq. (2.13), $\varphi(A_{ij} + B_{ij}) = \varphi(A_{ij}) + \varphi(B_{ij})$.

Claim 5 *For every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$, we have*

$$\varphi(A_{ii} + B_{ii}) = \varphi(A_{ii}) + \varphi(B_{ii}).$$

Assume $1 \leq i \neq j \leq 2$. Let

$$T = \varphi(A_{ii} + B_{ii}) - \varphi(A_{ii}) - \varphi(B_{ii}).$$

It follows from $[P_j, A_{ii}]_\diamond^\xi = 0$ that

$$\begin{aligned}[\varphi(P_j), A_{ii} + B_{ii}]_\diamond^\xi + [P_j, \varphi(A_{ii} + B_{ii})]_\diamond^\xi =&\varphi([P_j, A_{ii} + B_{ii}]_\diamond^\xi)\\ =&\varphi([P_j, A_{ii}]_\diamond^\xi) + \varphi([P_j, B_{ii}]_\diamond^\xi)\\ =&[\varphi(P_j), A_{ii} + B_{ii}]_\diamond^\xi + [P_j, \varphi(A_{ii}) + \varphi(B_{ii})]_\diamond^\xi.\end{aligned}$$

Then

$$[P_j, T]_\diamond^\xi = P_j T^* - \xi T P_j = 0. \tag{2.17}$$

Since $[\mathrm{i}P_j, A_{ii}]_\diamond^\xi = 0$, we have

$$\begin{aligned}[\varphi(\mathrm{i}P_j), A_{ii} + B_{ii}]_\diamond^\xi + [\mathrm{i}P_j, \varphi(A_{ii} + B_{ii})]_\diamond^\xi =&\varphi([\mathrm{i}P_j, A_{ii} + B_{ii}]_\diamond^\xi)\\ =&\varphi([\mathrm{i}P_j, A_{ii}]_\diamond^\xi) + \varphi([\mathrm{i}P_j, B_{ii}]_\diamond^\xi)\\ =&[\varphi(\mathrm{i}P_j), A_{ii} + B_{ii}]_\diamond^\xi + [\mathrm{i}P_j, \varphi(A_{ii}) + \varphi(B_{ii})]_\diamond^\xi.\end{aligned}$$

This implies that $[\mathrm{i}P_j, T]_\diamond^\xi = 0$, and so

$$P_j T^* + \xi T P_j = 0. \tag{2.18}$$

It follows from Eq. (2.17) and Eq. (2.18) that $T_{ij} = T_{jj} = 0$.

Since

$$[X_{ji}, A_{ii} + B_{ii}]_\diamond^\xi = X_{ji}A_{ii}^* + X_{ji}B_{ii}^* - \xi A_{ii}X_{ji}^* - \xi B_{ii}X_{ji}^*,$$

we have from Claim 3 and Claim 4 that

$$\begin{aligned}[\varphi(X_{ji}), A_{ii} + B_{ii}]_\diamond^\xi + [X_{ji}, \varphi(A_{ii} + B_{ii})]_\diamond^\xi =&\varphi([X_{ji}, A_{ii} + B_{ii}]_\diamond^\xi)\\ =&\varphi(X_{ji}A_{ii}^* - \xi A_{ii}X_{ji}^*) + \varphi(X_{ji}B_{ii}^* - \xi B_{ii}X_{ji}^*)\\ =&\varphi([X_{ji}, A_{ii}]_\diamond^\xi) + \varphi([X_{ji}, B_{ii}]_\diamond^\xi)\\ =&[\varphi(X_{ji}), A_{ii} + B_{ii}]_\diamond^\xi + [X_{ji}, \varphi(A_{ii}) + \varphi(B_{ii})]_\diamond^\xi.\end{aligned}$$

This implies that

$$[X_{ji}, T]_\diamond^\xi = X_{ji}T^* - \xi T X_{ji}^* = 0. \tag{2.19}$$

From

$$[\mathrm{i}X_{ji}, A_{ii} + B_{ii}]_\diamond^\xi = \mathrm{i}X_{ji}A_{ii}^* + \mathrm{i}X_{ji}B_{ii}^* + \mathrm{i}\xi A_{ii}X_{ji}^* + \mathrm{i}\xi B_{ii}X_{ji}^*,$$

