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New characterizations of the DMP inverse of matrices and its applications

M. El Bilali[a,*], M. Mouçouf[a]

[a]Chouaib Doukkali University, Faculty of sciences, Department of Mathematics,
Laboratory Fundamental Mathematics and their Applications, 24000 El Jadida, Morocco

Abstract. In this paper, we present some characterizations and several new properties of DMP inverse of a square matrix. We also consider some characterizations of the nonsingularity of matrices. Using the concept of DMP inverse to find a general solution for certain types of matrix equations.

1. Introduction and Notation.

Let $\mathbb{C}^{n\times m}$ and $\mathbb{N}$ denote the set of all $n \times m$ complex matrices and the set of all positive integers, respectively. The symbol I_n means the identity matrix in $\mathbb{C}^{n\times n}$. For $A \in \mathbb{C}^{n\times n}$, the symbols A^*, $rank(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ will stand for the conjugate transpose, the rank, the range space and the null space of A, respectively.

Recall that the smallest nonnegative integer k such that $rank(A^k) = rank(A^{k+1})$ is called the index of $A \in \mathbb{C}^{n\times n}$ and is denoted by $ind(A)$. The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^d \in \mathbb{C}^{n\times n}$ such that (see [2, 3]):

$$AA^d = A^dA, \quad A^dAA^d = A^d, \quad A^{k+1}A^d = A^k.$$

In addition, we denote $A^\pi = I_n - AA^d$ for any matrix $A \in \mathbb{C}^{n\times n}$.

The Drazin inverse of a square matrix is widely applied in many fields, such as singular differential or difference equations, Markov chains, iterative method and numerical analysis, which can be found in (see [2–4]). For this reason, M. Mouçouf and S. Zriaa [14, 18] studied the explicit formulas of the Drazin inverse of matrices and its nth powers.

For $A \in \mathbb{C}^{n\times m}$, the Moore–Penrose inverse of A is the unique matrix $A^\dagger \in \mathbb{C}^{m\times n}$ satisfying the following four equations (see [2, 3]):

$$A^\dagger AA^\dagger = A^\dagger, \quad AA^\dagger A = A, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$

The well-known class of EP matrices is defined by the square complex matrix A that commutes with its Moore–Penrose inverse $A^\dagger$, that is (see [2, 3]):

$$AA^\dagger = A^\dagger A.$$

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* Corresponding author: M. El Bilali
Email addresses: elbilali.mohammed@ucd.ac.ma (M. El Bilali), moucouf@hotmail.com (M. Mouçouf)
ORCID iD: https://orcid.org/0000-0002-3601-5525 (M. Mouçouf)

For $A \in \mathbb{C}^{n\times n}$, a matrix $X \in \mathbb{C}^{n\times n}$ satisfying $AXA = A$ is called an inner inverse of A and we denoted by $A\{1\} = \{X \in \mathbb{C}^{n\times n}, AXA = A\}$ all inner inverses of A. A matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X$ is called an outer inverse of A.

In 2014, Mallik and Thome (see [8]) introduced the concept of DMP inverse of a square matrix A of arbitrary index using Drazin inverse and Moore- Penrose inverse. In this case, for $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$, the unique matrix $G \in \mathbb{C}^{n\times n}$ satisfying

$$GAG = G, \quad A^kG = A^kA^\dagger, \quad GA = A^dA,$$

is called the DMP inverse of A and is denoted by $A^{d,\dagger}$. Moreover, it was proved that $A^{d,\dagger} = A^dAA^\dagger$. The authors introduced also another inverse associated to a square matrix, namely $A^{\dagger,d} = A^\dagger AA^d$ called dual DMP inverse of A.

A great popularity of the DMP inverse is confirmed by many recent published papers. In order to improve our motivation, we will present a short survey of main articles aimed to the DMP inverse. In [19], different characterizations of DMP inverse of matrices. Ferreyra et al. [5] were studied maximal classes of matrices determining the DMP inverse by Hartwig–Spindelböck decomposition of matrix. More details of the Hartwig–Spindelböck decomposition (see [6]). The DMP inverse for a Hilbert space operator was studied in [12] an extension of the DMP inverse for a square matrix. Some further extensions of the DMP inverse can be found in [9–11, 13, 15, 17]. A generalization of the DMP inverse for a square matrix was investigated in [7].

In this paper, we present new characterizations, expressions and several properties of the DMP inverse and the nonsingularity of some matrices. Finally, we apply the DMP and dual DMP inverse of matrix in solving some systems of linear equations.