Claim 3 and Claim 4, we have

$$\begin{aligned}[\varphi(\mathrm{i}X_{ji}), A_{ii} + B_{ii}]_\diamond^\xi + [\mathrm{i}X_{ji}, \varphi(A_{ii} + B_{ii})]_\diamond^\xi =&\varphi([\mathrm{i}X_{ji}, A_{ii} + B_{ii}]_\diamond^\xi)\\ =&\varphi(\mathrm{i}X_{ji}A_{ii}^* + \mathrm{i}\xi A_{ii}X_{ji}^*) + \varphi(\mathrm{i}X_{ji}B_{ii}^* + \mathrm{i}\xi B_{ii}X_{ji}^*)\\ =&\varphi([\mathrm{i}X_{ji}, A_{ii}]_\diamond^\xi) + \varphi([\mathrm{i}X_{ji}, B_{ii}]_\diamond^\xi)\\ =&[\varphi(\mathrm{i}X_{ji}), A_{ii} + B_{ii}]_\diamond^\xi + [\mathrm{i}X_{ji}, \varphi(A_{ii}) + \varphi(B_{ii})]_\diamond^\xi.\end{aligned}$$

It follows that $[\mathrm{i}X_{ji}, T]_\diamond^\xi = 0$, and so

$$X_{ji}T^* + \xi T X_{ji}^* = 0. \tag{2.20}$$

From Eq. (2.19), Eq. (2.20) and using the primeness of $\mathcal{A}$, we can see that $T_{ii} = T_{ji} = 0$. Consequently, $T = 0$.

Claim 6 *φ is additive on $\mathcal{A}$.*

For every $A, B \in \mathcal{A}$, we have $A = \sum_{i,j=1}^{2} A_{ij}$ and $B = \sum_{i,j=1}^{2} B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. It follows from Claim 3-5 that

$$\varphi(A+B) = \sum_{i,j=1}^{2} \varphi(A_{ij}+B_{ij}) = \sum_{i,j=1}^{2} \varphi(A_{ij}) + \sum_{i,j=1}^{2} \varphi(B_{ij}) = \varphi(A) + \varphi(B).$$

Hence φ is additive.

Claim 7 $\varphi(I) = 0$.

Since $[P_1, P_2]_\diamond^\xi = 0$, we have

$$0 = \varphi([P_1, P_2]_\diamond^\xi) = [\varphi(P_1), P_2]_\diamond^\xi + [P_1, \varphi(P_2)]_\diamond^\xi = \varphi(P_1)P_2 - \xi P_2\varphi(P_1)^* + P_1\varphi(P_2)^* - \xi\varphi(P_2)P_1. \tag{2.21}$$

Multiplying Eq. (2.21) by P_2 from both sides, we get

$$P_2\varphi(P_1)P_2 - \xi P_2\varphi(P_1)^*P_2 = 0. \tag{2.22}$$

Similarly,

$$0 = \varphi([P_2, P_1]_\diamond^\xi) = [\varphi(P_2), P_1]_\diamond^\xi + [P_2, \varphi(P_1)]_\diamond^\xi = \varphi(P_2)P_1 - \xi P_1\varphi(P_2)^* + P_2\varphi(P_1)^* - \xi\varphi(P_1)P_2. \tag{2.23}$$

Multiplying Eq. (2.23) by P_2 from both sides, we get

$$P_2\varphi(P_1)^*P_2 - \xi P_2\varphi(P_1)P_2 = 0. \tag{2.24}$$

It follows from Eq. (2.22) and Eq. (2.24) that

$$\xi P_2\varphi(P_1)^*P_2 = P_2\varphi(P_1)P_2 = \frac{1}{\xi}P_2\varphi(P_1)^*P_2.$$

If $P_2\varphi(P_1)^*P_2 \neq 0$, then $\xi = \frac{1}{\xi}$, and so $\xi = \pm 1$, which is a contradiction. Hence $P_2\varphi(P_1)P_2 = 0$. Similarly, we can show that $P_1\varphi(P_2)P_1 = 0$.

For any $X_{12} \in \mathcal{A}_{12}$, since $[P_1, X_{12}]_\diamond^\xi = 0$, we have

$$0 = \varphi([P_1, X_{12}]_\diamond^\xi) = [\varphi(P_1), X_{12}]_\diamond^\xi + [P_1, \varphi(X_{12})]_\diamond^\xi = \varphi(P_1)X_{12}^* - \xi X_{12}\varphi(P_1)^* + P_1\varphi(X_{12})^* - \xi\varphi(X_{12})P_1. \tag{2.25}$$

Multiplying Eq. (2.25) by P_1 from the left and by P_2 from the right, we get $-\xi X_{12}\varphi(P_1)^*P_2 + P_1\varphi(X_{12})^*P_2 = 0$. This together with the fact that $P_2\varphi(P_1)P_2 = 0$ yields that $P_2\varphi(X_{12})P_1 = 0$.