2. Preliminaries

For any matrix $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$ the Hartwig–Spindelböck decomposition is given by

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, ..., \sigma_t I_{r_t})$ is a diagonal matrix, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > ... > \sigma_t > 0$, $r_1 + r_2 + ... + r_t = r$ and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$ (see [6]). Note that if $A \in \mathbb{C}^{n\times n}$ is EP matrix if and only if $L = 0$ (see [1]).
If A is of the form (1), then the DMP inverse of A is as follows (see [8]).

$$A^{d,\dagger} = U\begin{pmatrix} (\Sigma K)^d & 0 \\ 0 & 0 \end{pmatrix}U^*, \quad A^d = U\begin{pmatrix} (\Sigma K)^d & ((\Sigma K)^d)^2\Sigma L \\ 0 & 0 \end{pmatrix}U^* \quad \text{and } A^\dagger = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix}U^*. \tag{2}$$

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$. Denote $G = A^{d,\dagger}$ and $H = A^{\dagger,d}$. Then the following statements hold:*

1. *$AG = A^mG^m$ and $HA = H^mA^m$, for any $m \in \mathbb{N} \setminus \{0\}$.*
2. *$G = A^mG^{m+1}$, for any $m \in \mathbb{N}$.*

Proof. Let $A \in \mathbb{C}^{n\times n}$. Since $AG^2 = G$ and $H^2A = H$. Then we have

1. $AG = A^2G^2 = A^2(AG^2)G = A^3G^3 = ... = A^mG^m$, for any $m \in \mathbb{N} \setminus \{0\}$. Similarly, we have $HA = H^mA^m$, for any $m \in \mathbb{N} \setminus \{0\}$.
2. $A^mG^{m+1} = A^mG^mG = AG^2 = G$, for any $m \in \mathbb{N}$.

□

Lemma 2.2. *Let $A \in \mathbb{C}^{n\times n}$. Denote $G = A^{d,\dagger}$, then the matrix $A^m - A^{m+1}G$ is nilpotent for any $m \in \mathbb{N} \setminus \{0\}$.*

Proof. For $m \in \mathbb{N} \setminus \{0\}$ and $ind(A) = k$, we have
$A^m - A^{m+1}G = A^m(I_n - AG) = A^m P$ with $P = I_n - AG$.
Since $PA^m = A^m A^\pi$, where $A^\pi = I_n - AA^d$. Then, $(A^m P)^k = A^{km} A^\pi P = 0$.
Therefore, $A^m - A^{m+1}G$ is nilpotent for any $m \in \mathbb{N} \setminus \{0\}$. □

Lemma 2.3. *Let $E, F \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C} \setminus \{0\}$ such that $E^2F = E$, $EF^2 = F$ and $(EF)^2 = EF$. Let $H = I_n - EF$, then the matrix $E + zH$ is nonsingular.*

Proof. Since $(E + zH)(F + \frac{1}{z}H) = I_n$, then $E + zH$ is nonsingular. □

Lemma 2.4. *Let $E, F \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C} \setminus \{0\}$ such that $EF^2 = F$, $(EF)^2 = EF$ and EH is nilpotent with $H = I_n - EF$. Then the matrix $E + zH$ is nonsingular.*

Lemma 2.5. *Let $A \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C}$ such that $z(z + 1) \neq 0$, then the matrix $AA^d + zI_n$ is nonsingular and $(AA^d + zI_n)^{-1} = \dfrac{-1}{z(z+1)}AA^d + \dfrac{1}{z}I_n$.*

3. Some characterizations of the DMP inverse

We will give several different characterizations and properties of the DMP inverse of matrix $A \in \mathbb{C}^{n\times n}$.

Property 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Denote $G = A^{d,\dagger}$ and $H = A^{\dagger,d}$. Then the following statements hold:*

1. *G^m and H^m are inner inverse of A^m, for any $m \in \mathbb{N}$ such that $m \geq k$.*
2. *$G^m = (A^d)^m AA^\dagger$ and $H^m = A^\dagger A(A^d)^m$, for any $m \in \mathbb{N} \setminus \{0\}$.*
3. *G^m and H^m are outer inverse of A^m, for any $m \in \mathbb{N} \setminus \{0\}$.*
4. *$G = A^m(A^m)^\dagger G$ and $H = H(A^m)^\dagger A^m$, for any $m \in \mathbb{N}$.*

Proof. Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Denote $G = A^{d,\dagger}$ and $H = A^{\dagger,d}$.

1. For $m \geq k$, we have, $A^m G^m A^m = A^m GA = A^m AA^d = A^{m+1}A^d = A^m$. Similarly, we can also prove $A^m H^m A^m = A^m$. Then, G^m and H^m are inner inverse of A^m.
2. We prove this identity by induction on m. The identity is true for $m = 2$, since

$$G^2 = A^d AA^\dagger AA^d A^\dagger = A^d A^\dagger = (A^d)^2 AA^\dagger.$$

Now assume the identity is true for m, that is, $G^m = (A^d)^m AA^\dagger$. Now

$$\begin{aligned} G^{m+1} &= G.G^m \\ &= A^d AA^\dagger (A^d)^m AA^\dagger \\ &= A^d A(A^d)^m A^\dagger \\ &= (A^d)^{m+1} AA^\dagger. \end{aligned}$$

Therefore,

$$(A^{d,\dagger})^m = (A^d)^m AA^\dagger, \quad m \geq 2.$$

Similarly, we have $(A^{\dagger,d})^m = A^\dagger A(A^d)^m$.

3. For any $m \geq 1$, we have

$$\begin{aligned} G^m A^m G^m &= A^d A(A^d)^m AA^\dagger \\ &= A^2(A^d)^{m+1} A^\dagger \\ &= A(A^d)^m A^\dagger \\ &= G^m. \end{aligned}$$

Similarly, we can also prove $H^m A^m H^m = H^m$. Then, G^m and H^m are outer inverse of A^m, for any $m \in \mathbb{N} \setminus \{0\}$.

4. For $m \geq 0$, we have $A^m(A^m)^\dagger G = A^m(A^m)^\dagger A^m G^{m+1} = A^m G^{m+1} = G$ and $H(A^m)^\dagger A^m = A^\dagger(A^d)^m A^m (A^m)^\dagger A^m = A^\dagger(A^d)^m A^m = H$.

□

Proposition 3.2. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

(1) $A^{d,\dagger} = G$

(2) $GWG = G, \quad WG = WA^\dagger, \quad GW = A^d A$ *with* $W = A^2A^d$.

Proof. (1) ⇒ (2). Suppose that $A^{d,\dagger} = G$. Then

(i) $GWG = GA^2A^dG = GA(AA^dG) = GAG = G$.

(ii) $WG = A^2A^dG = AG = WA^\dagger$.

(iii) $GW = A^dA^2A^d = A^dA$.

(2) ⇒ (1). Let $ind(A) = k$ and $W = A^2A^d$. Since

(i) $A^kG = A^dA^{k+1}G = A^{k-1}WG = A^{k-1}WA^\dagger = A^kA^\dagger$.

(ii) $GA = G(WG)A = GWA^\dagger A = A^dAA^\dagger A = AA^d$.

(iii) $GAG = A^dAG = GWG = G$.

Then, $G = A^{d,\dagger}$. □

We will discuss some equivalent conditions for $A^{d,\dagger}$ and $A^{\dagger,d}$ to be an inner inverse of A.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

i) $A^{d,\dagger}$ *is an inner inverse of* A,

ii) $ind(A) \leq 1$,

iii) $A^{\dagger,d}$ *is an inner inverse of* A.

Proof. *i)* ⟺ *ii)* Since

$$\begin{aligned} AA^{d,\dagger}A = A &\iff AA^dAA^\dagger A = A \\ &\iff AA^dA = A \\ &\iff ind(A) \leq 1. \end{aligned}$$

ii) ⟺ *iii)* Since

$$\begin{aligned} ind(A) \leq 1 &\iff AA^dA = A \\ &\iff AA^\dagger AA^dA = AA^\dagger A \\ &\iff AA^{\dagger,d}A = A. \end{aligned}$$

Therefore, the above conditions are equivalent. □

The following theorem gives some equivalent characterizations.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

i) $A^{d,\dagger}$ *is idempotent,*

ii) A^d *is idempotent,*

iii) $A^{\dagger,d}$ *is idempotent.*

Proof. Since $(A^{d,\dagger})^2 = (A^d)^2AA^\dagger$ and $(A^{\dagger,d})^2 = A^\dagger A(A^d)^2$, we get

i) $\Longleftrightarrow$ *ii)* Since

$$\begin{aligned} (A^{d,\dagger})^2 = A^{d,\dagger} &\Longleftrightarrow (A^d)^2AA^\dagger = A^dAA^\dagger \\ &\Longleftrightarrow A^d = A^dA \\ &\Longleftrightarrow (A^d)^2 = A^d. \end{aligned}$$

ii) $\Longleftrightarrow$ *iii)* Since

$$\begin{aligned} (A^d)^2 = A^d &\Longleftrightarrow A^\dagger A(A^d)^2 = A^\dagger AA^d \\ &\Longleftrightarrow (A^{\dagger,d})^2 = A^{\dagger,d}. \end{aligned}$$

Then, the above conditions are equivalent. □

Remark 3.5. *Note that if $A \in \mathbb{C}^{n\times n}$ is idempotent, then $A^{d,\dagger}$ and $A^{\dagger,d}$ are also idempotent.*

Solving some type of matrix equations, we present the DMP inverse of square matrix.

Theorem 3.6. *Let $A \in \mathbb{C}^{n\times n}$. The system*

$$XAX = X, \quad A^dX = A^dA^\dagger, \quad XA = A^dA, \tag{3}$$

is consistent. It has a unique solution given by $X = A^{d,\dagger}$.

Proof. It is easily seen that $X = A^{d,\dagger}$ is a solution of the system (3). Now we will prove the uniqueness. Suppose there exists X_1 and X_2 which satisfies the equations. Then

$$X_1 = X_1AX_1 = A^dAX_1 = A^dAA^\dagger = A^dAX_2 = X_2AX_2 = X_2.$$

□

Necessary and sufficient conditions for a square matrix to be the DMP inverse are given now.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$ with ind$(A) = k$. Then the following conditions are equivalent:*