From $[P_1, X_{21}]_\diamond^\xi = X_{21}^* - \xi X_{21}$ and the additivity of φ, we have

$$\begin{aligned}\varphi(X_{21}^*) - \varphi(\xi X_{21}) =& \varphi([P_1, X_{21}]_\diamond^\xi) \\ =& [\varphi(P_1), X_{21}]_\diamond^\xi + [P_1, \varphi(X_{21})]_\diamond^\xi \\ =& \varphi(P_1)X_{21}^* - \xi X_{21}\varphi(P_1)^* + P_1\varphi(X_{21})^* - \xi\varphi(X_{21})P_1.\end{aligned} \tag{2.26}$$

Multiplying Eq. (2.26) by P_2 from the left and by P_1 from the right, then using the fact that $P_2\varphi(X_{12})P_1 = 0$, we get

$$P_2\varphi(\xi X_{21})P_1 = \xi X_{21}\varphi(P_1)^*P_1 + \xi P_2\varphi(X_{21})P_1. \tag{2.27}$$

From $[P_2, X_{12}]_\diamond^\xi = X_{12}^* - \xi X_{12}$, we have

$$\begin{aligned}\varphi(X_{12}^*) - \varphi(\xi X_{12}) =& \varphi([P_2, X_{12}]_\diamond^\xi) \\ =& [\varphi(P_2), X_{12}]_\diamond^\xi + [P_2, \varphi(X_{12})]_\diamond^\xi \\ =& \varphi(P_2)X_{12}^* - \xi X_{12}\varphi(P_2)^* + P_2\varphi(X_{12})^* - \xi\varphi(X_{12})P_2.\end{aligned} \tag{2.28}$$

Multiplying Eq. (2.28) by P_2 from the left and by P_1 from the right, and using the fact that $P_2\varphi(X_{12})P_1 = 0$, we have

$$P_2\varphi(X_{12}^*)P_1 = P_2\varphi(P_2)X_{12}^* + P_2\varphi(X_{12})^*P_1. \tag{2.29}$$

It follows from $[X_{12}, P_2]^{\xi}_{\diamond} = X_{12} - \xi X^*_{12}$ that

$$\begin{aligned}\varphi(X_{12}) - \varphi(\xi X^*_{12}) =&\varphi([X_{12}, P_2]^{\xi}_{\diamond})\\ =&[\varphi(X_{12}), P_2]^{\xi}_{\diamond} + [X_{12}, \varphi(P_2)]^{\xi}_{\diamond}\\ =&\varphi(X_{12})P_2 - \xi P_2\varphi(X_{12})^* + X_{12}\varphi(P_2)^* - \xi\varphi(P_2)X^*_{12}.\end{aligned} \tag{2.30}$$

Multiplying Eq. (2.30) by P_2 from the left and by P_1 from the right, we have

$$P_2\varphi(\xi X^*_{12})P_1 = \xi P_2\varphi(X_{12})^*P_1 + \xi P_2\varphi(P_2)X^*_{12}.$$

This together with Eq. (2.27) gives

$$\xi P_2\varphi(X_{12})^*P_1 + \xi P_2\varphi(P_2)X^*_{12} = P_2\varphi(\xi X^*_{12})P_1 = \xi X^*_{12}\varphi(P_1)^*P_1 + \xi P_2\varphi(X^*_{12})P_1.$$

It follows that

$$P_2\varphi(X_{12})^*P_1 + P_2\varphi(P_2)X^*_{12} = X^*_{12}\varphi(P_1)^*P_1 + P_2\varphi(X^*_{12})P_1. \tag{2.31}$$

Comparing Eq. (2.29) and Eq. (2.31), we obtain that $X^*_{12}\varphi(P_1)^*P_1 = 0$ for all $X_{12} \in \mathcal{A}_{12}$, which implies $P_1\varphi(P_1)P_1 = 0$ by the primeness of $\mathcal{A}$. Similarly, we can show that $P_2\varphi(P_2)P_2 = 0$.