i) $G = A^{d,\dagger}$,

ii) *G satisfies the equations*

$$A^dAGAA^\dagger = G \quad \text{and} \quad A^kGA = A^k.$$

Proof. *i)* $\Rightarrow$ *ii)*. For $G = A^{d,\dagger}$ we have,

$$A^dAGAA^\dagger = A^dAA^dAA^\dagger AA^\dagger = A^dAA^\dagger = G.$$

And

$$A^kGA = A^kA^dAA^\dagger A = A^k.$$

ii) $\Rightarrow$ *i)*. We have,

$$\begin{aligned} GA &= A^dAGAA^\dagger A \\ &= A^dAGA \\ &= (A^d)^kA^kGA \\ &= (A^d)^kA^k \\ &= AA^d. \end{aligned}$$

$$\begin{aligned} A^kG &= A^dA^{k+1}GAA^\dagger \\ &= A^kGAA^\dagger \\ &= A^kA^\dagger. \end{aligned}$$

$$\begin{aligned} GAG &= A^dAG \\ &= A^dAA^dAGAA^\dagger \\ &= A^dAGAA^\dagger \\ &= G. \end{aligned}$$

Then, $G = A^dAA^\dagger$. □

In an analogous way to Theorem 3.7. We can state the next theorem which holds for dual DMP inverse of matrix.

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Then the following conditions are equivalent:*

i) $H = A^{\dagger,d}$,

ii) H satisfies the equations

$$A^\dagger AHAA^d = H \quad and \quad AHA^k = A^k.$$

Proof. Proof is similar too Theorem 3.7. □

Maximal classes of complex matrices for which the representation of the Drazin-Moore-Penrose inverse of matrix is still valid are established in the next theorem.

Theorem 3.9. *Let $A, X \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent:*

i) $A^{d,\dagger} = XAA^\dagger$,

ii) $XA = A^dA$,

iii) $X = A^d + V(I_n - AA^\dagger)$, *with* $V \in \mathbb{C}^{n\times n}$.

Proof. *i*) ⇒ *ii*). Since $A = AA^\dagger A$, we have

$$XA = XAA^\dagger A = A^{d,\dagger}A = A^dA.$$

ii) ⇒ *iii*) Hence, by applying Theorem 1. ([2].p.52), we have

$$X = A^d + V(I_n - AA^\dagger),$$

with $V \in \mathbb{C}^{n\times n}$.

iii) ⇒ *i*). For arbitrary $V \in \mathbb{C}^{n\times n}$, we have

$$\begin{aligned} XAA^\dagger &= (A^d + V(I_n - AA^\dagger))AA^\dagger \\ &= A^dAA^\dagger + V(I_n - AA^\dagger)AA^\dagger \\ &= A^{d,\dagger}. \end{aligned}$$

Follows the desired result. □

If A represented as in (1) with $rank(A) = r$, the matrix X is in Theorem 3.9 for some $V \in \mathbb{C}^{n\times n}$ can be expressed as

$$X = U\begin{pmatrix} (\Sigma K)^d & Z_{12} \\ 0 & Z_{22} \end{pmatrix} U^*,$$

for arbitrary $Z_{12} \in \mathbb{C}^{r\times(n-r)}$ and $Z_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$. By making the following partition with blocks of adequate sizes

$$V = U\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} U^*.$$

From (2), we have $I_n - AA^\dagger = U\begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} U^*$. A direct computation gives

$$X = A^d + V(I_n - AA^\dagger) = U\begin{pmatrix} (\Sigma K)^d & ((\Sigma K)^d)^2 \Sigma L + X_{12} \\ 0 & X_{22} \end{pmatrix} U^*.$$

Since X_{12} and X_{22} are arbitrary, we have that

$$X = U\begin{pmatrix} (\Sigma K)^d & Z_{12} \\ 0 & Z_{22} \end{pmatrix} U^*,$$

for arbitrary $Z_{12} \in \mathbb{C}^{r\times(n-r)}$ and $Z_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$.

Theorem 3.10. *Let $A, X \in \mathbb{C}^{n\times n}$ with $ind(A) = k$. Then the following conditions are equivalent:*

i) $A^{d,\dagger} = A^d AX$,

ii) $A^k X = A^k A^\dagger$,

iii) $X = A^\dagger + (I_n - (A^k)^\dagger A^k)V$, *with* $V \in \mathbb{C}^{n\times n}$.

Proof. *i*) ⇒ *ii*). Pre-multiplying $A^{d,\dagger} = A^d AX$ by A^k we get

$$A^k AA^d X = A^k X = A^k A^\dagger.$$

ii) ⇒ *iii*) Hence, by applying Theorem 1. ([2].p.52), we have

$$X = A^\dagger + (I_n - (A^k)^\dagger A^k)V,$$

with $V \in \mathbb{C}^{n\times n}$.

iii) ⇒ *i*). For arbitrary $V \in \mathbb{C}^{n\times n}$, we have

$$\begin{aligned} A^d AX &= A^d AA^\dagger + A^d A(I_n - (A^k)^\dagger A^k)V \\ &= A^d AA^\dagger + (A^d)^k A^k (I_n - (A^k)^\dagger A^k)V \\ &= A^{d,\dagger}. \end{aligned}$$

This completes the proof. □

We will give the condition under which the Moore–Penrose inverse coincides with the DMP inverse.

Proposition 3.11. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1). *Then the following conditions are equivalent:*

i) $A^{d,\dagger} = A^\dagger$,

ii) A is EP matrix,

iii) $A^d = A^\dagger$.

Proof. The proof can be checked directly by (2). □

The following example shows that the Proposition 3.11 is false when we substitute $A^\dagger$ with A^d in *i*).

Example 3.12. *Let* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

We have that $A^d = A^{d,\dagger} = 0$ *and* $A^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *However,*

$$AA^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$A^\dagger A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $AA^\dagger \neq A^\dagger A$, *then* A *is not EP matrix.*

Remark 3.13. *Let* $A \in \mathbb{C}^{n\times n}$.

1. *If* A *is EP matrix, then* $A^{d,\dagger}$ *and* $A^{\dagger,d}$ *are a polynomial in* A.
2. $A^{d,\dagger}$ *is a polynomial in* A *if and only if* $A^{d,\dagger} = A^d$.

We now characterize the $A^{d,\dagger}$ and $A^{\dagger,d}$ from a characteristic polynomial of A. For $A \in \mathbb{C}^{n\times n}$ we denote the characteristic polynomial of A by χ_A.

Theorem 3.14. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $ind(A) = k$, *the characteristic polynomial of* A *be,*

$$\chi_A(X) = X^n + a_{n-1}X^{n-1} + ... + a_1 X.$$

Then,

i) $A^{d,\dagger} + a_{n-1}(A^{d,\dagger})^2 + ... + a_1(A^{d,\dagger})^n = 0.$

ii) $A^{\dagger,d} + a_{n-1}(A^{\dagger,d})^2 + ... + a_1(A^{\dagger,d})^n = 0.$

Proof. i) Its well known that,

$$A^d + a_{n-1}(A^d)^2 + ... + a_1(A^d)^n = 0. \tag{4}$$

Post-multiplying (4) by $AA^\dagger$ we get

$$A^d AA^\dagger + a_{n-1}(A^d)^2 AA^\dagger + ... + a_1(A^d)^n AA^\dagger = 0.$$

Since $(A^{d,\dagger})^m = (A^d)^m AA^\dagger$ for any nonnegative integer $m \geq 1$.Then

$$A^{d,\dagger} + a_{n-1}(A^{d,\dagger})^2 + ... + a_1(A^{d,\dagger})^n = 0.$$

ii) It is obvious by pre-multiplying (4) by $A^\dagger A$ we get

$$A^{\dagger,d} + a_{n-1}(A^{\dagger,d})^2 + ... + a_1(A^{\dagger,d})^n = 0.$$

This completes the proof. □

Next, we give an example to verify Theorem 3.14.

Example 3.15. *Let* $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.
It is easy to check that $ind(A) = 2$ *and*

$$A^{\dagger} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, A^{d} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^{d,\dagger} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad A^{\dagger,d} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have,

$$\chi_A(X) = X^4 - X^3.$$

Therefore,

$$(A^{d,\dagger})^3 - (A^{d,\dagger})^4 = A^{d,\dagger} - A^{d,\dagger} = 0,$$
$$(A^{\dagger,d})^3 - (A^{\dagger,d})^4 = A^{\dagger,d} - A^{\dagger,d} = 0.$$

Then the Theorem 3.14 is verifed.

Theorem 3.16. *Let* $A \in \mathbb{C}^{n\times n}$ *be of the form* (1) *with* $rank(A) = r$. *Then*

$$\chi_{A^{d,\dagger}}(X) = (-X)^{n-r}\chi_{(\Sigma K)^d}(X).$$

Proof. Let A be represented as in (1) with $rank(A) = r$. By $A^{d,\dagger} = U\begin{pmatrix} (\Sigma K)^d & 0 \\ 0 & 0 \end{pmatrix}U^*$. Thus, we get

$$\begin{aligned} \chi_{A^{d,\dagger}}(X) &= det(A^{d,\dagger} - XI_n) \\ &= \begin{vmatrix} (\Sigma K)^d - XI_r & 0 \\ 0 & -XI_{n-r} \end{vmatrix} \\ &= (-X)^{n-r}det((\Sigma K)^d - XI_r) \\ &= (-X)^{n-r}\chi_{(\Sigma K)^d}(X). \end{aligned}$$

□

4. The nonsingularity of $(A^{d,\dagger})^m + z(I_n - AA^{d,\dagger})$ and $A^m + z(I_n - AA^{d,\dagger})$ for nonnegative integer $m \geq 1$

Motivated by [16], which show the nonsingularity of some matrices.