Since $[P_1, P_1]^{\xi}_{\diamond} = (1 - \xi)P_1$, we have

$$\begin{aligned}\varphi((1 - \xi)P_1) =&\varphi([P_1, P_1]^{\xi}_{\diamond})\\ =&[\varphi(P_1), P_1]^{\xi}_{\diamond} + [P_1, \varphi(P_1)]^{\xi}_{\diamond}\\ =&\varphi(P_1)P_1 - \xi P_1\varphi(P_1)^* + P_1\varphi(P_1)^* - \xi\varphi(P_1)P_1.\end{aligned} \tag{2.32}$$

It follows from Eq. (2.32) and $[(1 - \xi)P_1, P_2]^{\xi}_{\diamond} = 0$ that

$$\begin{aligned}0 =&\varphi([(1 - \xi)P_1, P_2]^{\xi}_{\diamond})\\ =&[\varphi((1 - \xi)P_1), P_2]^{\xi}_{\diamond} + [(1 - \xi)P_1, \varphi(P_2)]^{\xi}_{\diamond}\\ =&(\varphi(P_1)P_1 - \xi P_1\varphi(P_1)^* + P_1\varphi(P_1)^* - \xi\varphi(P_1)P_1)P_2 - \xi P_2(\varphi(P_1)P_1 - \xi P_1\varphi(P_1)^* + P_1\varphi(P_1)^* - \xi\varphi(P_1)P_1)^*\\ &+ (1 - \xi)P_1\varphi(P_2)^* - \xi(1 - \overline{\xi})\varphi(P_2)P_1\\ =& - \xi P_1\varphi(P_1)^*P_2 + P_1\varphi(P_1)^*P_2 + \xi\overline{\xi}P_2\varphi(P_1)P_1 - \xi P_2\varphi(P_1)P_1 + (1 - \xi)P_1\varphi(P_2)^*\\ &- \xi(1 - \overline{\xi})\varphi(P_2)P_1.\end{aligned} \tag{2.33}$$

Multiplying Eq. (2.33) by P_1 from the left and by P_2 from the right, we have

$$(1 - \xi)P_1\varphi(P_1)^*P_2 + (1 - \xi)P_1\varphi(P_2)^*P_2 = 0.$$

Then

$$P_2\varphi(P_1)P_1 + P_2\varphi(P_2)P_1 = 0. \tag{2.34}$$

Similarly, we can obtain that

$$P_1\varphi(P_2)P_2 + P_1\varphi(P_1)P_2 = 0. \tag{2.35}$$

By the fact that $P_2\varphi(P_1)P_2 = P_1\varphi(P_2)P_1 = P_1\varphi(P_1)P_1 = P_2\varphi(P_2)P_2 = 0$, Eq. (2.34) and Eq. (2.35), we have

$$\varphi(I) = \varphi(P_1) + \varphi(P_2) = P_1\varphi(P_1)P_2 + P_2\varphi(P_1)P_1 + P_1\varphi(P_2)P_2 + P_2\varphi(P_2)P_1 = 0.$$

Claim 8 *$\varphi(M)^* = \varphi(M)$ for all $M \in \mathcal{M}$.*

Let $M \in \mathcal{M}$. Then $[I, M]^{\xi}_{\diamond} = M - \xi M$. It follows from Claim 7 that

$$\varphi(M) - \varphi(\xi M) = \varphi([I, M]^{\xi}_{\diamond}) = [\varphi(I), M]^{\xi}_{\diamond} + [I, \varphi(M)]^{\xi}_{\diamond} = \varphi(M)^* - \xi\varphi(M), \tag{2.36}$$

which gives that

$$\varphi(\xi M) = -\varphi(M)^* + (1+\xi)\varphi(M). \tag{2.37}$$

From Claim 7 and $[M, I]_\diamond^\xi = M - \xi M$, we have

$$\varphi(M) - \varphi(\xi M) = \varphi([M, I]_\diamond^\xi) = [\varphi(M), I]_\diamond^\xi + [M, \varphi(I)]_\diamond^\xi = \varphi(M) - \xi\varphi(M)^*.$$

It follows that

$$\varphi(\xi M) = \xi\varphi(M)^*. \tag{2.38}$$

By Eq. (2.37) and Eq. (2.38), we obtain that $(1+\xi)\varphi(M)^* = (1+\xi)\varphi(M)$. Consequently, $\varphi(M)^* = \varphi(M)$.