Theorem 4.1. *Let* $A \in \mathbb{C}^{n\times n}$. *Denote* $P = I_n - AA^{d,\dagger}$. *Then the matrix* $(A^{d,\dagger})^m + zP$ *is nonsingular for* $z \in \mathbb{C} \setminus \{0\}$ *and* $m \in \mathbb{N} \setminus \{0\}$. *In addition, the following identity hold:*

$$A^{d,\dagger} = ((A^{d,\dagger})^m + zP)^{-1}(A^d)^m A^{\dagger}.$$

Proof. We take, $E = (A^{d,\dagger})^m$, $F = A^{m+1}A^{d,\dagger}$ and $H = I_n - EF$, then we have

- $EF = (A^{d,\dagger})^m A^{m+1}A^{d,\dagger} = AA^{d,\dagger}$, $(EF)^2 = EF$ and $H = I_n - AA^{d,\dagger} = P$.
- $EF^2 = AA^{d,\dagger}A^{m+1}A^{d,\dagger} = A^{m+1}A^{d,\dagger} = F$.
- $E^2F = (A^{d,\dagger})^m AA^{d,\dagger} = (A^{d,\dagger})^m = E$.

By Lemma 2.3, we deduce that $(A^{d,\dagger})^m + zP$ is nonsingular.
By $((A^{d,\dagger})^m + zP)A^{d,\dagger} = (A^{d,\dagger})^{m+1} + zA^{d,\dagger}P = (A^{d,\dagger})^{m+1}$. Then,

$$A^{d,\dagger} = ((A^{d,\dagger})^m + zP)^{-1}(A^d)^m A^\dagger.$$

□

Corollary 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and $P = I_n - AA^{d,\dagger}$. Then the matrix $A^{d,\dagger} + P$ is nonsingular. Furthermore the following identity hold:*

$$(A^{d,\dagger} + P)^{-1} = A^2A^{d,\dagger} + P.$$

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$. Denote $P = I_n - AA^{d,\dagger}$. Then the matrix $A^m + zP$ is nonsingular for $z \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{0\}$. In addition, the following identity hold:*

$$A^{d,\dagger} = (A^m + zP)^{-1}A^mA^{d,\dagger}.$$

Proof. We take, $E = A^m$, $F = (A^{d,\dagger})^m$ and $H = I_n - EF$, then we have

- $EF = A^m(A^{d,\dagger})^m = AA^{d,\dagger}$, $(EF)^2 = EF$ and $H = I_n - AA^{d,\dagger} = P$.
- $EF^2 = A^m(A^{d,\dagger})^{2m} = A(A^{d,\dagger})^{m+1} = (A^{d,\dagger})^m = F$.
- The matrix $EH = A^m - A^{m+1}(A^{d,\dagger})$ is nilpotent (by Lemma 2.2).

By Lemma 2.4, we deduce that $A^m + zP$ is nonsingular.
By $(A^m + zP)A^{d,\dagger} = A^mA^{d,\dagger} + zPA^{d,\dagger} = A^mA^{d,\dagger}$. Then,

$$A^{d,\dagger} = (A^m + zP)^{-1}A^mA^{d,\dagger}.$$

□

Remark 4.4. *Let $A \in \mathbb{C}^{n\times n}$. If $m \geq ind(A)$, we have*

$$A^{d,\dagger} = (A^m + zP)^{-1}A^mA^\dagger.$$

Corollary 4.5. *Let $A \in \mathbb{C}^{n\times n}$, $P = I_n - AA^{d,\dagger}$, $z \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{0\}$. Then the following identity hold,*

$$A^{d,\dagger} = A^mA^d(A^m + zP)^{-1}.$$

Proof. Notice that the matrix $A^m + zP$ is nonsingular and by, $A^{d,\dagger}(A^m + zP) = A^{d,\dagger}A^m + zA^{d,\dagger}P = A^{d,\dagger}A^m = A^mA^d$. Therefore,

$$A^{d,\dagger} = A^mA^d(A^m + zP)^{-1}.$$

□

Remark 4.6. *Let $A \in \mathbb{C}^{n\times n}$. If $m \geq ind(A) + 1$, we have*

$$A^{d,\dagger} = A^m(A^m + zP)^{-1}.$$

Theorem 4.7. *Let $A \in \mathbb{C}^{n\times n}$. Then the matrix $(A^{d,\dagger})^m + zA^\pi$ is nonsingular for $z \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{0\}$. In addition, the following identity hold:*

$$A^{d,\dagger} = ((A^{d,\dagger})^m + zA^\pi)^{-1}(A^d)^mA^\dagger.$$

Proof. We take $E = (A^{d,\dagger})^m$, $F = A^{m+1}A^d$ and $H = I_n - EF$, then we have

- $EF = A^{d,\dagger}A^2A^d = A^dA$, $(EF)^2 = EF$ and $H = I_n - EF = A^\pi$,

- $EF^2 = A^dAA^{m+1}A^d = A^{m+1}A^d = F$.
- The matrix $EH = (A^{d,\dagger})^m A^\pi$ is nilpotent.

By Lemma 2.4, we deduce that $(A^{d,\dagger})^m + zA^\pi$ is nonsingular.
By, $((A^{d,\dagger})^m + zA^\pi)A^{d,\dagger} = (A^{d,\dagger})^{m+1} + zA^\pi A^{d,\dagger} = (A^{d,\dagger})^{m+1}$.
Therefore,

$$A^{d,\dagger} = ((A^{d,\dagger})^m + zA^\pi)^{-1}(A^d)^m A^\dagger.$$

□

Corollary 4.8. *Let $A \in \mathbb{C}^{n\times n}$. Then the matrix $A^{d,\dagger} + A^\pi$ is nonsingular. In addition, the following identity hold:*

$$A^{d,\dagger} = (A^{d,\dagger} + A^\pi)^{-1}A^dA^\dagger.$$

Theorem 4.9. *Let $A \in \mathbb{C}^{n\times n}$ be of the form (1) and $z \in \mathbb{C}$ such that $z(z+1) \neq 0$. Then the matrix $AA^{d,\dagger} + zI_n$ is nonsingular and*

$$(AA^{d,\dagger} + zI_n)^{-1} = \frac{-1}{z(z+1)}AA^{d,\dagger} + \frac{1}{z}I_n.$$

Proof. Let A be represented as in (1) with $rank(A) = r$ and $z \in \mathbb{C}$ such that $z(z+1) \neq 0$. Applying (2), then

$$AA^{d,\dagger} + zI_n = U\begin{pmatrix} (\Sigma K)(\Sigma K)^d + zI_r & 0 \\ 0 & zI_{n-r} \end{pmatrix}U^*.$$

By Lemma 2.5, the matrix $(\Sigma K)(\Sigma K)^d + zI_r$ is nonsingular, we deduce that $AA^{d,\dagger} + zI_n$ is nonsingular. In this case,

$$(AA^{d,\dagger} + zI_n)^{-1} = \frac{-1}{z(z+1)}AA^{d,\dagger} + \frac{1}{z}I_n.$$

□

Next, we give an example to verify Theorem 4.1, Theorem 4.3 and Theorem 4.7.

Example 4.10. *Let* $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It is easy to check that $ind(A) = 3$. The Moore-Penrose inverse $A^\dagger$ and the Drazin inverse A^d, the DMP inverse $A^{d,\dagger}$ are respectively,

$$A^\dagger = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^d = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A^{d,\dagger} = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $z \in \mathbb{C}^*$ we can get

$$(A^{d,\dagger})^3 + zP = \begin{pmatrix} 1 & 1-z & 2-2z & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, (A^{d,\dagger})^3 + zA^\pi = \begin{pmatrix} 1 & 1-z & 2-2z & -2z \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}$$

and $A^3 + zP = \begin{pmatrix} 1 & 1-z & 2-2z & 2 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}$.

Notice that the matrices $(A^{d,\dagger})^3 + zP$, $(A^{d,\dagger})^3 + zA^\pi$ and $A^3 + zP$ are nonsingular for $z \neq 0$.

Proposition 4.11. *Let $A \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C} \setminus \{0\}$. Then*

i) $A^{d,\dagger} = (A + zA^{\pi})^{-1}AA^{d,\dagger}$.
ii) $A^{\dagger,d} = A^{\dagger,d}A(A + zA^{\pi})^{-1}$.

Proof. Notice that the matrix $A + zA^{\pi}$ is nonsingular.
By, $(A + zA^{\pi})A^{d,\dagger} = AA^{d,\dagger} + zA^{\pi}A^{d,\dagger} = AA^{d,\dagger}$ and $A^{\dagger,d}(A + zA^{\pi}) = A^{\dagger,d}A + zA^{\dagger,d}A^{\pi} = A^{\dagger,d}A$.
Therefore,

$$A^{d,\dagger} = (A + zA^{\pi})^{-1}AA^{d,\dagger},$$
$$A^{\dagger,d} = A^{\dagger,d}A(A + zA^{\pi})^{-1}.$$

□

5. Applications of the DMP inverse

In this section, we apply the DMP inverse and dual DMP inverse of matrix $A \in \mathbb{C}^{n\times n}$ in solving some systems of linear equations.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$, $z \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}^n$ with ind$(A) = k \leq m$. Then the system*

$$(A^m + zA^{\pi})x = A^mA^{\dagger}b,$$

has a unique solution given by

$$x = A^{d,\dagger}b.$$

Proof. Since the matrix $(A^m + zA^{\pi})$ is nonsingular with $(A^m + zA^{\pi})^{-1} = (A^d)^m + \frac{1}{z}A^{\pi}$, then the system has a unique solution given by

$$x = (A^m + zA^{\pi})^{-1}A^mA^{\dagger}b = A^dAA^{\dagger}b = A^{d,\dagger}b.$$

□

Theorem 5.2. *Let $A \in \mathbb{C}^{n\times n}$ with ind$(A) = k \leq m$ and $b \in \mathbb{C}^n$. Then the system,*

$$A^mx = A^mA^{\dagger}b,$$

is consistent and its general solution is given by

$$x = A^{d,\dagger}b + A^{\pi}y,$$

where $y \in \mathbb{C}^n$ is arbitrary. Moreover,

$$x = A^{d,\dagger}b$$

is the unique solution to the system in $\mathcal{R}(A^k)$.