Claim 9 $\varphi(N)^* = -\varphi(N)$ *for all* $N \in \mathcal{N}$.

Let $N \in \mathcal{N}$. Then $[N, I]_\diamond^\xi = N + \xi N$. By Claim 7, we have

$$\varphi(N) + \varphi(\xi N) = \varphi([N, I]_\diamond^\xi) = [\varphi(N), I]_\diamond^\xi = \varphi(N) - \xi\varphi(N)^*. \tag{2.39}$$

It follows from Claim 7 and $[I, N]_\diamond^\xi = -N - \xi N$ that

$$-\varphi(N) - \varphi(\xi N) = \varphi([I, N]_\diamond^\xi) = [I, \varphi(N)]_\diamond^\xi = \varphi(N)^* - \xi\varphi(N). \tag{2.40}$$

By Eq. (2.39) and Eq. (2.40), we get $(1-\xi)(\varphi(N) + \varphi(N)^*) = 0$. Hence $\varphi(N)^* = -\varphi(N)$.

Claim 10 $\varphi(\mathrm{i}I) = 0$.

On the one hand, it follows from $\mathrm{i}I \in \mathcal{N}$ and Claim 9 that

$$\varphi((1-\xi)I) = \varphi([\mathrm{i}I, \mathrm{i}I]_\diamond^\xi) = [\varphi(\mathrm{i}I), \mathrm{i}I]_\diamond^\xi + [\mathrm{i}I, \varphi(\mathrm{i}I)]_\diamond^\xi = -2\mathrm{i}(1-\xi)\varphi(\mathrm{i}I). \tag{2.41}$$

On the other hand, it follows from Claim 7 that

$$\varphi((1-\xi)I) = \varphi([I, I]_\diamond^\xi) = [\varphi(I), I]_\diamond^\xi + [I, \varphi(I)]_\diamond^\xi = 0. \tag{2.42}$$

Comparing Eq. (2.41) and Eq. (2.42), we obtain that $\varphi(\mathrm{i}I) = 0$.

Claim 11 $\varphi(\mathrm{i}A) = \mathrm{i}\varphi(A)$ *for all* $A \in \mathcal{A}$.

Let $M \in \mathcal{M}$. Then by Eq. (2.36) and Claim 8, we have

$$\varphi(\xi M) = \xi\varphi(M) \tag{2.43}$$

for all $M \in \mathcal{M}$. Since $[\mathrm{i}I, \mathrm{i}M]_\diamond^\xi = M - \xi M$, we have from Eq. (2.43), Claim 9 and Claim 10 that

$$\varphi(M) - \xi\varphi(M) = \varphi([\mathrm{i}I, \mathrm{i}M]_\diamond^\xi) = [\mathrm{i}I, \varphi(\mathrm{i}M)]_\diamond^\xi = -\mathrm{i}\varphi(\mathrm{i}M) + \mathrm{i}\xi\varphi(\mathrm{i}M).$$

Consequently, we get

$$\varphi(\mathrm{i}M) = \mathrm{i}\varphi(M) \tag{2.44}$$

for all $M \in \mathcal{M}$. Let $A \in \mathcal{A}$. Then $A = H + \mathrm{i}K$, where $H, K \in \mathcal{M}$. Thus, we have from Eq. (2.44) that

$$\varphi(\mathrm{i}A) = \varphi(\mathrm{i}H) - \varphi(K) = \mathrm{i}\varphi(H) + \mathrm{i}\varphi(\mathrm{i}K) = \mathrm{i}\varphi(H + \mathrm{i}K) = \mathrm{i}\varphi(A).$$

Claim 12 $\varphi(A^*) = \varphi(A)^*$ *for all* $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$. Then $A = H + \mathrm{i}K$, where $H, K \in \mathcal{M}$. It follows from Claim 8 and Claim 11 that

$$\varphi(A^*) = \varphi(H) - \mathrm{i}\varphi(K) = (\varphi(H) + \mathrm{i}\varphi(K))^* = \varphi(H + \mathrm{i}K)^* = \varphi(A)^*.$$

Claim 13 $\varphi(\xi A) = \xi\varphi(A)$ *for all* $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$. Since $[I, A]_\diamond^\xi = A^* - \xi A$, we have from Claim 7 that

$$\varphi(A^*) - \varphi(\xi A) = \varphi([I, A]_\diamond^\xi) = [I, \varphi(A)]_\diamond^\xi = \varphi(A)^* - \xi\varphi(A).$$