Proof. For $x = A^{d,\dagger}b + A^{\pi}y$, we have

$$A^mx = A^m(A^{d,\dagger}b + A^{\pi}y) = A^mA^{d,\dagger}b + A^mA^{\pi}y = A^mA^{d,\dagger}b = A^mA^{\dagger}b.$$

So, x is a solution of $A^mx = A^mA^{\dagger}b$.

Conversly, assume x satisfies $A^mx = A^mA^{\dagger}b$. Then,

$$A^dAx = (A^dA)^mx = (A^d)^mA^mA^{\dagger}b = A^dAA^{\dagger}b = A^{d,\dagger}b,$$

so that,

$$x = A^{d,\dagger}b + x - A^dAx = A^{d,\dagger}b + A^{\pi}x,$$

which is of the form given.

To prove that $x = A^{d,\dagger}b$ is the unique solution to the system in $\mathcal{R}(A^k)$, suppose that there is another solution $x_1 \in \mathcal{R}(A^k)$ of the system, then, $x - x_1 \in \mathcal{N}(A^k)$, therefore,

$$x - x_1 \in \mathcal{R}(A^k) \cap \mathcal{N}(A^k) = \{0\}.$$

That is,

$$x = x_1.$$

□

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$y = e^{-A^{\dagger,d}t}v,$$

is a solution of $Ax' + AA^dx = 0$. For every column vector $v \in \mathbb{C}^n$.

Proof. Let $y = e^{-A^{\dagger,d}t}v$. Then

$$\begin{aligned} Ay' &= A(-A^{\dagger,d})e^{-A^{\dagger,d}t}v \\ &= -A^dAe^{-A^{\dagger,d}t}v \\ &= -A^dAy. \end{aligned}$$

as desired. □

Theorem 5.4. *Let $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^n$ such that $Av = v$. Then,*

$$y = e^{-A^{\dagger,d}t}\int e^{A^{\dagger,d}t}v(t)dt$$

is a particular solution of $Ax' + AA^dx = v$.

References

[1] O. M. Baksalary, G. Trenkler, *On the projectors $FF^\dagger$ and $F^\dagger F$*, Appl. Math. Comput. **217** (2011), 10213-10223.
[2] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd Edition. Springer Verlag, New York. (2003).
[3] S.L. Campbell, C.D. Meyer, *Generalized Inverse of Linear Transformations (Siam)*, USA: North Carolina State University Raleigh, North Carolina. (1979).
[4] S.L. Campbell, C.D. Meyer, N.J. Rose, *Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients*, J. App. Math. (1976), 411-425.
[5] D.E. Ferreyra, F.E. Levis, N.Thome, *Maximal classes of matrices determining generalized inverses*, Appl. Math. Comput. **333** (2018), 42–52.
[6] R.E. Hartwig, K. Spindelbock, *Matrices for which A^* and $A^\dagger$ conmmute*, Linear Multilinear Algebra. **14** (1984), 241-256.
[7] M.V. Hernndez, M.B. Lattanzi, N. Thome, *GDMP-inverses of a matrix and their duals*, Linear Multilinear Algebra. **70** (2020), 3923-3935.
[8] S.B. Malik, N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Appl. Math. Comput. **226** (2014), 575-580.
[9] L. Meng, *The DMP inverse for rectangular matrices*, Filomat. **31** (2017), 6015–6019.
[10] D. Mosić, *Weighted gDMP inverse of operators between Hilbert spaces*, Bull. Korean Math. Soc, **55** (2018) , 1263–1271.
[11] D. Mosić, *Drazin-Star and Star-Drazin Matrices*, Results Math. **75** (2020), 1-21.
[12] D. Mosić, D.S. Djordjević, *The gDMP inverse of Hilbert space operators*, J. Spectr. Theory. **8** (2018), 555–573.
[13] D, Mosić, P.S. Stanimirović, S.D. Mourtas, *Minimal Rank Properties of Outer Inverses with Prescribed Range and Null Space*, Mathematics. **11** (2023), 1732. https://doi.org/10.3390/math11071732
[14] M. Moucouf, *P-canonical forms and Drazin inverses of matrices*, Filomat. **38** (2024), 2399–2417.
[15] F.P. Romo, *On the coincidence of the Drazin inverse and the Drazin-Moore-Penrose inverses*, Linear Multilinear Algebra. **70** (2020), 3617-3632. 10.1080/03081087.2020.1849004.
[16] J. Wu, K. Zuo, J. Gao, L. Zhengb, *Characterizations and properties of the matrices A such that $AA^{-1}_{(L)} - A^{-1}_{(L)}A$ are nonsingular*, Filomat. **37** (2023), 5333–5346.
[17] H. Zhu, *On DMP inverses and m-EP elements in rings*, Linear Multilinear Algebra. **67** (2018).
[18] S. Zriaa, M. Mouçouf, *The explicit formulas of the Drazin inverse of matrices and its nth powers*, Rend. Circ. Mat. Palermo (2). **2** (2023).
[19] K.Z. Zuo, D.S. Cvetković Ilić, Y.J. Chen, *Different characterizations of DMP-inverse of matrices*, Linear Multilinear Algebra. **70** (2020), https://doi.org/10.1080/03081087.2020.1729084.