This together with Claim 12 yields that $\varphi(\xi A) = \xi\varphi(A)$.
Claim 14 *φ is an additive $*$-derivation on $\mathcal{A}$.*

By Claim 6 and Claim 12, we only need to prove that $\varphi(AB) = \varphi(A)B + A\varphi(B)$ for all $A, B \in \mathcal{A}$. Let $H, K \in \mathcal{M}$. Then $[H,K]_\diamond^\xi = HK - \xi KH$. Thus, we have from Claim 12 and Claim 13 that

$$\begin{aligned}\varphi(HK) - \xi\varphi(KH) =&\varphi([H,K]_\diamond^\xi)\\ =&[\varphi(H),K]_\diamond^\xi + [H,\varphi(K)]_\diamond^\xi\\ =&\varphi(H)K - \xi K\varphi(H) + H\varphi(K) - \xi\varphi(K)H.\end{aligned} \tag{2.45}$$

Since $[H, \mathrm{i}K]_\diamond^\xi = -\mathrm{i}HK - \mathrm{i}\xi KH$, we have from Claim 11-13 that

$$\begin{aligned}-\mathrm{i}\varphi(HK) - \mathrm{i}\xi\varphi(KH) =&\varphi([H,\mathrm{i}K]_\diamond^\xi)\\ =&[\varphi(H),\mathrm{i}K]_\diamond^\xi + [H,\varphi(\mathrm{i}K)]_\diamond^\xi\\ =&-\mathrm{i}\varphi(H)K - \mathrm{i}\xi K\varphi(H) - \mathrm{i}H\varphi(K) - \mathrm{i}\xi\varphi(K)H,\end{aligned}$$

which implies that

$$\varphi(HK) + \xi\varphi(KH) = \varphi(H)K + \xi K\varphi(H) + H\varphi(K) + \xi\varphi(K)H. \tag{2.46}$$

It follows from Eq. (2.45) and Eq. (2.46) that

$$\varphi(HK) = \varphi(H)K + H\varphi(K) \tag{2.47}$$

for all $H, K \in \mathcal{M}$.

For any $A, B \in \mathcal{A}$, we have $A = A_1 + \mathrm{i}A_2, B = B_1 + \mathrm{i}B_2$, where $A_i, B_i \in \mathcal{M}$ $(i = 1, 2)$. It follows from Eq. (2.47) and Claim 11 that

$$\begin{aligned}\varphi(AB) =&\varphi(A_1B_1 + \mathrm{i}A_1B_2 + \mathrm{i}A_2B_1 - A_2B_2)\\ =&\varphi(A_1B_1) + \mathrm{i}\varphi(A_1B_2) + \mathrm{i}\varphi(A_2B_1) - \varphi(A_2B_2)\\ =&\varphi(A_1)B_1 + A_1\varphi(B_1) + \mathrm{i}\varphi(A_1)B_2 + \mathrm{i}A_1\varphi(B_2) + \mathrm{i}\varphi(A_2)B_1 + \mathrm{i}A_2\varphi(B_1) - \varphi(A_2)B_2 - A_2\varphi(B_2)\\ =&(\varphi(A_1) + \mathrm{i}\varphi(A_2))(B_1 + \mathrm{i}B_2) + (A_1 + \mathrm{i}A_2)(\varphi(B_1) + \mathrm{i}\varphi(B_2))\\ =&\varphi(A)B + A\varphi(B).\end{aligned}$$

Therefore, φ is an additive $*$-derivation on $\mathcal{A}$. □

Let H be a complex Hilbert space, $B(H)$ be the algebra of all bounded linear operators on H, and $\mathcal{A} \subseteq B(H)$ be a von Neumann algebra. Recall that $\mathcal{A}$ is a factor if its center contains only the scalar operators. It's well known that $\mathcal{A}$ is prime.

Corollary 2.2. *Let $\mathcal{A}$ be a factor von Neumann algebra acting on a complex Hilbert space H with $\dim\mathcal{A} > 1$ and ξ be a nonzero scalar. A map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $\varphi([A,B]_\diamond^\xi) = [\varphi(A),B]_\diamond^\xi + [A,\varphi(B)]_\diamond^\xi$ for all $A, B \in \mathcal{A}$ if and only if φ is an additive $*$-derivation and $\varphi(\xi A) = \xi\varphi(A)$ for all $A \in \mathcal{A}$.*

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